Semistability of Nonlinear Dynamical Systems

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1 Introduction

The notion of *Semistability* was introduced for the cases where there is a continuum of equilibrium. As in these cases every neighbourhood of a non-isolated equilibrium point contains another non-isolated equilibrium point and hence they cannot be asymptotically stable. To characterize these points two notions of *convergence* and *semistability* are used. Convergence is the property where every system solution converges to a limit point which may depend on the initial condition of the system. Semistability requires an additional condition that all solutions should converge to limit points that are *Lyapunov* stable.
Important Note: Semistability $\not\equiv$ Asymptotic Stability. i.e. it is possible to converge to a set of equilibria without converging to any one equilibrium point. Semistability lies in between Lyapunov stability and asymptotically stable equilibrium in the sense that an asymptotically stable equilibrium is also semistable, while a semistable equilibrium is also Lyapunov stable.

2 Necessary and sufficient conditions for semistability

Consider nonlinear dynamical systems $\mathcal{G}$ of the form :

$$\dot{x}(t) = f(x(t)), \qquad x(0) = x_0, \qquad t \in \mathcal{I}_{x_0}, \tag{1}$$

where $x(t) \in \mathcal{D} \subseteq \mathbb{R}^n$, is the system state vector, $\mathcal{D}$ is an open set, $f : \mathcal{D} \to \mathbb{R}^n$ is Lipschitz continuous on $\mathcal{D}, f^{-1}(0) \triangleq x \in \mathcal{D} : f(x) = 0$ is non-empty, and $\mathcal{I}_{x_0} = [0, \tau_{x_0}), 0 \leq \tau_{x_0} \leq \infty$, is the maximum interval of existence for the solution $x(.)$ of (1). Here we assume that for every initial condition $x_0 \in \mathcal{D}$, (1) has a unique solution defined on $[0, \infty)$, and hence, the solutions of (1) define a continuous global semiflow on $\mathcal{D}$.

Definition: An equilibrium point $x \in \mathcal{D}$(of(1)) is *semistable* if it is *Lyapunov* stable and there exists an open subset $\mathcal{Q}$ of $\mathcal{D}$ containing x such that for all initial conditions in $\mathcal{Q}$, the trajectory of (1) converges to a *Lyapunov* stable equilibrium point, that is, $\lim_{t\to\infty} s(t, x) = y$, where $y \in \mathcal{D}$ is a *Lyapunov* stable equilibrium point of (1) and $x \in \mathcal{Q}$

Definition: The domain of semistability is the set of points $x_0 \in \mathcal{D}$ such that if $x(t)$ is a solution to (1) with $x(0) = x_0, t \geq 0$, then $x(t)$ converges to a *Lyapunov* stable equilibrium point in $\mathcal{D}$.

Theorem 1: Consider the nonlinear dynamical system (1). Let $\mathcal{Q}$ be an open neighbourhood of $f^{-1}(0)$ and assume that there exists a continuously differentiable function $V : \mathcal{Q} \to \mathbb{R}$ such that:

$$V'(x)f(x) < 0, \qquad x \in \mathcal{Q} \setminus f^{-1}(0) \tag{2}$$

If (1) is *Lyapunov* stable then (1) is *semistable*.
Proof: We assume (1) is Lyapunov stable. Therefore, for every $z \in f^{-1}(0)$, there exists an open neighborhood $\mathcal{V}_z$ of z such that $s([0, \infty) \times \mathcal{V}_z)$ is bounded and contained in $\mathcal{Q}$. The set $\mathcal{V} \triangleq \bigcup_{z\in f^{-1}(0)} \mathcal{V}_z$ is an open neighborhood of $f^{-1}(0)$ contained in $\mathcal{Q}$. Consider $x \in \mathcal{V}$ so that there exists $z \in f^{-1}(0)$ such that $x \in \mathcal{V}_z$ and $s(t, x) \in \mathcal{V}_z, t \geq 0$. Since $\mathcal{V}_z$ is bounded it follows that the positive limit set of x is nonempty and invariant. Furthermore, it follows from (2) that $\dot{V}(s(t, x)) \leq 0, t \geq 0$, and hence, it follows from LaSalle's Theorem that $s(t, x) \to \mathcal{M}$ as $t \to \infty$, where $\mathcal{M}$ is the largest invariant set contained in the set $\mathcal{R} = \{y \in \mathcal{V}_z : V'(y)f(y) = 0\}$. Also, $\mathcal{R} = f^{-1}(0)$ is invariant, and hence, $\mathcal{M} = \mathcal{R}$, which implies that $\lim_{t\to\infty} dist(s(t, x), f^{-1}(0)) = 0$. Since, every point in $f^{-1}(0)$ is Lyapunov stable, $\lim_{t\to\infty} s(t, x) = x^*$, where $x^* \in f^{-1}(0)$ is Lyapunov stable. Hence, by definition (1) is semistable.

