261; HOMEWORK #1

1. Show the conservation equation for momentum, Eq I.23. To derive the V_iV_j term (Reynolds stress) you need to first show that $\rho dV_i/dt = \partial(\rho V_i)/\partial t + \partial(\rho V_iV_j)/\partial x_j$ using the continuity equation, then realize that the second term will come from transforming the pressure tensor from the laboratory (satellite) frame to the plasma frame.

Euler: $$\rho \frac{dV_i}{dt} = -\frac{\partial P_{ij}}{\partial x_j} + \rho_c E_i + (\vec{J} \times \vec{B})_i + \rho g_i \qquad \text{I.23}$$

Starting from Boltzmann:

$$\frac{\partial f}{\partial t} + \nabla\cdot(f\vec{v}) + \nabla_v\cdot\left(\frac{\vec{F}}{m}f\right) = \left(\frac{\partial f}{\partial t}\right)_{coll}$$

Remember: $M_n\left[\left(\frac{\partial f_a}{\partial t}\right)_{coll}\right] = 0 \qquad n = 0,1,2$ only

because collisions conserve momentum, this is true for $n=1$, after summation over species. (Electron mom. gain is ion loss)

Remember: We discussed in class how the last term (force term) n-moment can be simplified to get: $M_n\left[\vec{\nabla}_v\left(\frac{\vec{F}}{m}f\right)\right] = -\sum_a \frac{1}{n!}\int n\, V_j^{n-1} F_j^a f^a d^3v$

For $n=1$, get $M_1\left[\vec{\nabla}_v\left(\frac{\vec{F}}{m}f\right)\right] = -\sum_a \int F_j^a f^a d^3v = -\langle F\rangle$

where $\langle F\rangle$ is $\sum_a\int q^a E_j f d^3v + \sum_a\int q^a \vec{V}_j\times\vec{B} f d^3v = \underbrace{\rho_c\vec{E} + \vec{J}\times\vec{B}}_{\text{Force density}}$

1st term: $M_1\left[\partial f/\partial t\right] = \sum_a m_a\int v_j \partial f^a/\partial t\, d^3v =$
$\partial/\partial t\left\{\sum_a m_a\int v_j f d^3v\right\} = \partial/\partial t(\rho V_j)$ ✓
$= \partial/\partial t(\rho\vec{v})$

2^{nd} term: $M_1\left[\vec{\nabla}(f\vec{v})\right] = \sum_a m_a \int v_i \frac{\partial}{\partial x_j}(f^a v_j)\, d^3v =$ (now i index refers to vector in Euler and j index is internal sum)

$$= \sum_a m_a \int \frac{\partial}{\partial x_j}\left\{ f^a v_i v_j \right\} d^3v$$ (note: $\frac{\partial v_i}{\partial x_j} = 0$: independent variables)

Now consider the velocity in the fluid center of mass with fluid velocity V (capitalized v).

Then $\bar{v} - \vec{V}$ is the velocity in that frame.

$$f^a v_i v_j = f^a (v_i - V_i + V_i)(v_j - V_j + V_j) =$$

$$f^a (v_i - V_i)(v_j - V_j) + \underbrace{f^a (v_i - V_i)}V_j + \underbrace{f^a (v_j - V_j)}V_i + f^a V_i V_j$$

integral over d^3v of these will be $\emptyset$

Left with: $$\frac{\partial}{\partial x_j}\sum_a m_a \underbrace{\int f^a (v_i - V_i)(v_j - V_j)\, d^3v}_{P_{ij}} + \frac{\partial}{\partial x_j}(\rho V_i V_j)$$

$$= \frac{\partial}{\partial x_j}\left[P_{ij} + \rho V_i V_j\right] = \vec{\nabla}\left\{ \overleftrightarrow{P} + \rho \vec{V}\vec{V} \right\}$$

