

Development Of A Wheeled Inverted Pendulum

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April 8th, 2020

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1 Dynamic Model

1.1 Derivation

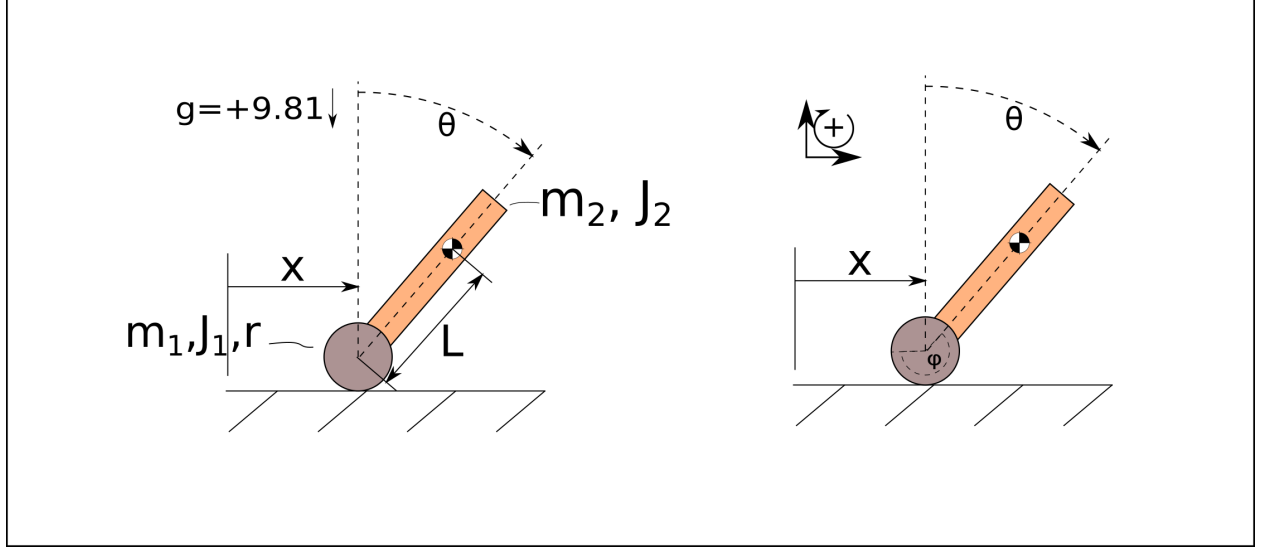


Figure 1: Overview and Definitions

As shown in 1 the configuration of the system is given by x and θ . The variable ϕ is the rotation of the wheel relative to the vertical axis and is completely determined by x :

$$x = r\phi \text{ and } \dot{x} = r\dot{\phi}$$

Positive translations are to the right and position rotations are clockwise. Also a positive torque is one which tends to make the WIP accelerate in the positive direction (and cause a negative rotation).

Since there are two bodies in this system the Euler-Lagrange equation will be used to derive the equations of motion for the WIP:

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \right) - \frac{\partial L}{\partial q_i} = F_i \quad (1)$$

where $q = \{x, \theta\}$ are the generalized coordinates defining the configuration of the system and $F = \{\frac{\tau}{r}, -\tau\}$ are the generalized forces acting on each axis. The Lagrangian is also defined as the difference between the kinetic and potential energies of the system:

$$L = K - U \quad (2)$$

The Kinetic energy can be expressed as the sum of the translational and rotational kinetic energies of each body:

$$K = \frac{1}{2}m_1|\dot{\vec{x}}_1|^2 + \frac{1}{2}m_2|\dot{\vec{x}}_2|^2 + \frac{1}{2}J_1\dot{\phi}^2 + \frac{1}{2}J_2\dot{\theta}^2$$

To continue $|\dot{\vec{x}}_1|^2$ and $|\dot{\vec{x}}_2|^2$ must be evaluated. Given the definitions of these vectors in 2:

$$\vec{x}_1 = x\hat{i}$$

$$\dot{\vec{x}}_1 = \dot{x}\hat{i}$$

$$|\dot{\vec{x}}_1|^2 = \dot{x}^2$$

$$\vec{x}_2 = (x + L\sin(\theta))\hat{i} + L\cos(\theta)\hat{j}$$

$$\dot{\vec{x}}_2 = (\dot{x} + L\cos(\theta)\dot{\theta})\hat{i} - L\sin(\theta)\dot{\theta}\hat{j}$$

