
DM545 – Linear and Integer Programming

Answers to the Take-home Assignment, Winter 2025

In this assignment I'm sometimes using functions that I've defined in previous subtasks. I'm mentioning it because of the split-assignments-by-subtask rule. I try to indicate when I do that, but I am not defining them again, as I think it would make the assignment unnecessarily lengthy.

Additionally, I'm using the tableau function from the course github files.

Task 1

Subtask 1.a

We notice that the variables are bounded on both sides, so as the first inspection we can put those variables - whose coefficients in the objective function are positive - to their maximum value, and vice-versa for the negative ones, and check if the constraints are satisfied.

$$\begin{aligned}x_1 &= 5, x_2 = 1, x_3 = -2 \\4 \cdot 5 + 5 \cdot 1 - 7 \cdot (-2) &= 39 \\-5 - 1 + (-2) &= -8 \leq 2 \\-5 \cdot 5 + 10 \cdot (-2) &= -45 \leq 10\end{aligned}$$

As we can see - all constraints are satisfied and the objective function is maximized for all possible values of \mathbf{x} , therefore $x_1 = 5, x_2 = 1, x_3 = -2$ is a feasible and optimal solution for this problem.

Subtask 1.b

1. $s = 2, t = 1$
2. $s = 1, t = 1$
3. impossible
4. $s = 0, t = 0$

The three examples can be easily checked, but I suspect a few more words should be said about the third point.

For positive values s and t - x 's can just be put to 0 which gives a feasible solution.

For negative values s and t - the constraint will be satisfied by any pair of non-negative x 's.

For one positive and one negative - similarly, the x with the positive coefficient can be put to 0 and the other can be anything.

Task 2

Subtask 2.a

In order to write the tableau, we need to bring the problem into the equational standard form:

$$\begin{array}{ll} \min & z = 3x_1 + 2x_2 + 7x_3 \\ \text{s.t.} & -x_1 + x_2 = 10 \\ & 2x_1 - x_2 + x_3 \geq 10 \\ & x_1, x_2, x_3 \geq 0 \end{array} \rightarrow \begin{array}{ll} \max & w = -3x_1 - 2x_2 - 7x_3 \\ \text{s.t.} & -x_1 + x_2 + 0x_3 = 10 \\ & -2x_1 + x_2 - x_3 + s_1 = -10 \\ & x_1, x_2, x_3, s_1 \geq 0 \end{array}$$

```
import numpy as np

T = np.array([
    [-1, 1, 0, 0, 0, 10],
    [-2, 1, -1, 1, 0, -10],
    [-3, -2, -7, 0, 1, 0]
])

from util import tableau
tableau(T)
```

x1	x2	x3	x4	-z	b
-1	1	0	0	0	10
-2	1	-1	1	0	-10
-3	-2	-7	0	1	0

However, to be able to say anything about the optimality the tableau needs to be in canonical form - we can achieve that by bringing eg. x_2 in the basis:

```
def pivot(T, row, col):
    T[row] /= T[row][col] # normalize to 1
    for j in range(len(T)):
        if j != row:
            T[j] -= T[row] * T[j][col] # all other rows in col -> 0

pivot(T, 0, 1) # row 1 col 2
tableau(T)
```

x1	x2	x3	x4	-z	b
-1	1	0	0	0	10

	-1	0	-1	1	0	-20	
-----+-----+-----+-----+-----+							
	-5	0	-7	0	1	20	
-----+-----+-----+-----+-----+							

For a maximization problem (which it is in this form) the optimality condition is that all reduced costs must be negative (no further improvement). We can see that it is the case.

However, the current tableau is clearly not optimal (negative b-term) - we would need to apply dual simplex (basically just pivot on row 2 column 1) and check if the feasibility has been retrieved.

Subtask 2.b

According to the largest pivoting rule x_6 would have to be brought in the basis, however - it turns out that there is no most constraining constraint for the new value of x_6 .

More formally:

$$\max_{\theta} \{\bar{b} - a_{.6}\theta \geq 0\} = \infty$$

Which means that the problem is unbounded.

