Predator-Prey Systems

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Math 1270

October 29, 2024

The Predator Prey System

-The Lotka–Volterra predator–prey model is a pair of two variable, second and first-order nonlinear differential equations.

-The populations change through time according to the prey of equations $\frac{dx}{dt} = 9x - \alpha x^2 - 3xy$ and the population changes of predator is $\frac{dy}{dt} = -2y + xy$ where the variable x is the population (in some scaled units) of prey and y is the population of predators.

-Parameter α, where $0 \leq \alpha$, may affect prey's population other than predator. We may treat it as an unknown exogenous condition that affects the prey population.

Equilibrium Points

-In order to get equilibrium points, we have to find the nullcline first, which means we have to treat this two differential equations to be equal zero such that $\frac{dx}{dt} = 9x - \alpha x^2 - 3xy = 0$ and $\frac{dy}{dt} = -2y + xy = 0$.
-For $x^{'}$=0, $x(9 - \alpha x - 3y) = 0$, then x=0 or y=3-$\frac{\alpha}{3}x$. For $y^{'}$=0, $y(-2 + x) = 0$ then we have $x = 2$ or $y = 0$

-The way we find equilibrium points is to combine the solution during when they intersect. Such that we can treat y both equals to 0 and 3-$\frac{\alpha}{3}x$.
-Then we know that 3-$\frac{\alpha}{3}x = 0$, then x can only be equal to $\frac{9}{\alpha}$. However, we can not treat x to equal 2 and 0 simultaneously. Then, we can find that there are three equilibrium points (0,0) ($\frac{9}{\alpha}$, 0) (2, $\frac{9-2\alpha}{3}$).

Jacobian matrix

-The purpose of the Jacobian is to determine the stability of equilibria for systems of differential equations. It can approximate the behavior near an equilibrium point.
-After we get the Jacobian matrix, we can plug the number (for x and y) we want into the matrix and calculate the trace and the determinant, which can tell us the value for 'T-D plane'

-In order to get Jacobian matrix, we first treat $\frac{dx}{dt}$ as f(x) and $\frac{dy}{dt}$ as g(x).
-Then the Jacobian Matrix for the entire system is $\begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{bmatrix}$ which also equals to $\begin{bmatrix} 9 - 2\alpha x - 3y & -3x \\ y & -2 + x \end{bmatrix}$

Equilibrium at x = 0 and y = 0

-Using linearization, the Jacobian for this system at (0,0) becomes:

$$\begin{bmatrix} 9 & 0 \\ 0 & -2 \end{bmatrix}$$

-And so T = 7, and D = -18, which is a saddle

-The equilibrium point at x = 0 and y = 0 occurs for all values of α.

-At this point, x and y are the respective populations of the prey and predator, both populations are zero.

-When initial conditions are on either axis, we see differing behavior.

-When $x_i = 0$, the solutions tend to (0,0). (No Prey)

-But, when $y_i = 0$, the solutions tend to infinity as $t \to \infty$, when there is no α term. (No Predators)

A special Case when $\alpha = 0$

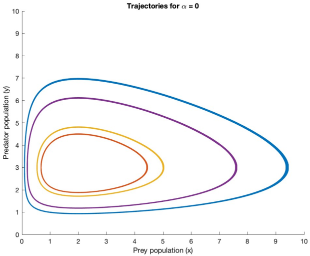
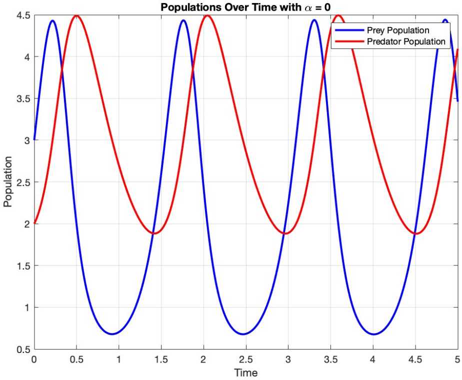


Figure: Phase portrait

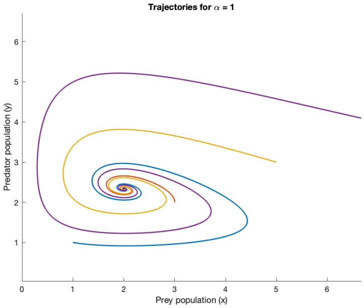
-The equilibrium points in this case occur at (0,0) and (2,3).

