Optimal Estimation
Homework 0

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MECH 7710

1. A control law for a simple rotation table is to be designed. The table has a rotational moment of inertia (J) of 10 $[kg \cdot m^2]$ and rotational damping (b) of 1 $\left[\frac{N \cdot m \cdot s}{rad}\right]$. Torque is commanded to the motor and the table's position is measured using a rotary encoder.

   (a) Derive the simple differential equation for the system.
   Ans: If we understand the commanded torque to be rotating the table in the counter-clockwise direction and the rotational damping to counteract the torque in the clockwise direction, we can write a simple equation of motion equal to the sum of moments for the system.

$$\begin{aligned} \Sigma M = \tau - b\dot{\theta} = J\ddot{\theta} \\ \frac{\tau}{J} = \ddot{\theta} + \frac{b}{J}\dot{\theta} \end{aligned} \tag{1}$$

   (b) Convert the system into a state-space format.
   Ans: We can define a state-space system with the equation

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$$

   Expanding this to matrix form provides us with a state-space form from the differential equation defined previously.

$$\begin{bmatrix} \dot{\theta} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -\frac{b}{J} \end{bmatrix} \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{J} \end{bmatrix} \tau \tag{2}$$

   (c) What are the eigenvalues of the system?
   Ans: Assuming the system to be in equilibrium, we can define the equation of motion to be

$$\ddot{\theta} + \frac{1}{10}\dot{\theta} - \frac{0}{10} \tag{3}$$

   Then, we substitute θ for s and solve for the characteristic equation, giving us the following eigenvalues

$$\begin{aligned} 0 = s^2 + \frac{1}{10}s \\ s = 0, -\frac{1}{10} \end{aligned} \tag{4}$$

2. Design an observer for the above system.
   (a) Show that the system is observable.
   Ans: The system can be proven observable by using the such called 'Observability matrix' and identifying that it has no zero columns.

$$\begin{bmatrix} \mathbf{C} \\ \mathbf{C\,A} \\ \mathbf{C\,A}^2 \\ \vdots \\ \mathbf{C\,A}^{n-1} \end{bmatrix} \tag{5}$$

   where $n = dim(\mathbf{x}$, or in our case, $n = 2$. We define $\mathbf{C}$ as

$$\mathbf{C} = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

   and proceed by substituting our $\mathbf{A}$ and $\mathbf{C}$ matrices as follows:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{6}$$

   We can see from the observability matrix that the observability matrix has all non-zero columns, meaning the system is fully observable.

   (b) Design L such that the error dynamics have:

$$\omega_n = 50\,Hz \quad , \quad \zeta = 0.7$$

   Ans: In order to design $\mathbf{L}$ to show the specified error parameters, we need to first define a modified $\mathbf{A}$ matrix, $\mathbf{A}_{esti}$.

$$\begin{aligned} \mathbf{A}_{esti} &= (\mathbf{A} - \mathbf{LC}) \\ &= \left(\begin{bmatrix} 0 & 1 \\ 0 & -\frac{1}{10} \end{bmatrix} - \begin{bmatrix} L_1 \\ L_2 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} \right) \\ \mathbf{A}_{esti} &= \begin{bmatrix} -L_1 & 1 \\ -L_2 & -\frac{1}{10} \end{bmatrix} \end{aligned} \tag{7}$$

   Solving for $det\,(s\mathbf{I} - \mathbf{A}_{esti})$ and comparing the equation to the ideal characteristic equation will give us values of $\mathbf{L}$ with the specified error parameters.
   Our desired characteristic equation is:

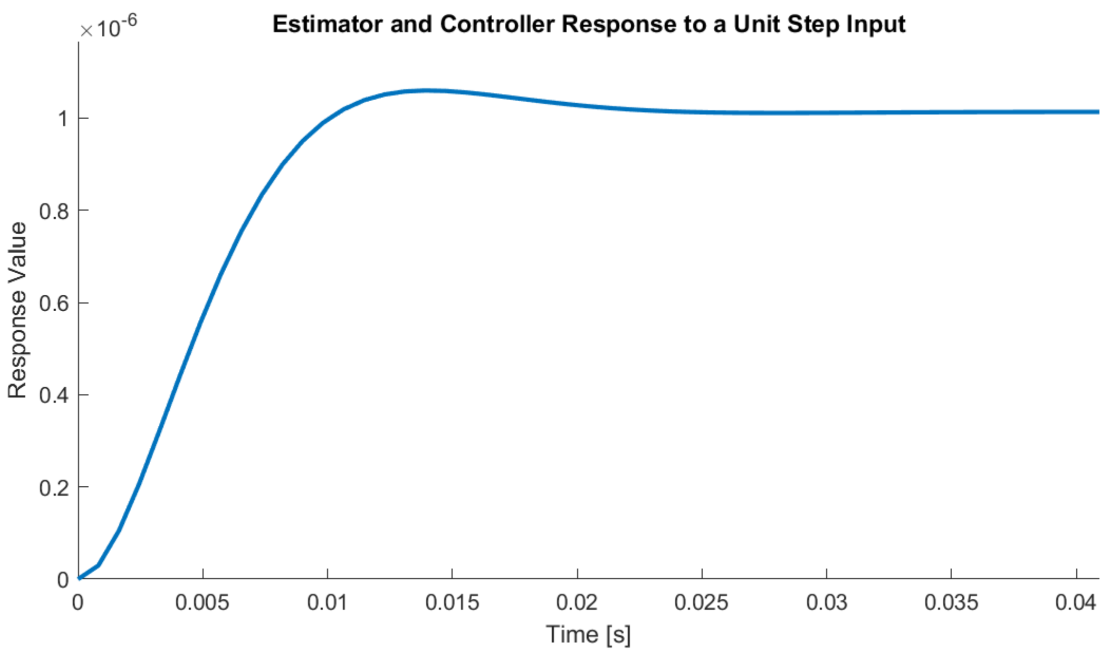
$$s^2 + 2\zeta\omega_n s + \omega_n^2 \;\rightarrow\; s^2 + 2(0.7)(50 \cdot 2\pi)s + (50 \cdot 2\pi)^2 \;\rightarrow\; s^2 + 314.2s + 98,696$$

$$
\begin{aligned}
det\left(s\mathbf{I} - \mathbf{A}_{esti}\right) &= det\left(\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} -L_1 & 1 \\ -L_2 & -\frac{1}{10} \end{bmatrix}\right) \\
&= det\left(\begin{bmatrix} s + L_1 & -1 \\ L_2 & s + \frac{1}{10} \end{bmatrix}\right) \\
&= s^2 + \left(L_1 + \frac{1}{10}\right)s + \left(\frac{1}{10}L_1 + L_2\right)
\end{aligned} \tag{8}
$$

After a bit of math, we get the values for L_1 and L_2.

$$L_1 = 439.72 \quad L_2 = 98,652$$

(c) Provide a plot of the step response of the estimator.
Ans:



3. Design a state-feedback controller for the table

   (a) Show that the system is controllable.

   Ans: The system can be proven controllable using the said "controllability matrix"(equation 9). If the matrix has a rank equal to the number of dimensions in $\mathbf{x}$, then we can say the system is controllable.

$$rank([\mathbf{B} \quad \mathbf{AB} \quad \mathbf{A}^{n-1}\mathbf{B} \quad \mathbf{A}^{n-2}\mathbf{B}]) \equiv dim(\mathbf{x}) \tag{9}$$

   Plugging in our known $\mathbf{A}$ and $\mathbf{B}$ matrices into equation 9 gives us the following controllability matrix.

$$\begin{aligned} rank([\mathbf{B}\,\mathbf{AB}]) &= rank\left(\begin{bmatrix} 0 & 0.1 \\ 0.1 & -0.01 \end{bmatrix}\right) \\ rank([\mathbf{B}\,\mathbf{AB}]) &= 2 \end{aligned} \tag{10}$$

   With the controllability matrix being full rank, we can confidently say that the system is fully controllable.