Below, is a more generalized version of the semistability Theorem 1 where we do not assume that all points in $\dot{V}^{-1}(0)$ are Lyapunov stable but rather we assume that all points in the largest invariant subset of $\dot{V}^{-1}(0)$ are Lyapunov stable.

Theorem 2: Consider the nonlinear dynamical system (1) and let $\mathcal{Q}$ be an open neighborhood of $f^{-1}(0)$. Suppose the orbit $\mathcal{O}_x$ of (1) is bounded for all $x \in \mathcal{Q}$ and assume that there exists a continuously differentiable function $V : \mathcal{Q} \to \mathbb{R}$ such that,

$$V'(x)f(x) \leq 0, \qquad x \in \mathcal{Q} \tag{3}$$

If every point in the largest invariant subset $\mathcal{M}$ of $\{x \in \mathcal{Q} : V'(x)f(x) = 0\}$ is Lyapunov stable, then (1) is semistable.
Proof: Since every solution of (1) is bounded, it follows from the hypotheses on $V(.)$ that, for every $x \in \mathcal{Q}$, the positive limit set $\omega(x)$ of (1) is nonempty and contained in the largest invariant subset $\mathcal{M}$ of $\{x \in \mathcal{Q} : V'(x0f(x) = 0\}$.

Since every point in $\mathcal{M}$ is a Lyapunov stable equilibrium, it follows that $\omega(x)$ contains a single point for every $x \in \mathcal{Q}$ and $\lim_{t\to\infty} s(t,x)$ exists for every $x \in \mathcal{Q}$. Now, since $\lim_{t\to\infty} s(t,x) \in \mathcal{M}$ is Lyapunov stable for every $x \in \mathcal{Q}$, we also get the system to be semistable.

3 Examples of semistable equilibrium

Example 1: Consider the nonlinear dynamical system given by:

$$\dot{x_1}(t) = \sigma_{12}(x_2(t)) - \sigma_{21}(x_1(t)), \qquad x_1(0) = x_{10}, \qquad t \geq 0, \tag{4}$$

$$\dot{x_2}(t) = \sigma_{21}(x_1(t)) - \sigma_{12}(x_2(t)), \qquad x_2(0) = x_{20}, \tag{5}$$

where $x_1, x_2 \in \mathbb{R}, \sigma_{ij}(.), i, j = 1, 2, i \neq j$, are Lipschitz continuous, $\sigma_{12}(x_2) - \sigma_{21}(x_1) = 0$ if and only if $x_1 = x_2$, and $(x_1 - x_2)(\sigma_{12}(x_2) - \sigma_{21}(x_1)) \leq 0, x_1, x_2 \in \mathbb{R}$. Note that $f^{-1}(0) = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 = x_2 = \alpha, \alpha \in \mathbb{R}\}$.
We will show that the system governed by the equations (4) and (5) is semistable.
Consider a Lyapunov function candidate: $V(x_1, x_2) = \dfrac{1}{2}(x_1 - \alpha)^2 + \dfrac{1}{2}(x_2 - \alpha)^2$, where $\alpha \in \mathbb{R}$.
Differentiating along the trajectories of the system,

