ГЛАВА 5

ИНТЕГРАЛИ ОТ РАЦИОНАЛНИ ФУНКЦИИ на $\sin x$ и $\cos x$

И при този клас интеграли (както в глава 4) избираме такава смяна на интеграционната променлива $x = \varphi(t)$, чрез която пресмятането на интеграла се свежда до интеграл от рационална функция.

Разглеждаме интеграл от вида

$$\int R(\sin x, \cos x)\mathrm{d}x, \tag{5.1}$$

където $R(\ldots,\ldots)$ е рационална функция на $\sin x$ и $\cos x$.

I клас

Нека $R(-\sin x, \cos x) = -R(\sin x, \cos x)$, т.е. функцията R е *нечетна* относно $\sin x$. Тогава

$$\textit{полагаме} \quad \cos x = t \Longrightarrow x = \mathrm{arccos} t \Longrightarrow \mathrm{d}x = -\frac{\mathrm{d}t}{\sqrt{1-t^2}}$$

$$\sin x = \sqrt{1-\cos^2 t} = \sqrt{1-t^2}$$

$$\text{и } \textit{заместваме} \Longrightarrow I = -\int R(\sqrt{1-t^2}, t)\frac{\mathrm{d}t}{\sqrt{1-t^2}} \tag{5.2}$$

Пример 5.1. Решете интеграла $\displaystyle\int \frac{\sin x\mathrm{d}x}{2+3\cos^2 x}$.

Решение. От условието $R(\sin x, \cos x) = \dfrac{\sin x}{2+3\cos^2 x}$. Тогава $R(-\sin x, \cos x)$ $= \dfrac{(-\sin x)}{2+3\cos^2 x} = -R(\sin x, \cos x)$. От (5.2) имаме:

$$I = \int \frac{\sqrt{1-t^2}}{2+3t^2}\left(\frac{-\mathrm{d}t}{\sqrt{1-t^2}}\right) = -\int \frac{\mathrm{d}t}{2+3t^2} = -\frac{1}{2}\int \frac{\mathrm{d}t}{1+\frac{3t^2}{2}} = -\frac{1}{2}\sqrt{\frac{2}{3}}\int \frac{\mathrm{d}\left(t\sqrt{\frac{3}{2}}\right)}{1+\left(t\sqrt{\frac{3}{2}}\right)^2}$$

$$= -\frac{1}{\sqrt{6}}\mathrm{arctg}\left(t\sqrt{\frac{3}{2}}\right) + C = -\frac{1}{\sqrt{6}}\mathrm{arctg}\left(\sqrt{\frac{3}{2}}\cos x\right) + C.$$

Забележка. Интегралът може да се реши и непосредствено:

$$I = \int \frac{\sin x \mathrm{d}x}{2+3\cos^2 x} = -\int \frac{\mathrm{d}(\cos x)}{2+3\cos^2 x} = -\frac{1}{2}\int \frac{\mathrm{d}(\cos x)}{1+\left(\sqrt{\frac{3}{2}}\cos x\right)^2}$$

$$= -\frac{1}{2}\cdot\sqrt{\frac{2}{3}}\int \frac{\mathrm{d}\left(\sqrt{\frac{3}{2}}\cos x\right)}{1+\left(\sqrt{\frac{3}{2}}\cos x\right)^2} = -\frac{1}{\sqrt{6}}\mathrm{arctg}\left(\sqrt{\frac{3}{2}}\cos x\right) + C.$$

II клас

Нека $R(\sin x, -\cos x) = -R(\sin x, \cos x)$, т.е. функцията R е *нечетна* относно $\cos x$. Тогава

$$\textit{полагаме} \quad \sin x = t \Longrightarrow x = \mathrm{arcsin} t \Longrightarrow \mathrm{d}x = \frac{\mathrm{d}t}{\sqrt{1-t^2}}$$

$$\cos x = \sqrt{1-\sin^2 t} = \sqrt{1-t^2}$$

$$\text{и } \textit{заместваме} \Longrightarrow I = \int R(t, \sqrt{1-t^2})\frac{\mathrm{d}t}{\sqrt{1-t^2}}. \tag{5.3}$$

Пример 5.2. Решете интеграла $\displaystyle\int \frac{\mathrm{d}x}{\sin x \sin 2x}$.

