Discrete Structures - HW1 — Nitishree Supekar

Section 1.1

Q2) For any proposition, there must be a clear true/false (a) It is a command, so not a PROPOSITION

(b) It is a question, so not a proposition

(c) It is a proposition since it has an answer (true/false)

(d) Since we don't know x, it is not a proposition

(e) It is a proposition since it ~~has~~ is false.

(f) It is not a proposition as we don't know 'n'.

Q10) (a) $\neg p$ in the questions means the election is not decided

(b) $p \vee q$ means the election is decided or the votes have been counted

(c) $\neg p \vee q$ means the election is not decided and the votes have been counted

(d) $q \rightarrow p$ means If the votes have been counted then the election is decided.

(e) $\neg q \rightarrow \neg p$ means if the votes have not been counted then the election is not decided.

(f) $\neg p \rightarrow \neg q$ means if the election is not decided then the votes have not been counted

(g) $p \leftrightarrow q$ means ~~If~~ the election is decided if and only if the votes have been counted

(h) $\neg q \vee (\neg p \vee q)$ means The votes have not been counted or the votes have been counted but the election is not decided.

Q18) (a) If $1+1=3$ then unicorns exist

Ans) If then else ($p \rightarrow q$) is false only when p is true and q is false. Otherwise it is true so,

a) $1+1=3$ is false, unicorns exist = false $\therefore p \rightarrow q =$ true

(b) $1+1=3$ is false, dogs can fly = false $\therefore p \rightarrow q =$ true

(c) $1+1=2$ is true, dogs can fly = false $\therefore p \rightarrow q =$ false

(d) $2+2=4$ is true, $1+2=3$ = ~~fal~~ true, $\therefore p \rightarrow q =$ true

Section 1.2

Q10) Assume p = "The system software is being upgraded" and q = "Users can access the file system".
r = "Users can save new files"

so we can write the question as :

$\boxed{p \to q}$ and $\boxed{q \to r}$ and $\boxed{-r \to -p}$

for If then else ~~for~~ eg $a \to b$, this becomes false only if a is true and b is false, otherwise it is true. Similarly,

the given system is consistent and the combinations are : if any x, y are true then if z is false. or if x and y are false, and z is true. these combinations make the system consistent.

Section 1.3

Q 8) Using De-morgan's law : $-(a \vee b) = (-a \wedge -b)$

(a) kwame will take a job in industry or go to graduate school. ⇒ kwame will not take a job in industry and not go to graduate school.

(b) Yoshiko knows java and calculus ⇒ yoshiko doesn't know java or calculus

(c) James is young and strong ⇒ James is not young or strong.

(d) Rita will move to Oregon or Washington ⇒ Rita will not move to Oregon and Washington.

Q12) Laws =

(a) $[-p \wedge (p \vee q)] \to q$ (by decomposition law)

$\equiv -(-p \wedge (p \vee q)) \vee q$ (by De-morgan's law)

$\equiv [p \vee (-p \wedge -q)] \vee q$

$\equiv [\underbrace{(p \vee -p)}_{\text{tautology}} \wedge (p \vee -q)] \vee q$ [by distribution law]

$\equiv [T \wedge (p \vee -q)] \vee q$ [by negation]

$\equiv (p \vee -q) \vee q$ [by Identity law]

$\equiv p \vee \underbrace{(-q \vee q)}_{\text{negation}}$ (by associative law)

$\equiv p \vee T \equiv \boxed{T}$ (by Identity law)

b) $[(p \to q) \land (q \to r)] \to (p \to r)$ (by decomposition law)

$\equiv -[(p \to q) \land (q \to r)] \lor (p \to r)$

$\equiv -(-p \lor q) \lor -(-q \lor r) \lor (-p \lor r)$ (De-Morgan's law)

$\equiv (p \land -q) \lor (q \land -r) \lor (-p \lor r)$ (Commutative law)

$\equiv -p \lor (p \land -q) \lor (q \land -r) \lor r$ (Distributive law)

$\equiv (-p \lor p) \land (-p \lor -q) \lor (q \lor r) \land (-r \lor r)$ (by negation)

$\equiv [T \land (-p \lor -q)] \lor ((q \lor r) \land T)$ (identity law)

$\equiv (-p \lor -q) \lor (q \lor r)$ (associative law)

$\equiv -p \lor (-q \lor q) \lor r$ (by negation)

$\equiv (-p \lor r) \lor T \equiv \boxed{T}$ (by identity law)

(c) $[p \land (p \to q)] \to q$ (by decomposition)

$\equiv -[p \land (p \to q)] \lor q \equiv [-p \lor -(-p \lor q)] \lor q$

(by de-morgan's law) $\equiv (-p \lor p \lor -q) \lor q$

(by associative law) $\equiv (-p \lor q) \lor (-q \lor q)$

(by negation) $\equiv T \lor T \equiv \boxed{T}$

(d) $[(p \lor q) \land (p \to r) \land (q \to r)] \to r$ $\equiv$ ~~$-[(p \lor q) \land (-p \lor r) \land (q$~~ $\lor r)$

(by decomposition law:

$\equiv -[(p \lor q) \land (-p \lor r) \land (-q \lor -r)] \lor r$ $\equiv$ ~~$-(p \lor q) \lor -(p \lor r) \lor r$~~

$\equiv -(p \lor q) \lor -(-p \lor r) \lor -(q \lor r) \lor r$ (De Morgan's law)

$\equiv (-p \land q) \lor (p \land -r) \lor (q \land -r) \lor r$ (De-morgan's)

$\equiv (-p \land -q) \lor (p \land -r) \lor [(q \lor r) \land (-r \lor r)]$ [Distributive law

$\equiv [-p \land -q) \lor (p \land -r) \lor [(q \lor r) \land T]$ (by Negation)

$\equiv [-p \land -q) \lor (p \land -r) \lor (q \lor r)$ (identity law)

$\equiv (-p \land -q) \lor q \lor (p \land -r) \lor r$ (associative law).

$\equiv [(-p \lor q) \land (-q \lor q)] \lor [(p \lor r) \land (-r \lor r)]$ (distributive law

$\equiv ([-p \lor q) \land T] \lor ((p \lor r) \land T)]$ (by Negation)

$\equiv (-p \lor q) \lor (p \lor r)$ (by identity law)

$\equiv (-p \lor p) \lor (q \lor r)$ (by associative law)

$\equiv T \lor q \lor r \equiv \boxed{T}$ (by domination)

Section 1.4

Q8) (a) $\forall x\,(R(x) \rightarrow H(x))$ becomes - If an animal is rabbit then the animal hops.

(b) $\forall x\,(R(x) \wedge H(x))$ becomes - Every animal is a rabbit and it hops.

(c) $\exists x\,(R(x) \rightarrow H(x))$ becomes - There exists an animal such that if it is a rabbit it hops.

(d) $\exists x\,[R(x) \wedge H(x)]$ - becomes - There exists an animal such that it is a rabbit and hops.

Q20) (d) $P(1) \vee P(3) \vee P(5)$

(e) $[-p[-5] \vee -p(-3) \vee -p(-1) \vee -p(1) \vee -P(3) \vee -p(5)] \wedge [P(-5) \wedge P(-3) \wedge P(-1)]$

$\equiv [-p(1) \vee -p(3) \vee -p(5)] \wedge [p(-5) \wedge P(-3) \wedge P(-1)]$

Q50) To prove $\forall x\, P(x) \vee \forall x\, Q(x)$ and $\forall x[P(x) \vee Q(x)]$ are not logically equivalent.

Assume $p(x)$ is a proposition that is either true or false and $Q(x)$ be a propositional function that is false then $\forall x[P(x) \rightarrow Q(x)]$ is false when $P(x)$ is true but $\forall x[P(x) \rightarrow \forall x\, Q(x)]$ is True because $\forall x\, P(x)$ is a false.

Section 1.5

Q22) (a) $\forall x \forall y\,(((x \neq 0) \wedge y < 0)) \rightarrow (xy > 0))$

(b) $\forall x \forall y\,(((x > 0) \wedge (y > 0)) \rightarrow ((x+y)/2 > 0)$

(c) $\exists x \exists y\,(((x < 0) \wedge (y < 0)) \wedge (x - y) \geq 0)$

(d) $\forall x \forall y\,(|x+y| \leq |x| + |y|)$

Q30) (a) $\forall x \forall y - P(x, y)$

(b) $\exists x \forall y - P(x, y)$

(c) $\forall y\, [-Q(y) \vee \exists x\, R(x, y)]$

(d) $\forall y\, [\forall x - R(x, y) \wedge \exists x - S(x, y)]$

(e) $\forall y\, [\exists x \forall z - T(x, y, z) \wedge \forall x \exists z - V(x, y, z)]$

Section 1.6

Q14)(a) $c(x)$: x is in class ; $r(x)$ = x owns a red convertible

$t(x)$: x has gotten a speeding ticket

1) $\forall x\,(r(x) \to t(x))$ hypothesis
2) $r(\text{Linda}) \to t(\text{Linda})$ Universal instantiation
3) $r(\text{linda})$ hypothesis
4) $t(\text{Linda})$ hypothesis
5) $c(\text{Linda})$ Modes ponens (2) & (3)
(6) $c(\text{linda}) \wedge t(\text{Linda})$ conjunction
(7) $\exists x\,(c(x) \wedge t(x))$ Existential generalization

(b) $a(x)$: x is one of the five roommates

$b(x)$: x has taken a course discrete

$c(x)$: x has taken a course algord

1) $\forall x\,(a(x) \to b(x))$ hypothesis
2) $a(k) \to b(k)$ Universal instantiation
3) $\forall x\,(b(x) \to c(x))$ hypothesis
4) $b(k) \to c(k)$ Universal instantiation
5) $a(k) \to c(k)$ hypothetical syllogism (2) & (4)
(6) $\forall x\,(a(x) \to c(x))$ Universal generalization

(c) Let $a(x)$ x is a movie produced by sayles.

$b(x)$: x is about coal mines

$c(x)$: x is wonderful.

1) $\exists x\,(a(x) \wedge b(x))$ hypothesis
2) $a(k) \wedge b(k)$ existential instantiation
3) $a(k)$ simplification
4) $\forall x\,(a(x) \to c(x))$ hypothesis
5) $a(k) \to c(k)$ Universal instantion
6) $c(k)$ modes ponens (3) (5)
7) $b(k)$ simplification (2)
8) $c(k) \wedge b(k)$ conjunction
9) $\exists x\,(c(x) \wedge b(x))$ Existential generalization

(d) $c(x)$: x is in class
$f(x)$: x has been to France
$l(x)$: x has visited Louvre

1) $\exists x (c(x) \wedge f(x))$ — hypothesis
2) $c(s) \wedge f(s)$ — Existential instantiation
3) $c(s)$ — Simplification
4) $f(s)$ — Simplification
5) $\forall x (f(x) \rightarrow L(x))$ — hypothesis
6) $f(s) \rightarrow L(S)$ — Universal instantiation
7) $L(S)$ — Modus ponens (4) & (6)
(8) $c(S) \wedge L(S)$ — conjunction (3) & (7)
(9) $\exists x (c(x) \wedge L(x))$ — Existential generalization

Q27)
(1) $\forall x (P(x) \vee Q(x))$ — hypothesis
(2) $P(c) \vee Q(c)$ — Universal instantiation
(3) $\forall x (\neg P(x) \wedge Q(x) \rightarrow R(x))$ — hypothesis
(4) $\neg P(c) \wedge Q(c) \rightarrow R(c)$ — Universal Instantiation
(5) $\neg(\neg P(c) \wedge Q(c)) \vee R(c)$ — Decomposition
(6) $P(c) \vee \neg Q(c) \vee R(c)$ — De morgan's law
(7) $\neg P(c) \vee R(c)$ — Resolution (2) & (6)
(8) $\neg(\neg R(c)) \vee P(c)$ — Rewriting
(9) $\neg R(c) \rightarrow P(c)$ — Universal generalization
(10) $\forall x (\neg R(x) \rightarrow P(x))$

Q28) In the step 3 and 5, we can see that $p(c)$ and $q(c)$ are existential quantifiers from (2) and (4)
(2) $\exists x P(x)$ simplification from (1)
(4) $\exists x q(x)$ simplification from (1)
We know that x does exist but there is no proof that these x are same values.

Hence, 3 & 5 cannot be performed.

Section 1.7

Q28) given $m^2 = n^2 \Rightarrow m^2 - n^2 = 0 \Rightarrow (m-n)(m+n) = 0$

Hence $m - n = 0$, and $\boxed{m = n}$

or $m + n = 0$; $\boxed{m = -n}$

If $m = n$, then $m^2 = (m)(m) = (n)(n) = n^2$

and $m = -n$ then $m^2 = (m)(m) = (-n)(-n) = n^2$

so we get $m^2 = n^2$

Q34) for The Equation : $\sqrt{2x^2 - 1} = x$

(1) square : $2x^2 - 1 = x^2$

(2) $x^2 - 1 = 0$ (substraction)

(3) $(x-1)(x+1) = 0$

(4) $x = 1$ or $x = -1$

but this solution doesn't match with the equation given with value $= -1$.

at $x = 1 \Rightarrow \sqrt{2(1)^2 - 1} = 1 = x$

at $x = -1 \Rightarrow \sqrt{2(-1)^2 - 1} = 1 \neq x$

So only the checking value in initial equation step is missing.

Section 1.8

Q8) For proving there is an + integer that equals the sum of positive integers not exceeding it :-

let n = a positive integer

so $n = n + (n-1) + (n-2) + \dots + 2 + 1$

Sum^n of $n = \boxed{\frac{n(n+1)}{2}} = n$ (since it is given)

$\therefore$

$2n = n^2 + n$ (by simplification)

$n^2 - n = 0 \Rightarrow n(n-1) = 0$

So, $n = 0$ or $n = 1$ but $n > 0$.

for this solution $n = 1$ is (constructive)

Q10) To prove: $2 \cdot 10^{500} + 15$ or $2 \cdot 10^{500} + 16$ is not a perfect square

let $p = 2 \cdot 10^{500} + 15$

$q = 2 \cdot 10^{500} + 16$

by this $q = p + 1$

Assuming p and q are perfect squares so,

$p = x^2$ and $q = y^2$

and $q = p + 1$

$y^2 = x^2 + 1 \Rightarrow y^2 - x^2 = 1 \Rightarrow (y-x)(y+x) = 1$

so $y - x = 1$

and $y + x = 1$

$y = 1$, $x = 0$

and hence $p = 0$ and $q = 1$

so from above, we can conclude that atleast one amongst them is not a perfect square

Section 2.1

Q2) (a) Set Builder notation :-

(a) $\{0, 3, 6, 9, 12\} \equiv \{x \mid x = 3a,\ a = 0, 1, 2, 3, 4\}$

(b) $\{-3, -2, -1, 0, 1, 2, 3\} \equiv \{x \in Z,\ -3 \leq x \leq 3\}$

(c) $\{m, n, o, p\} \equiv \{x \mid x = m \vee x = n \vee x = o \vee x = p\}$

Q10) (a) True since ϕ is empty set so no value

(b) true since ϕ can be written as $\{\phi$ and $\{\phi\}\}$

(c) false

(d) true

(e) true

(f) true

(g) true

Q30) given : $A \times B = \phi$ and A, B are sets.

$\because A \times B = \phi \Rightarrow |A \times B| = 0$

so either $|A|$ or $|B| = 0$

Hence either A or B is an empty set.

Section 2.2

Q18) (e) $(B-A) \cup (C-A) = (B \cup C) - A$

by set identities :- $x \in (B-A) \cup (C-A)$

$\because x \in (B-A) \cup x \in (C-A)$ (Union property)

$\because (x \in B \wedge \sim(x \in A) \vee (x \in C \wedge \sim(x \in A))$

(by difference rule)

$\because (x \in B \vee x \in C) \wedge \sim(x \in A)$ (distributive rule)

$\because x \in B \cup C) \wedge \sim(x \in A)$ (by union property)

$\because x \in (B \cup C) - A$ (difference rule)

Q30) (a) $A \cup B = B \cup C$ since A & B are not same sets so we can't conclude $A = B$

(b) $A \cap C = B \cap C$

For this take an example :-

$A = \{1, 3, 5\}$ since here we can

$C = \{1, 5\}$ see $A \cap C = B \cap C$

$B = \{1, 2, 5\}$ but $A \neq B$.

(c) $A \cup C = B \cup C$ and $A \cap C = B \cap C$

→ ~~For~~ ~~let~~ $x \in A$

$\Rightarrow x \in A \cup C$ (~~by~~ take union)

$\Rightarrow x \in B \cup C$ ($\because A \cup C = B \cup C$)

assuming $x \in C$

$\therefore x \in A \cap C$ (intersection)

$\Rightarrow x \in B \cap C$ ($\because A \cap C = B \cap C$)

$x \in B$

so ~~A~~ $x \in B$ also $x \in B$ $\therefore$ $\boxed{A \subseteq B}$

For $x \in B$

$\Rightarrow x \in B \cup C$ (union)

$x \in A \cup C$ ($B \cup C = A \cup C$)

Assume $x \in C$

$\Rightarrow x \in B \cap C$ (intersection)

$\Rightarrow x \in A \cap C$ ($B \cap C = A \cap C$)

$\Rightarrow x \in A$ (intersection)

$B \subseteq A$

$A = B$

Section 2.3

Q10) ~~for~~ To determine functions are one-to-one.

(a) $f(a) = b$, $f(b) = a$; $f(c) = c$; $f(d) = d$
~~these~~ since no other has same set ~~to~~ as to which one refers eg b is output of $f(a)$ & ~~same~~ b is not output of others $\therefore$ One to One.

(b) $f(a) = b$, $f(B) = b$, $f(c) = d$, $f(d) = c$
since the output "(b)" is repeated more than once so they are not one-to-one

(c) $f(a) = d$, $f(b) = b$, $f(c) = c$; $f(d) = d$
Since here output 'd' repeats twice, so these functions are not one to one.

Q34) given: f and g are one-to-one
to ~~prove~~ investigate if g is one-to-one

since f and g are one to one
and $g: A \to B$ and $f: B \to C$
suppose $g(a) = g(b)$
taking function f of both sides
$f(g(a)) = f(g(b))$
by the definition of composition:-
$(f \cdot g)(a) = (f \cdot g)(b)$
since $f \cdot g$ is one to one
then $a = b$
so $\Rightarrow g(a) = g(b)$
so
by definition g is one-to-one function