No matter what other pivoting rules we might apply, the conclusion will either be the same, or to bring x_3 in the basis (as it is the only other non-basic variable with a positive reduced cost) - so let's see what happens then:

For the iteration I will use the pivot function, which I've defined in the previous subtask.

```
T = np.array([
    [0, 0, 0, 1, 0, 0, 0],
    [0, 1, 1, 2, 0, -1, 0, 30],
    [1, 0, 1, 1, 0, -1, 0, 20],
    [0, 0, 2, -7, 0, 5, 1, -120]
])
pivot(T, 2, 2) # column 3 row 3 because of ratio test
tableau(T)
```

x1	x2	x3	x4	x5	x6	-z	b
0	0	0	1	1	0	0	0
-1	1	0	1	0	0	0	10
1	0	1	1	0	-1	0	20
-2	0	0	-9	0	7	1	-160

And the conclusion is the same - The only variable left with a positive reduced cost is x_6 and its value upon bringing into the basis is unconstrained.

Task 3

Subtask 3.a

- solution: $x_1 = 5, x_2 = 15$, objective value: 110
- reduced costs: $\bar{c} = [0, 0, -2, -2]^T$
- dual variables: $y_1 = 2, y_2 = 2$
- shadow prices: equal to the dual variables - $[2, 2]^T$
- no slack-capacity (as all slack variables are not in the basis, or because the dual variables are non-zero \Rightarrow from the complementary slackness theorem all constraints are binding)

Task 4

Subtask 4.a

Integrate the constraints into the objective function:

$$\begin{aligned}
 P = & \begin{array}{ll} \min & \sum_i c_i y_i \\ \forall_i & \sum_j a_j x_{ij} \leq b y_i \\ \forall_j & \sum_i x_{ij} = 1 \\ \forall_i & y_i \leq 1 \\ \forall_{ij} & x_{ij} \geq 0 \\ \forall_i & y_i \geq 0 \end{array} \rightarrow PR(\alpha, \beta, \gamma) = \min_{x,y} \left\{ \begin{array}{l} \sum_i c_i y_i \\ + \sum_i \alpha_i (b y_i - \sum_j a_j x_{ij}) \\ + \sum_j \beta_j (1 - \sum_i x_{ij}) \\ + \sum_i \gamma_i (1 - y_i) \end{array} \right\} \\
 & \forall_{ij} \quad x_{ij} \geq 0 \\
 & \forall_i \quad y_i \geq 0
 \end{aligned}$$

Now we want the objective function of $PR(\alpha, \beta, \gamma)$ to be always \leq than the original objective function. For that we need:

- $\alpha \leq \mathbf{0}$
- $\beta \in \mathbf{R}^m$
- $\gamma \leq \mathbf{0}$

Because then:

$$\begin{aligned}
 & \sum_i \alpha_i (b y_i - \sum_j a_j x_{ij}) \leq 0 \\
 \wedge \quad & \sum_j \beta_j (1 - \sum_i x_{ij}) = 0 \\
 \wedge \quad & \sum_i \gamma_i (1 - y_i) \leq 0
 \end{aligned} \implies \begin{aligned}
 & + \sum_i \alpha_i (b y_i - \sum_j a_j x_{ij}) \\
 & + \sum_j \beta_j (1 - \sum_i x_{ij}) \\
 & + \sum_i \gamma_i (1 - y_i)
 \end{aligned} \leq \sum_i c_i y_i$$

$$\implies opt(PR(\alpha, \beta, \gamma)) \leq opt(P)$$

Which means that the maximum value (with respect to α, β, γ like stated above) of $opt(PR(\alpha, \beta, \gamma))$ is also $\leq opt(P)$, therefore we can try maximizing $opt(PR(\alpha, \beta, \gamma))$ to “approximate” $opt(P)$ from below.

$$\begin{aligned}
 & \max_{\alpha, \beta, \gamma} \{opt(PR(\alpha, \beta, \gamma))\} = \\
 & = \max_{\alpha, \beta, \gamma} \left\{ \min_{x,y} \left\{ \begin{array}{l} \sum_i c_i y_i \\ + \sum_i \alpha_i (b y_i - \sum_j a_j x_{ij}) \\ + \sum_j \beta_j (1 - \sum_i x_{ij}) \\ + \sum_i \gamma_i (1 - y_i) \end{array} \right\} \right\} = \max_{\alpha, \beta, \gamma} \left\{ \min_{x,y} \left\{ \begin{array}{l} \sum_j \beta_j + \sum_i \gamma_i \\ + \sum_i \sum_j (-\alpha_i a_j - \beta_j) x_{ij} \\ + \sum_i (c_i + \alpha_i b - \gamma_i) y_i \end{array} \right\} \right\}
 \end{aligned}$$

In order to maximize this quantity we need:

- $\forall_{ij}(-\alpha_i a_j - \beta_j \geq 0)$
- $\forall_i(c_i + \alpha_i b - \gamma_i \geq 0)$

Because else:

$$\begin{aligned} & \exists_i(c_i + \alpha_i b - \gamma_i < 0) \\ \Rightarrow & \min_{x,y}\{(c_i + \alpha_i b - \gamma_i)y_i + \dots\} = -\infty \\ \Rightarrow & \max_{\alpha,\beta,\gamma}\{\min_{x,y}\{\dots\}\} = -\infty \end{aligned}$$

For $y_i = \infty$, and vice-versa for the coefficient next to x_{ij} .