   (b) Design K such that the controller has:

$$f_n = 10\,Hz \quad , \quad \zeta = 0.7$$

   Ans: We can design $\mathbf{K}$ to fit the desired parameters by modifying our observer state space model from the previous question and comparing our equation with the desired characteristic equation.

$$\begin{aligned} \dot{\hat{\mathbf{x}}} &= \mathbf{A}\hat{\mathbf{x}} + \mathbf{B}u + \mathbf{L}(y - \hat{y}) \\ u &= -\mathbf{K}\hat{\mathbf{x}} \\ \hat{y} &= \mathbf{C}\hat{\mathbf{x}} \\ \dot{\hat{\mathbf{x}}} &= \mathbf{A}\hat{\mathbf{x}} - \mathbf{BK}\hat{\mathbf{x}} + \mathbf{L}y - \mathbf{LC}\hat{\mathbf{x}} \\ \dot{\hat{\mathbf{x}}} &= (\mathbf{A} - \mathbf{BK} - \mathbf{LC})\,\hat{\mathbf{x}} + \mathbf{L}y \end{aligned} \tag{11}$$

   The only variable not defined here is $\mathbf{K}$ (and of course $\hat{\mathbf{x}}$). We can modify our system model and isolate $\mathbf{A}_c$ to solve for $\mathbf{K}$.

$$\mathbf{A}_c = \mathbf{A} - \mathbf{BK}$$

   Like we did in problem 2, calculating $det\,(s\mathbf{I} - \mathbf{A}_c)$ and comparing this calculated equation to the desired characteristic equation will give us values for $\mathbf{K}$. Our desired characteristic equation is:

$$s^2 + 2\zeta\omega_n s + \omega_n^2 \;\rightarrow\; s^2 + 2(0.7)(10 \cdot 2\pi)s + (10 \cdot 2\pi)^2 \;\rightarrow\; s^2 + 87.965s + 3947.8$$

$$
\begin{aligned}
det\,(s\mathbf{I}-\mathbf{A}_c) &= \frac{k_1}{10}+\frac{s}{10}+\frac{k_2\,s}{10}+s^2 \\
&= s^2+\left(\frac{k_2}{10}+\frac{1}{10}\right)s+\frac{k_1}{10}
\end{aligned} \tag{12}
$$

Comparing the equation gives the following values for the closed loop controller

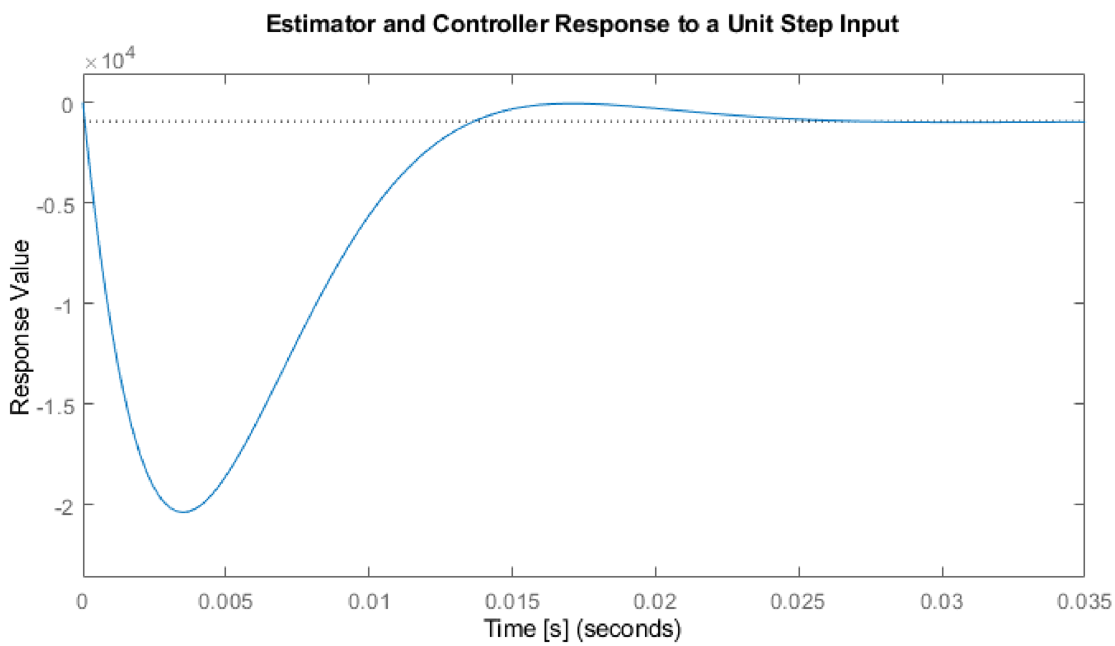
$$k_1 = 39,478 \qquad k_2 = 878.65$$

With the indices of $\mathbf{K}$ solved, we can return to equation 11 and finalize our controller and estimator state space model

