Convex Optimization EE5121 - Assignment

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1 Introduction

The report contains the optimization problem formulation as well as the optimal values for each of the problems in the given assignment. The code has been written in python with the cvxpy library. Please install cvxpy according to the instructions given here. The terminal outputs for each problem have been given at the end of the report.

2 Recovering a piecewise constant signal from its noisy measurement

We have been given a noisy signal and we want to reconstruct the original signal which is known to be a piecewise constant function.
Define $e := y - \tilde{x}$ which is the error between the estimated data $\tilde{x}$ and the data received y by us. We want to minimize the squared error, i.e., the 2-norm of e and to have as few jumps in the estimated signal $\tilde{x}$ as possible(not more than 20). This problem can be formulated in the epigraph form as follows:

$$\text{minimize } t$$

$$\text{subject to } w_1||y - \tilde{x}||_2 \leq w_1 t - w_2||A\tilde{x}||_1 + 20w_2.$$

where,

$$A = \begin{bmatrix} -1 & 1 & 0 & \dots & 0 & 0 \\ 0 & -1 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & -1 & 1 \end{bmatrix}$$

Here, $A\tilde{x}$ gives a vector which is the difference between the consecutive values of the estimated signal $\tilde{x}$. Thus if we evaluate the cardinality of $A\tilde{x}$ we will get the number of jumps in the estimated signal $\tilde{x}$. We approximate $card(A\tilde{x})$ with $||A\tilde{x}||_1$ to bring it in the convex form.

The optimal value for the mean squared error obtained was '14.175727791108283'.

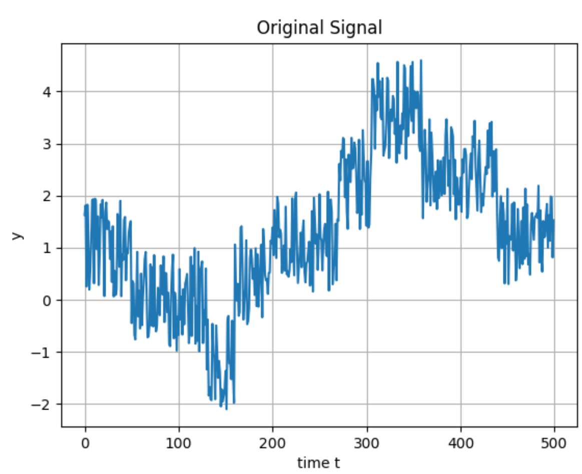


Figure 1: The original signal given

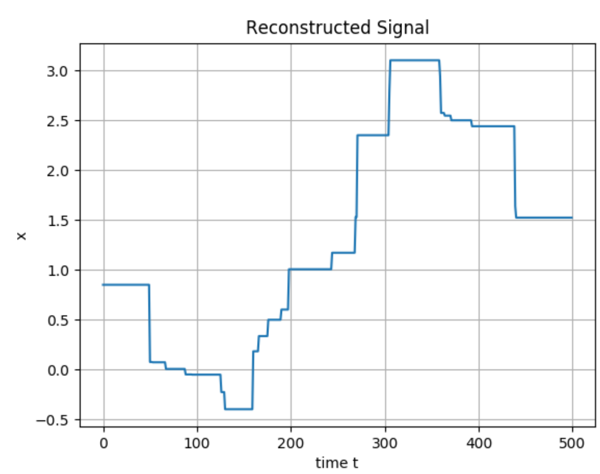


Figure 2: The reconstructed signal which is approximately piecewise constant

3 Activity Levels with discounts

With the given data:

$$r_j = \begin{cases} p_j x_j, & 0 \leq x_j \leq q_j \\ p_j q_j + p_j^{disc}(x_j - q_j), & x_j \geq q_j \end{cases}$$

We have to maximize $\sum_{j=1}^{n} r_j$ subject to: $x \succeq 0$ and $c^{max} \succeq Ax$ where $x = [x_1, x_2, \dots, x_n]$.

With the given formulation of r_j, we have $r_j = \min\left(p_j x_j, p_j q_j + p_j^{disc}(x_j - q_j)\right) \forall j \in (1, 2, \dots, n)$.

Therefore we have r_j concave in x_j. So we have,$r_j \geq u_j$ if and only if $p_j x_j \geq u_j$, $p_j q_j + p_j^{disc}(x_j - q_j) \geq 0$.

So now we can form the Linear Program as:

$$\text{maximize } \mathbf{1}^T u$$

$$\text{subject to } x \succeq 0$$

$$c^{max} \succeq Ax$$

$$p_j x_j \geq u_j, \;\; p_j q_j + p_j^{disc}(x_j - q_j) \geq u_j, \;\; j = 1, 2, ..., n$$

Solving the above Linear Program with the given data we get,

The optimal activity levels $x_{opt} = [4.00, 22.50, 31.00, 1.50]$

The optimal revenues generated from each of the activity $r_{opt} = [12.00, 32.50, 139.00, 9.00]$

The total revenue $R_{tot} = 192.50$

The average price for each of the activity $avg_{opt} = [3.00, 1.44, 4.48, 6.00]$.

Comments on the activity levels obtained: The 3rd activity level is the highest because it has the highest basic price as well as a high discounted price. The contribution from the 3rd activity is also the highest. The 4th activity level is the lowest because it has a relatively high basic price and low discount price.

4 Matrix Completion Problem: Netflix Data

We have been given a matrix X where the rows correspond to the users and the columns correspond to the movies. We have to complete the given matrix. The original optimization problem can be written as:

$$\text{minimize } rank(\tilde{X})$$

$$\text{subject to } \tilde{X}_{ij} = X_{ij} \forall (i,j) \in J$$

where J is the set of indices with known elements of X with $|J| <$No. of users $\times$ No. of movies.

