Lotka–Volterra Predator–Prey Model

Max Barrett, Neil Slavishak, Roberto Defazio, Wenxuan Lu, Yuang Wang

October 28, 2024

Abstract

The Lotka–Volterra predator-prey model is a pair of two variable second and first-order nonlinear differential equations which are frequently used to describe the dynamics of biological systems in which two species interact. One describes a predator and the other a prey species. This math model is designed ideally which means it only measures how the population of prey is related to the population of the predator and an abstract parameter α ranges from 0 to 5. α is considered as the index of environmental hazards that will greatly reduce the population of prey and may also affect the population of predators to some degree depending on each situation. There are a couple of ways to solve a second-order differential equation, but we will use the linearization technique in our model. We first need to compute the equilibrium points and the Jacobian Matrix, then plug in the value of equilibrium points and α into the Jacobian Matrix. From the result and graph, we can see α is the most crucial parameter that will affect the result of the outcome. When $\alpha = 0$, which also means that there are no environmental factors, the populations of all species will continuously oscillate forever as time t goes to ∞. The second case is that when $0 \leq \alpha \leq 2.7$, the populations spirally sink as t approaches infinity toward $(2, \frac{9-2\alpha}{3})$. The third case is that when $2.7 \leq \alpha \leq 4.5$, the populations will reach equilibrium much faster as it is a nodal sink. The last case is when $4.5 \leq \alpha$, the predator populations will reach extinction.

Although this is a good mathematical model that enables us to learn how these two differential equations intersect and how we can solve second-order nonlinear differential equations, this system is an ideal math model so it may fail to be accurate in real-world applications, since there are many more factors that can affect the ecology system and there are always more than one predators for any prey species. However, this model can be used as a simple tool for financial marketing analysis, as is touched upon later.

1 Introduction

In the regular predator-prey model, there is no natural disaster, climate change, or environmental factors considered. Prey are considered to have unlimited food supply and their population is not affected by competition, diseases, or old age. They can reproduce exponentially in whatever way they want and it is shameful to admit that the genetic adaptation is also considered as inconsequential.

The only factor that may reduce their population is predators. However, in this model, we have a parameter α where $\alpha \geq 0$ may affect the prey's population, and in turn the predator. The populations change through time according to the prey of equations $\frac{dx}{dt} = 9x - \alpha x^2 - 3xy$ where the variable x is the population (in some scaled units) of prey and y is the population of predators. Since α in the context of the equation has a negative sign in front of it, we may treat it as an unknown condition that indeed decreases the population of prey. Maybe α represents the competition between the prey or the living environment's hazardous conditions. Predators in this model are considered under ideal conditions, similarly to the original form of the Lotka-Volterra model. Their population changes according to the equation $\frac{dy}{dt} = -2y + xy$. The predator population will grow unbounded unless there is a lack of food to cause this growth (the prey). No competition will happen and the hazardous environment will not affect them. Prey are the only source of food for predators, and all of the predators are considered as infinite appetites. This means that as long as a predator encounters a prey, it will brutally hunt the prey and eat it. [2]

It is fair game to set a population of prey higher than predator initially to avoid great extinction. In this model, we will vary α from 0 to five to represent how the degree of hazardous changed the population of prey and predators.

2 Methods

In order to examine the changes to the system depending on alpha, the concept of linearization can be used to determine the behavior of each phase portrait at the system's equilibrium points. To find these equilibrium points, we must set both equations of the system equal to 0. Then, various solutions to the system of equations can be found, yielding the equilibrium points.

$$\frac{dx}{dt} = 9x - \alpha x^2 - 3xy = x(9 - \alpha x - 3y) = 0$$
$$\frac{dy}{dt} = -2y + xy = y(-2 + x) = 0$$

Solving these equations leave us with four lines, called nullclines:

$$x = 0,\ y = 0,\ y = 3 - \frac{\alpha}{3}x,\ x = 2$$

The equilibrium points can be found where these nullclines intersect. This gives us equilibrium points at $(0, 0)$, $(\frac{9}{\alpha})$, and $(2, \frac{9-2\alpha}{3})$.