$$
\begin{aligned}
\mathbf{A}_{comp} &= \mathbf{A}-\mathbf{BK}-\mathbf{LC} \\
\mathbf{A}_{comp} &= \begin{bmatrix} 0 & 1 \\ 0 & -0.1 \end{bmatrix} - \begin{bmatrix} 0 \\ 0.1 \end{bmatrix}\begin{bmatrix} k_1 & k_2 \end{bmatrix} - \begin{bmatrix} L_1 \\ L_2 \end{bmatrix}\begin{bmatrix} 1 & 0 \end{bmatrix} \\
\mathbf{A}_{comp} &= \begin{bmatrix} -439.7 & 1.0 \\ -98,752.1 & -14.0 \end{bmatrix}
\end{aligned} \tag{13}
$$

$$
\begin{aligned}
\dot{\hat{\mathbf{x}}} &= (\mathbf{A}_{comp})\,\hat{\mathbf{x}}+\mathbf{L}y \\
\dot{\hat{\mathbf{x}}} &= \begin{bmatrix} -439.7 & 1.0 \\ -98,752.1 & -14.0 \end{bmatrix}\hat{\mathbf{x}} + \begin{bmatrix} 439.7 \\ 98,652.1 \end{bmatrix} y
\end{aligned} \tag{14}
$$

(c) Provide a plot of the step response of the combined controller and estimator.
Ans:



4. Solve for the equivalent compensator for the system. **Ans:** Solving for the equivalent compensator of the system begins with finding the equivalent transfer function of our state space model (equation 15).

$$\frac{U(s)}{Y(s)} = -\mathbf{K}\left(s\mathbf{I} - \mathbf{A} + \mathbf{LC} + \mathbf{BK}\right)^{-1}\mathbf{L} \tag{15}$$

Plugging in our previously defined matrices give us a transfer function of

$$\frac{U(s)}{Y(s)} = \frac{-1.0404 \times 10^{8}\, s - 3.896 \times 10^{9}}{s^2 + 527.68\, s + 141,280}$$

(a) What kind of classical compensator does it resemble?
Ans: The system resembles a lead-lag compensator because of the second-order polynomial in the denominator.

(b) Calculator the closed loop transfer function.

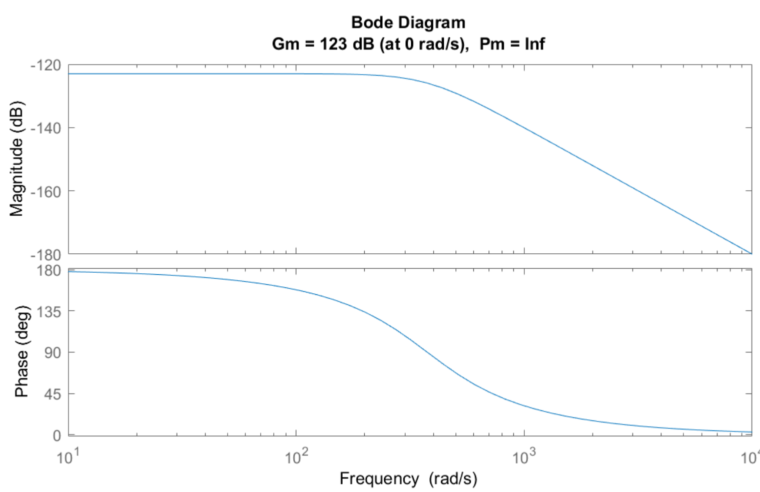
Similar to finding the equivalent compensator, we can find the closed-loop transfer function in the following fashion. (equation 16).