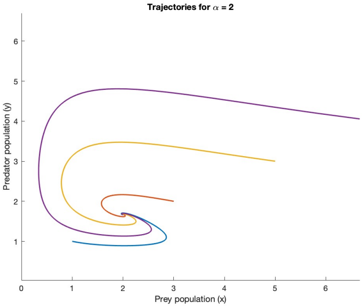
-The Jacobians for each are then $\begin{bmatrix} 9 & 0 \\ 0 & -2 \end{bmatrix}$ and $\begin{bmatrix} 0 & -6 \\ 3 & 0 \end{bmatrix}$. In the first, T $=$ 7 and D $=$ -18 which gives a saddle. The second gives T $=$ 0 and D $=$ 18, giving a center solution.

x(t) and y(t) when $\alpha = 0$



Solutions at $\alpha = 1$ and $\alpha = 2$

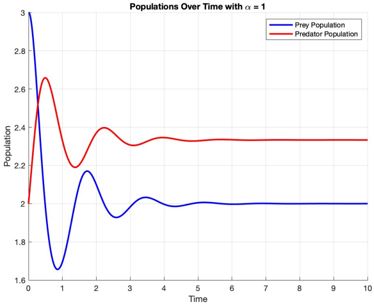


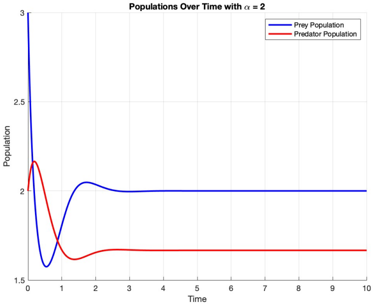


-When $\alpha = 1$ the equilibria aside from (0,0) are (9,0) which is a saddle and (2, $\frac{7}{3}$) which is a spiral sink.

-Similarly, when $\alpha = 2$ the equilibria are ($\frac{9}{2}$,0) which is a saddle and (2, $\frac{5}{3}$) which is a spiral sink.

x(t) and y(t) when $\alpha = 1$ and $\alpha = 2$





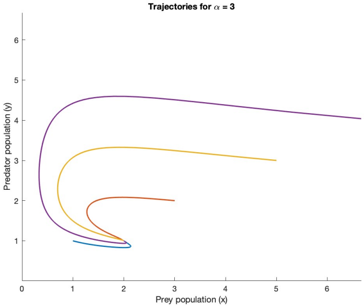
Bifurcation at 2.7

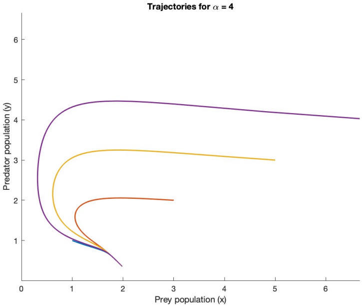
-At $\alpha \sim 2.7$, we see a bifurcation at the equilibrium point $(2, \frac{6}{5})$

-$J(2, \frac{6}{5}) = \begin{bmatrix} \frac{-27}{5} & -6 \\ \frac{6}{5} & 0 \end{bmatrix}$ with $T = \frac{-27}{5}, D = \frac{36}{5}$

-In this case, $\frac{T^2}{4} = \frac{(\frac{-27}{5})^2}{4} \sim \frac{36}{5} = D$, meaning the equilibrium point is crossing the parabola on the T-D Plane

