

# Math 317: Homework 1

NAME:

**General Hints:** When you are asked to “show” or “prove” something, you should make it a point to write down clearly the information you are given and what it is you are to show. One word of warning regarding the second part of Problem 1.1.22: To say that  $\mathbf{v}$  is a linear combination of  $\mathbf{v}_1, \dots, \mathbf{v}_k$  is to say that  $\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \dots + c_k\mathbf{v}_k$  for some scalars  $c_1, \dots, c_k$ . These scalars will surely be different when you express a different vector  $\mathbf{w}$  as a linear combination of  $\mathbf{v}_1, \dots, \mathbf{v}_k$ , so be sure you give the scalars for  $\mathbf{w}$  different names.

(“SA 1.1.21” means exercise 21 in Section 1.1 of textbook by Shifrin and Adams.)

*Problem 0* (SA 1.1.28). Carefully prove the following properties of vector arithmetic. Justify all steps. Optional: Give the geometric interpretation of each property in case  $n = 2$  or  $n = 3$ . You are required to turn proofs for e. f. g. (but you are encouraged to try them all).

- a. For all  $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ ,  $\mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x}$ .
- b. For all  $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^n$ ,  $(\mathbf{x} + \mathbf{y}) + \mathbf{z} = \mathbf{x} + (\mathbf{y} + \mathbf{z})$ .
- c. For all  $\mathbf{x} \in \mathbb{R}^n$ ,  $\mathbf{0} + \mathbf{x} = \mathbf{x}$  for all  $\mathbf{x} \in \mathbb{R}^n$ .
- d. For each  $\mathbf{x} \in \mathbb{R}^n$ , there is a vector  $-\mathbf{x}$  so that  $\mathbf{x} + (-\mathbf{x}) = \mathbf{0}$ .
- e. For all  $c, d \in \mathbb{R}$  and  $\mathbf{x} \in \mathbb{R}^n$ ,  $c(d\mathbf{x}) = (cd)\mathbf{x}$ .
- f. For all  $c \in \mathbb{R}$  and  $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ ,  $c(\mathbf{x} + \mathbf{y}) = c\mathbf{x} + c\mathbf{y}$ .
- g. For all  $c, d \in \mathbb{R}$  and  $\mathbf{x} \in \mathbb{R}^n$ ,  $(c + d)\mathbf{x} = c\mathbf{x} + d\mathbf{x}$ .
- h. For all  $\mathbf{x} \in \mathbb{R}^n$ ,  $1\mathbf{x} = \mathbf{x}$ .

(If you will use this pdf to complete your homework, please write your solutions to Problem 0 on the next page.)

e. **Claim:** For all  $c, d \in \mathbb{R}^n$  and  $\mathbf{x} \in \mathbb{R}^n$ ,  $c(d\mathbf{x}) = (cd)\mathbf{x}$ .

*Proof.*

□

f. For all  $c \in \mathbb{R}$  and  $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ ,  $c(\mathbf{x} + \mathbf{y}) = c\mathbf{x} + c\mathbf{y}$ .

*Proof.*

□

g. For all  $c, d \in \mathbb{R}$  and  $\mathbf{x} \in \mathbb{R}^n$ ,  $(c + d)\mathbf{x} = c\mathbf{x} + d\mathbf{x}$ .

*Proof.*

□

*Problem 1* (SA 1.1.21). Suppose  $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$  and  $c$  is a scalar. Prove that  $\text{Span}(\mathbf{v} + c\mathbf{w}, \mathbf{w}) = \text{Span}(\mathbf{v}, \mathbf{w})$ . (See the blue box on p. 12 of the textbook.)

*Problem 2* (SA 1.1.22). Suppose the vectors  $\mathbf{v}$  and  $\mathbf{w}$  are both linear combinations of  $\mathbf{v}_1, \dots, \mathbf{v}_k$ .

- a. Prove for any scalar  $c$  that  $c\mathbf{v}$  is a linear combination of  $\mathbf{v}_1, \dots, \mathbf{v}_k$ .
- b. Prove that  $\mathbf{v} + \mathbf{w}$  is a linear combination of  $\mathbf{v}_1, \dots, \mathbf{v}_k$ .

*Problem 3* (SA 1.1.25). Suppose  $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$  are nonparallel vectors. (Recall def, p.3 of text.)

- a. Prove that if  $s\mathbf{x} + t\mathbf{y} = \mathbf{0}$ , then  $s = t = 0$ . (Hint: Show neither  $s \neq 0$  nor  $t \neq 0$  is possible.)
- b. Prove that if  $a\mathbf{x} + b\mathbf{y} = c\mathbf{x} + d\mathbf{y}$ , then  $a = c$  and  $b = d$ . (Hint: Use part a.)

*Problem 4* (SA 1.2.11). Suppose  $\mathbf{x}, \mathbf{v}_1, \dots, \mathbf{v}_k \in \mathbb{R}^n$  and  $\mathbf{x}$  is orthogonal to each of the vectors  $\mathbf{v}_1, \dots, \mathbf{v}_k$ . Show that  $\mathbf{x}$  is orthogonal to every linear combination  $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \dots + c_k\mathbf{v}_k$ .

*Problem 5* (SA 1.2.18). Prove the triangle inequality: For all  $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ ,  $\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$ .  
(Hint: Use the dot product to calculate  $\|\mathbf{x} + \mathbf{y}\|^2$ .)

*Problem 6* (SA 1.3.12). Suppose  $\mathbf{a} \neq \mathbf{0}$  and  $\mathcal{P} \subset \mathbb{R}^3$  is the plane through the origin with normal vector  $\mathbf{a}$ . Suppose  $\mathcal{P}$  is spanned by  $\mathbf{u}$  and  $\mathbf{v}$ .

- a. Suppose  $\mathbf{u} \cdot \mathbf{v} = 0$ . Show that for every  $\mathbf{x} \in \mathcal{P}$ , we have  $\mathbf{x} = \text{proj}_{\mathbf{u}} \mathbf{x} + \text{proj}_{\mathbf{v}} \mathbf{x}$ .
- b. Suppose  $\mathbf{u} \cdot \mathbf{v} = 0$ . Show that for every  $\mathbf{x} \in \mathbb{R}^3$ , we have  $\mathbf{x} = \text{proj}_{\mathbf{a}} \mathbf{x} + \text{proj}_{\mathbf{u}} \mathbf{x} + \text{proj}_{\mathbf{v}} \mathbf{x}$ .  
(Hint: Apply part a. to the vector  $\mathbf{x} - \text{proj}_{\mathbf{a}} \mathbf{x}$ .)
- c. Give an example to show the result of part a is false when  $\mathbf{u}$  and  $\mathbf{v}$  are not orthogonal.