Other than equilibrium points, we also need to calculate the Jacobian Matrix before we can linearize it. Jacobian matrix determines the stability of equilibria for systems of differential equations. It can approximate the behavior near an equilibrium point. To get the Jacobian matrix, we first treat $\frac{dx}{dt}$ as f(x) and $\frac{dy}{dt}$ as g(x). Then the Jacobian Matrix for the entire system is $\begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{bmatrix}$ which also equals to $\begin{bmatrix} 9-2\alpha x-3y & -3x \\ y & -2+x \end{bmatrix}$. After we get the Jacobian matrix, we can plug those three equilibrium points ($(0,0)$, $(\frac{9}{\alpha})$, and $(2, \frac{9-2\alpha}{3})$) into the matrix and calculate the trace and the determinant, which can tell us the value for 'T-D plane'.

2.1 $\alpha = 0$

Now that we have the Jacobian matrix and the equilibrium points, we can begin to examine the behavior of the system at different values of alpha. When $\alpha = 0$, note that the equilibrium point $(\frac{9}{\alpha}, 0)$ is undefined. So, the only two equilibrium points for the system for this case is $(0,0)$ and $(2,3)$. We first compute the Jacobian at $(0,0)$:

$$J(0,0) = \begin{bmatrix} 9-3(0) & -3(0) \\ 0 & -2+0 \end{bmatrix} = \begin{bmatrix} 9 & 0 \\ 0 & -2 \end{bmatrix}$$

We can use the Trace-Determinant (T-D) Plane to decipher what type of solution is present at this equilibrium point. The trace of a matrix is the sum of all of the diagonal entries of a matrix. The determinant of a matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $D = ad - bc$. Figure 1 shows the relationship of these two computations, illustrating different phase portraits depending on the values of the trace and the determinant.

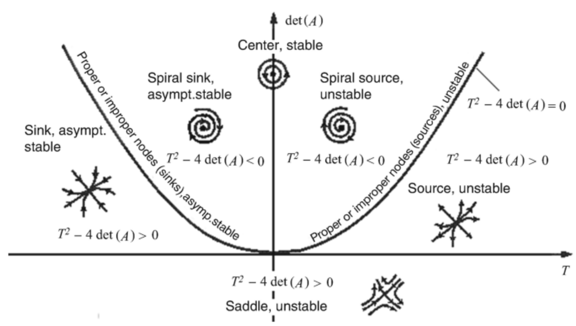

Figure 1: Trace-Determinant Plane [1]

By examining the Jacobian at (0,0), we computed a trace of 7 and a determinant of -18. Therefore, based on the T-D Plane, we have a saddle solution at $(0,0)$ since $D < 0$. We can further examine the phase portrait at this equilibrium point by finding the eigenvalues and eigenvectors of the Jacobian matrix.

$$det(J - \lambda I) = \begin{bmatrix} 9-\lambda & 0 \\ 0 & -2-\lambda \end{bmatrix} = \lambda^2 - 7\lambda - 18 = (\lambda - 9)(\lambda + 2) = 0$$

$$\lambda = 9, -2$$

$$0 = (J - 9I) = \begin{bmatrix} 0 & 0 \\ 0 & -11 \end{bmatrix} \to \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \to y = 0 \to v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$0 = (J + 2I) = \begin{bmatrix} 11 & 0 \\ 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \to x = 0 \to v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Since $9 > 0$ and $-2 < 0$, the solutions follow the vector $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ (y-axis) toward the equilibrium point and then get pushed out following the vector $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ (x-axis) away from the equilibrium point.

Now, we can examine the point $(2, 3)$ by computing its Jacobian:

$$J(2,3) = \begin{bmatrix} 9 - 3(3) & -3(2) \\ 3 & -2 + 2 \end{bmatrix} = \begin{bmatrix} 0 & -6 \\ 3 & 0 \end{bmatrix}$$

This matrix has a trace of 0 and a determinant of 18. By looking at the T-D Plane (Figure 1) again, we find that the solutions near this equilibrium point act like a center solution. We can examine the behavior at a specific point to determine the direction of rotation, specifcally at (1,1). Note that with linearization, the new system found by using the Jacobian is not centered around the point used to linearize, but the origin.

$$x' = \begin{bmatrix} 0 & -6 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -6 \\ 3 \end{bmatrix}$$

This means that the solution is moving up and to the left at (1,1), meaning the linearized system is moving counterclockwise around the origin. So, for the actual system, the solutions are rotating counterclockwise around (2,3). Now, we can combine the two phase portraits of the linearized systems together to get the phase portrait of the original system, as seen in Figure 2.