Summing all terms:

$$\text{L.H.S} = \partial/\partial t(\rho \bar{V}) + \bar{\nabla}(\rho \vec{V}\vec{V}) = -\bar{\nabla}\overleftrightarrow{P} + \rho_c \vec{E} + \vec{J}\times\vec{B}$$

Now use continuity eqn to deal with L.H.S.

$$\frac{\partial \rho}{\partial t} + \bar{\nabla}(\rho\vec{V}) = 0 \Rightarrow \vec{v}\frac{\partial\rho}{\partial t} + \bar{v}\,\underbrace{\bar{\nabla}\cdot(\rho\bar{v})}_{\text{scalar}} = 0 = \boxed{v_i \frac{\partial \rho}{\partial t} + v_i \frac{\partial(\rho v_j)}{\partial x_j}}$$

$$\text{Note that: } \rho\frac{d\vec{v}}{dt} = \rho\frac{\partial \vec{v}}{\partial t} + \rho(\vec{v}\cdot\vec{\nabla})\vec{v} = \overbrace{\rho\frac{\partial v_i}{\partial t} + \rho\left(v_j \frac{\partial}{\partial x_j}\right)v_i}^{\text{GOAL}}$$

$$\text{LHS} = \rho\frac{\partial v_i}{\partial t} + v_i\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_j}(\rho v_i v_j) = \rho\frac{\partial v_i}{\partial t} + \overbrace{\boxed{v_i\frac{\partial\rho}{\partial t} + v_i\frac{\partial}{\partial x_j}(\rho v_j)}}^{\text{from continuity} = 0} + \rho v_j\frac{\partial v_i}{\partial x_j}$$

$$= \rho\frac{\partial v_i}{\partial t} + \rho v_j\frac{\partial v_i}{\partial x_j} = \text{GOAL} = \boxed{\rho\frac{d\bar{v}}{dt} = -\bar{\nabla}\overleftrightarrow{P} + \rho_c\vec{E} + \vec{J}\times\vec{B}}$$ ✓

2. Derive the energy conservation Equation I.31 by taking the n=2 moment M_2 of the Boltzmann equation.

Boltzmann:

$$\frac{\partial f}{\partial t} + \nabla\cdot(f\vec{v}) + \nabla_v\cdot\left(\frac{\vec{F}}{m}f\right) = \left(\frac{\partial f}{\partial t}\right)_{coll}$$

Energy conservation:

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho V^2 + u\right) + \frac{\partial}{\partial x_j}\left[\left(\frac{1}{2}\rho V^2 + u\right)V_j + P_{jk}V_k + q_j\right] = J_j E_j + \rho V_j g_j$$

As before, the collision term gives: $M_2\left[(\partial f/\partial t)_{col}\right] = 0$ because the energy gain of ions is equal to the energy loss of e^- since energy is conserved (collisions are elastic, wave-particle interactions are included in f and F).

As before, the n=2 moment of the force term is simplified:

$$M_2\left[\bar{\nabla}_v\cdot\frac{\bar{F}}{m}f\right] = -\sum_a \frac{\cancel{m_a}}{n!}\int n\, v_j^{n-1}\frac{F_j}{\cancel{m_a}} f^a d^3v = -\sum_a \int \underbrace{v_j F_j}_{\vec{v}\cdot\vec{F}} f^a d^3v =$$

using $\vec{F}^a = q^a\vec{E} + \cancel{q^a\vec{v}\times\vec{B}} + m_a\vec{g}$, get: $\vec{v}\cdot(\vec{v}\times\vec{B}) = 0$

$$= \underbrace{-\sum_a \int q^a E_j v_j f^a d^3v}_{E_j J_j} - \underbrace{\left(\sum_a m_a \int f v_j d^3v\right)}_{\rho V_j} g_j = -\vec{J}\cdot\vec{E} - \rho\vec{V}\cdot\vec{g}$$