We use the following theorem:
Given $X \in \mathcal{R}^{m \times n}$, $rank(X) \leq r$ if and only if $\exists Y \in \mathbf{S}^m$ and $Z \in \mathbf{S}^n$ such that $rank(Y) + rank(Z) \leq 2r$ and $\left[\begin{smallmatrix} Y & X \\ X^T & Z \end{smallmatrix}\right] \succeq 0$, where Y and Z are symmetric matrices.

With the convex relaxation the problem can be formulated as:

$$\underset{Y, Z}{\text{minimize}} \; r$$

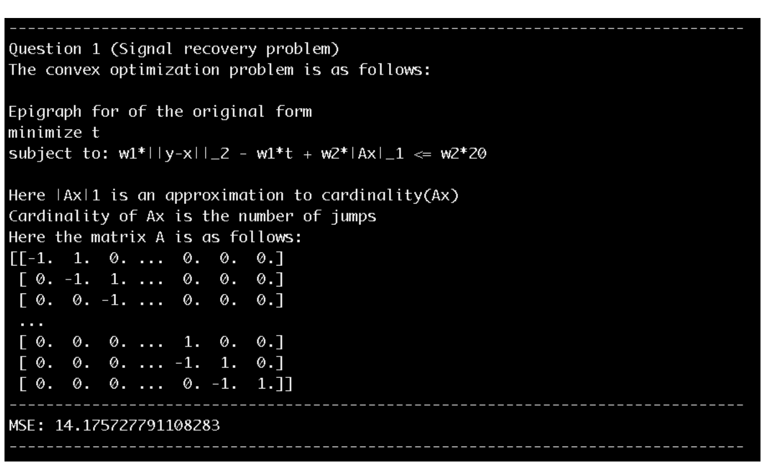
$$\text{subject to } \tilde{X}_{ij} = X_{ij} \;\; \forall (i,j) \in J$$

$$trace(Y) + trace(Z) \leq 2r$$

$$\left[\begin{smallmatrix} Y & X \\ X^T & Z \end{smallmatrix}\right] \succeq 0$$

The rank of the completed matrix is 19 which is equal to half of the number of movies.
Note: The matrices Y and Z are constrained to by positive semidefinite(PSD) matrices because the trace approximation to rank holds only when the matrix is PSD. Also the matrix obtained after filling, doesn't have integral values for the original ratings. This is because of the gradient based solvers in cvxpy.

5 Terminal outputs



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------------------------------------------------------------------------------
Question 1 (Signal recovery problem)
The convex optimization problem is as follows:

Epigraph for of the original form
minimize t
subject to: w1*||y-x||_2 - w1*t + w2*|Ax|_1 <= w2*20

Here |Ax|1 is an approximation to cardinality(Ax)
Cardinality of Ax is the number of jumps
Here the matrix A is as follows:
[[-1.  1.  0. ...  0.  0.  0.]
 [ 0. -1.  1. ...  0.  0.  0.]
 [ 0.  0. -1. ...  0.  0.  0.]
 ...
 [ 0.  0.  0. ...  1.  0.  0.]
 [ 0.  0.  0. ... -1.  1.  0.]
 [ 0.  0.  0. ...  0. -1.  1.]]
------------------------------------------------------------------------------
MSE: 14.175727791108283
------------------------------------------------------------------------------
```

Figure 3: Terminal output for question 1

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Question 2 (Activity Level Problems)
The convex optimization problem is as follows:

maximize 1T.u
subject to:

x >= 0
Ax <= c_max
px >= u
pq + p_disc(x-q) >= u
-----------------------------------------------------------------------------
The maximum revenue is 192.5
The optimal activity levels are:
[ 4.  22.5 31.   1.5]
The optimal revenues from each activity are:
[ 12.   32.5 139.    9. ]
The optimal average price per unit for each activity level are:
[3.         1.44444444 4.48387097 6.        ]
-----------------------------------------------------------------------------
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Figure 4: Terminal output for question 2

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Question 3 (Netflix Problem)
Ratings matrix is of the shape: (50, 38)
-----------------------------------------------------------------------------
The relaxed convex optimisation problem is as follows:

minimize: r
subject to:
Xij = Mij for all i,j in W(set of known elements in the Netflix data)
trace(Y)+trace(Z) <= 2*r
[Y   X]
[X.T Z] > 0
-----------------------------------------------------------------------------
Solving the problem...
-----------------------------------------------------------------------------
The optimal solution X is
[[2.0000142  1.00001419 1.00001419 ... 2.45128908 3.99997    1.00001419]
 [0.91754905 0.79326562 0.79326562 ... 1.37433838 1.11373712 0.79326562]
 [1.99998505 0.99998662 0.99998662 ... 1.00000645 2.99999835 0.99998662]
 ...
 [1.99999683 1.05268795 1.05268795 ... 3.99997578 1.9999925  1.05268795]
 [1.99999949 0.69905613 0.69905613 ... 0.99998317 1.11864372 0.69905613]
 [1.00000728 0.99998449 0.99998449 ... 1.36970703 1.99999096 0.99998449]]

Rank of the optimal matrix of X is: 19
-----------------------------------------------------------------------------
```

Figure 5: Terminal output for question 3

6 Errors and debugging

If you get the following error please check your installation of cvxpy once again or change the solver type. cvxpy.error.Solvererror: Solver 'SCS' failed. Try another solver with verbose=True for more information. Try recentering the problem data around 0 and rescaling to reduce the dynamic range.

7 References

Stephen Boyd and Lieven Vandenberghe. Convex Optimization. Cambridge University Press