Решение. От условието $R(\sin x, \cos x) = \dfrac{1}{2\sin^2 x\cos x}$. Тогава $R(\sin x, -\cos x)$ $= \dfrac{1}{2\sin^2 x(-\cos x)} = -\dfrac{1}{2\sin^2 x\cos x} = -R(\sin x, \cos x)$. От (5.3) имаме:

$$I = \int \frac{1}{2t^2\sqrt{1-t^2}}\frac{\mathrm{d}t}{\sqrt{1-t^2}} = \frac{1}{2}\int \frac{\mathrm{d}t}{t^2(1-t)(1+t)}$$

$$\frac{1}{t^2(1-t)(1+t)} = \frac{A}{t^2} + \frac{B}{t} + \frac{C}{1-t} + \frac{D}{1+t}$$

$$\Longrightarrow 1 = (-B+C-D)t^3 + (-A+C+D)t^2 + Bt + A$$

$$\Longrightarrow \left|\begin{array}{r} -B+C-D=0 \\ -A+C+D=0 \\ B=0 \\ A=1 \end{array}\right. \Longrightarrow \begin{array}{l} A=1 \\ B=0 \\ C=1/2 \\ D=-1/2 \end{array}$$

$$\Longrightarrow I = \int \frac{\mathrm{d}t}{t^2} + \frac{1}{2}\int \frac{\mathrm{d}t}{1-t} - \frac{1}{2}\int \frac{\mathrm{d}t}{1+t} = -\frac{1}{t} - \frac{1}{2}\ln|1-t^2| + C$$

$$= -\frac{1}{\sin x} - \ln|\sqrt{1-\sin^2 x}| + C = -\frac{1}{\sin x} - \ln|\cos x| + C.$$

Забележка. До същия интеграл от рационална функция се достига и след като подинтегралната функция се преобразува по следния начин:

$$I = \int \frac{\mathrm{d}x}{2\sin^2 x \cos x} = \frac{1}{2}\int \frac{\cos x \mathrm{d}x}{\sin^2 x \cos^2 x} = \frac{1}{2}\int \frac{\mathrm{d}\sin x}{\sin^2 x(1-\sin^2 x)}.$$

Полагаме $\sin x = t$, $x = \mathrm{arcsin}t$, $\mathrm{d}x = \dfrac{\mathrm{d}t}{\sqrt{1-t^2}} \Longrightarrow I = \dfrac{1}{2}\displaystyle\int \frac{\mathrm{d}t}{t^2(1-t^2)}$.

III клас

Нека $R(-\sin x, -\cos x) = R(\sin x, \cos x)$, т.е. функцията R е *четна* относно $\sin x$ и $\cos x$ едновременно. Тогава

$$\textit{полагаме}\quad \mathrm{tg}x = t \Longrightarrow x = \mathrm{arctg}t \Longrightarrow \mathrm{d}x = \frac{\mathrm{d}t}{1+t^2}$$

$$\sin x = \frac{\mathrm{tg}x}{\sqrt{1+\mathrm{tg}^2 x}} = \frac{t}{\sqrt{1+t^2}},$$

$$\cos x = \frac{1}{\sqrt{1+\mathrm{tg}^2 x}} = \frac{1}{\sqrt{1+t^2}}$$

$$\text{и } \textit{заместваме} \Longrightarrow I = \int R\left(\frac{t}{\sqrt{1+t^2}}, \frac{1}{\sqrt{1+t^2}}\right)\frac{\mathrm{d}t}{1+t^2} \tag{5.4}$$

Пример 5.3. Решете интеграла $\displaystyle\int \frac{\sin x \mathrm{d}x}{\sin^3 x + \cos^3 x}$.

Решение. От условието $R(\sin x, \cos x) = \dfrac{\sin x}{\sin^3 x + \cos^3 x}$. Тогава $R(-\sin x, -\cos x) = \dfrac{(-\sin x)}{(-\sin x)^3 + (-\cos x)^3} = R(\sin x, \cos x)$. От (5.4) имаме:

$$I = \int \frac{\dfrac{t}{\sqrt{1+t^2}} \cdot \dfrac{\mathrm{d}t}{1+t^2}}{\dfrac{t^3+1}{(1+t^2)\sqrt{1+t^2}}} = \int \frac{t\mathrm{d}t}{t^3+1}$$

$$\frac{t}{(t+1)(t^2-t+1)} = \frac{A}{t+1} + \frac{Bt+C}{t^2-t+1}$$

$$\Longrightarrow t = (A+B)t^2 + (-A+B+C)t + A + C$$

$$\Longrightarrow \left|\begin{aligned} A+B&=0\\ -A+B+C&=1\\ A+C&=0\end{aligned}\right. \Longrightarrow \begin{aligned} A&=-1/3\\ B&=1/3\\ C&=1/3\end{aligned}$$

$$\Longrightarrow I = -\frac{1}{3}\int \frac{\mathrm{d}(t+1)}{t+1} + \frac{1}{3}\int \frac{t+1}{t^2-t+1}\mathrm{d}t = -\frac{1}{3}\ln|t+1| + I_1.$$