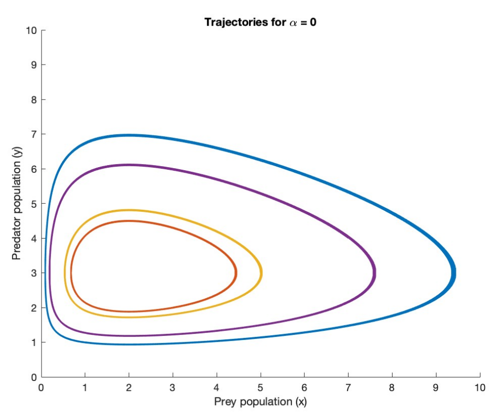

Figure 2: Phase portrait when $\alpha = 0$

Notice how at the points $(0, 0)$ and $(2, 3)$, we can see behavior similar to the behavior we saw when using linearization. We can also examine the behavior of the predator and prey populations over time to futher understand the qualities of this system.

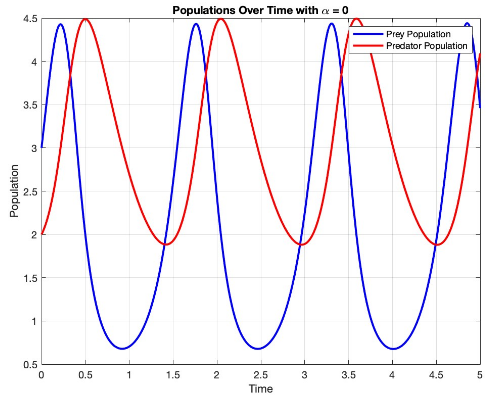

Figure 3: x(t) and y(t) when $\alpha = 0$

2.2 $\alpha = 1$

For all $\alpha \neq 0$, we begin to see much different behavior. We, again, can use linearization to see this. In this case, the equilibrium points are $(0,0)$, $(9,0)$, and $(2, \frac{7}{3})$. We have already determined the Jacobian for the point (0,0), so there is no need to compute and analyze it again. For all cases, there will always be a saddle solution at (0,0). We now compute the Jacobian for the point $(9,0)$:

$$J(9,0) = \begin{bmatrix} 9 - 2(1)(9) - 3(0) & -3(9) \\ 0 & -2+9 \end{bmatrix} = \begin{bmatrix} -9 & -27 \\ 0 & 7 \end{bmatrix}$$

This matrix has a trace of -2 and a determinant of -56. On the T-D Plane, since $D < 0$, we have another saddle solution. Like before, we can use the Jacobian's eigenvalues and eigenvectors to better understand the systems behavior.

$$det(J - \lambda I) = \begin{bmatrix} -9-\lambda & -27 \\ 0 & 7-\lambda \end{bmatrix} = \lambda^2 + 2\lambda - 63 = (\lambda + 9)(\lambda - 7) = 0$$

$$\lambda = -9, 7$$

$$0 = (J + 9I) = \begin{bmatrix} 0 & -27 \\ 0 & 16 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \rightarrow y = 0 \rightarrow v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$0 = (J - 7I) = \begin{bmatrix} -16 & 0 \\ -27 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & \frac{27}{16} \\ 0 & 0 \end{bmatrix} \rightarrow x = \frac{-27}{16} y \rightarrow v_2 = \begin{bmatrix} \frac{-27}{16} \\ 1 \end{bmatrix}$$