$$\begin{aligned}
\dot{V}(x_1, x_2) &= (x_1 - \alpha)[\sigma_{12}(x_2) - \sigma_{21}(x_1)] + (x_2 - \alpha)[\sigma_{21}(x_1) - \sigma_{12}(x_1)] \\
&= x_1[\sigma_{12}(x_2) - \sigma_{21}(x_1)] + x_2[\sigma_{21}(x_1) - \sigma_{12}(x_1)] \\
&= (x_1 - x_2)[\sigma_{12}(x_2) - \sigma_{21}(x_2)] \\
&\leq 0, (x_1, x_2) \in \mathbb{R} \times \mathbb{R}
\end{aligned}$$

This implies that $x_1 = x_2 = \alpha$ is Lyapunov stable.
Let, $\mathbb{R} \triangleq \{(x_1, x_2) \in \mathbb{R}^2 : \dot{V}(x_1, x_2) = 0\} = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 = x_2 = \alpha, \alpha \in \mathbb{R}\}$. Since $\mathbb{R}$ consists of equilibrium points, it follows that $\mathbb{M} = \mathbb{R}$. Hence, it follows from Theorem 2 that $x_1 = x_2 = \alpha$ is semistable for all $\alpha \in \mathbb{R}$.

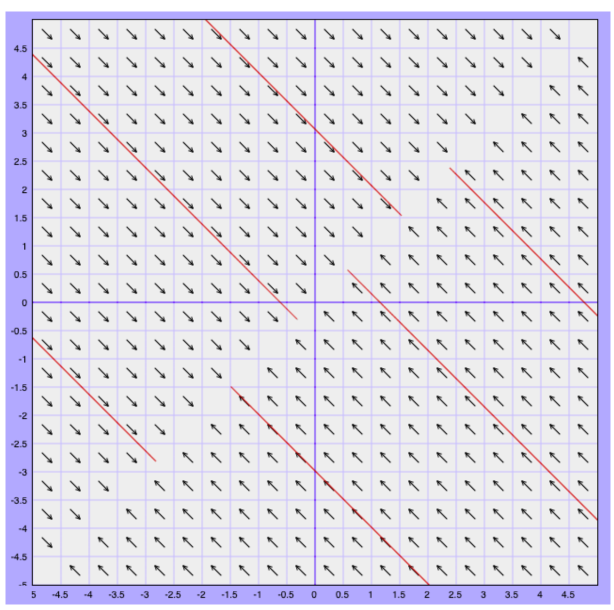

Figure 1: Phase portrait for system of equations (4) and (5) with $\sigma_{21}(x_1(t)) = x_1(t)$ and $\sigma_{12}(x_2(t)) = x_2(t)$. The horizontal axis is x_1 and vertical axis is x_2. The line $x_1 = x_2$ is a continuum of equilibrium points. Depending on the initial conditions, the system will converge at different equilibrium points(shown by the red lines).

Example 2: Consider the following dynamical system: $\dot{y}(t) = f(y(t))$, where $f : \mathbb{R}^2 \to \mathbb{R}^2$ is a continuous vector field given by,

$$f(x) = sign(x_1^2 + x_2^2 - 1)|x_1^2 + x_2^2 - 1|^{\alpha} f_r(x) + sign(x_1^2 + x_2^2 - 1)|x_1^2 + x_2^2 - 1|^{\beta} f_\theta(x)$$

with $\alpha, \beta \geq 1$ and the vector fields f_r and f_θ given by,

$$f_r(x) = \begin{bmatrix} -x_1 \\ -x_2 \end{bmatrix}, f_\theta(x) = \begin{bmatrix} x_2 \\ -x_1 \end{bmatrix}$$