Now we can observe that with these additional constraints:

$$\begin{aligned} \forall_{ij}(-\alpha_i a_j - \beta_j \geq 0) \wedge \forall_i(c_i + \alpha_i b - \gamma_i \geq 0) & \implies \arg \min_{x,y} \left\{ \begin{array}{l} \sum_j \beta_j + \sum_i \gamma_i \\ + \sum_i \sum_j (-\alpha_i a_j - \beta_j)x_{ij} \\ + \sum_i (c_i + \alpha_i b - \gamma_i)y_i \end{array} \right\} = (\mathbf{0}, \mathbf{0}) \\ \implies \max_{\alpha,\beta,\gamma} \{opt(PR(\alpha, \beta, \gamma))\} & = \max_{\alpha,\beta,\gamma} \left\{ \sum_j \beta_j + \sum_i \gamma_i \right\} \end{aligned}$$

And therefore we're left with the final dual problem:

$$D = \begin{array}{ll} \max & \sum_j \beta_j + \sum_i \gamma_i \\ \forall_{ij} & \alpha_i a_j + \beta_j \leq 0 \\ \forall_i & -\alpha_i b + \gamma_i \leq c_i \\ \forall_i & \alpha_i \leq 0 \\ \forall_j & \beta_j \in \mathbf{R} \\ \forall_i & \gamma_i \leq 0 \end{array}$$

Q.E.D.

Task 5

Subtask 5.a

To determine if the solution is optimal, we need to calculate the reduced costs corresponding to given non-basis variables. For that we can use the formula from the lecture:

$$\bar{c}_N^T = c_N^T - c_B^T A_B^{-1} A_N$$

And first let's bring the problem into maximization equational standard form for it's relaxation:

$$\begin{array}{ll} \min & x_1 + x_2 \\ -3x_1 + 7x_2 \leq -21 & \\ x_1 - 4x_2 \leq 4 & \\ x_1, x_2 \in \mathbf{Z}^+ & \end{array} \rightarrow \begin{array}{ll} \max & -x_1 - x_2 \\ -3x_1 + 7x_2 + s_1 = -21 & \\ x_1 - 4x_2 + s_2 = 4 & \\ x_1, x_2, s_1, s_2 \geq 0 & \end{array}$$

```
n = 2
m = 2

# model in standard form
A = np.array([
    [-3, 7],
    [1, -4]
])
b = np.array([-21, 4])
c = np.array([-1, -1])

# equational standard form - add slacks
A = np.hstack([A, np.identity(m)])
c = np.hstack([c, np.zeros(m)])

# new basis
B = [0, 1] # 1, 2
N = [2, 3] # 3, 4

# not perfectly efficient, but correct way to calculate c_bar
A_Binv = np.linalg.inv(A[:, B])
c_bar = c[N] - c[B] @ A_Binv @ A[:, N] # c bar
```

c_bar

['-1', '-2']

Since this is a maximization problem and all non-basic reduced costs are negative, the solution associated with basis $B = \{1, 2\}$ is optimal.

Subtask 5.b

To make life easier I will print the tableau corresponding to the solution, using some variables defined in the previous subtask:

```
# A and c - basic
T[:m, B] = np.identity(m)
T[-1, B] = np.zeros(m)

# A and c - nonbasic
T[:m, N] = A_Binv @ A[:, N]
T[-1, N] = c[N] - c[B] @ T[:m, N]

# b and d
T[:m, -1] = A_Binv @ b
T[-1, -1] = -c[B] @ T[:m, -1]
```

```
tableau(T)
```

x1	x2	x3	x4	-z	b
1	0	-4/5	-7/5	0	56/5
0	1	-1/5	-3/5	0	9/5
0	0	-1	-2	1	13

Now we have a clear view at what's going on. We're in a basic optimal solution, but infeasible for the integer problem and both basic variables are fractional.

