

Math 317: Homework 1

NAME:

General Hints: When you are asked to “show” or “prove” something, you should make it a point to write down clearly the information you are given and what it is you are to show. One word of warning regarding the second part of Problem 1.1.22: To say that \mathbf{v} is a linear combination of $\mathbf{v}_1, \dots, \mathbf{v}_k$ is to say that $\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \dots + c_k\mathbf{v}_k$ for some scalars c_1, \dots, c_k . These scalars will surely be different when you express a different vector \mathbf{w} as a linear combination of $\mathbf{v}_1, \dots, \mathbf{v}_k$, so be sure you give the scalars for \mathbf{w} different names.

(“SA 1.1.21” means exercise 21 in Section 1.1 of textbook by Shifrin and Adams.)

Problem 0 (SA 1.1.28). Carefully prove the following properties of vector arithmetic. Justify all steps. Optional: Give the geometric interpretation of each property in case $n = 2$ or $n = 3$. You are required to turn proofs for e. f. g. (but you are encouraged to try them all).

- a. For all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, $\mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x}$.
- b. For all $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^n$, $(\mathbf{x} + \mathbf{y}) + \mathbf{z} = \mathbf{x} + (\mathbf{y} + \mathbf{z})$.
- c. For all $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{0} + \mathbf{x} = \mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^n$.
- d. For each $\mathbf{x} \in \mathbb{R}^n$, there is a vector $-\mathbf{x}$ so that $\mathbf{x} + (-\mathbf{x}) = \mathbf{0}$.
- e. For all $c, d \in \mathbb{R}$ and $\mathbf{x} \in \mathbb{R}^n$, $c(d\mathbf{x}) = (cd)\mathbf{x}$.
- f. For all $c \in \mathbb{R}$ and $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, $c(\mathbf{x} + \mathbf{y}) = c\mathbf{x} + c\mathbf{y}$.
- g. For all $c, d \in \mathbb{R}$ and $\mathbf{x} \in \mathbb{R}^n$, $(c + d)\mathbf{x} = c\mathbf{x} + d\mathbf{x}$.
- h. For all $\mathbf{x} \in \mathbb{R}^n$, $1\mathbf{x} = \mathbf{x}$.

(If you will use this pdf to complete your homework, please write your solutions to Problem 0 on the next page.)

e. **Claim:** For all $c, d \in \mathbb{R}^n$ and $\mathbf{x} \in \mathbb{R}^n$, $c(d\mathbf{x}) = (cd)\mathbf{x}$.

Proof.

□

f. For all $c \in \mathbb{R}$ and $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, $c(\mathbf{x} + \mathbf{y}) = c\mathbf{x} + c\mathbf{y}$.

Proof.

□

g. For all $c, d \in \mathbb{R}$ and $\mathbf{x} \in \mathbb{R}^n$, $(c + d)\mathbf{x} = c\mathbf{x} + d\mathbf{x}$.

Proof.

□

Problem 1 (SA 1.1.21). Suppose $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ and c is a scalar. Prove that $\text{Span}(\mathbf{v} + c\mathbf{w}, \mathbf{w}) = \text{Span}(\mathbf{v}, \mathbf{w})$. (See the blue box on p. 12 of the textbook.)

Problem 2 (SA 1.1.22). Suppose the vectors \mathbf{v} and \mathbf{w} are both linear combinations of $\mathbf{v}_1, \dots, \mathbf{v}_k$.

- a. Prove for any scalar c that $c\mathbf{v}$ is a linear combination of $\mathbf{v}_1, \dots, \mathbf{v}_k$.
- b. Prove that $\mathbf{v} + \mathbf{w}$ is a linear combination of $\mathbf{v}_1, \dots, \mathbf{v}_k$.

Problem 3 (SA 1.1.25). Suppose $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ are nonparallel vectors. (Recall def, p.3 of text.)

- a. Prove that if $s\mathbf{x} + t\mathbf{y} = \mathbf{0}$, then $s = t = 0$. (Hint: Show neither $s \neq 0$ nor $t \neq 0$ is possible.)
- b. Prove that if $a\mathbf{x} + b\mathbf{y} = c\mathbf{x} + d\mathbf{y}$, then $a = c$ and $b = d$. (Hint: Use part a.)

Problem 4 (SA 1.2.11). Suppose $\mathbf{x}, \mathbf{v}_1, \dots, \mathbf{v}_k \in \mathbb{R}^n$ and \mathbf{x} is orthogonal to each of the vectors $\mathbf{v}_1, \dots, \mathbf{v}_k$. Show that \mathbf{x} is orthogonal to every linear combination $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \dots + c_k\mathbf{v}_k$.

Problem 5 (SA 1.2.18). Prove the triangle inequality: For all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, $\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$.
(Hint: Use the dot product to calculate $\|\mathbf{x} + \mathbf{y}\|^2$.)

Problem 6 (SA 1.3.12). Suppose $\mathbf{a} \neq \mathbf{0}$ and $\mathcal{P} \subset \mathbb{R}^3$ is the plane through the origin with normal vector \mathbf{a} . Suppose \mathcal{P} is spanned by \mathbf{u} and \mathbf{v} .

- a. Suppose $\mathbf{u} \cdot \mathbf{v} = 0$. Show that for every $\mathbf{x} \in \mathcal{P}$, we have $\mathbf{x} = \text{proj}_{\mathbf{u}} \mathbf{x} + \text{proj}_{\mathbf{v}} \mathbf{x}$.
- b. Suppose $\mathbf{u} \cdot \mathbf{v} = 0$. Show that for every $\mathbf{x} \in \mathbb{R}^3$, we have $\mathbf{x} = \text{proj}_{\mathbf{a}} \mathbf{x} + \text{proj}_{\mathbf{u}} \mathbf{x} + \text{proj}_{\mathbf{v}} \mathbf{x}$.
(Hint: Apply part a. to the vector $\mathbf{x} - \text{proj}_{\mathbf{a}} \mathbf{x}$.)
- c. Give an example to show the result of part a is false when \mathbf{u} and \mathbf{v} are not orthogonal.