Based on these eigenvalues and eigenvectors, we can determine the paths of the solutions. Because $-9 < 0$ and $7 > 0$, the solutions will follow the vector $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ toward the equilibrium point and then follow vector $\begin{bmatrix} \frac{-27}{16} \\ 1 \end{bmatrix}$ away from the equilibrium point. Now we compute the Jacobian for the point $(2, \frac{7}{3})$:

$$J(2, \tfrac{7}{3}) = \begin{bmatrix} 9 - 2(1)(2) - 3(\frac{7}{3}) & -3(2) \\ \frac{7}{3} & -2+2 \end{bmatrix} = \begin{bmatrix} -2 & -6 \\ \frac{7}{3} & 0 \end{bmatrix}$$

This matrix has a trace of -2 and a determinant of 14. Because $D < \frac{T^2}{4}$ and $T < 0$, this matrix falls in the spiral sink section of the T-D plane. By combining the behavior of the solutions at all three equilibrium points, we obtain the phase portrait of the system seen in Figure 4.

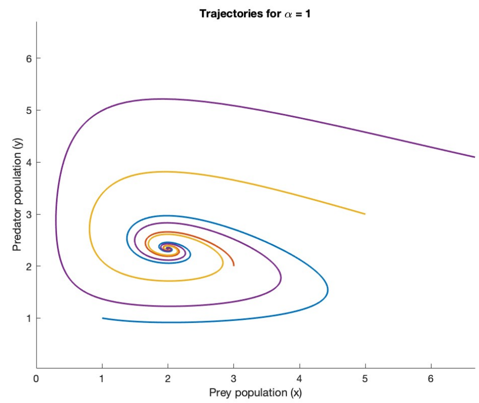

Figure 4: Phase Portrait when $\alpha = 1$

As we can see, the phase portrait illustrates the behavior derived from linearizing the system at the equilibrium points, as solutions bounce away from $(0, 0)$ and $(9, 0)$ and spirally tend toward $(2, \frac{7}{3})$. By examining the predator and prey populations graphs (Figure 5), we can see this sink illustrated better.

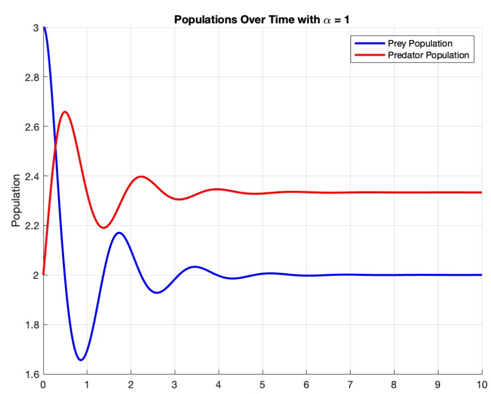

Figure 5: x(t) and y(t) when $\alpha = 1$

As seen in the figure, the system experiences a damped nature because of the existence of α. Over time, the population sizes will "level out" at the equilibrium point $(2, \frac{9-2\alpha}{3})$ as the oscillations decay.

2.3 $\alpha = 2$

We see very similar behavior in the solutions when $\alpha = 2$. In this case, the equilibrium points are found at $(0,0)$, $(\frac{9}{2},0)$, and $(2,\frac{5}{3})$. For $(\frac{9}{2},0)$, the Jacobian is $\begin{bmatrix} -9 & \frac{-27}{2} \\ 0 & \frac{5}{2} \end{bmatrix}$, which has a trace of $\frac{-13}{2}$ and a determinant of $\frac{-45}{2}$, again yielding a saddle solution. For $(2,\frac{5}{3})$, the Jacobian is $\begin{bmatrix} -4 & -6 \\ \frac{5}{3} & 0 \end{bmatrix}$, which has a trace of -4 and a determinant of -10, which is a spiral sink. By examining the phase portrait (Figure 6), we see that not much has changed to the solutions except for a shift in equilibrium points.

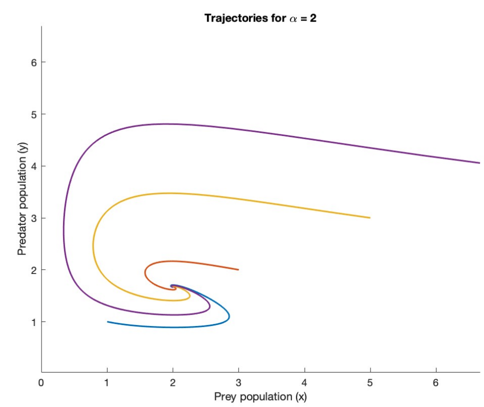

Figure 6: Phase Portrait when $\alpha = 2$

We also see similar behavior in the populations graphs (Figure 7).