$$\frac{U(s)}{Y(s)} = -\mathbf{C}\left(s\mathbf{I} - \mathbf{A} + \mathbf{LC} + \mathbf{BK}\right)^{-1}\mathbf{B} \tag{16}$$

Plugging in our previously defined matrices give us a transfer function of

$$\frac{U(s)}{Y(s)} = \frac{0.1}{s^2 + 527.68\, s + 141,280}$$

(c) Plot the bode plot of the closed-loop system.
Ans:



(d) Find the gain and phase margin.
Ans: Using >> *margin* in MATLAB provides us with a gain margin of **123 dB** and an **infinite** phase margin.

5. Design the controller in the discrete domain assuming a 1 $[kHz]$ sample rate.

   (a) Discretize the state space model. Where are the eigenvalues?
   Ans: To design our controller in the discrete domain we can reintroduce our state space model and then use >> *c2d* from MATLAB.

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 0 & -0.1 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 0 \\ 0.1 \end{bmatrix} \quad \mathbf{C} = \begin{bmatrix} 1 & 0 \end{bmatrix} \quad \mathbf{D} = 0$$

$$>> c2d(ss(\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}), 0.001,' tustin') \tag{17}$$

   Executing the command in MATLAB produces the system in the discrete domain.

$$\mathbf{A} = \begin{bmatrix} 1 & 0.001 \\ 0 & 0.999 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 5 \times 10^{-8} \\ 0.0001 \end{bmatrix} \quad \mathbf{C} = \begin{bmatrix} 1 & 0.0005 \end{bmatrix} \quad \mathbf{D} = 2.5 \times 10^{-8} \tag{18}$$

   The eigenvalues for the discrete system are

$$z_{1,2} = 1, 0.9999$$

   (b) Design the L to provide the same response as problem #2.
   Ans: To design **L** in the discrete domain, we must first convert the eigenvalues of the desired response to discrete.

$$\begin{aligned} z_1 &= e^{s_1 t} & z_2 &= e^{s_2 t} \\ z_1 &= e^{(-157.1+272.06j)\frac{1}{1000}} & z_2 &= e^{(-157.1-272.06j)\frac{1}{1000}} \\ z_1 &= 0.82319 + 0.22965j & z_2 &= 0.82319 - 0.22965j \end{aligned} \tag{19}$$

   With our discrete eigenvalues for a desired observer response, we can use >> *place* to find the discrete values for **L**

$$\mathbf{L} \;=>> place(\mathbf{A}_z', \mathbf{C}_z', [\, z_1 \quad z_2 \,]) \tag{20}$$

$$\mathbf{L} = \begin{bmatrix} 0.31154 \\ 83.975 \end{bmatrix} \tag{21}$$

   (c) Design **K** to provide the same response as problem #3.
   Ans: In order to design **K** we need to convert the desired response to the discrete domain. This can be done in a similar fashion to the desired response for the observer gains, **L**.