The Gomory cuts are of form:

$$\sum_{j \in N} (\bar{a}_{ij} - \lfloor \bar{a}_{ij} \rfloor) x_i \geq \bar{b}_i - \lfloor \bar{b}_i \rfloor$$

Which in the equational standard form will become:

$$\sum_{j \in N} (-\bar{a}_{ij} + \lfloor \bar{a}_{ij} \rfloor) x_i + s_{m+i} = -\bar{b}_i + \lfloor \bar{b}_i \rfloor$$

```
def gomory(T, i):
    T = np.column_stack([ # new column for slack
        T[:, :-2],
        np.zeros_like(T[:, 0]),
        T[:, -2:]
    ])
```

```

cut = -T[i, :] + np.floor(T[i, :]) # gomory cut row
cut[-3] = 1 # for slack

T = np.vstack([ # insert row as last
    T[:-1, :],
    cut,
    T[-1, :]
])
return T

T = gomory(T, 0)
T = gomory(T, 1)
m += 2

```

```
tableau(T)
```

x1	x2	x3	x4	x5	x6	-z	b
1	0	-4/5	-7/5	0	0	0	56/5
0	1	-1/5	-3/5	0	0	0	9/5
0	0	-1/5	-3/5	1	0	0	-1/5
0	0	-4/5	-2/5	0	1	0	-4/5
0	0	-1	-2	0	0	1	13

And here we have the updated tableau with the Chvatal-Gomory cuts. As expected - it is infeasible.

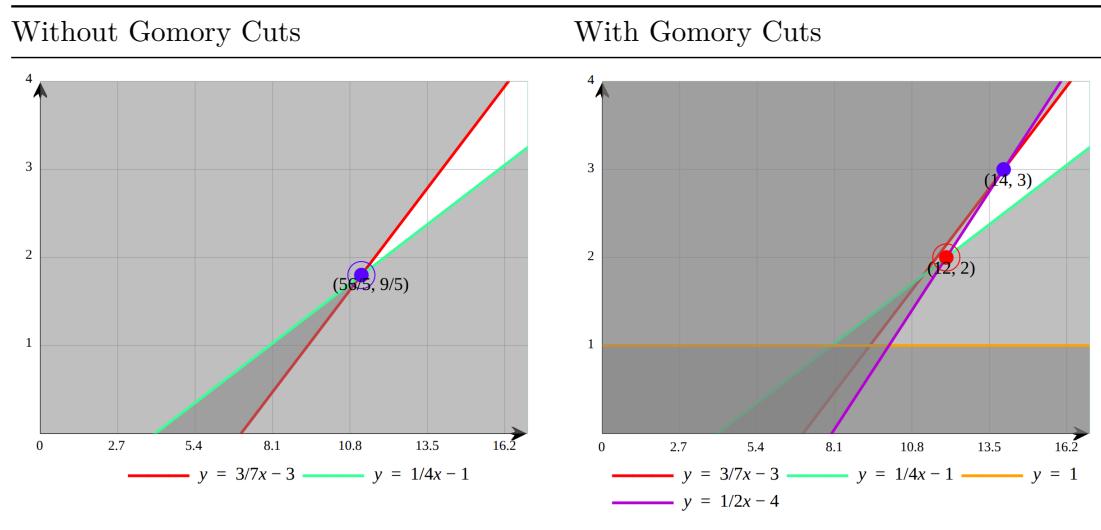
Subtask 5.c

We can easily extract the inequalities corresponding to the Gomory cuts by bringing the full tableau from the previous subtask into the initial form, where only the slack variables are in the basis:

```
tableau(T)
pivot(T, 0, 2)
pivot(T, 1, 3)
tableau(T)
```

x1	x2	x3	x4	x5	x6	-z	b
1	0	-4/5	-7/5	0	0	0	56/5
0	1	-1/5	-3/5	0	0	0	9/5
0	0	-1/5	-3/5	1	0	0	-1/5
0	0	-4/5	-2/5	0	1	0	-4/5
0	0	-1	-2	0	0	1	13
x1	x2	x3	x4	x5	x6	-z	b
-3	7	1	0	0	0	0	-21
1	-4	0	1	0	0	0	4
0	-1	0	0	1	0	0	-2
-2	4	0	0	0	1	0	-16
-1	-1	0	0	0	0	1	0

Now we can clearly see the original constraints and the new ones, and we can plot them using any tool.



Task 6**Subtask 6.a**