$\underline{1^{st}\text{ term}}$: $M_2\left[\partial f/\partial t\right] = \frac{\partial}{\partial t}\sum_a \frac{m_a}{2}\int f^a v^2 d^3v$; in plasma frame ($\vec{V}$) it is: $v_j = (v_i - V_i) + V_i$

$$v_i v_i = (v_i - V_i)^2 + V_i^2 - 2\cancel{(v_i - V_i)}\cdot V_i$$

implied sum

This will give 0 when integrated over f.

$$= \frac{\partial}{\partial t}\left\{\sum_a m_a \int f^a\left[(v_i - V_i)^2 + V_i^2\right]d^3v\right\}$$

sum implied

This is internal energy, u

$$= \frac{\partial}{\partial t}\left\{u + \frac{1}{2}\rho\vec{V}^2\right\}$$

$\underline{2^{nd}\text{ term}}$: $M_2\left[\bar\nabla(f\bar v)\right] = \sum_a \frac{m_a}{2}\int v^2 \frac{\partial}{\partial x_j}(f^a v_j)\,d^3v$; again replace $v_i \to (v_i - V_i) + V_i$ to move in the fluid frame.

This time ($n=2$) we have 3 velocities: (indices: $i\,j\,k$)

so $v^2 \cdot v_j \to \left[(v_i-V_i)^2 + V_i^2 + 2(v_i-V_i)\cdot V_i\right]\cdot\left[(v_j-V_j)+V_j\right]$

implied summation

This will produce 6 terms but 2 terms will be zero because they involve $(v_i - V_i)$ times a fluid velocity V_j which can be gotten out of the integral, and $(v_i - V_i)$ integrates to zero. The 4 terms are:

$$\underbrace{(v_i-V_i)^2\cdot(v_j-V_j)}_{\text{heat flux}} + \underbrace{(v_i-V_i)^2 V_j}_{\text{internal energy}} + \underbrace{2(v_i-V_i)(v_j-V_j)\cdot V_i}_{\text{pressure tensor}} + \underbrace{V_i^2 V_j}_{\text{constant, will give }\rho}$$

summation implied (this is a scalar)

Performing the integration and summation, get:

$$\underbrace{\sum_a \frac{m_a}{2}\frac{\partial}{\partial x_j}\left\{\int f^a(v_i-V_i)^2(v_j-V_j)\,d^3v\right\}}_{\frac{\partial}{\partial x_j}\{q_j\}} + \underbrace{\sum_a \frac{m_a}{2}\frac{\partial}{\partial x_j}\int f(v_i-V_i)^2\,d^3v\; V_j}_{\frac{\partial}{\partial x_j}(u\cdot V_j)}$$

$$+ \underbrace{\sum_a m_a \frac{\partial}{\partial x_j}\int f^a(v_i-V_i)(v_j-V_j)\,d^3v\cdot V_i}_{\frac{\partial}{\partial x_j}P_{ij}V_i = \frac{\partial}{\partial x_j}P_{jk}V_k} + \underbrace{\sum_a \frac{m_a}{2}\int f^a\,d^3v\; V_i^2 V_j}_{\frac{1}{2}\rho\,\vec V^2\cdot V_j}$$

$P_{ij} = P_{ji}$ rename index $i \to k$

Summing all terms we get:

$$\boxed{\frac{\partial}{\partial t}\left\{\frac{1}{2}\rho V^2 + u\right\} + \frac{\partial}{\partial x_j}\left\{\left(\frac{1}{2}\rho V^2 + u\right)V_j + P_{jk}V_k + q_j\right\} = J_j E_j + \rho V_j g_j}$$

3. Prove that the Maxwell Stress tensor T_{ij} is indeed given by Equation I.38, by showing that the electromagnetic force is its divergence: $\partial T_{ij}/\partial x_j$, in other words prove Eq. I.40.

Explain why the Poynting vector term appears in Eq. I.40, but not in Eq. I.23. (Note that both equations describe conservation laws, one for the field, and the other for the fluid).

