Notes on Gaussian Quadrature with Application to Multiple Dimensions of Selection

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1 Goal

For the motivation of this model, see the "Panel Data Models Controlling for Selection into the GED and Whether or Not the Agent Works" by James Heckman.

Our eventual object of interest is to evaluate integrals of the form:

$$E\left[V|\,V>a,M>b\right]=\frac{\int_a^\infty V\int_b^\infty f\left(V,M\right)dVdM}{\int_a^\infty\int_b^\infty f\left(V,M\right)dVdM}$$

where the random variables V and M are distributed bivariate normal $N\left(0,\Sigma_{VM}\right)$ with

$$\Sigma_{VM}=\begin{bmatrix}1 & . \\ \rho & 1\end{bmatrix}$$

.

In the following sections, we develop the methods necessary to implement this indefinite multidimensional integral using the Gaussian quadrature method.

2 Solution for Single Dimension of Integration

Let $V\sim N\left(\mu,\sigma^2\right)$. Redefine the integral problem as

$$\begin{aligned}E\left[V|\,V>a\right] &= \frac{\int_a^\infty vf\left(v\right)dv}{\int_a^\infty f\left(v\right)dv}\\ &= \frac{\int_a^\infty v\frac{1}{\sigma}\phi\left(\frac{v-\mu}{\sigma}\right)dv}{\int_a^\infty \frac{1}{\sigma}\phi\left(\frac{v-\mu}{\sigma}\right)dv}\end{aligned}$$

$$
\begin{aligned}
&= \frac{\int_{\frac{a-\mu}{\sigma}}^{\infty} \left[U\sigma + \mu\right] \phi\left(U\right) \sigma dU}{\int_{\frac{a-\mu}{\sigma}}^{\infty} \phi\left(U\right) \sigma dU} \\
&= \frac{\int_{\frac{a-\mu}{\sigma}}^{\infty} \left[U\sigma + \mu\right] \phi\left(U\right) dU}{1 - \Phi\left(\frac{a}{\sigma} - \mu\right)} \\
&= \frac{\left\{\sigma \int_{\frac{a-\mu}{\sigma}}^{\infty} U\phi\left(U\right) dU + \mu \int_{\frac{a-\mu}{\sigma}}^{\infty} \phi\left(U\right) dU\right\}}{1 - \Phi\left(\frac{a}{\sigma} - \mu\right)} \\
&= \sigma \frac{\int_{\frac{a-\mu}{\sigma}}^{\infty} U\phi\left(U\right) dU}{1 - \Phi\left(\frac{a-\mu}{\sigma}\right)} + \mu
\end{aligned}
$$

The third line uses the change of variables $v = (U\sigma + \mu)$ with $dv = \sigma dU$. Notice that if our variable $V \sim N(0,1)$ this expression simplifies to

$$
E\left[V \mid V > a\right] = \frac{\int_{a}^{\infty} U\phi\left(U\right) dU}{1 - \Phi\left(a\right)}
$$

This special case is commonly encountered, and the solution is well known as the inverse Mills ratio $\lambda(a) \equiv \phi(a) / [1 - \Phi(a)]$. Because our two-dimensional problem does not have a closed-form solution, we motivate the manipulations necessary for numerical integration with this simpler case.

We will show how this quantity can be approximated using the Gaussian quadrature method. Gaussian quadrature uses the approximation

$$
\int_{-1}^{1} f\left(x\right) dx \approx \sum_{i=1}^{n} \omega_i f\left(x_i\right) \tag{1}
$$

where the x_i are select points within the interval $[-1,1]$ and ω_i are unique solutions to the problem of producing an exact integral of a polynomial of degree $2n-1$.

The first necessary step is to transform our integral $\sigma \frac{\int_{\frac{a-\mu}{\sigma}}^{\infty} U\phi(U)dU}{1-\Phi\left(\frac{a-\mu}{\sigma}\right)} + \mu$ into a definite integral. We use the change of variables $U = \Phi^{-1}(u)$ and $du = \phi(U)\, dU$. The bounds of integration will be $\left[\Phi\left(\frac{a-\mu}{\sigma}\right), \Phi(\infty)\right] = \left[\Phi\left(\frac{a-\mu}{\sigma}\right), 1\right]$. Using these transformations we obtain

$$
\begin{aligned}
\sigma \frac{\int_{\frac{a-\mu}{\sigma}}^{\infty} U\phi\left(U\right) dU}{1 - \Phi\left(\frac{a-\mu}{\sigma}\right)} + \mu &= \sigma \int_{\Phi\left(\frac{a-\mu}{\sigma}\right)}^{1} \Phi^{-1}\left(u\right) \frac{\phi\left(\Phi^{-1}\left(u\right)\right)}{1 - \Phi\left(\frac{a-\mu}{\sigma}\right)} \frac{du}{\phi\left(\Phi^{-1}\left(u\right)\right)} + \mu \\
&= \sigma \int_{\Phi\left(\frac{a-\mu}{\sigma}\right)}^{1} \frac{\Phi^{-1}\left(u\right)}{1 - \Phi\left(\frac{a-\mu}{\sigma}\right)} du + \mu
\end{aligned} \tag{2}
$$

In order to use the quadrature method, we perform one more change of variables to change the bounds of integration to $[-1,1]$. We want p and q such that $u = px + q$ and$-1 \le x \le 1 \Rightarrow -p + q \le u \le p + q$. Given that u has bounds a and b, this is a two-equation system where it is straightforward to show that $q = \frac{b+a}{2}$ and $p = \frac{b-a}{2}$.