Solutions at $\alpha = 3$ and $\alpha = 4$

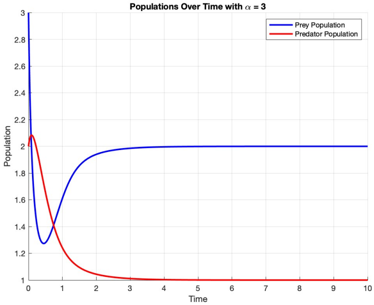


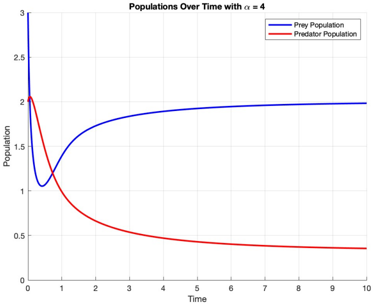


-When $\alpha = 3$ the equilibria aside from (0,0) are (3,0) which is a saddle and (2, 1) which is a nodal sink.

-Similarly, when $\alpha = 4$ the equilibria are ($\frac{9}{4}$,0) which is a saddle and (2, $\frac{1}{3}$) which is a nodal sink.

x(t) and y(t) when $\alpha = 3$ and $\alpha = 4$





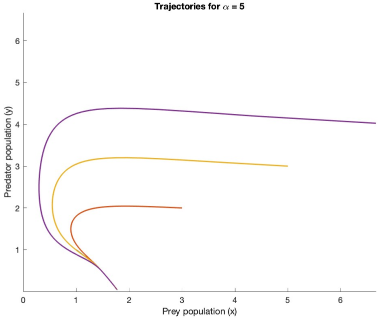
Bifurcation at 4.5

-At $\alpha = 4.5$, we see changes to the solutions at both equilibrium points

- Since $9 - 2\alpha \leq 0$ for $\alpha \geq 4.5$, the point $(2, \frac{9-2\alpha}{3})$ leaves the first quadrant

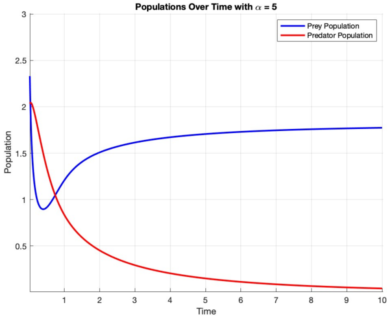
- For $(\frac{9}{\alpha}, 0)$, $J(2,0) = \begin{bmatrix} -9 & -6 \\ 0 & 0 \end{bmatrix}$, which yields $T = -9$ and $D = 0$. So the point crosses the D-axis, meaning that the equilibrium point goes from a saddle to a nodal sink

The Solution at $\alpha = 5$



-When $\alpha = 5$ the equilibria aside from (0,0) are $(\frac{9}{5},0)$ which is a nodal sink and $(2, \frac{-1}{3})$ which is unimportant

x(t) and y(t) when $\alpha = 5$



Bifurcation Analysis

- At $\alpha = 0$, the populations of all species will continuously oscillate forever

-At this point, there are no environmental factors, the populations compete for all t.

- At $0 \leq \alpha \leq 2.7$, the populations spirally sink as t approaches infinity toward $(2, \frac{9-2\alpha}{3})$

- At $2.7 \leq \alpha \leq 4.5$, the populations will reach equilibrium much faster as it is a nodal sink

- At $4.5 \leq \alpha$, the predator populations will reach extinction

Conclusions about this Model

- Unless there's no prey to begin with, since each equilibrium is either a saddle point, nodal sink, or spiral sink, the predators and prey never actually reach 0
-The Lotka-Volterra model is useful framework for understanding how two species may interact, and can also be used in economics by tracking two direct competitors within the market
- This system may fail to be accurate in real-world applications due to many other animals sharing the same environment, the density of the space that these animals live in, and constant predator/prey interactions over time