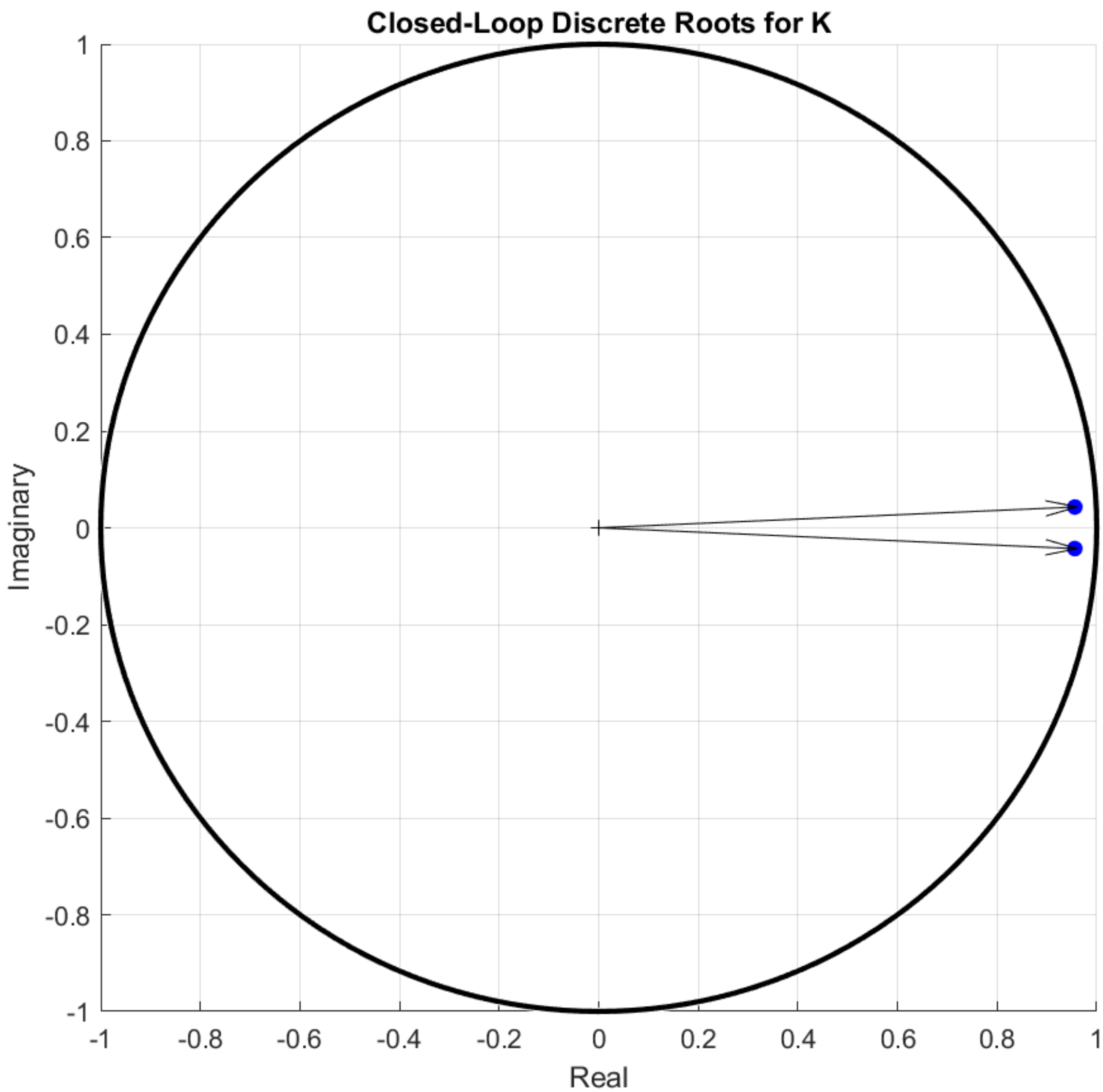
$$\begin{aligned} z_1 &= e^{s_1 t} & z_2 &= e^{s_2 t} \\ z_1 &= e^{(-43.983+44.87j)\frac{1}{1000}} & z_2 &= e^{(-43.983-44.87j)\frac{1}{1000}} \\ z_1 &= 0.95601 + 0.042925j & z_2 &= 0.95601 - 0.042925j \end{aligned} \tag{22}$$

