Linear Upper Bound on a Segment XOR Cardinality

Nika Skybytska, n.skybytskyi@knu.ua

Taras Shevchenko National University of Kyiv

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Bitwise XOR (exclusive or) is a fundamental operation in theoretical computer science.

- We study properties of XOR applied to sets.
- Defined as $X \oplus Y = \{x \oplus y \mid x \in X, y \in Y\}$.
- Focus on XORs of segments of consecutive integers.

Let $X = \{1, 2\}$ and $Y = \{3, 4\}$. $X \oplus Y = \{1, 2, 5, 6\}$:

$\oplus$	3	4
1	2	5
2	1	6

Let $X = \{1, 2, 3\}$ and $Y = \{2, 3, 4\}$. $X \oplus Y = \{0, 1, 2, 3, 5, 6, 7\}$:

$\oplus$	2	3	4
1	3	2	5
2	**0**	*1*	6
3	*1*	**0**	7

Definition

$[x, x+k) = \{x, x+1, \ldots, x+k-1\}$.

Problem

Given a positive integer k, solve $|[x, x+k) \oplus [y, y+k)| \to \max_{x,y}$ over nonnegative integers x, y.

Remark

Two subproblems: establish an **upper bound** and provide an efficient **construction**. Both experimental and analytical methods are fine.

Theorem

$|[x, x+k) \oplus [y, y+k)| \le 5(k-2)$ for all $k \ge 5$.

Remark

Our result improves on the naive quadratic upper bound:

$$|X \oplus Y| \le |X| \cdot |Y| = k^2.$$

Proposition

$|[x, x+k) \oplus [y, y+k)| \leq 4(k-1)$ for all $k > 1$.

Remark

This bound is tight for infinitely many values of $k = 2^m + 2$. The construction is $x = 2^m - 1$, $y = 3 \cdot 2^m$.

Lemma

For any fixed k, the optimization problem

$$g(k; x, y) = |[x, x+k) \oplus [y, y+k)| \to \max_{x,y}$$

has an optimal solution (x_0, y_0) with $x_0, y_0 \leq 4k$.

Remark

Hence, base cases can be established computationally.

Lemma

If $f(k) = \max_{x,y} g(k; x, y)$, *then:*

$$f(2k) \leq 2f(k+1), \quad f(2k+1) \leq 2f(k+1).$$

Remark

Inductive proof follows from these inequalities.

- We established a linear upper bound on segment XOR cardinality.
- It is stronger than naive $O(k^2)$ bound.
- Future work: refinement of constant factors.

References

- R. P. Stanley. *Enumerative Combinatorics*. Cambridge University Press, 1997.
- E. Bach, J. Shallit. *Algorithmic Number Theory*. MIT Press, 1996.
- R. Sedgewick, K. Wayne. *Algorithms*. Addison-Wesley, 2011.