Математика 2 (МФ - ИД)	24.09.2014	11.30 h

1. **(10 точки)** Намерете локалните екстремуми на функцията $y = \ln\dfrac{x-1}{x^2}$.

2. Решете интегралите

а/ **(5 точки)** $\displaystyle\int_0^1 x\ln\left(1+x^2\right)dx$ и б/ **(5 точки)** $\displaystyle\int\frac{\sin 2x}{1+\sin^2 x}dx$.

3. **(10 точки)** Намерете екстремумите на функцията $z = f\left(x, y\right) = x^2 + x^2 y + y^2 + 4$ и определете вида им .

4. **(10 точки)** Намерете общото решение на диференциалното уравнение:

$$\frac{xy' - y}{x} = tg\frac{y}{x}$$

5. **(10 точки)** Намерете лицето на фигурата, ограничена от параболата: $y = 4 - x^2$ и оста Ox.

6. **(10 точки)** Изследвайте за сходимост чрез критерия на Даламбер числовия ред $\displaystyle\sum_{n=1}^{\infty}\frac{3^n}{(2n)!}$.

Решения

1. $y = \ln\frac{x-1}{x^2}$ → локални екстремуми

*) $y' = \frac{1}{\frac{x-1}{x^2}} \cdot \frac{1 \cdot x^2 - 2x(x-1)}{x^4} = \frac{\cancel{x^2}}{x-1} \cdot \frac{x^2-2x^2+2x}{x^{\cancel{4}2}} =$

$= \frac{1}{x-1} \cdot \frac{-x^2+2x}{x^2} = \frac{1}{x-1} \cdot \frac{2-x}{x} = \frac{2-x}{x(x-1)}$

**) $y' = 0 \to \frac{2-x}{x(x-1)} = 0$

$x = 2$

***) $y'' = \left(\frac{2-x}{x(x-1)}\right)' = \frac{-1(x^2-x)-(2x-1)(2-x)}{x^2(x-1)^2} = \frac{-x^2+x-(4x-2-2x^2+x)}{x^2(x-1)^2} =$

$= \frac{-x^2 \cancel{+x} -4x+2+2x^2 \cancel{-x}}{x^2(x-1)^2} = \frac{x^2-4x+2}{x^2(x-1)^2}$

$y''(2) = \frac{2^2-4.2+2}{2^2(2-1)^2} = \frac{-2}{2^2} < 0$

⇒ в т. $x=2$ функцията има максимум.

$y_{max} = \ln\frac{2-1}{2^2} = \ln\frac{1}{4}$

2. a) $\int_0^1 x\ln(1+x^2)dx = \frac{1}{2}\int_0^1 \ln(1+x^2)dx^2 =$

$= \frac{1}{2}\int_0^1 \ln(1+x^2)dx^2 = \frac{1}{2}\left(x^2\ln(1+x^2)\Big|_0^1 -\right.$

(по части!)

$\left.-\int_0^1 x^2 \cdot \frac{1}{1+x^2} \cdot 2x\,dx\right) = \frac{1}{2}\left((1 \cdot \ln 2 - 0) - \cancel{2}\frac{1}{\cancel{2}}\int_0^1 \frac{x^2}{1+x^2}dx^2\right) =$

$\ldots \left(\int \frac{x^2+1}{\ldots} dx^2 - \int \frac{d(x^2+1)}{\ldots}\right) =$

②

$= \frac{1}{2}\left(\ln 2 - \left(\frac{x^3}{3}\Big|_0^1 - \ln(x^2+1)\Big|_0^1\right)\right) =$

$= \frac{1}{2}\left(\ln 2 - \frac{1}{3} - (\ln 2 - 0)\right) =$

$= \frac{1}{2}\left(\cancel{\ln 2} - \frac{1}{3} - \cancel{\ln 2}\right) = -\frac{1}{6}$

2.б) $\int \frac{\sin 2x}{1+\sin^2 x}\,dx =$

об $\cos 2x = \cos^2 x - \sin^2 x \Rightarrow$

$\cos 2x = 1 - \sin^2 x - \sin^2 x = 1 - 2\sin^2 x$

$\Rightarrow \sin^2 x = \frac{1-\cos 2x}{2}$

$\int \frac{\sin 2x}{1+\frac{1-\cos 2x}{2}}\,dx = \int \frac{\sin 2x}{\frac{2+1-\cos 2x}{2}} = 2\int \frac{\sin 2x}{3-\cos 2x}\,dx =$

$= 2\cdot\frac{1}{2}\int \frac{\sin 2x}{3-\cos 2x}\,d2x = -\int \frac{d\cos 2x}{3-\cos 2x} = \int \frac{d(3-\cos 2x)}{3-\cos 2x}$