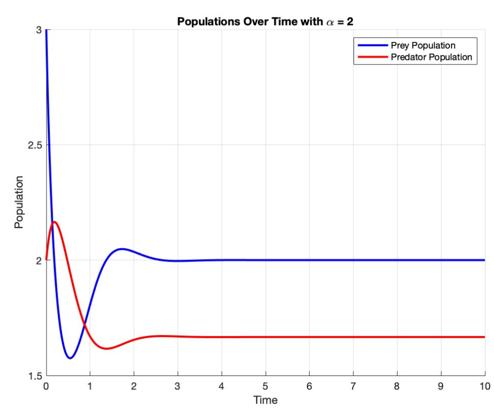

Figure 7: x(t) and y(t) when $\alpha = 2$

2.4 Birfurcation Point: $\alpha \sim 2.7$

At $\alpha \sim 2.7$, we found our first bifurcation point of the system, specifically with the equilibrium point $(2,\frac{9-2\alpha}{3})$. As we discussed before, the solutions had previously been spirally sinking toward this equilibrium point because for the Jacobians, $D > \frac{T^2}{4}$. However, when $\alpha = 2.7$, we see that this inequality changes. At $\alpha = 2.7$, the Jacobian is $\begin{bmatrix} \frac{-27}{5} & -6 \\ \frac{6}{5} & 0 \end{bmatrix}$, which has a trace of $\frac{-27}{5}$ and a determinant of $\frac{36}{5}$. We can then compute where on the T-D Plane this point falls.

$$\frac{T^2}{4} = \frac{(\frac{-27}{5})^2}{4} = 7.29 \sim \frac{36}{5} = D$$

Since $\frac{T^2}{4} \sim D$, we can infer that the point $(2, \frac{9-2\alpha}{3})$ is leaving the inside of the parabola and moving outside of it, illustrating a shift from a spiral sink to a nodal sink. This change is illustrated in the next case.

2.5 $\alpha = 3, 4$

The next two cases illustrate the shift in behavior. For $\alpha = 3$, the equilibrium points are $(0, 0)$, $(3, 0)$, and $(2, 1)$. For $(3, 0)$, the Jacobian is $\begin{bmatrix} -9 & -9 \\ 0 & 1 \end{bmatrix}$, which has a trace of -8 and a determinant of -9, which is a saddle solution. For $(2, 1)$, the Jacobian is $\begin{bmatrix} -6 & -6 \\ 1 & 0 \end{bmatrix}$, which has a trace of -6 and a determinant of 6. Since $D < \frac{T^2}{4}$, this point is a nodal sink. Looking at the phase portrait (Figure 8), we can see the change in behavior from $\alpha = 2$ to $\alpha = 3$.

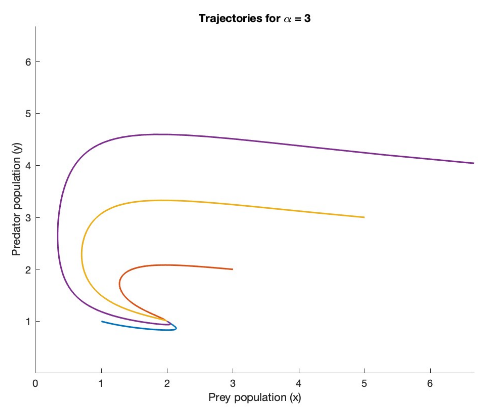

Figure 8: Phase Portrait when $\alpha = 3$

As we can see, solutions no longer spiral around the equilibrium point, but instead tend straight into the point. By examining this change, as well as looking at the population graphs (Figure 9), it seems like solutions tend much faster toward the equilibrium due to this shift.

