Определен интеграл

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$$\int_a^b f(x)\,dx = F(b)\Big|_a^b = F(b) - F(a)$$

$$\int_a^b u(x)\,d(v(x)) = u(x)v(x)\Big|_a^b - \int_a^b v(x)\,d(u(x))$$

Да се пресметне определения интеграл:

Задача 1

$$\int_{-1}^{2} x^2 dx = \frac{x^3}{3}\Big|_{-1}^{2} = \frac{1}{3}\left(2^3 - (-1)^3\right) = \frac{1}{3}\left(8 - (-1)\right) = \frac{9}{3} = 3$$

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Задача 2

$$\int_0^{\pi} \sin 5x dx = \frac{1}{5}\int_0^{\pi} \sin 5x d\left(5x\right) = -\frac{\cos 5x}{5}\Bigg|_0^{\pi} = -\frac{1}{5}\left(-1-1\right) = \frac{2}{5}$$

$$\cos 5\pi = -1 \qquad \cos 0 = 1$$

Задача 3

$$\int_1^2 \frac{e^x}{e^x-1}dx = \int_1^2 \frac{1}{e^x-1}d\left(e^x\right) = \int_1^2 \frac{1}{e^x-1}d\left(e^x-1\right) = \ln\left(e^x-1\right)\Big|_1^2 =$$

$$= \ln\left(e^2-1\right) - \ln\left(e^1-1\right) = \ln\left|\frac{e^2-1}{e-1}\right| = \ln\left|\frac{e^2-1}{e-1}\frac{e+1}{e+1}\right| =$$

$$= \ln\left|\frac{\left(e^2-1\right)\left(e+1\right)}{e^2-1}\right| = \ln\left(e+1\right)$$

Задача 4

$$\int_0^{\frac{1}{2}} x\sqrt{1-x^2}\,dx = \frac{1}{2}\int_0^{\frac{1}{2}} \sqrt{1-x^2}\,d\left(x^2\right) = -\frac{1}{2}\int_0^{\frac{1}{2}} \sqrt{1-x^2}\,d\left(1-x^2\right) =$$

$$= -\frac{1}{2}\left.\frac{\left(1-x^2\right)^{\frac{3}{2}}}{\frac{3}{2}}\right|_0^{\frac{1}{2}} = -\frac{1}{3}\left(1-\frac{1}{4}\right)^{\frac{3}{2}} - \left(-\frac{1}{3}(1-0)^{\frac{3}{2}}\right) =$$

$$= -\frac{1}{3}\left(\frac{3}{4}\right)^{\frac{3}{2}} + \frac{1}{3} = \frac{1}{3}\left(1-\left(\frac{3}{4}\right)^{\frac{3}{2}}\right)$$

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Задача 5

$$\int_0^1 \frac{\operatorname{arctg} x}{1+x^2}dx = \int_0^1 \operatorname{arctg} x d\left(\operatorname{arctg} x\right) = \left.\frac{\operatorname{arctg}^2 x}{2}\right|_0^1 = \frac{\operatorname{arctg}^2 1}{2} - \frac{\operatorname{arctg}^2 0}{2} = \frac{\pi^2}{32}$$

$$\operatorname{arctg} 0 = 0 \qquad \operatorname{arctg} 1 = \frac{\pi}{4}$$

Задача 6

$$\int_0^{\frac{\pi}{2}} \sqrt{\cos x - \cos^3 x}\,dx = \int_0^{\frac{\pi}{2}} \sqrt{\cos x\left(1-\cos^2 x\right)}dx = \int_0^{\frac{\pi}{2}} |\sin x|\sqrt{\cos x}\,dx =$$

$$= \int_0^{\frac{\pi}{2}} \sin x\sqrt{\cos x}\,dx = -\int_0^{\frac{\pi}{2}} \sqrt{\cos x}\,d\cos x = -\left.\frac{2\left(\cos x\right)^{\frac{3}{2}}}{3}\right|_0^{\frac{\pi}{2}} = \frac{2}{3}$$

$$\cos\frac{\pi}{2} = 0 \qquad \cos 0 = 1$$

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Задача 7

$$\int_0^1 x\ln\left(1+x^2\right)dx = \frac{1}{2}\int_0^1 \ln\left(1+x^2\right)d\left(x^2+1\right) =$$

$$= \frac{1}{2}\left(x^2+1\right)\ln\left(1+x^2\right)\Big|_0^1 - \frac{1}{2}\int_0^1 \left(x^2+1\right)d\ln\left(1+x^2\right) =$$