The vector fields f_r and f_θ point in the radial and circumferential directions, respectively, and thus the parameters α and β determine the rates at which solutions move in these direction, respectively. By writing the above two equations in polar form $r = \sqrt{x_1^2 + x_2^2}$ and $\theta = \arctan(x_2/x_1)$:

$$\dot{r} = -r sign(r^2 - 1)|r^2 - 1|^{\alpha} \tag{6}$$

$$\dot{\theta} = -sign(r^2 - 1)|r^2 - 1|^{\beta} \tag{7}$$

From equation (6) and (7), we can see that the set of equilibria $f^{-1}(0)$ consists of the origin $x = 0$ and the unit circle $S = \{x \in \mathcal{R}^2 : x_1 + x_2 = 1\}$. All solutions of the system starting from nonzero initial conditions $y(0)$ that are not on the unit circle approach the unit circle. Solutions starting outside the unit circle spiral in clockwise toward the unit circle, while solutions starting inside the unit circle spiral out counterclockwise. Consequently, all solutions are bounded, and, for every choice of α and β, all solutions converge to the set of equilibria. However, if $\alpha \geq \beta+1$, then the system is not convergent. This can be seen by using (6) and (7) to obtain

$$\frac{dr}{d\theta} = r|r^2 - 1|^{\alpha-\beta} \tag{8}$$

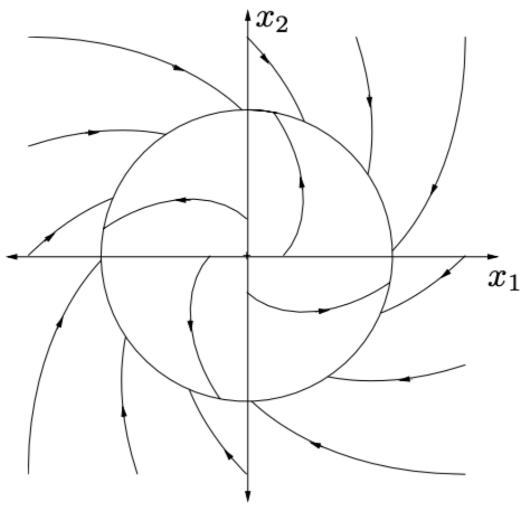

Figure 2: Phase portrait for system of equations (6) and (7) with $\alpha = \beta = 1$.

For initial values $r(0) > 1$, solutions of (8) converge to the equilibrium value $r = 1$ for decreasing θ, while for initial values $r(0) < 1$, solutions converge to $r = 1$ for increasing θ. For $\alpha \geq \beta + 1$, the right side of (8) is locally Lipschitz in r. Thus, (8) has a unique maximally defined solution given by $r \equiv 1$ for the initial condition $r(0) = 1$ for increasing as well as decreasing values of θ. For $\alpha = \beta = 1$ the trajectories approach the unit circle nontangentially and the system is convergent and all equilibria points on the unit circle are semistable. In a general case, the system defined by (5) and (6) are semistable if $\alpha \leq \beta$.

4 References

Wassim M. Haddad, Vijaysekhar Chellaboina. Nonlinear Dynamical Systems and Control: A Lyapunov-based Approach(2008)

Nontangency-based Lyapunov tests for convergence and stability in systems having a continuum of equilibria. Sanjay P. Bhat, Dennis S. Bernstein. Proceedings of the American Control Conference, Arlington, VA, 2001

Phase portrait Plotter: https://aeb019.hosted.uark.edu/pplane.html

5 Bibliography:

LaSalle's Theorem: Consider a dynamical system:

$$\dot{x}(t) = f(x(t)), \qquad x(0) = x_0, \qquad t \in \mathcal{I}_{x_0} \tag{8}$$

assume that $\mathcal{D}_c \subseteq \mathcal{D}$ is a compact positively invariant set with respect to (8) and assume that there exists a continuously differentiable function $V : \mathcal{D}_c \to \mathbb{R}$ such that $V'(x)f(x) \leq 0, x \in \mathcal{D}_c$. Let $\mathcal{R} \triangleq \{x \in \mathcal{D}_c : V'(x)f(x) = 0\}$ and let $\mathcal{M}$ be the largest invariant set contained in $\mathcal{R}$. If $x(0) \in \mathcal{D}_c$, then $x(t) \to \mathcal{M}$ as $t \to \infty$.