Maxwell Stress Tensor:

$$T_{ij} \equiv \varepsilon_o E_i E_j + \frac{1}{\mu_o} B_i B_j - \left(\frac{\varepsilon_o}{2} E^2 + \frac{1}{2\mu_o} B^2\right) \delta_{ij}$$

Show;

$$\frac{\partial T_{ij}}{\partial x_j} = \varepsilon_o \mu_o \frac{\partial S_i}{\partial t} + \rho_c E_i + (\vec{J} \times \vec{B})_i$$

Take $\bar{E}$-field terms, recognize similarity with gravitational stress tensor we did in class:

$$\frac{\partial}{\partial x_j}\left(\epsilon_o E_i E_j - \frac{\epsilon_o}{2} E^2 \delta_{ij}\right) = \epsilon_o E_i \underbrace{\frac{\partial E_j}{\partial x_j}}_{\text{Divergence, use Gauss}} + \epsilon_o E_j \frac{\partial}{\partial x_j} E_i - \frac{\epsilon_o}{2}\frac{\partial E^2}{\partial x_i}$$

$$= \rho_c \cdot E_i + \epsilon_o \Big\{ \underbrace{E_j \frac{\partial E_i}{\partial x_j} - \frac{1}{2}\frac{\partial E^2}{\partial x_i}}_{} \Big\}$$

Use identity: $\bar{\nabla}\left(\frac{A^2}{2}\right) = (\bar{A}\cdot\bar{\nabla})\bar{A} + \bar{A}\times(\bar{\nabla}\times\bar{A})$

This is $(\bar{E}\cdot\bar{\nabla})\bar{E} - \frac{1}{2}\bar{\nabla}(\vec{E}^2) = -\bar{E}\times\underbrace{(\bar{\nabla}\times\bar{E})}_{\text{use Faraday}} = \bar{E}\times\frac{\partial \bar{B}}{\partial t}$

$$= \rho_c E_i + \epsilon_o \left(\bar{E}\times\frac{\partial \bar{B}}{\partial t}\right)_i$$

Now, for $\bar{B}$-field terms:

$$\frac{\partial}{\partial x_j}\left(\frac{B_i B_j}{\mu_o} - \frac{B^2}{2\mu_o}\delta_{ij}\right) = \frac{1}{\mu_o} B_i \underbrace{\cancel{\frac{\partial B_j}{\partial x_j}}}_{\text{Divergence, use Gauss} \to 0} + \frac{1}{\mu_o} B_j \frac{\partial B_i}{\partial x_j} - \frac{1}{2\mu_o}\frac{\partial B^2}{\partial x_i}$$

$$= \frac{1}{\mu_o}\left\{ B_j \frac{\partial B_i}{\partial x_j} - \frac{1}{2}\frac{\partial B^2}{\partial x_j} \right\} \rightarrow \text{This is } \frac{1}{\mu_o}\left[(\bar{B}\cdot\bar{\nabla})\bar{B} - \bar{\nabla}\frac{B^2}{2}\right] = -\frac{1}{\mu_o}\bar{B}\times(\bar{\nabla}\times\bar{B})$$

use Max. Amp law

$$= -\frac{1}{\mu_o}\left\{\bar{B}\times\left(\mu_o\bar{J} + \epsilon_o\mu_o\frac{\partial\bar{E}}{\partial t}\right)\right\}_i = \left\{+\bar{J}\times\bar{B} + \epsilon_o\frac{\partial\bar{E}}{\partial t}\times\bar{B}\right\}_i$$