$$= \left[\frac{1}{2}\left(1^2+1\right)\ln\left(1+1^2\right) - \frac{1}{2}\left(0^2+1\right)\ln\left(1+0^2\right)\right] - \frac{1}{2}\int_0^1 \frac{\left(x^2+1\right)}{\left(x^2+1\right)}2xdx =$$

$$= \left[\frac{2}{2}\ln 2 - \frac{1}{2}\ln 1\right] - \int_0^1 xdx = \ln 2 - \frac{x^2}{2}\Big|_0^1 = \qquad \ln 1 = 0$$

$$= \ln 2 - \left[\frac{1}{2} - 0\right] = \ln 2 - \frac{1}{2}$$

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Изчисление с Mathematica

Задача: Да се пресметне определения интеграл:

$$\int_0^{\frac{1}{2}} x\sqrt{1-x^2}\,dx$$

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f[x_] := ∫_0^(1/2) x √(1 - x^2) ⅆx
f[x]
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$$\frac{1}{3}-\frac{\sqrt{3}}{8}$$

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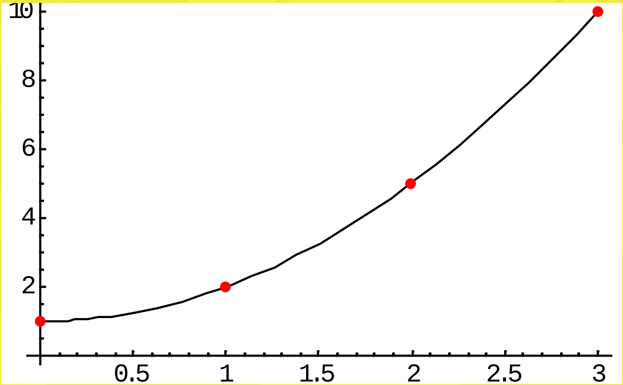
Геометрични приложения на определен интеграл

Пресмятане на дължина на равнинна крива

$\boxed{L: y = f(x), a \le x \le b}$, то $\boxed{l = \int_a^b \sqrt{1 + \left(f(x)'\right)^2}\, dx}$

Задача 1 Да се намери дължината на крива, която има уравнение $L: y = x^2 + 1$, от точка $x = 0$ до точка $x = 3$

x	0	1	2	3
y	1	2	5	10



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$$l=\int_0^3\sqrt{1+\left(f\left(x\right)'\right)^2}\,dx=\int_0^3\sqrt{1+\left(\left(x^2+1\right)'\right)^2}\,dx=\underset{\substack{\shortparallel\\ I}}{\int_0^3\sqrt{1+\left(2x\right)^2}\,dx}=$$

$$=x\sqrt{1+4x^2}\,\Big|_0^3-\int_0^3 xd\left(\sqrt{1+4x^2}\right)=3\sqrt{37}-\int_0^3\frac{x8x}{2\sqrt{1+4x^2}}\,dx=$$

$$=3\sqrt{37}-\int_0^3\frac{4x^2+1-1}{\sqrt{1+4x^2}}\,dx=3\sqrt{37}-\int_0^3\frac{1+4x^2}{\sqrt{1+4x^2}}\,dx+\int_0^3\frac{1}{\sqrt{1+4x^2}}\,dx=$$

$$=3\sqrt{37}-\underset{\substack{\shortparallel\\ I}}{\int_0^3\sqrt{1+4x^2}\,dx}+\frac{1}{2}\int_0^3\frac{1}{\sqrt{1+\left(2x\right)^2}}\,d2x$$

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$$I=3\sqrt{37}-I+\frac{1}{2}\ln\left|2x+\sqrt{1+4x^2}\right|\Big|_0^3=$$

$$2I=3\sqrt{37}+\frac{1}{2}\ln\left|6+\sqrt{37}\right|+\frac{1}{2}\ln 1$$

$$\ln 1=0$$

$$I=\frac{6\sqrt{37}+\ln\left|6+\sqrt{37}\right|}{4}$$

Пресмятане на лице на равнинна фигура

$$D\begin{cases} a \le x \le b \\ f_1(x) \le y \le f_2(x) \end{cases}$$, то $$\boxed{S(D) = \int_a^b [f_2(x) - f_1(x)]dx}$$