Thus, we can use this second change of variables with $u = \frac{1-\Phi\left(\frac{a-\mu}{\sigma}\right)}{2}x + \frac{1+\Phi\left(\frac{a-\mu}{\sigma}\right)}{2}$ and $du = \frac{1-\Phi\left(\frac{a-\mu}{\sigma}\right)}{2}dx$ in combination with equation (2) to obtain

$$\begin{aligned}
\sigma \int_{\Phi\left(\frac{a}{\sigma}-\mu\right)}^{1} \frac{\Phi^{-1}(u)}{1-\Phi\left(\frac{a-\mu}{\sigma}\right)} du + \mu &= \sigma \int_{-1}^{1} \frac{\Phi^{-1}\left(\frac{1-\Phi\left(\frac{a-\mu}{\sigma}\right)}{2}x + \frac{1+\Phi\left(\frac{a-\mu}{\sigma}\right)}{2}\right)}{1-\Phi\left(\frac{a-\mu}{\sigma}\right)} \left(\frac{1-\Phi(a)}{2}dx\right) + \mu \\
&= \sigma \int_{-1}^{1} \frac{1}{2}\Phi^{-1}\left(\frac{1-\Phi\left(\frac{a-\mu}{\sigma}\right)}{2}x + \frac{1+\Phi\left(\frac{a-\mu}{\sigma}\right)}{2}\right) dx + \mu
\end{aligned}$$

Thus we can apply equation (1) where $f(x_i) = \frac{1}{2}\Phi^{-1}\left(\frac{1-\Phi\left(\frac{a-\mu}{\sigma}\right)}{2}x_i + \frac{1+\Phi\left(\frac{a-\mu}{\sigma}\right)}{2}\right)$.

3 Solution for Two Dimensional Integration

The double integral reflecting dual truncation is:

$$\begin{aligned}
E[V|V>a, M>b] &= \int_a^\infty V \int_b^\infty f(V,M)\, dM dV \\
&= \int_a^\infty V \int_b^\infty \phi_2(V,M,\rho)\, dM dV \Big/ \left[\int_a^\infty \int_b^\infty \phi_2(V,M,\rho)\, dV dM\right]
\end{aligned}$$

where $\phi_2(x,y,\rho)$ is the bivariate normal density function with correlation parameter ρ, where

$$\phi(x,y,\rho) = \frac{1}{2\pi\sqrt{1-\rho^2}} exp\left\{-\frac{x^2+y^2+2\rho xy}{2(1-\rho^2)}\right\}$$

We will use the dual change of variables of $u = \Phi(V)$, $w = \Phi(M)$.[1] The relationships we use for substitution are $V = \Phi^{-1}(u)$, $M = \Phi^{-1}(w)$, $du = \phi(V)\, dV$, $dw = \phi(M)$, and the bounds of the integrals will be $[\Phi(a), \Phi(\infty)] = [\Phi(a), 1]$ for u and $[\Phi(b), \Phi(\infty)] = [\Phi(b), 1]$ for w. This transforms our integral to

$$\int_{\Phi(a)}^{1} \Phi^{-1}(u) \left[\int_{\Phi(b)}^{1} \phi_2\left(\Phi^{-1}(u), \Phi^{-1}(w), \rho\right) \frac{dw}{\phi\left(\Phi^{-1}(w)\right)}\right] \frac{du}{\phi\left(\Phi^{-1}(u)\right)} \Big/ \left[\int_a^\infty \int_b^\infty \phi_2(V,M,\rho)\, dV dM\right]$$

We do not need to transform the denominator since the value of that integral is well known.

We apply one more set of variable substitution to transform the bounds to $[-1,1]$ for both integrals. The substitution is $u = \frac{1-\Phi(a)}{2}x + \frac{1+\Phi(a)}{2}$ and $w = \frac{1-\Phi(b)}{2}y + \frac{1+\Phi(b)}{2}$, and $du = \frac{1-\Phi(a)}{2}dx$ and $dw = \frac{1-\Phi(b)}{2}dy$. Applying it yields:

[1]Recall that both V and M are standard normal random variables in our application.

$$... = \int_{-1}^{1} \Phi^{-1}\left(\frac{1-\Phi(a)}{2}x + \frac{1+\Phi(a)}{2}\right)\left[\int_{-1}^{1} \phi_2\left(\Phi^{-1}\left(\frac{1-\Phi(a)}{2}x + \frac{1+\Phi(a)}{2}\right), \Phi^{-1}\left(\frac{1-\Phi(b)}{2}y + \frac{1+\Phi(b)}{2}\right), \rho\right) \frac{1-\Phi(b)}{2} \frac{dy}{\phi\left(\Phi^{-1}\left(\frac{1-\Phi(b)}{2}y+ \right.\right.}\right.$$

Simplifying the expression:

$$... = \frac{(1-\Phi(b))(1-\Phi(a))}{4}\left[\int_{-1}^{1}\int_{-1}^{1}\frac{p\phi_2(p,q,\rho)}{\phi(q)\phi(p)}dydx\right] \Big/ \left[\int_{a}^{\infty}\int_{b}^{\infty}\phi_2(V,M,\rho)\,dVdM\right]$$

where

$$\begin{aligned} p &= \Phi^{-1}\left(\frac{1-\Phi(a)}{2}x + \frac{1+\Phi(a)}{2}\right) \\ q &= \Phi^{-1}\left(\frac{1-\Phi(b)}{2}y + \frac{1+\Phi(b)}{2}\right) \end{aligned}$$