За да решим I_1, полагаме $t = u - b/2a = u + 1/2$, $\mathrm{d}t = \mathrm{d}u$

$$I_1 = \int \frac{u+\frac{1}{2}+1}{u^2+u+\frac{1}{4}-u-\frac{1}{2}+1}\mathrm{d}u = \int \frac{u+\frac{3}{2}}{u^2+\frac{3}{4}}\mathrm{d}u = \int \frac{u\mathrm{d}u}{u^2+\frac{3}{4}} + \frac{3}{2}\int \frac{\mathrm{d}u}{u^2+\frac{3}{4}}$$
$$= \frac{1}{2}\int \frac{\mathrm{d}(u^2+\frac{3}{4})}{u^2+\frac{3}{4}} + \frac{3}{2}\cdot\frac{4.\sqrt{3}}{3.2}\int \frac{\mathrm{d}(2u/\sqrt{3})}{(2u/\sqrt{3})^2+1} = \frac{1}{2}\ln\left|u^2+\frac{3}{4}\right| + \sqrt{3}\mathrm{arctg}\frac{2u}{\sqrt{3}}$$
$$= \frac{1}{2}\ln\left|\left(t-\frac{1}{2}\right)^2+\frac{3}{4}\right| + \sqrt{3}\mathrm{arctg}\frac{2(t-\frac{1}{2})}{\sqrt{3}} = \frac{1}{2}\ln|t^2-t+1| + \sqrt{3}\mathrm{arctg}\frac{2t-1}{\sqrt{3}}.$$

Тогава

$$I = -\frac{1}{3}\ln|\mathrm{tg}x+1| + \frac{1}{3}\ln\sqrt{\mathrm{tg}^2x-\mathrm{tg}x+1} + \frac{\sqrt{3}}{3}\mathrm{arctg}\frac{2\mathrm{tg}x-1}{\sqrt{3}} + C.$$

Забележка. Интегралът се свежда до интеграл от рационална функция чрез елементарни преобразувания и съответното полагане:

$$I = \int \frac{\sin x}{\cos^3 x\left(\frac{\sin^3 x}{\cos^3 x}+1\right)}\mathrm{d}x = \int \frac{\frac{\sin x}{\cos x}}{\frac{\sin^3 x}{\cos^3 x}+1}\frac{\mathrm{d}x}{\cos^2 x} = \int \frac{\mathrm{tg}x\mathrm{d}\mathrm{tg}x}{\mathrm{tg}^3x+1}.$$

Полагаме $\mathrm{tg}x = t$, $x = \mathrm{arctg}t \Longrightarrow I = \int \frac{t\mathrm{d}t}{t^3+1}$.

И така при интеграли от I, II и III клас може да се стигне до интеграл от рационална функция чрез елементарни преобразувания и подходяща субституция.

IV клас

Ако интегралът (5.1) *не е* от изброените три класа, прилагаме *класическата субституция*:

$$\textit{полагаме}\quad \mathrm{tg}\frac{x}{2} = t \Longrightarrow x = 2\mathrm{arctg}t \Longrightarrow \mathrm{d}x = \frac{2\mathrm{d}t}{1+t^2}$$
$$\sin x = \frac{2\mathrm{tg}\frac{x}{2}}{1+\mathrm{tg}^2\frac{x}{2}} = \frac{2t}{1+t^2},$$
$$\cos x = \frac{1-\mathrm{tg}^2\frac{x}{2}}{1+\mathrm{tg}^2\frac{x}{2}} = \frac{1-t^2}{1+t^2}$$
$$\text{и }\textit{заместваме} \Longrightarrow I = 2\int R\left(\frac{2t}{1+t^2}, \frac{1-t^2}{1+t^2}\right)\frac{\mathrm{d}t}{1+t^2}. \tag{5.5}$$

Пример 5.4. Решете интеграла $\displaystyle\int \frac{\mathrm{d}x}{2+\cos x}$.