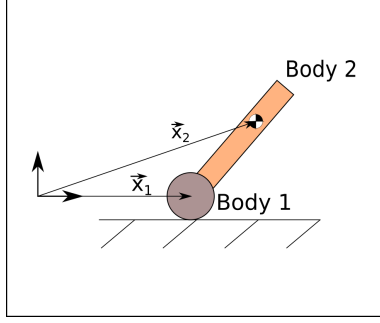


Figure 2: Definitions of \dot{x}_1 and \dot{x}_2

$$|\dot{x}_2| = \sqrt{(\dot{x} + L\cos(\theta)\dot{\theta})^2 + (-L\sin(\theta)\dot{\theta})^2}$$

$$|\dot{x}_2| = \sqrt{\dot{x}^2 + 2L\cos(\theta)\dot{x}\dot{\theta} + L^2\cos^2(\theta)\dot{\theta}^2 + L^2\sin^2(\theta)\dot{\theta}^2}$$

$$|\dot{x}_2|^2 = \dot{x}^2 + 2L\cos(\theta)\dot{x}\dot{\theta} + L^2\dot{\theta}^2$$

Substituting back into the appropriate terms in the Kinetic energy along with the definition of ϕ :

$$K = \frac{1}{2}m_1\dot{x}^2 + \frac{1}{2}m_2(\dot{x}^2 + 2L\cos(\theta)\dot{x}\dot{\theta} + L^2\dot{\theta}^2) + \frac{1}{2}J_1\frac{\dot{x}^2}{r^2} + \frac{1}{2}J_2\dot{\theta}^2$$

$$K = \frac{1}{2}(m_1 + m_2 + \frac{J_1}{r^2})\dot{x}^2 + m_2L\cos(\theta)\dot{x}\dot{\theta} + \frac{1}{2}(m_2L^2 + J_2)\dot{\theta}^2$$

This equation is split into three terms, linear acceleration, angular acceleration, and an interaction term between linear and angular velocity. Therefore it seems reasonable to define an effective mass and moment of inertia as $M = m_1 + m_2 + \frac{J_1}{r^2}$ and $I = m_2L^2 + J_2$.

Substituting these into the Kinetic energy gives:

$$K = \frac{1}{2}M\dot{x}^2 + m_2L\cos(\theta)\dot{x}\dot{\theta} + \frac{1}{2}I\dot{\theta}^2 \quad (3)$$

Potential energy can be found as:

$$U = m_2gL\cos(\theta) \quad (4)$$

Substituting 3 and 4 into 2 gives the full Lagrangian for the system:

$$L = \frac{1}{2}M\dot{x}^2 + m_2L\cos(\theta)\dot{x}\dot{\theta} + \frac{1}{2}I\dot{\theta}^2 - m_2gL\cos(\theta) \quad (5)$$

The Lagrangian must now be substituted into 1 for each generalized coordinate. Starting with x :

$$\frac{\partial L}{\partial x} = 0$$

$$\frac{\partial L}{\partial \dot{x}} = M\dot{x} + m_2L\cos(\theta)\dot{\theta}$$

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}} \right) = M\ddot{x} + m_2L\cos(\theta)\ddot{\theta} - m_2L\sin(\theta)\dot{\theta}^2$$

$$M\ddot{x} + m_2L\cos(\theta)\ddot{\theta} - m_2L\sin(\theta)\dot{\theta}^2 = \frac{\tau}{r} \quad (6)$$

Equation 6 is the first equation of motion for the system. Repeating this for θ will give the second:

$$\frac{\partial L}{\partial \theta} = -m_2L\sin(\theta)\dot{x}\dot{\theta} + m_2gL\sin(\theta)$$

$$\frac{\partial L}{\partial \dot{\theta}} = m_2 L \cos(\theta) \dot{x} + I \dot{\theta}$$

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\theta}} \right) = m_2 L \cos(\theta) \ddot{x} - m_2 L \sin(\theta) \dot{x} \dot{\theta} + I \ddot{\theta}$$

$$m_2 L \cos(\theta) \ddot{x} - m_2 L \sin(\theta) \dot{x} \dot{\theta} + I \ddot{\theta} + m_2 L \sin(\theta) \dot{x} \dot{\theta} - m_2 g L \sin(\theta) = -\tau \quad (7)$$

Equations 6 and 7 both depend on \ddot{x} and $\ddot{\theta}$ and must be solved in terms of these simultaneously. Doing this yields:

$$\ddot{x} = -\frac{m_2 L I \sin(\theta)}{m_2^2 L^2 \cos(\theta) - M I} \dot{\theta}^2 + \frac{m_2^2 L^2 g}{m_2^2 L^2 \cos(\theta) - M I} \cos(\theta) \sin(\theta) - \left[\frac{I + r m_2 L \cos(\theta)}{r(m_2^2 L^2 \cos(\theta) - M I)} \right] \tau \quad (8)$$

$$\ddot{\theta} = \frac{m_2^2 L^2 \sin(\theta)}{m_2^2 L^2 \cos(\theta) - M I} \dot{\theta}^2 - \frac{M m_2 g L \sin(\theta)}{m_2^2 L^2 \cos(\theta) - M I} + \left[\frac{r M + m_2 L}{r(m_2^2 L^2 \cos(\theta) - M I)} \right] \tau \quad (9)$$

These two non-linear equations describe the motion of the system in the configuration space. To make simulation easier these are converted into the state space by adding the velocities associated with each coordinate:

$$\begin{bmatrix} \dot{x} \\ \ddot{x} \\ \dot{\theta} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} \dot{x} \\ -\frac{m_2 L I \sin(\theta)}{m_2^2 L^2 \cos(\theta) - M I} \dot{\theta}^2 + \frac{m_2^2 L^2 g}{m_2^2 L^2 \cos(\theta) - M I} \cos(\theta) \sin(\theta) \\ \dot{\theta} \\ \frac{m_2^2 L^2 \sin(\theta)}{m_2^2 L^2 \cos(\theta) - M I} \dot{\theta}^2 - \frac{M m_2 g L \sin(\theta)}{m_2^2 L^2 \cos(\theta) - M I} \end{bmatrix} + \begin{bmatrix} 0 \\ -\frac{I + r m_2 L \cos(\theta)}{r(m_2^2 L^2 \cos(\theta) - M I)} \\ 0 \\ \frac{r M + m_2 L}{r(m_2^2 L^2 \cos(\theta) - M I)} \end{bmatrix} \tau \quad (10)$$

1.2 Linearization

Equation 10 can be easily simulated with tools such as Scipy's ODEint function to create a reasonable representation of the system. It is not, however, easy to develop control laws for a non-linear system and so it is necessary to linearize it. This will be done about the upwards stationary point $\vec{x}^* = [x, 0, 0, 0]^T$. There is an x in this state since there is a symmetry in x. Currently the system is of the form:

$$\dot{\vec{x}} = \vec{f}(\vec{x}, u), \text{ where in this case } u = \tau$$

Linearizing would put the system into the form:

$$\dot{\vec{x}} = A \vec{x} + B u$$

To generate A and B the following formula must be applied: $A = \left[\frac{\partial \vec{f}(\vec{x}, u)}{\partial \vec{x}} \right]_{x^*, u^*}$ and $B = \left[\frac{\partial \vec{f}(\vec{x}, u)}{\partial u} \right]_{x^*, u^*}$

Doing so yields:

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{m_2^2 L^2 g}{m_2^2 L^2 - M I} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -\frac{M m_2 g L}{m_2^2 L^2 - M I} & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 \\ -\frac{I + r m_2 L}{r(m_2^2 L^2 - M I)} \\ 0 \\ \frac{r M + m_2 L}{r(m_2^2 L^2 - M I)} \end{bmatrix} \quad (11)$$

The output equation will initially return only θ and so will be:

$$y = \vec{C} \vec{x} + D u, \text{ where } \vec{C} = [0, 0, 1, 0] \text{ and } D = 0$$

1.3 Transfer Function

Due to some quirks in the Python-Controls library with regards to analyzing state space models, the state space representation derived will be converted into a transfer function to make analysis easier to interpret. Doing so uses:

$$\frac{\theta(s)}{\tau(s)} = \vec{C} (sI - A)^{-1} \vec{B} + D \quad (12)$$

Applying this to the current system gives:

$$\frac{\theta(s)}{\tau(s)} = \frac{a}{s^2 - b} \quad (13)$$

$$\text{where } a = \frac{rM+m_2L}{r(m_2^2L^2-MI)} \text{ and } b = -\frac{Mm_2gL}{m_2^2L^2-MI}$$