$= \ln|(3-\cos 2x)| + C$

3) $z = f(x,y) = x^2 + x^2 y + y^2 + 4 \rightarrow$ екстремуми!

x) $\frac{\partial z}{\partial x} = 2x + 2xy$

$\frac{\partial z}{\partial y} = x^2 + 2y$

$\frac{\partial z}{\partial x} = 0 \Rightarrow 2x + 2xy = 0 \rightarrow 2x(1+y) = 0 \Rightarrow$

$\frac{\partial z}{\partial y} = 0 \Rightarrow x^2 + 2y = 0 \Rightarrow y = -\frac{x^2}{2}$

$(\ \ x^2) = 0$

$x_1 = 0, \; y = 0$
$x_2 = \sqrt{2}, \; y = -1$ } стаціонарні точки
$x_3 = -\sqrt{2}, y = -1$

$**)\; \frac{\partial^2 z}{\partial x^2} = 2 + 2y$

$\frac{\partial^2 z}{\partial y^2} = 2$

$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = 2x$

$***)\quad \Delta = \frac{\partial^2 z}{\partial x^2} \cdot \frac{\partial^2 z}{\partial y^2} - \left(\frac{\partial^2 z}{\partial x \partial y}\right)^2 =$

$= (2+2y) \cdot 2 - (2x)^2 =$

$= 4 + 4y - 4x^2$

$\Delta_{0,0} = 4 \neq 0, > 0 \Rightarrow$ т. $(0,0)$ є екстремальна

$\Delta_{\sqrt{2},-1} = 4 - 4 - 4 \cdot 2 < 0 \Rightarrow$ т. $(\sqrt{2}, -1)$ не є екстремальна

$\Delta_{-\sqrt{2},-1} = 4 - 4 - 4 \cdot 2 < 0 \Rightarrow$ т. $(-\sqrt{2}, -1)$ не є екстремальна

$****)\; \frac{\partial^2 z}{\partial x^2} = 2 + 2y$ за $(0,0) = 2 > 0$

$\Rightarrow$ в т. $(0,0)$ функція має мінімум.

$*****)\; z_{min} = 4$

$4.)\; \frac{xy' - y}{x} = tg\frac{y}{x}$

$y' - \frac{y}{x} = tg\frac{y}{x}$

$z = \frac{y}{}$

$y = x \cdot u$
$y' = u + u'x$

$\cancel{u} + u'x \cancel{- u} = \operatorname{tg} u$

$\frac{du}{dx} \cdot x = \operatorname{tg} u$

$\frac{du}{\operatorname{tg} u} = \frac{dx}{x}$

$\int \frac{du}{\operatorname{tg} u} = \int \frac{dx}{x}$

$\int \frac{du}{\frac{\sin u}{\cos u}} = \ln x$

$\int \frac{\cos u \, du}{\sin u} = \ln x$

$\int \frac{d \sin u}{\sin u} = \ln x$

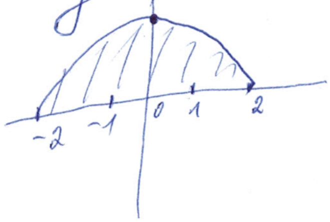
$\ln |\sin u| = \ln x$

$\frac{1}{\sin u} = x$

$\frac{1}{\sin \frac{y}{x}} = x$

⑤ Лице на фигура, ограничена от парабола

$y = 4 - x^2$ и оста OX



$y = 0 \Rightarrow 4 - x^2 = 0$

$x = \pm 2$

$$\int_{-2}^{2} (4 - x^2)\,dx$$

четна функция
интегралът е със симетрични граници

$$\Rightarrow \int_{-2}^{2} (4-x^2)\,dx = 2\int_{0}^{2} (4-x^2)\,dx =$$

$$= 2\left(\int_0^2 4\,dx - \int_0^2 x^2\,dx\right) =$$

$$= 2\left(4x\Big|_0^2 - \frac{x^3}{3}\Big|_0^2\right) =$$

$$= 2\left(8 - \frac{8}{3}\right) = 2 \cdot \frac{16}{3} = \frac{32}{3}$$

⑥ Сходимост с критерия на Даламбер

$$\sum_{n=1}^{\infty} \frac{3^n}{(2n)!}$$

$$\lim_{n\to\infty} \frac{\frac{3^{n+1}}{(2(n+1))!}}{\frac{3^n}{(2n)!}} = \lim_{n\to\infty} \frac{\frac{\cancel{3^n} \cdot 3}{\cancel{(2n)!}\,(2n+2)}}{\frac{\cancel{3^n}}{\cancel{(2n)!}}} =$$

$$= \lim_{n\to\infty} \frac{3}{2n+2} = 0 < 1$$

$\Rightarrow$ редът е сходящ!