Решение. От (5.5) имаме

$$I=\int \frac{\dfrac{2\mathrm{d}t}{1+t^2}}{2+\dfrac{1-t^2}{1+t^2}}=2\int\frac{\mathrm{d}t}{2+2t^2+1-t^2}=2\int\frac{\mathrm{d}t}{t^2+3}=\frac{2}{3}\int\frac{\mathrm{d}t}{\dfrac{t^2}{3}+1}$$
$$=\frac{2\sqrt{3}}{3}\int\frac{\mathrm{d}\left(\dfrac{t}{\sqrt{3}}\right)}{\left(\dfrac{t}{\sqrt{3}}\right)^2+1}=\frac{2\sqrt{3}}{3}\mathrm{arctg}\frac{t}{\sqrt{3}}+C=\frac{2\sqrt{3}}{3}\mathrm{arctg}\left(\frac{\mathrm{tg}\frac{x}{2}}{\sqrt{3}}\right)+C.$$

Пример 5.5. Решете интеграла $\displaystyle\int \frac{\cos 2x\mathrm{d}x}{\cos^4 x+\sin^4 x}$.

Решение. Преобразуваме подинтегралната функция:

$$I=\int\frac{\cos 2x}{(\cos^2 x+\sin^2 x)^2-2\sin^2 x\cos^2 x}\mathrm{d}x=\int\frac{\cos 2x\mathrm{d}x}{1-\frac{1}{2}\sin^2 2x}.$$
$$R(\sin 2x,\cos 2x)=\frac{2\cos 2x}{2-\sin^2 2x},$$
$$R(\sin 2x,-\cos 2x)=-\frac{2\cos 2x}{2-\sin^2 2x}=-R(\sin 2x,\cos 2x).$$

Следователно полагаме (вж. (5.3))

$$\sin 2x=t,\quad x=\frac{1}{2}\mathrm{arcsin}t,\quad \mathrm{d}x=\frac{1}{2\sqrt{1-t^2}},\quad \cos 2x=\sqrt{1-t^2}$$
$$\Longrightarrow I=\int\frac{2\sqrt{1-t^2}}{2-t^2}\cdot\frac{1}{2\sqrt{1-t^2}}\mathrm{d}t=\int\frac{\mathrm{d}t}{2-t^2}=\frac{1}{2\sqrt{2}}\ln\left|\frac{\sqrt{2}+t}{\sqrt{2}-t}\right|+C$$
$$=\frac{1}{2\sqrt{2}}\ln\left|\frac{\sqrt{2}+\sin 2x}{\sqrt{2}-\sin 2x}\right|+C.$$

Пример 5.6. Решете интеграла $\displaystyle\int\frac{(\cos 2x-3)\mathrm{d}x}{\cos^4 x\sqrt{4-\mathrm{cotg}^2 x}}$.

Решение. От условието $R(\sin x,\cos x)=\dfrac{\cos^2 x-\sin^2 x-3}{\cos^4 x\sqrt{4-\dfrac{\cos^2 x}{\sin^2 x}}}$

$R(-\sin x,-\cos x)=R(\sin x,\cos x)$ (функцията е четна спрямо $\sin x$ и $\cos x$).

Полагаме $\text{cotg}x = t$ (вместо $\text{tg}x = t$),

$$x = \text{arccotg}t, \quad dx = -\frac{dt}{1+t^2}, \quad \sin^2 x = \frac{1}{1+t^2}, \quad \cos^2 x = \frac{t^2}{1+t^2}$$

$$I = \int \frac{\frac{t^2-1}{t^2+1} - 3}{\frac{t^4}{(1+t^2)^2}\sqrt{4-t^2}}\left(-\frac{dt}{1+t^2}\right) = \int \frac{2t^2+4}{t^4\sqrt{4-t^2}}dt.$$

Правим второ полагане (вж. пр. 4.1) $t = 2\sin z, \quad dt = 2\cos z dz, \quad z = \arcsin\frac{t}{2}$

$$\Longrightarrow I = \int \frac{8\sin^2 z + 4}{16\sin^4 z\sqrt{4-4\sin^2 z}}(2\cos z)dz = \frac{1}{4}\int \frac{\sin^2 z + 1}{\sin^4 z}dz$$
$$= \frac{1}{4}\int \frac{2\sin^2 z + \cos^2 z}{\sin^4 z}dz = \frac{1}{2}\int \frac{dz}{\sin^2 z} + \frac{1}{4}\int \frac{\cos^2 z}{\sin^2 z}\cdot\frac{dz}{\sin^2 z}$$
$$= -\frac{1}{2}\text{cotg}z - \frac{1}{4}\int \text{cotg}^2 z d(\text{cotg}z) = -\frac{1}{2}\text{cotg}z - \frac{1}{12}\text{cotg}^3 z + C.$$

От $\sin z = \frac{t}{2} \Longrightarrow \text{cotg}z = \frac{\sqrt{1-\sin^2 z}}{\sin z} = \frac{\sqrt{4-t^2}}{2t} = \frac{\sqrt{4-\text{cotg}^2 x}}{2\text{cotg}x}$

$$\Longrightarrow I = -\frac{1}{2}\text{cotg}\frac{\sqrt{4-\text{cotg}^2 x}}{2\text{cotg}x} - \frac{1}{12}\text{cotg}^3\frac{\sqrt{4-\text{cotg}^2 x}}{2\text{cotg}x} + C.$$

Пример 5.7. Да се реши $\int \frac{1+\cos x}{(\cos x + \sin x + 2)^3}dx.$

Решение. $R(\sin x, \cos x) = \frac{1+\cos x}{(\cos x + \sin x + 2)^3}$. Полагаме $t = \text{tg}\frac{x}{2}$. От (5.5) имаме

$$I = \int \frac{1 + \frac{1-t^2}{1+t^2}}{\left(\frac{1-t^2}{1+t^2} + \frac{2t}{1+t^2} + 2\right)^3}\cdot\frac{2dt}{1+t^2} = 4\int \frac{(1+t^2)dt}{(t^2+2t+3)^3}$$
$$= 4\int \frac{t^2+2t+3-2t-2}{(t^2+2t+3)^3}dt = 4\int \frac{dt}{(t^2+2t+3)^2} - 8\int \frac{t+1}{(t^2+2t+3)^3}dt$$
$$= 4\int \frac{dt}{[(t+1)^2+2]^2} - 8\int \frac{t+1}{[(t+1)^2+2]^3}dt.$$

Полагаме $t+1=u$, $t=u-1$, $\mathrm{d}t=\mathrm{d}u$

$$\Longrightarrow I=4\int\frac{\mathrm{d}u}{(u^2+2)^2}-8\int\frac{u\mathrm{d}u}{(u^2+2)^3}=2\int\frac{2+u^2-u^2}{(u^2+2)^2}\mathrm{d}u-4\int\frac{\mathrm{d}(u^2+2)}{(u^2+2)^3}$$

$$=2\int\frac{\mathrm{d}u}{u^2+2}-\int\frac{u\mathrm{d}(u^2+2)}{(u^2+2)^2}-4\frac{(u^2+2)^{-2}}{-2}=2\int\frac{\mathrm{d}u}{u^2+2}$$

$$+\int u\mathrm{d}\frac{1}{u^2+2}+\frac{2}{(u^2+2)^2}=2\int\frac{\mathrm{d}u}{u^2+2}+\frac{u}{u^2+2}-\int\frac{\mathrm{d}u}{u^2+2}+\frac{2}{(u^2+2)^2}$$

$$=\frac{\sqrt{2}}{2}\int\frac{\mathrm{d}\left(\frac{u}{\sqrt{2}}\right)}{\left(\frac{u}{\sqrt{2}}\right)^2+1}+\frac{u(u^2+2)+2}{(u^2+2)^2}=\frac{1}{\sqrt{2}}\operatorname{arctg}\frac{u}{\sqrt{2}}+\frac{u^3+2u+2}{(u^2+2)^2}+C.$$

От $u=t+1=\operatorname{tg}\frac{x}{2}+1\Longrightarrow I=\frac{1}{\sqrt{2}}\operatorname{arctg}\frac{\operatorname{tg}\frac{x}{2}+1}{\sqrt{2}}+\frac{\operatorname{tg}^3\frac{x}{2}+2\operatorname{tg}\frac{x}{2}+2}{(\operatorname{tg}^2\frac{x}{2}+2\operatorname{tg}\frac{x}{2}+3)^2}+C.$

Пример 5.8. Решете интегралите

а) $I_1=\int\frac{\cos x}{\cos 3x}\mathrm{d}x$ б) $I_2=\int\frac{\sin x}{\sin 3x}\mathrm{d}x$

Решение. Прилагаме формулите $\cos 3x=4\cos^3x-3\cos x$; $\sin 3x=-4\sin^3x+3\sin x$.