Sumgin all components get:

$$\frac{\partial}{\partial x_j} T_{ij} = \rho_c E_i + (\bar{J} \times \bar{B})_i + \epsilon_0 \left\{ \bar{E} \times \frac{\partial \bar{B}}{\partial t} + \frac{\partial \bar{E}}{\partial t} \times \bar{B} \right\}_i =$$

$$= \rho_c E_i + (\bar{J} \times \bar{B})_i + \epsilon_0 \frac{\partial}{\partial t} \underbrace{\left(\frac{\bar{E} \times \bar{B}}{\mu_0} \right)_i}_{S_i} \cdot \mu_0 = \rho_c E_i + (\bar{J} \times \bar{B})_i + \epsilon_0 \mu_0 \frac{\partial}{\partial t} \left(\frac{\bar{E} \times \bar{B}}{\mu_0} \right)_i$$

$$\boxed{\frac{\partial}{\partial x_j} T_{ij} = \rho_c E_i + (\bar{J} \times \bar{B})_i + \epsilon_0 \mu_0 \frac{\partial S_i}{\partial t}}$$

Note that $\frac{\vec{S}}{c^2}$ is momentum density and appears in this equation for the EM momentum equation. The Poynting flux, which is energy flux density appears only because it is related to the momentum density as $\frac{\vec{S}}{c^2}$. There is no reason to appear in the fluid momentum conservation eqn.

4. Show the conservation of gravitational energy, $\rho\phi$, Eq. I.45

$$\frac{\partial}{\partial t} (\rho \phi) + \nabla \cdot (\rho \phi \vec{V}) = - \rho \vec{V} \cdot \vec{g} + \rho \frac{\partial \phi}{\partial t} \qquad \text{(I.45)}$$

Following Siscoe's notes, we start from $\vec{\nabla}(\rho \phi \vec{v})$:

$$\bar{\nabla}(\rho \phi \vec{v}) = \rho \phi \bar{\nabla} \cdot \vec{v} + (\vec{v} \cdot \bar{\nabla}) \rho \phi = - \frac{d\rho}{dt} \cdot \phi + \rho (\bar{v} \cdot \bar{\nabla}) \phi + \phi (\bar{v} \cdot \bar{\nabla}) \rho$$

replace from continuity eqn.: $\frac{d\rho}{dt} + \rho \bar{\nabla} \bar{v} = 0$

$$= - \phi \frac{\partial \rho}{\partial t} - \cancel{\phi \cdot \bar{v} \bar{\nabla} \rho} + \rho (\bar{v} \cdot \bar{\nabla}) \phi + \cancel{\phi \cdot (\bar{v} \cdot \bar{\nabla}) \rho}$$

$$= - \frac{\partial (\rho \phi)}{\partial t} + \rho \frac{\partial \phi}{\partial t} + \rho \underbrace{(\bar{v} \cdot \bar{\nabla}) \phi}_{-\vec{g}}$$

Reerranging, we get:

$$\boxed{\frac{\partial (\rho \phi)}{\partial t} + \bar{\nabla} (\rho \phi \bar{v}) = - \rho \bar{V} \cdot \vec{g} + \rho \frac{\partial \phi}{\partial t}}$$

5. You are given a 1-D distribution f(u)= a / $[1+(u-v)^2/w^2]^2$ where a, v, and w are constants.

a. Sketch f(u) and indicate the meanings of a, v, and w.

b. Show that:

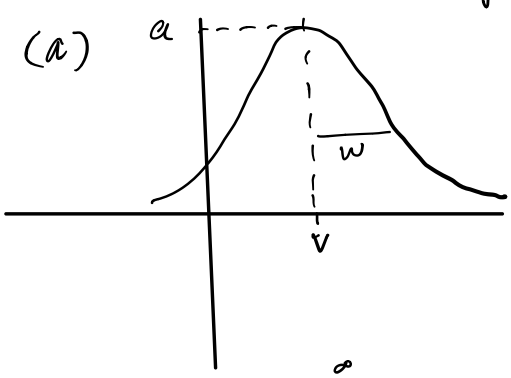
i. the number density is n= (π/2) a w

ii. the convection speed is v

iii. the pressure is r w^2, where ρ=nm and m=the mass of the particles

ρ

(a)