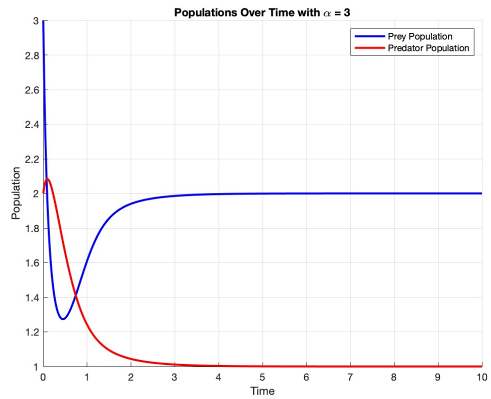

Figure 9: x(t) and y(t) when $\alpha = 3$

For $\alpha = 4$, we see the same behavior. In this case, the equilibrium points are $(0,0)$, $(\frac{9}{4}, 0)$, and $(2, \frac{1}{3})$. For $(\frac{9}{4}, 0)$, the Jacobian is $\begin{bmatrix} -9 & \frac{-27}{4} \\ 0 & \frac{1}{4} \end{bmatrix}$, which has a trace of $\frac{-27}{4}$ and a determinant of $\frac{-9}{4}$, which is a saddle solution. For $(2, \frac{1}{3})$, the Jacobian is $\begin{bmatrix} -8 & -6 \\ \frac{1}{3} & 0 \end{bmatrix}$, which has a trace of -8 and a determinant of 2, which is a nodal sink solution. The phase portrait for this system illustrates these findings (Figure 10).

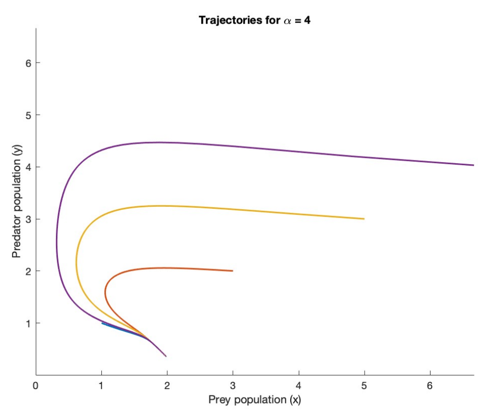

Figure 10: Phase Portrait when $\alpha = 4$

The population graphs also illustrate similar behavior (Figure 11).

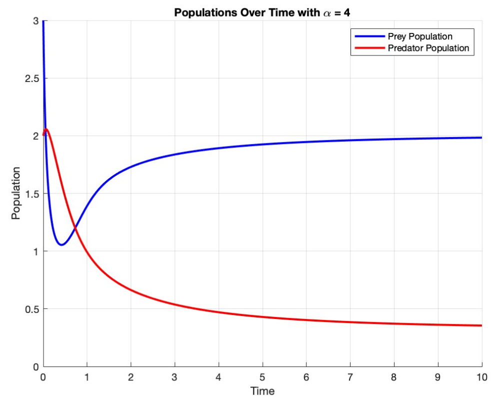

Figure 11: x(t) and y(t) when $\alpha = 4$

The nature of these solutions in comparison to the earlier cases allow for inferences to be made about the system in a real-life scenario, as will be discussed in the results.

2.6 Bifurcation Point: $\alpha = 4.5$

When $\alpha = 4.5$, we see another shift in behavior in the system, this time at both equilibrium points. First of all, when $\alpha > 4.5$, since $9 - 2\alpha < 0$, we find that $(2, \frac{9-2\alpha}{3})$ leaves the first quadrant. In this case, since populations cannot be negative, solutions will not be able to tend toward this equilibrium point, so the point essentially becomes unimportant in the system. Also, when $\alpha = 4.5$, the other equilibrium point is $(2, 0)$ which has a Jacobian of $\begin{bmatrix} -9 & -6 \\ 0 & 0 \end{bmatrix}$. This matrix has a determinant of 0, meaning that this equilibrium point is moving from below the T-axis to above it. So, the behavior at this point changes from a saddle to a nodal sink. Putting both of these findings together, we will see in the next case that solutions tend to the equilibrium point $(\frac{9}{\alpha}, 0)$ instead of the point $(2, \frac{9-2\alpha}{3})$ like we saw in the previous cases.