а) $I_1=\int\frac{\cos x}{4\cos^3x-3\cos x}\mathrm{d}x=\int\frac{\mathrm{d}x}{4\cos^2x-3}$ и от (5.4) имаме (полагаме $\operatorname{tg}x=t$):

$$I_1=\int\frac{\frac{1}{1+t^2}}{4\frac{1}{1+t^2}-3}\mathrm{d}t=\int\frac{\mathrm{d}t}{1-3t^2}=\frac{1}{\sqrt{3}}\int\frac{\mathrm{d}(t\sqrt{3})}{1-(t\sqrt{3})^2}=\frac{1}{2\sqrt{3}}\ln\left|\frac{1+t\sqrt{3}}{1-t\sqrt{3}}\right|+C$$

$$=\frac{1}{2\sqrt{3}}\ln\left|\frac{1+\sqrt{3}\operatorname{tg}x}{1-\sqrt{3}\operatorname{tg}x}\right|+C=\frac{1}{2\sqrt{3}}\ln\left|\frac{\operatorname{ctg}x+\sqrt{3}}{\operatorname{ctg}x-\sqrt{3}}\right|+C;$$

б) $I_2=\int\frac{\sin x}{3\sin x-4\sin^3x}\mathrm{d}x=\int\frac{\mathrm{d}x}{3-4\sin^2x}$ и аналогично на а) получаваме

$$I_2=\int\frac{\frac{1}{1+t^2}}{3-\frac{4t^2}{1+t^2}}\mathrm{d}t=\int\frac{\mathrm{d}t}{3-t^2}=\frac{1}{2\sqrt{3}}\ln\left|\frac{\sqrt{3}+t}{\sqrt{3}-t}\right|+C$$

$$=\frac{1}{2\sqrt{3}}\ln\left|\frac{\sqrt{3}+\operatorname{tg}x}{\sqrt{3}-\operatorname{tg}x}\right|+C.$$

Пример 5.9. Решете интеграла $\displaystyle\int \frac{\mathrm{tg}x}{\sqrt{1+\sin^2 x}}\mathrm{d}x$.

Решение. Полагаме

$$\sqrt{1+\sin^2 x} = t \Longrightarrow 1+\sin^2 x = t^2 \Longrightarrow \sin x = \sqrt{t^2-1}; \quad x = \arcsin\sqrt{t^2-1}$$

$$\Longrightarrow \mathrm{d}x = \frac{1}{\sqrt{1-t^2+1}}\cdot\frac{1}{2\sqrt{t^2-1}}\cdot 2t\mathrm{d}t = \frac{t\mathrm{d}t}{\sqrt{2-t^2}\sqrt{t^2-1}};$$

$$\mathrm{tg}x = \frac{\sin x}{\sqrt{1-\sin^2 x}} = \frac{\sqrt{t^2-1}}{\sqrt{2-t^2}}$$

$$\Longrightarrow I = \int \frac{\frac{\sqrt{t^2-1}}{\sqrt{2-t^2}}}{t}\cdot\frac{t\mathrm{d}t}{\sqrt{2-t^2}\sqrt{t^2-1}} = \int\frac{\mathrm{d}t}{\left(\sqrt{2-t^2}\right)^2} = \int\frac{\mathrm{d}t}{2-t^2}$$

$$= \frac{1}{2\sqrt{2}}\ln\left|\frac{\sqrt{2}+t}{\sqrt{2}-t}\right| + C = \frac{1}{2\sqrt{2}}\ln\frac{(\sqrt{2}+t)^2}{|2-t^2|}+C$$

$$= \frac{1}{2\sqrt{2}}\ln\frac{(\sqrt{2}+\sqrt{1+\sin^2 t})^2}{2-1-\sin^2 t}+C = \frac{1}{2\sqrt{2}}\ln\frac{(\sqrt{2}+\sqrt{1+\sin^2 t})^2}{\cos^2 t}+C$$

$$= \frac{1}{\sqrt{2}}\ln\frac{\sqrt{2}+\sqrt{1+\sin^2 t}}{|\cos t|}+C.$$

Пример 5.10. Да се реши интегралът $I_{m.n} = \displaystyle\int \cos^m x\sin^n x\mathrm{d}x$, $n.m \in \mathbb{N}$.

Решение.