Задача 2 Да се намери лицето на фигурата, заградена от кривите:

$$D\begin{cases} y = 2x - x^2 \\ x + y = 2 \end{cases}$$

-> парабола $2x - x^2 = 0 \Rightarrow x(2 - x) = 0 \quad x_1 = 0 \quad x_2 = 2$

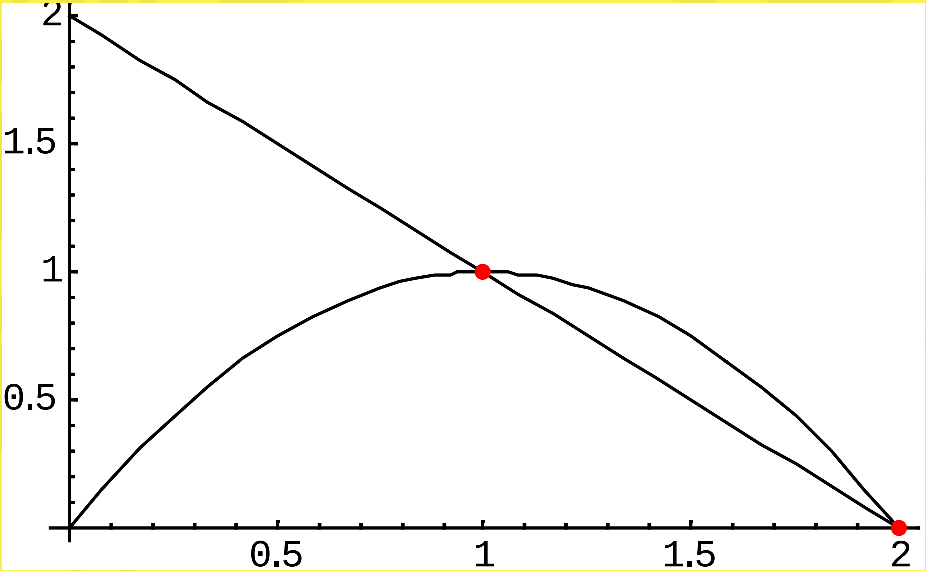
-> права

парабола

x	0	1	2
y	0	1	0

права

x	0	1	2
y	2	1	0



Търсим пресечните точки:

$$\begin{cases} y = 2x - x^2 \\ y = 2 - x \end{cases}$$

$$2 - x = 2x - x^2$$

$$x^2 - 3x + 2 = 0$$

$$D = 9 - 8 = 1$$

$$x_{1,2} = \frac{3 \pm \sqrt{1}}{2}$$

$$x_1 = 1 \quad x_2 = 2$$

$$y_1 = 1 \quad y_2 = 0$$

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$$D\begin{cases}1\le x\le 2\\ 2-x\le y\le 2x-x^2\end{cases}$$

$$S(D)=\int_1^2\left[\left(2x-x^2\right)-(2-x)\right]dx=\int_1^2\left(2x-x^2-2+x\right)dx=$$

$$=\int_1^2\left(-x^2+3x-2\right)dx=-\int_1^2 x^2dx+3\int_1^2 xdx-2\int_1^2 dx=$$

$$=-\frac{x^3}{3}\Big|_1^2+3\frac{x^2}{2}\Big|_1^2-2x\Big|_1^2=-\left(\frac{8}{3}-\frac{1}{3}\right)+3\left(\frac{4}{2}-\frac{1}{2}\right)-2(2-1)=$$

$$=-\frac{7}{3}+\frac{9}{2}-2=\frac{-14+27-12}{6}=\frac{1}{6}$$

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Пресмятане на обем на ротационно тяло

$$D\begin{cases} a \le x \le b \\ 0 \le y \le f(x) \end{cases}$$

, то

$$V(T) = \pi\int_a^b \left(f(x)\right)^2 dx$$

Задача 2 Да се намери обема на ротационното тяло (Т) получено при въртене на кривата $y = 2x - x^2$ $(1 \le x \le 2)$ около оста Ox.

$$V(T) = \pi\int_1^2 \left(2x - x^2\right)^2 dx = \pi\int_1^2 \left(4x^2 - 4x^3 + x^4\right) dx =$$

$$= \pi\left(\int_1^2 x^4 dx - 4\int_1^2 x^3 dx + 4\int_1^2 x^2 dx\right) = \pi\left(\frac{x^5}{5}\Bigg|_1^2 - 4\frac{x^4}{4}\Bigg|_1^2 + 4\frac{x^3}{3}\Bigg|_1^2\right) =$$

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$$=\pi\left(\frac{1}{5}(32-1)-4\left(\frac{16}{4}-\frac{1}{4}\right)+4\left(\frac{8}{3}-\frac{1}{3}\right)\right)=\pi\left(\frac{31}{5}-15+\frac{28}{3}\right)=$$

$$=\pi\left(\frac{140-225+93}{15}\right)=\frac{8\pi}{15}.$$