Assignment 1

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1 Question 1:

1.1 Part A:

$\|f(x^*) - f(x)\| \leq \alpha \|x^* - x\| \rightarrow$ contraction mapping definition
$\|(f(x^*) - x) - (f(x) - x)\| \leq \alpha \|x^* - x\|$
$\big| \|f(x^*) - x\| - \|f(x) - x\| \big| \leq \|(f(x^*) - x) - (f(x) - x)\| \leq \alpha \|x^* - x\|$
$\therefore -\alpha \|x^* - x\| \leq \|f(x^*) - x\| - \|f(x) - x\| \leq \alpha \|x^* - x\|$
$\therefore -\alpha \|x^* - x\| \leq \|x^* - x\| - \|f(x) - x\| \leq \alpha \|x^* - x\| \rightarrow \text{since} f(x^*) = x^*$
$\therefore -(\alpha + 1) \|x^* - x\| \leq - \|f(x) - x\| \leq (\alpha - 1) \|x^* - x\|$
$\therefore (\alpha + 1) \|x^* - x\| \geq \|f(x) - x\| \geq (1 - \alpha) \|x^* - x\|$
$\therefore \|x^* - x\| \leq \dfrac{1}{1-\alpha} \|f(x) - x\|$

1.2 Part B:

2 Question 2: Energetic Salesman

Writing the Bellman Equation for the problem, we get,
$J(A) = \min\limits_{a \in \{\text{stay,change}\}} \Big[r_a + \alpha J(A), -c + \alpha J(B) \Big]$
$J(B) = \min\limits_{a \in \{\text{stay,change}\}} \Big[r_b + \alpha J(B), -c + \alpha J(A) \Big]$
Actions $\rightarrow \{a_1 : \text{stay}, a_2 : \text{change}\}$

2.1 When the discount factor $\alpha \rightarrow 0$

We will apply policy iteration to get the optimal policy:
Let $\Pi_0(A) = a_2$ and $\Pi_0(B) = a_2$

3 Question 3:

3.1 Part A:

$\tilde{p_{ij}} = \dfrac{p_{ij} - m_j}{1 - \sum_{k=1}^{n} m_k}$
$\therefore \sum_{j=1}^{n} \tilde{p_{ij}} = \dfrac{\sum_{j=1}^{n} p_{ij} - \sum_{j=1}^{n} m_j}{1 - \sum_{k=1}^{n} m_k}$
Since $\sum\limits_{j=1}^{n} p_{ij} = 1$
$\therefore \sum_{j=1}^{n} \tilde{p_{ij}} = \dfrac{1 - \sum_{j=1}^{n} m_j}{1 - \sum_{k=1}^{n} m_k} = 1$

Therefore $\tilde{p_{ij}}$ are indeed transition probabilities.

3.2 Part B:

Using Bellman's Equation:
$\tilde{J}(i) = \min\limits_{a \in A} \Big[g(i,a) + \tilde{\alpha} \sum\limits_{j=1}^{n} \tilde{p_{ij}}(a) \tilde{J}(j) \Big] \forall i$
Substituting the values of $\tilde{\alpha}$ and $\tilde{p_{ij}}(a)$,
$\tilde{J}(i) = \min\limits_{a \in A} \Big[g(i,a) + \alpha(1 - \sum\limits_{k=1}^{n} m_k) \sum\limits_{j=1}^{n} \dfrac{p_{ij}(a) - m_j}{1 - \sum_{k=1}^{n} m_k} \tilde{J}(j) \Big]$
$\tilde{J}(i) = \min\limits_{a \in A} \Big[g(i,a) + \alpha \sum\limits_{j=1}^{n} (p_{ij}(a) - m_j) \tilde{J}(j) \Big]$
$\tilde{J}(i) = \min\limits_{a \in A} \Big[g(i,a) + \alpha \sum\limits_{j=1}^{n} p_{ij}(a) \tilde{J}(j) - \alpha \sum\limits_{k=1}^{n} m_k \tilde{J}(k) \Big]$
Since the min function is only on actions:a we can write:

$$\tilde{J}(i) + \frac{\alpha \sum_{k=1}^{n} m_k \tilde{J}(k)}{1-\alpha} e = \min_{a \in A} \Big[g(i,a) + \alpha \sum_{j=1}^{n} p_{ij}(a) \tilde{J}(j) - \alpha \sum_{k=1}^{n} m_k \tilde{J}(k) + \frac{\alpha \sum_{k=1}^{n} m_k \tilde{J}(k)}{1-\alpha} \Big]$$

$$\tilde{J}(i) + \frac{\alpha \sum_{k=1}^{n} m_k \tilde{J}(k)}{1-\alpha} e = \min_{a \in A} \Big[g(i,a) + \alpha \sum_{j=1}^{n} p_{ij}(a) \tilde{J}(j) - \alpha \sum_{k=1}^{n} m_k \tilde{J}(k) \big(1 - \frac{1}{1-\alpha}\big) \Big]$$

$$\tilde{J}(i) + \frac{\alpha \sum_{k=1}^{n} m_k \tilde{J}(k)}{1-\alpha} e = \min_{a \in A} \Big[g(i,a) + \alpha \sum_{j=1}^{n} p_{ij}(a) \tilde{J}(j) + \alpha \frac{\alpha \sum_{k=1}^{n} m_k \tilde{J}(k)}{1-\alpha} \Big]$$

Since $\sum_{j=1}^{n} p_{ij} = 1$

$$\tilde{J}(i) + \frac{\alpha \sum_{k=1}^{n} m_k \tilde{J}(k)}{1-\alpha} e = \min_{a \in A} \Big[g(i,a) + \alpha \sum_{j=1}^{n} p_{ij}(a) \tilde{J}(j) + \alpha \sum_{j=1}^{n} p_{ij}(a) \frac{\alpha \sum_{k=1}^{n} m_k \tilde{J}(k)}{1-\alpha} \Big]$$

$$\tilde{J}(i) + \frac{\alpha \sum_{k=1}^{n} m_k \tilde{J}(k)}{1-\alpha} e = \min_{a \in A} \left[g(i,a) + \alpha \sum_{j=1}^{n} p_{ij}(a) \Big(\tilde{J}(j) + \frac{\alpha \sum_{k=1}^{n} m_k \tilde{J}(k)}{1-\alpha} \Big) \right]$$

Now for the original problem, we have the Bellman equation:

$$J^*(i) = \min_{a \in A} \Big[g(i,a) + \alpha \sum_{j=1}^{n} p_{ij}(a) J^*(j) \Big] \forall i$$

Therefore by comparing the above two equations we get: $J^*(i) = \tilde{J}(i) + \frac{\alpha \sum_{k=1}^{n} m_k \tilde{J}(k)}{1-\alpha} e \ \forall i$ as they satisfy the above equations.

4 References:

Question 1:
Question 2:
Question 3:
Question 4: