Development Of A Wheeled Inverted Pendulum

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Contents

1 Dynamic Model

1.1 Derivation

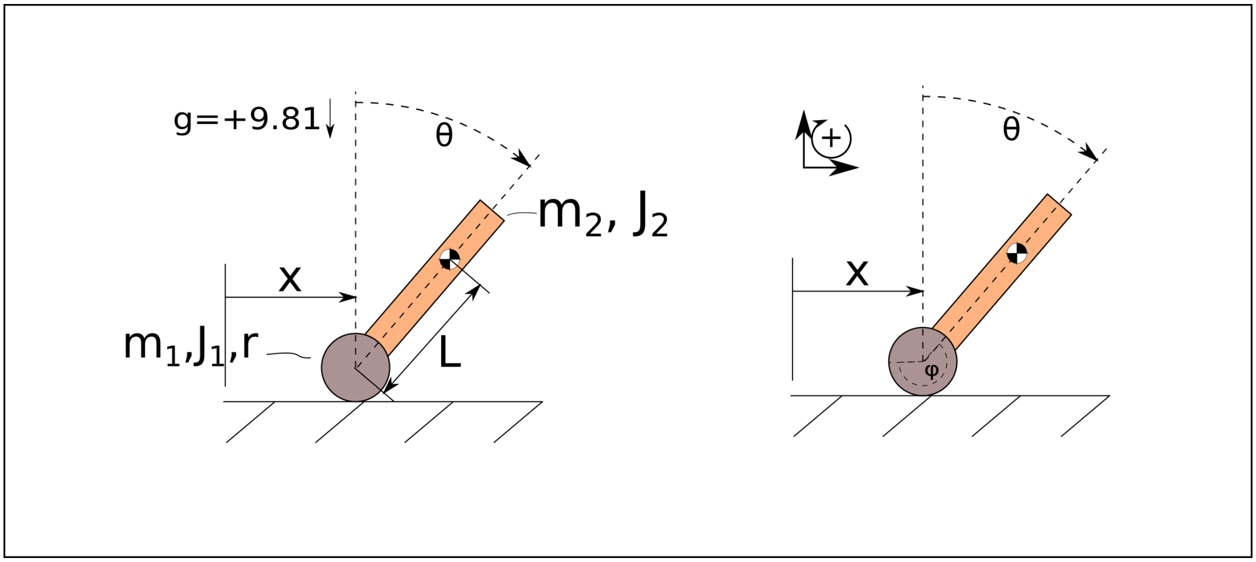

Figure 1: Overview and Definitions

Since there are two bodies in this system the Euler-Lagrange equation will be used to derive the equations of motion for the WIP:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = F_i \tag{1}$$

where $q = \{x, \theta\}$ are the generalized coordinates of the system and $F = \{\frac{\tau}{r}, -\tau\}$ are the generalized forces acting on each axis. The Lagrangian is also defined as the difference between the kinetic and potential energies of the system:

$$L = K - U \tag{2}$$

The Kinetic energy can be expressed as the sum of the translational and rotational kinetic energies of each body:

$$K = \frac{1}{2}m_1|\dot{\vec{x}}_1|^2 + \frac{1}{2}m_2|\dot{\vec{x}}_2|^2 + \frac{1}{2}J_1\dot{\phi}^2 + \frac{1}{2}J_2\dot{\theta}^2$$

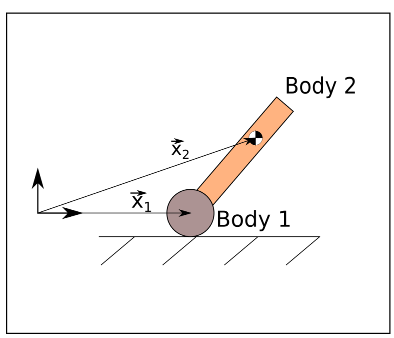

Figure 2: Definitions of $\dot{\vec{x}}_1$ and $\dot{\vec{x}}_2$

To continue $|\dot{\vec{x}}_1|^2$ and $|\dot{\vec{x}}_2|^2$ must be evaluated:

$$\vec{x}_1 = x\hat{i}$$

$$\dot{\vec{x}}_1 = \dot{x}\hat{i}$$
$$|\dot{\vec{x}}_1|^2 = \dot{x}^2$$

$$\vec{x}_2 = (x + Lsin(\theta)\hat{i} + Lcos(\theta)\hat{j}$$
$$\dot{\vec{x}}_2 = (\dot{x} + Lcos(\theta)\dot{\theta})\hat{i} - Lsin(\theta)\dot{\theta}\hat{j}$$
$$|\dot{\vec{x}}_2| = \sqrt{(\dot{x} + Lcos(\theta)\dot{\theta})^2 + (-Lsin(\theta)\dot{\theta})^2}$$
$$|\dot{\vec{x}}_2| = \sqrt{\dot{x}^2 + 2Lcos(\theta)\dot{x}\dot{\theta} + L^2cos^2(\theta)\dot{\theta}^2 + L^2sin^2(\theta)\dot{\theta}^2}$$
$$|\dot{\vec{x}}_2|^2 = \dot{x}^2 + 2Lcos(\theta)\dot{x}\dot{\theta} + L^2\dot{\theta}^2$$

Substituting back into the appropriate terms in the Kinetic energy along with the definition of ϕ:

$$K = \frac{1}{2}m_1\dot{x}^2 + \frac{1}{2}m_2(\dot{x}^2 + 2Lcos(\theta)\dot{x}\dot{\theta} + L^2\dot{\theta}^2) + \frac{1}{2}J_1\frac{\dot{x}^2}{r^2} + \frac{1}{2}J_2\dot{\theta}^2$$
$$K = \frac{1}{2}(m_1 + m_2 + \frac{J_1}{r^2})\dot{x}^2 + m_2Lcos(\theta)\dot{x}\dot{\theta} + \frac{1}{2}(m_2L^2 + J_2)\dot{\theta}^2$$

This equation is split into three terms, linear acceleration, angular acceleration, and an interaction term between linear and angular velocity. Therefore it seems reasonable to define an effective mass and moment of inertia as $M = m_1 + m_2 + \frac{J_1}{r^2}$ and $I = m_2L^2 + J_2$.

Substituting these into the Kinetic energy gives:

$$K = \frac{1}{2}M\dot{x}^2 + m_2Lcos(\theta)\dot{x}\dot{\theta} + \frac{1}{2}I\dot{\theta}^2 \tag{3}$$

Potential energy can be found as:

$$U = m_2gLcos(\theta) \tag{4}$$

Substituting 3 and 4 into 2 gives the full Lagrangian for the system:

$$L = \frac{1}{2}M\dot{x}^2 + m_2Lcos(\theta)\dot{x}\dot{\theta} + \frac{1}{2}I\dot{\theta}^2 - m_2gLcos(\theta) \tag{5}$$

The Lagrangian must now be substituted into 1 for each generalized coordinate. Starting with x:

$$\frac{\partial L}{\partial x} = 0$$

$$\frac{\partial L}{\partial \dot{x}} = M\dot{x} + m_2Lcos(\theta)\dot{\theta}$$
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = M\ddot{x} + m_2Lcos(\theta)\ddot{\theta} - m_2Lsin(\theta)\dot{\theta}^2$$
$$M\ddot{x} + m_2Lcos(\theta)\ddot{\theta} - m_2Lsin(\theta)\dot{\theta}^2 = \frac{\tau}{r} \tag{6}$$

Equation 6 is the first equation of motion for the system. Repeating this for θ will give the second:

$$\frac{\partial L}{\partial \theta} = -m_2Lsin(\theta)\dot{x}\dot{\theta} + m_2gLsin(\theta)$$

$$\frac{\partial L}{\partial \dot{\theta}} = m_2Lcos(\theta)\dot{x} + I\dot{\theta}$$
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = m_2Lcos(\theta)\ddot{x} - m_2Lsin(\theta)\dot{x}\dot{\theta} + I\ddot{\theta}$$

$$m_2Lcos(\theta)\ddot{x} - m_2Lsin(\theta)\dot{x}\dot{\theta} + I\ddot{\theta} + m_2Lsin(\theta)\dot{x}\dot{\theta} - m_2gLsin(\theta) = -\tau \tag{7}$$

Equations 6 and 7 both depend on $\ddot{x}$ and $\ddot{\theta}$ and must be solved in terms of these simultaneously. Doing this yields:

$$\ddot{x} = -\frac{m_2LIsin(\theta)}{m_2^2L^2cos(\theta) - MI}\dot{\theta}^2 + \frac{m_2^2L^2g}{m_2^2L^2cos(\theta) - MI}cos(\theta)sin(\theta) - \left[\frac{I + rm_2Lcos(\theta)}{r(m_2^2L^2cos(\theta) - MI)}\right]\tau \tag{8}$$

$$\ddot{\theta} = \frac{m_2^2L^2sin(\theta)}{m_2^2L^2cos(\theta) - MI}\dot{\theta}^2 - \frac{Mm_2gLsin(\theta)}{m_2^2L^2cos(\theta) - MI} + \left[\frac{rM + m_2L}{r(m_2^2L^2cos(\theta) - MI)}\right]\tau \tag{9}$$

Putting these into state space representation:

$$\begin{bmatrix} \dot{x} \\ \ddot{x} \\ \dot{\theta} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} \dot{x} \\ -\frac{m_2LIsin(\theta)}{m_2^2L^2cos(\theta)-MI}\dot{\theta}^2 + \frac{m_2^2L^2g}{m_2^2L^2cos(\theta)-MI}cos(\theta)sin(\theta) \\ \dot{\theta} \\ \frac{m_2^2L^2sin(\theta)}{m_2^2L^2cos(\theta)-MI}\dot{\theta}^2 - \frac{Mm_2gLsin(\theta)}{m_2^2L^2cos(\theta)-MI} \end{bmatrix} + \begin{bmatrix} 0 \\ -\frac{I+rm_2Lcos(\theta)}{r(m_2^2L^2cos(\theta)-MI)} \\ 0 \\ \frac{rM+m_2L}{r(m_2^2L^2cos(\theta)-MI)} \end{bmatrix}\tau \tag{10}$$

1.2 Linearization

1.3 Transfer Function

2 First Section

This is text

2.1 Begining Things

Some beginning things