(b)

$$n = \int_{-\infty}^{\infty} f(u)\, du = \int \frac{a\, du}{\left[1 + \left(\frac{u-v}{w}\right)^2\right]^2} \qquad \text{set } x = \frac{u-v}{w}$$

$$= wa \int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^2} \qquad \text{set } x = \tan\theta \quad -\infty \to +\infty \quad \theta: -\pi/2 \to +\pi/2$$

$$dx = \frac{1}{\cos^2\theta}\, d\theta$$

$$= wa \int_{-\pi/2}^{\pi/2} \frac{d\theta}{\cos^2\theta} \left[\cos^2\theta\right]^2 \qquad \frac{1}{[1+x^2]^2} = \frac{1}{\left[\frac{1}{\cos^2\theta}\right]^2} = \left[\cos^2\theta\right]^2$$

$$= wa \underbrace{\int_{-\pi/2}^{\pi/2} \cos^2\theta\, d\theta}_{\frac{1}{2} \text{ of } \int_{-\pi/2}^{\pi/2} [\sin^2\theta + \cos^2\theta]\, d\theta = \int_{-\pi/2}^{\pi/2} 1 \cdot d\theta = \pi} = \frac{\pi}{2} \cdot wa$$

$$\Rightarrow \boxed{n = \frac{\pi}{2} a w}$$

$$V_{conv} = \frac{1}{n}\int u \cdot f(u)\,du = \frac{a}{n}\int_{-\infty}^{\infty} \frac{u\,du}{\left[1+\left(\frac{u-v}{w}\right)^2\right]^2} = \qquad x = \frac{u-v}{w}$$

$$\frac{aw^2}{n}\int_{-\infty}^{\infty}\left\{\frac{(x+\frac{v}{w})\,dx}{[1+x^2]^2}\right\} = \frac{aw^2}{n}\left\{\cancel{\int_{-\infty}^{\infty}\frac{x}{[1+x^2]^2}dx} + \frac{v}{w}\int_{-\infty}^{\infty}\frac{dx}{[1+x^2]^2}\right\}$$

odd function has zero integral

$$= \frac{avw}{n}\underbrace{\int_{-\infty}^{\infty}\frac{dx}{[1+x^2]^2}}_{\text{as before } = \pi/2} = \frac{\pi}{2}aw\frac{v}{\cancel{n}}$$

n

$$\Rightarrow \boxed{V_{conv} = V}$$

NEXT SHOW: $P = (\overset{\rho}{n \cdot m}) \cdot w^2$

$$P = m\int (u-v)^2 f(u)\,du = maw^3\int\left(\frac{u-v}{w}\right)^2 \frac{d\left(\frac{u-v}{w}\right)}{\left[1+\left(\frac{u-v}{w}\right)^2\right]^2} = \qquad \text{set } x = \frac{u-v}{w}$$

$$= maw^3\int\frac{x^2\,dx}{[1+x^2]^2} = maw^3\int_{-\infty}^{\infty}\frac{[1+x^2]-1}{[1+x^2]^2}dx$$

$$= maw^3\left\{\int_{-\infty}^{\infty}\frac{dx}{[1+x^2]} - \int_{-\infty}^{\infty}\frac{dx}{[1+x^2]^2}\right\}$$

$$x \to \tan\theta, \quad dx \to \frac{1}{\cos^2\theta}d\theta$$

$$= maw^3\left\{\underbrace{\int_{-\pi/2}^{\pi/2} d\theta}_{\pi} - \underbrace{\int_{-\pi/2}^{\pi/2}\cos^2\theta\,d\theta}_{\pi/2}\right\} = \frac{\pi}{2}maw^3$$

$$\Rightarrow P = \left(\frac{\pi}{2}aw\right)m w^2 \Rightarrow \boxed{P = (nm)w^2}$$

n