1) При $m < n$:

$$I_{m,n} = \int\cos^{m-1}x\sin^n x\mathrm{d}(\sin x) = \frac{1}{n+1}\int\cos^{m-1}x\mathrm{d}(\sin^{n+1}x)$$

$$= \frac{1}{n+1}\cos^{m-1}x\sin^{n+1}x - \frac{1}{n+1}\int\sin^{n+1}x\mathrm{d}(\cos^{m-1}x)$$

$$= A + \frac{m-1}{n+1}\int\sin^{n+2}x\cos^{m-2}x\mathrm{d}x$$

$$\Longrightarrow I_{m,n} = \frac{\cos^{m-1}x\sin^{n+1}x}{n+1} + \frac{m-1}{n+1}I_{m-2,n+2}.$$

Забележка. Рекурентната формула се прилага неколкократно, като при $m = 2k$ се стига до интеграл от вида $\int\sin^n x\mathrm{d}x$ (вж. пример 2.14.б), а при $m = 2k+1$ до

$$\int\sin^n x\cos x\mathrm{d}x = \int\sin^n x\mathrm{d}(\sin x) = \frac{\sin^{n+1}x}{n+1}.$$

2) При $n < m$:

$$
\begin{aligned}
I_{m,n} &= -\int \cos^m x \sin^{n-1} x \mathrm{d}(\cos x) = -\frac{1}{m+1}\int \sin^{n-1} x \mathrm{d}(\cos^{m+1} x) \\
&= -\frac{1}{m+1}\sin^{n-1} x \cos^{m+1} x + \frac{1}{m+1}\int \cos^{m+1} x \mathrm{d}(\sin^{n-1} x) \\
&= A + \frac{n-1}{m+1}\int \cos^{m+2} x \sin^{n-2} x \mathrm{d}x \\
&\Longrightarrow I_{m,n} = -\frac{\sin^{n-1} x \cos^{m+1} x}{m+1} + \frac{n-1}{m+1} I_{m+2,n-2}.
\end{aligned}
$$

Забележка. Рекурентната формула се прилага неколкократно, като при $n = 2k$ се стига до интеграл от вида $\int \cos^n x \mathrm{d}x$ (вж. пример 2.14.а), а при $n = 2k+1$ до

$$\int \cos^n x \sin x \mathrm{d}x = -\int \cos^n x \mathrm{d}(\cos x) = -\frac{\cos^{n+1} x}{n+1}.$$

ЗАДАЧИ

Решете *интегралите*:

1. $\displaystyle\int \frac{\sin x \mathrm{d}x}{\cos x + \sin^2 x}$, пол. $\cos x = t$ Отг. $\displaystyle\frac{1}{\sqrt{5}} \ln\left|\frac{2\cos x - 1 + \sqrt{5}}{2\cos x - 1 - \sqrt{5}}\right| + C$

2. $\displaystyle\int \frac{\sin^3 x \mathrm{d}x}{1 + \cos^2 x}$ Отг. $\cos x - 2\mathrm{arctg}(\cos x) + C$

3. $\displaystyle\int \frac{\sin x \mathrm{d}x}{2 - \sin^2 x - \cos x}$ Отг. $-\frac{4}{3}\mathrm{arctg}\displaystyle\frac{2\cos x - 1}{\sqrt{3}} + C$

4. $\displaystyle\int \frac{\sin 2x \mathrm{d}x}{\cos^3 x + \sin^2 x + 1}$ Отг. $\frac{1}{5}\ln\displaystyle\frac{(1+\cos x)^2}{\cos^2 x - 2\cos x + 2} - \frac{6}{5}\mathrm{arctg}(\cos x - 1) + C$

5. $\displaystyle\int \frac{\mathrm{d}x}{\sqrt{\cos x(1 - \cos x)}}$ Отг. $\displaystyle\frac{1}{\sqrt{2}}\ln\frac{\sqrt{1+\cos x} - \sqrt{2\cos x}}{\sqrt{1+\cos x} + \sqrt{2\cos x}} + C$

6. $\displaystyle\int \frac{\cos x \mathrm{d}x}{3 + \cos^2 x}$, пол. $\sin x = t$ Отг. $\displaystyle\frac{1}{2\sqrt{2}}\ln\frac{\sqrt{2} + \sin x}{\sqrt{2} - \sin x} + C$

7. $\displaystyle\int \sin^{10} x \cos^5 x \mathrm{d}x$ Отг. $\displaystyle\frac{\sin^{11} x}{11} - 2\frac{\sin^{13} x}{13} + \frac{\sin^{15} x}{15} + C$

8. $\displaystyle\int \frac{\mathrm{d}x}{\cos x \sin^3 x}$ Отг. $\ln|\mathrm{tg}x| - \displaystyle\frac{1}{2\sin^2 x} + C$

9. $\displaystyle\int \frac{\sin^2 x \cos x}{1 + \sin^2 x}\mathrm{d}x$ Отг. $\sin x - \mathrm{arctg}(\sin x) + C$

10. $\displaystyle\int \frac{\mathrm{d}x}{\sin 2x \sin x + \cos x}$ Отг. $\frac{1}{6}\ln\displaystyle\frac{1+\sin x}{1-\sin x} + \frac{2}{3\sqrt{2}}\mathrm{arctg}(\sqrt{2}\sin x) + C$

11. $\displaystyle\int \frac{\sin x dx}{\sin^3 x - \cos^3 x}$, пол.$\mathrm{tg}x = t$

Отг. $\frac{1}{3}\ln\dfrac{|\mathrm{tg}x - 1|}{\sqrt{\mathrm{tg}^2x + \mathrm{tg}x + 1}} + \frac{1}{\sqrt{3}}\mathrm{arctg}\dfrac{2\mathrm{tg}x + 1}{\sqrt{3}} + C$

12. $\displaystyle\int \frac{\sin x + \cos x}{\sin x \cos^2 x} dx$ Отг. $\mathrm{tg}x + \ln|\mathrm{tg}x| + C$

13. $\displaystyle\int \frac{\cos 2x dx}{\sin^2 x \cos^2 x}$ Отг. $-(\mathrm{ctg}x + \mathrm{tg}x) + C$

14. $\displaystyle\int \frac{\cos 2x dx}{1 + \sin x \cos x}$ Отг. $\ln\dfrac{\mathrm{tg}^2x + \mathrm{tg}x + 1}{\mathrm{tg}^2x + 1} + C$

15. $\displaystyle\int \frac{\sin 2x dx}{\sin^4 x + \cos^4 x}$ Отг. $\mathrm{arctg}(\mathrm{tg}^2x) + C$

16. $\displaystyle\int \frac{1 + \mathrm{tg}^2x}{\mathrm{tg}^2x(4 + \mathrm{tg}^2x)} dx$ Отг. $-\frac{1}{4}\mathrm{ctg}x - \frac{1}{8}\mathrm{arctg}\left(\dfrac{\mathrm{tg}x}{2}\right) + C$

17. $\displaystyle\int \frac{dx}{\sin^2 x + \mathrm{tg}^2x}$ Отг. $\frac{1}{4}\ln\dfrac{\mathrm{tg}^2x}{2 + \mathrm{tg}^2x} + C$

18. $\displaystyle\int \frac{\sin x dx}{2 + \sin x + 2\cos x}$, пол. $\mathrm{tg}\frac{x}{2} = t$ Отг. $\frac{2}{5}\ln\dfrac{\mathrm{tg}^2\frac{x}{2} + 1}{(\mathrm{tg}^2\frac{x}{2} + 2)^2} + \frac{2}{5}\mathrm{arctg}(\mathrm{tg}\frac{x}{2}) + C$

19. $\displaystyle\int \frac{dx}{\cos x(2 + \sin x + \cos x)}$ Отг. $\frac{1}{3}\ln\left|\dfrac{(1 + \mathrm{tg}\frac{x}{2})^3}{(1 - \mathrm{tg}\frac{x}{2})(\mathrm{tg}^2\frac{x}{2} + 2\mathrm{tg}\frac{x}{2} + 3)}\right| + C$

20. $\displaystyle\int \frac{1 + \sin x}{\sin x(1 + \cos x)} dx$ Отг. $\frac{1}{4}\mathrm{tg}^2\frac{x}{2} + \mathrm{tg}\frac{x}{2} + \frac{1}{2}\ln|\mathrm{tg}\frac{x}{2}| + C$

21. $\displaystyle\int \frac{dx}{1 - \sin x + \cos x}$ Отг. $-\ln|1 - \mathrm{tg}\frac{x}{2}| + C$

22. $\displaystyle\int \frac{dx}{\sin 2x - 2\sin x}$ Отг. $\frac{1}{3}\ln\dfrac{3\mathrm{tg}^2\frac{x}{2} + 1}{|\mathrm{tg}^3\frac{x}{2}|} + C$