2.7 $\alpha = 5$

In this case, the equilibrium points are $(0, 0)$, $(\frac{9}{5}, 0)$, and $(2, \frac{-1}{3})$. Again, we do not need to examine the equilibrium point not in the first quadrant. as for $(\frac{9}{5}, 0)$, its Jacobian is $\begin{bmatrix} -9 & \frac{-27}{5} \\ 0\frac{-1}{5} & \end{bmatrix}$, which has a trace of $\frac{-46}{5}$ and a determinant of $\frac{9}{5}$, which yields a nodal sink solution. In the phase portrait (Figure 12), we can visually examine the changes to the behavior of the system.

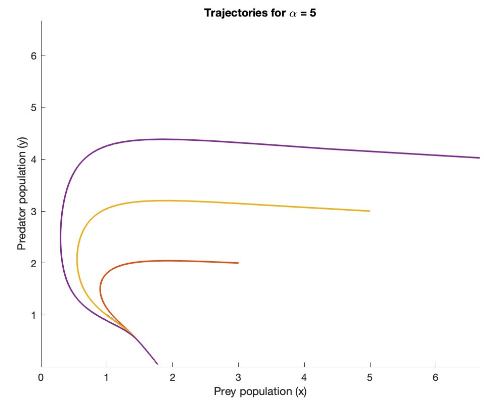

Figure 12: Phase portrait when $\alpha = 5$

The major change in the behavior is that all solutions tend toward the x-axis, illustrating that the predator population will eventually tend toward 0. We can see this observation clearly in the population graphs (Figure 13).

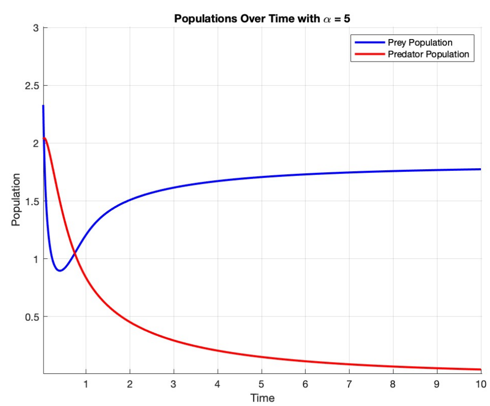

Figure 13: x(t) and y(t) when $\alpha = 5$

As with the other bifurcation point, the change in behavior can lead to observations about the system in a real-life scenario.

2.8 Notes on Modeling

All of the graphs and phase portraits were made using MATLAB. By using Euler's method, we were able to obtain an estimate of the solutions over time. When attempting to obtain these solutions, we found that using a very small time step with a very large maximum time was necessary for producing the results needed, especially for the $\alpha = 0$ case.

3 Results

The Results found using the Methods discussed above yield some insight into the parameter α and what it could represent at different values. The same analysis was conducted on multiple values of α as was shown in the methods section. As is made evident by the equations that make up the Lotka-Voltera system, α is a parameter that negatively affects the Prey equation. Which in turn, affects the predator. We find that when increasing alpha from its initial value of 0, the solutions to this system take on new behaviors.

Prior to exploring the parameter and its potential meanings, we feel it necessary to touch on the unique equilibrium that occurs at all values of α. The point (0,0) is an equilibrium regardless of α due to the properties of linearization. This equilibrium and its real world meaning are very intuitive as when the population of both predators and prey in a system are at 0, of course there is no interaction between the two species and no resulting fluctuation in their populations. This represents a special constant case for any circumstance that may characterize this system, regardless of any exogenous parameters such as the one discussed in the coming results.

Beginning with the situation where $\alpha = 0$, we see that near the equilibrium point (2,3), the solutions approximate a center (Figure 2). Looking at this solution further , when graphing x(t) and y(t) in Figure 3 we see an oscillation that lasts for all t with a constant amplitude. It is also clear that this oscillation is slightly out of sync, where when prey reach their max the predators grow even faster. This abundance of predators then causes the prey population to decrease, and the predator population to follow suit shortly after. This constant fluctuation represents the constant competition that occurs between the populations in the real world, and is an intuitive way to understand the system. This case then represents a unique behavior for the parameter α, an ideal situation. This parameter seems to represent some exogenous factor affecting the system, specifically the prey population. This could have to do with human behavior, food growth, or another environmental constraint. Based upon this, when $\alpha = 0$, this effect does not exist and we have an ideal Predator-Prey system. The species will compete and their populations will oscillate for all time.

Next, we delve into the cases where $\alpha = 1$ and $\alpha = 2$. In these cases, the parameter takes on values that affect the growth rate for the prey and in turn, predator populations. In addition to (0,0), there is another saddle at (9, 0) and $(\frac{9}{2}, 0)$ for each value (Figures 4 and 6). We see that the equilibriums at $(2, \frac{7}{3})$ and $(2, \frac{5}{3})$ for each value approximate spiral sinks. And so the populations slowly decrease, with dampened oscillations as seen in the x(t) and y(t) graphs (Figures 5 and 7). Now, the parameter does have a real effect on the system, causing oscillations to die out at a point, and send each population to equilibrium.

At the point where $\alpha = 2.7$ we see a bifurcation, where the system changes behavior. By varying α, the Jacobian matrix changes and so the solution given by its trace and determinant changes. After 2.7, α becomes 'hazardous', meaning that it causes solutions to decay toward equilibrium much faster. We can see that when $\alpha = 3$ and $\alpha = 4$ the spiral sink solutions are now nodal sinks (Figures 8 and 10), decaying to equilibrium much faster than before. These nodal sink equilibriums occur at (2,1) and (2, $\frac{1}{3}$) for each value. However, in addition to (0,0) there is another saddle at (3, 0) and ($\frac{9}{4}$, 0) for each value. In addition, the x(t) and y(t) plots illustrate the behavior of the system after $\alpha = 2.7$ (Figures 9 and 11), with the values going to an equilibrium much faster than before.

At $\alpha = 4.5$ there is another bifurcation, where we see solutions change from a saddle to a nodal sink, and the nodal sink from before leaves the first quadrant, making it nonexistent in the context of this model. It is interesting to note that this nodal sink occurs at a point on the x axis, where we had a saddle prior, as seen in Figure 12 where we use $\alpha = 5$ to visualize solutions after this bifurcation. This means that the population decays to an equilibrium where y = 0, the predator population goes extinct. After this point, the parameter α has an effect on the prey population that is so large, it cannot support a predator population. We can see the population fluctuations die out to equilibrium very quickly in the x(t) and y(t) plots for this value of α (Figure 13).

These values of α where bifurcations occur completely change the location of each Jacobian matrix on the T-D plane, therefore changing the behavior of the solutions at equilibrium. These bifurcations can be looked at as thresholds of a sort, where the prey populations behavior over time has to change to accommodate α, and this will then affect the predators as well. These results suggest that α is a parameter that negatively effects the population of prey, causing it to go to equilibrium much faster as it increases. Through this analysis, we have isolated the effect of the parameter, and derived some meaning from it based on how it affects the system.

4 Concluding Remarks and Discussion of Further Study

We can now see how the result changes if we apply different α values to our system. Based on this analysis, we can conclude that α is a parameter capturing some exogenous factor of this system that affects both population growth rates in a negative way. At different values, the parameter effects the system much differently, allowing us to think about its true meaning in a real world application. As mentioned before, due to its lack of accuracy for the real ecological system application, we may not directly apply this model to animal population analysis. However, we can still use this model in some economic systems. In markets such as stock or future exchanges markets, which are dynamic and periodically oscillate, consumers can be modeled and analyzed by the economists and investment bankers unless abnormal fluctuations occur. This is especially true in some categories with high demand such as food or gasoline. For example, we can treat x (population of prey) as the value for a certain product, we can treat y (predator) as the artificial cyclical adjustment of market prices, and we can treat α as the abnormal fluctuations for the price. Although further studies are still needed if we apply the predator-prey model to the financial market, the consumer is able to be manipulated and much easier to analyze since they abide by regulations and laws.

References

[1] Henner, V., Nepomnyashchy, A., Belozerova, T., Khenner, M. (2023). *Qualitative Methods and Stability of ODE Solutions.* Springer

[2] Wikipedia contributors. (2024, March 19). Lotka–Volterra equations. Wikipedia. https://en.m.wikipedia.org/wiki/Lotka