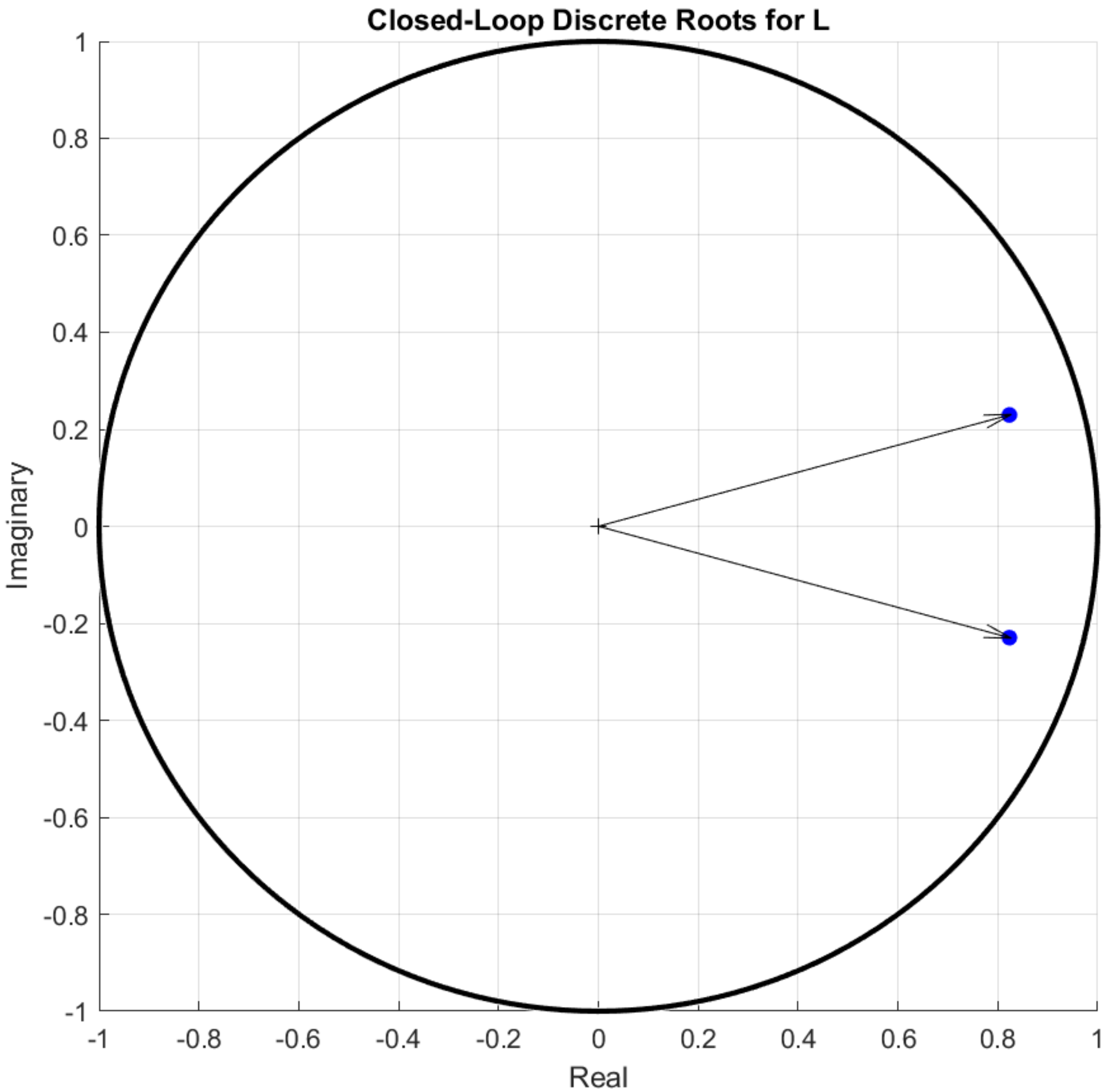
With our discrete eigenvalues for a desired observer response, we can use >> *place* to find the discrete values for **K**

$$\mathbf{K} \;=>> place(\mathbf{A}_z, \mathbf{B}_z, [\, z_1 \quad z_2 \,]) \tag{23}$$

$$\mathbf{K} = \begin{bmatrix} 840058.5 \\ 3115.4 \end{bmatrix} \tag{24}$$

(d) Where are the closed loop estimator and controller poles located?
Ans:




Closed-Loop Discrete Roots for L
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
Imaginary
-1
-0.8
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
0.8
1
Real

(e) Solve for the equivalent compensator transfer function.
Ans: Finding the transfer function for the discrete system can easily be found using $>> tf$ in MATLAB.

$$>> tf(\mathbf{A}_z, \mathbf{B}_z, \mathbf{C}_z, \mathbf{D}_z, 0.001) = \frac{-5.233e5\, z + 4.528e5}{z^2 - 1.293\, z + 0.5335} \tag{25}$$

6. Compare continuous and discrete response using simulation and using an equivalent compensator. Plot the both responses on a single figure.
   Ans: