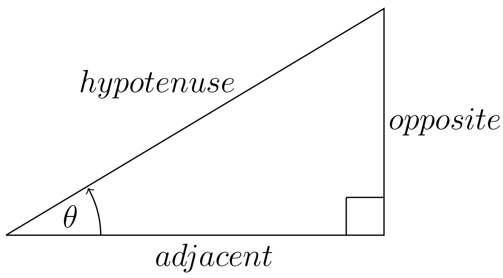
# Trigonometric Formula Sheet
## Definition of the Trig Functions

### Right Triangle Definition

Assume that:

$0 < \theta < \frac{\pi}{2}$ or $0^\circ < \theta < 90^\circ$



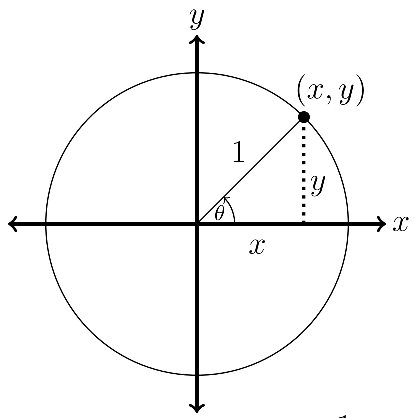

$$\sin\theta = \frac{opp}{hyp} \qquad \csc\theta = \frac{hyp}{opp}$$

$$\cos\theta = \frac{adj}{hyp} \qquad \sec\theta = \frac{hyp}{adj}$$

$$\tan\theta = \frac{opp}{adj} \qquad \cot\theta = \frac{adj}{opp}$$

### Unit Circle Definition

Assume $\theta$ can be any angle.




$$\sin\theta = \frac{y}{1} \qquad \csc\theta = \frac{1}{y}$$

$$\cos\theta = \frac{x}{1} \qquad \sec\theta = \frac{1}{x}$$

$$\tan\theta = \frac{y}{x} \qquad \cot\theta = \frac{x}{y}$$

## Domains of the Trig Functions

$\sin\theta, \quad \forall\ \theta \in (-\infty, \infty)$

$\cos\theta, \quad \forall\ \theta \in (-\infty, \infty)$

$\tan\theta, \quad \forall\ \theta \neq \left(n + \frac{1}{2}\right)\pi,\ where\ n \in \mathbb{Z}$

$\csc\theta, \quad \forall\ \theta \neq n\pi,\ where\ n \in \mathbb{Z}$

$\sec\theta, \quad \forall\ \theta \neq \left(n + \frac{1}{2}\right)\pi,\ where\ n \in \mathbb{Z}$

$\cot\theta, \quad \forall\ \theta \neq n\pi,\ where\ n \in \mathbb{Z}$

## Ranges of the Trig Functions

$$-1 \leq \sin\theta \leq 1$$
$$-1 \leq \cos\theta \leq 1$$
$$-\infty \leq \tan\theta \leq \infty$$

$$\csc\theta \geq 1\ and\ \csc\theta \leq -1$$
$$\sec\theta \geq 1\ and\ \sec\theta \leq -1$$
$$-\infty \leq \cot\theta \leq \infty$$

## Periods of the Trig Functions

The period of a function is the number, T, such that f $(\theta + \mathrm{T}) =$ f $(\theta)$ .
So, if $\omega$ is a fixed number and $\theta$ is any angle we have the following periods.

$$\sin(\omega\theta) \ \Rightarrow\ T = \frac{2\pi}{\omega} \qquad \csc(\omega\theta) \ \Rightarrow\ T = \frac{2\pi}{\omega}$$

$$\cos(\omega\theta) \ \Rightarrow\ T = \frac{2\pi}{\omega} \qquad \sec(\omega\theta) \ \Rightarrow\ T = \frac{2\pi}{\omega}$$

$$\tan(\omega\theta) \ \Rightarrow\ T = \frac{\pi}{\omega} \qquad \cot(\omega\theta) \ \Rightarrow\ T = \frac{\pi}{\omega}$$

# Identities and Formulas

## Tangent and Cotangent Identities

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cot\theta = \frac{\cos\theta}{\sin\theta}$$

## Reciprocal Identities

$$\sin\theta = \frac{1}{\csc\theta} \qquad \csc\theta = \frac{1}{\sin\theta}$$

$$\cos\theta = \frac{1}{\sec\theta} \qquad \sec\theta = \frac{1}{\cos\theta}$$

$$\tan\theta = \frac{1}{\cot\theta} \qquad \cot\theta = \frac{1}{\tan\theta}$$

## Pythagorean Identities

$$\sin^2\theta + \cos^2\theta = 1$$

$$\tan^2\theta + 1 = \sec^2\theta$$

$$1 + \cot^2\theta = \csc^2\theta$$

## Even and Odd Formulas

$$\sin(-\theta) = -\sin\theta \qquad \csc(-\theta) = -\csc\theta$$

$$\cos(-\theta) = \cos\theta \qquad \sec(-\theta) = \sec\theta$$

$$\tan(-\theta) = -\tan\theta \qquad \cot(-\theta) = -\cot\theta$$

## Periodic Formulas

If n is an integer

$$\sin(\theta + 2\pi n) = \sin\theta \qquad \csc(\theta + 2\pi n) = \csc\theta$$

$$\cos(\theta + 2\pi n) = \cos\theta \qquad \sec(\theta + 2\pi n) = \sec\theta$$

$$\tan(\theta + \pi n) = \tan\theta \qquad \cot(\theta + \pi n) = \cot\theta$$

## Double Angle Formulas

$$\sin(2\theta) = 2\sin\theta\cos\theta$$

$$\begin{aligned}\cos(2\theta) &= \cos^2\theta - \sin^2\theta \\ &= 2\cos^2\theta - 1 \\ &= 1 - 2\sin^2\theta\end{aligned}$$

$$\tan(2\theta) = \frac{2\tan\theta}{1 - \tan^2\theta}$$

## Degrees to Radians Formulas

If $x$ is an angle in degrees and $t$ is an angle in radians then:

$$\frac{\pi}{180^\circ} = \frac{t}{x} \qquad \Rightarrow \qquad t = \frac{\pi x}{180^\circ} \quad \textbf{and} \quad x = \frac{180^\circ t}{\pi}$$

## Half Angle Formulas

$$\sin\theta = \pm\sqrt{\frac{1 - \cos(2\theta)}{2}}$$

$$\cos\theta = \pm\sqrt{\frac{1 + \cos(2\theta)}{2}}$$

$$\tan\theta = \pm\sqrt{\frac{1 - \cos(2\theta)}{1 + \cos(2\theta)}}$$

## Sum and Difference Formulas

$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$$

$$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$$

$$\tan(\alpha \pm \beta) = \frac{\tan\alpha \pm \tan\beta}{1 \mp \tan\alpha\tan\beta}$$

## Product to Sum Formulas

$$\sin\alpha\sin\beta = \frac{1}{2}[\cos(\alpha - \beta) - \cos(\alpha + \beta)]$$

$$\cos\alpha\cos\beta = \frac{1}{2}[\cos(\alpha - \beta) + \cos(\alpha + \beta)]$$

$$\sin\alpha\cos\beta = \frac{1}{2}[\sin(\alpha + \beta) + \sin(\alpha - \beta)]$$

$$\cos\alpha\sin\beta = \frac{1}{2}[\sin(\alpha + \beta) - \sin(\alpha - \beta)]$$

## Sum to Product Formulas

$$\sin\alpha + \sin\beta = 2\sin\left(\frac{\alpha + \beta}{2}\right)\cos\left(\frac{\alpha - \beta}{2}\right)$$

$$\sin\alpha - \sin\beta = 2\cos\left(\frac{\alpha + \beta}{2}\right)\sin\left(\frac{\alpha - \beta}{2}\right)$$

$$\cos\alpha + \cos\beta = 2\cos\left(\frac{\alpha + \beta}{2}\right)\cos\left(\frac{\alpha - \beta}{2}\right)$$

$$\cos\alpha - \cos\beta = -2\sin\left(\frac{\alpha + \beta}{2}\right)\sin\left(\frac{\alpha - \beta}{2}\right)$$

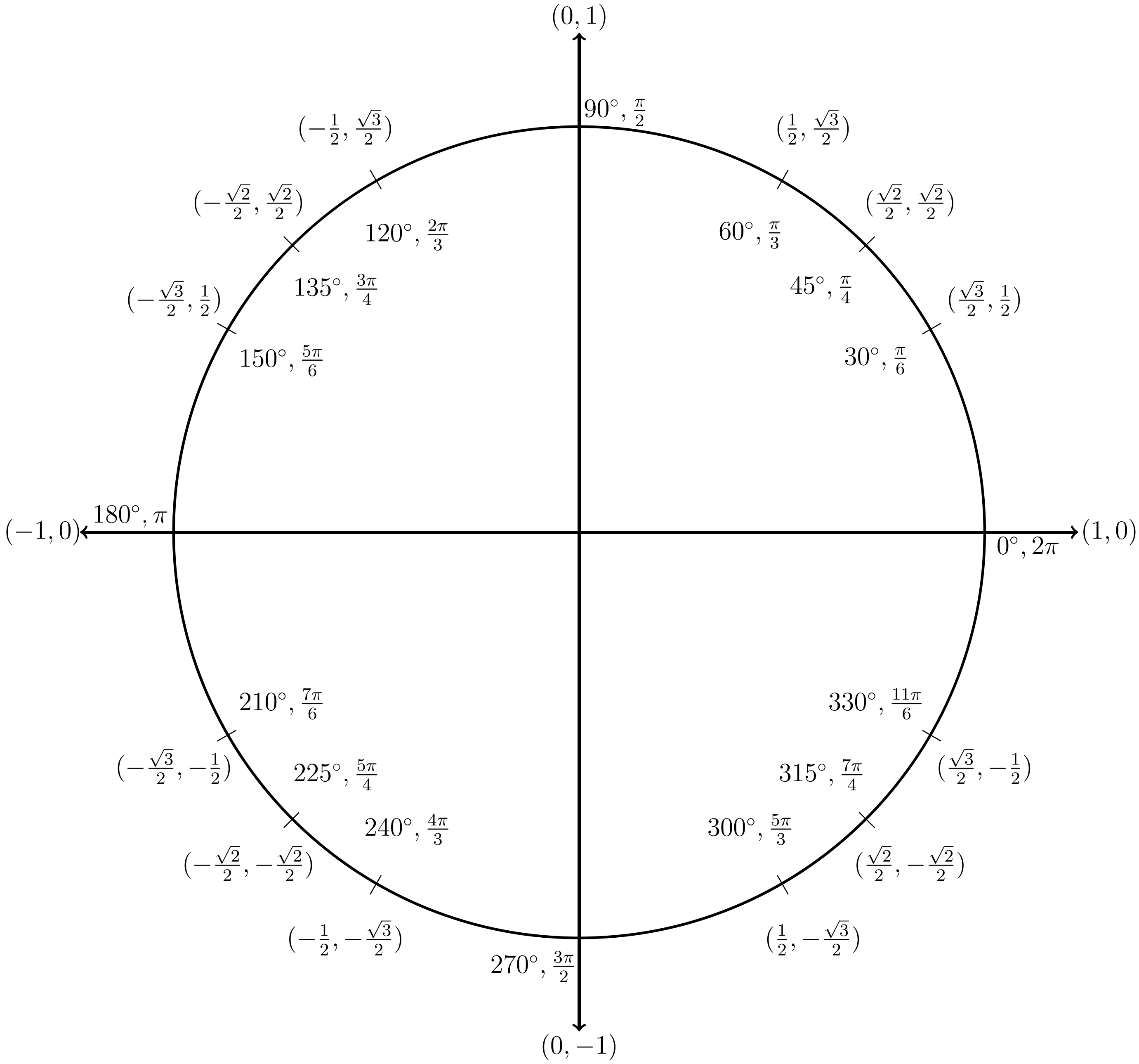
## Cofunction Formulas

$$\sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta \qquad \cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta$$

$$\csc\left(\frac{\pi}{2} - \theta\right) = \sec\theta \qquad \sec\left(\frac{\pi}{2} - \theta\right) = \csc\theta$$

$$\tan\left(\frac{\pi}{2} - \theta\right) = \cot\theta \qquad \cot\left(\frac{\pi}{2} - \theta\right) = \tan\theta$$

# Unit Circle




*For any ordered pair on the unit circle* $(x, y)$ : $\cos\theta = x$ *and* $\sin\theta = y$

## Example

$$\cos\left(\frac{7\pi}{6}\right) = -\frac{\sqrt{3}}{2} \qquad \sin\left(\frac{7\pi}{6}\right) = -\frac{1}{2}$$

# Inverse Trig Functions

### Definition

$\theta = \sin^{-1}(x)$ **is equivalent to** $x = \sin\theta$

$\theta = \cos^{-1}(x)$ **is equivalent to** $x = \cos\theta$

$\theta = \tan^{-1}(x)$ **is equivalent to** $x = \tan\theta$

### Domain and Range

| Function | Domain | Range |
|---|---|---|
| $\theta = \sin^{-1}(x)$ | $-1 \le x \le 1$ | $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$ |
| $\theta = \cos^{-1}(x)$ | $-1 \le x \le 1$ | $0 \le \theta \le \pi$ |
| $\theta = \tan^{-1}(x)$ | $-\infty \le x \le \infty$ | $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ |

### Inverse Properties

These properties hold for x in the domain and $\theta$ in the range

$\sin(\sin^{-1}(x)) = x$ $\quad$ $\sin^{-1}(\sin(\theta)) = \theta$

$\cos(\cos^{-1}(x)) = x$ $\quad$ $\cos^{-1}(\cos(\theta)) = \theta$

$\tan(\tan^{-1}(x)) = x$ $\quad$ $\tan^{-1}(\tan(\theta)) = \theta$

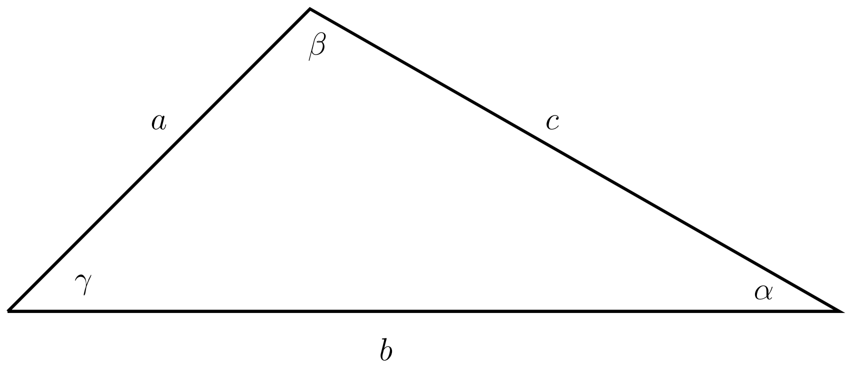
### Other Notations

$\sin^{-1}(x) = \arcsin(x)$

$\cos^{-1}(x) = \arccos(x)$

$\tan^{-1}(x) = \arctan(x)$

# Law of Sines, Cosines, and Tangents




### Law of Sines

$$\frac{\sin\alpha}{a} = \frac{\sin\beta}{b} = \frac{\sin\gamma}{c}$$

### Law of Cosines

$$a^2 = b^2 + c^2 - 2bc\cos\alpha$$

$$b^2 = a^2 + c^2 - 2ac\cos\beta$$

$$c^2 = a^2 + b^2 - 2ab\cos\gamma$$

### Law of Tangents

$$\frac{a-b}{a+b} = \frac{\tan\frac{1}{2}(\alpha-\beta)}{\tan\frac{1}{2}(\alpha+\beta)}$$

$$\frac{b-c}{b+c} = \frac{\tan\frac{1}{2}(\beta-\gamma)}{\tan\frac{1}{2}(\beta+\gamma)}$$

$$\frac{a-c}{a+c} = \frac{\tan\frac{1}{2}(\alpha-\gamma)}{\tan\frac{1}{2}(\alpha+\gamma)}$$

## Complex Numbers

$$i = \sqrt{-1} \qquad i^2 = -1 \qquad i^3 = -i \qquad i^4 = 1$$

$\sqrt{-a} = i\sqrt{a}, \quad a \geq 0$

$(a+bi)+(c+di) = a+c+(b+d)i$

$(a+bi)-(c+di) = a-c+(b-d)i$

$(a+bi)(c+di) = ac-bd+(ad+bc)i$

$(a+bi)(a-bi) = a^2+b^2$

$|a+bi| = \sqrt{a^2+b^2}$ **Complex Modulus**

$\overline{(a+bi)} = a-bi$ **Complex Conjugate**

$\overline{(a+bi)}(a+bi) = |a+bi|^2$

## DeMoivre's Theorem

Let $z = r(\cos\theta + i\sin\theta)$, and let $n$ be a positive integer.
Then:

$$z^n = r^n(\cos n\theta + i\sin n\theta).$$

**Example:** Let $z = 1-i$, find $z^6$.

Solution: First write $z$ in polar form.

$r = \sqrt{(1)^2+(-1)^2} = \sqrt{2}$

$\theta = arg(z) = \tan^{-1}\left(\frac{-1}{1}\right) = -\frac{\pi}{4}$

Polar Form: $z = \sqrt{2}\left(\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right)$

Applying DeMoivre's Theorem gives :

$$\begin{aligned} z^6 &= \left(\sqrt{2}\right)^6\left(\cos\left(6\cdot-\frac{\pi}{4}\right) + i\sin\left(6\cdot-\frac{\pi}{4}\right)\right) \\ &= 2^3\left(\cos\left(-\frac{3\pi}{2}\right) + i\sin\left(-\frac{3\pi}{2}\right)\right) \\ &= 8(0+i(1)) \\ &= 8i \end{aligned}$$

## Finding the $nth$ roots of a number using DeMoivre's Theorem

**Example:** Find all the complex fourth roots of 4. That is, find all the complex solutions of $x^4 = 4$.

We are asked to find all complex fourth roots of 4.
These are all the solutions (including the complex values) of the equation $x^4 = 4$.

For any positive integer $n$ , a nonzero complex number $z$ has exactly $n$ distinct $n$th roots. More specifically, if $z$ is written in the trigonometric form $r(\cos\theta + i\sin\theta)$, the $n$th roots of $z$ are given by the following formula.

$$(*)\ r^{\frac{1}{n}}\left(\cos\left(\frac{\theta}{n}+\frac{360^\circ k}{n}\right)+i\sin\left(\frac{\theta}{n}+\frac{360^\circ k}{n}\right)\right), \quad for \quad k = 0, 1, 2, ..., n-1.$$

Remember from the previous example we need to write 4 in trigonometric form by using:

$$r = \sqrt{(a)^2 + (b)^2} \qquad \text{and} \qquad \theta = arg(z) = \tan^{-1}\left(\frac{b}{a}\right).$$

So we have the complex number $a + ib = 4 + i0$.

Therefore $a = 4$ and $b = 0$

So $r = \sqrt{(4)^2 + (0)^2} = 4$ and
$\theta = arg(z) = \tan^{-1}\left(\frac{0}{4}\right) = 0$
Finally our trigonometric form is $4 = 4(\cos 0^\circ + i\sin 0^\circ)$

Using the formula $(*)$ above with $n = 4$, we can find the fourth roots of $4(\cos 0^\circ + i\sin 0^\circ)$

- For $k = 0$, $4^{\frac{1}{4}}\left(\cos\left(\frac{0^\circ}{4}+\frac{360^\circ * 0}{4}\right)+i\sin\left(\frac{0^\circ}{4}+\frac{360^\circ * 0}{4}\right)\right) = \sqrt{2}\left(\cos(0^\circ)+i\sin(0^\circ)\right) = \sqrt{2}$
- For $k = 1$, $4^{\frac{1}{4}}\left(\cos\left(\frac{0^\circ}{4}+\frac{360^\circ * 1}{4}\right)+i\sin\left(\frac{0^\circ}{4}+\frac{360^\circ * 1}{4}\right)\right) = \sqrt{2}\left(\cos(90^\circ)+i\sin(90^\circ)\right) = \sqrt{2}i$
- For $k = 2$, $4^{\frac{1}{4}}\left(\cos\left(\frac{0^\circ}{4}+\frac{360^\circ * 2}{4}\right)+i\sin\left(\frac{0^\circ}{4}+\frac{360^\circ * 2}{4}\right)\right) = \sqrt{2}\left(\cos(180^\circ)+i\sin(180^\circ)\right) = -\sqrt{2}$
- For $k = 3$, $4^{\frac{1}{4}}\left(\cos\left(\frac{0^\circ}{4}+\frac{360^\circ * 3}{4}\right)+i\sin\left(\frac{0^\circ}{4}+\frac{360^\circ * 3}{4}\right)\right) = \sqrt{2}\left(\cos(270^\circ)+i\sin(270^\circ)\right) = -\sqrt{2}i$

Thus all of the complex roots of $x^4 = 4$ are:

$\mathbf{\sqrt{2}, \sqrt{2}i, -\sqrt{2}, -\sqrt{2}i}$ .