Phys 657, Asst 2, Question 2
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50/60

$$F(n) = \frac{1}{L^2}\int d\vec{r}\left\{ n\left[\frac{B^\ell}{2} + \frac{B^x}{2}(2\nabla^2 + \nabla^4)\right]n - \frac{tn^3}{3} + \frac{vn^4}{4}\right\}$$

$$= \frac{1}{L^2}\left\{\int d\vec{r}\left[\frac{B^\ell}{2}n^2 - \frac{t}{3}n^3 + \frac{v}{4}n^4\right] + \underbrace{\int d\vec{r}\,\frac{B^x}{2}\,n(2\nabla^2+\nabla^4)n}_{\text{Gradient terms}}\right\}$$

Dealing with the gradient term first:

$$\frac{B^x}{2}\int d\vec{r}\; n(\vec{r})\;(2\nabla^2+\nabla^4)\,n(\vec{r})$$

$$\Rightarrow \frac{B^x}{2}\int d\vec{r}\, n(\vec{r})\int d\vec{r}'\,\delta(\vec{r}'-\vec{r})\,(2\nabla^2+\nabla^4)\,n(\vec{r}') \Rightarrow \frac{B^x}{2}\int d\vec{r}\, n(\vec{r})\int d\vec{r}'\, C_2(|\vec{r}-\vec{r}'|)\,n(\vec{r}')$$

where $(2\nabla^2+\nabla^4)\delta(\vec{r}-\vec{r}')$ is the 2-pt correlation function $C_2(|\vec{r}-\vec{r}'|)$ and

$$C_2(|\vec{r}-\vec{r}'|) = \int d\vec{k}\;\hat{C}_2(|\vec{k}|)\,e^{i\vec{k}\cdot(\vec{r}-\vec{r}')}$$

substituting this in (& for convenience dropping explicit dependence of $\hat{C}_2$ on $\vec{k}$)

$$\Rightarrow \frac{B^x}{2}\int d\vec{r}\, n(\vec{r})\int d\vec{r}'\int d\vec{k}\;\hat{C}_2\; n(\vec{r}')\,e^{i\vec{k}\cdot(\vec{r}-\vec{r}')}$$

Substituting in the form of $n(\vec{r}')$ that satisfies a triangular (hexagonal) lattice structure in 2D

$$n(\vec{r}) = \sum_{j=1}^{3}\eta_j(\vec{r})\,e^{i\vec{G}_j\cdot\vec{r}} + \sum_{j=1}^{3}\eta_j^\dagger(\vec{r})\,e^{-i\vec{G}_j\cdot\vec{r}}$$

where $\vec{G}_1 = -\frac{1}{2}(\sqrt{3}\,\hat{x} + \hat{y})$
$\vec{G}_2 = \hat{y}$
$\vec{G}_3 = \frac{\sqrt{3}}{2}\hat{x} - \frac{1}{2}\hat{y}$

$$\Rightarrow \frac{B^x}{2}\int d\vec{r}\, n(\vec{r})\int d\vec{r}'\int d\vec{k}\;\hat{C}_2\left(\sum_j^3\eta_j(\vec{r}')e^{i\vec{G}_j\cdot\vec{r}'} + \sum_j^3\eta_j^\dagger(\vec{r}')e^{-i\vec{G}_j\cdot\vec{r}'}\right)e^{i\vec{k}\cdot(\vec{r}-\vec{r}')}$$

Now assume the amplitudes η_j are slow varying compared to our volume-averaging scale. take a Taylor expansion of the η_j's (to 2nd order) around $\vec{r}' = \vec{r}$:

$$\eta_j(\vec{r}')\Big|_{\vec{r}} = \eta_j(\vec{r}) + (\vec{r}_1'-\vec{r}_1)\,\partial_{\vec{r}_1}\eta_j(\vec{r}) + \frac{1}{2}(\vec{r}_2'-\vec{r}_2)(\vec{r}_1'-\vec{r}_1)\,\partial_{\vec{r}_2}\partial_{\vec{r}_1}\eta_j(\vec{r})$$

$$\Rightarrow \frac{B^x}{2}\int d\vec{r}\, n(\vec{r})\int d\vec{r}'\int d\vec{k}\;\hat{C}_2\Bigg[\sum_j^3\left\{\eta_j(\vec{r}) + (\vec{r}_1'-\vec{r}_1)\partial_{\vec{r}_1}\eta_j(\vec{r}) + \frac{1}{2}(\vec{r}_2'-\vec{r}_2)(\vec{r}_1'-\vec{r}_1)\partial_{\vec{r}_2}\partial_{\vec{r}_1}\eta_j(\vec{r})\right\}e^{i\vec{G}_j\cdot\vec{r}'}$$

$$+\sum_j^3\left\{\eta_j^\dagger(\vec{r}) + (\vec{r}_1'-\vec{r}_1)\partial_{\vec{r}_1}\eta_j^\dagger(\vec{r}) + \frac{1}{2}(\vec{r}_2'-\vec{r}_2)(\vec{r}_1'-\vec{r}_1)\partial_{\vec{r}_2}\partial_{\vec{r}_1}\eta_j^\dagger(\vec{r})\right\}e^{-i\vec{G}_j\cdot\vec{r}'}\Bigg]e^{i\vec{k}\cdot(\vec{r}-\vec{r}')}$$

$$\Rightarrow \frac{\beta^2}{2}\int d\vec{r}\, n(\vec{r}) \sum_{j}^{3} \Big\{ \int d\vec{r}' \int d\vec{k}\, \hat{C}_2\, \eta_j(\vec{r})\, e^{i\vec{G}_j\cdot\vec{r}'} e^{i\vec{k}\cdot(\vec{r}-\vec{r}')} + \int d\vec{r}' \int d\vec{k}\, \hat{C}_2 (\vec{r}_1'-\vec{r}_1)\partial_{r_1}\eta_j(\vec{r}) e^{i\vec{G}_j\cdot\vec{r}'} e^{i\vec{k}\cdot(\vec{r}-\vec{r}')}$$

$$+\frac{1}{2}\int d\vec{r}' \int d\vec{k}\, \hat{C}_2 (\vec{r}_2'-\vec{r}_2)(\vec{r}_1'-\vec{r}_1)\partial_{r_2}\partial_{r_1}\eta_j(\vec{r}) e^{i\vec{G}_j\cdot\vec{r}} e^{i\vec{k}\cdot(\vec{r}-\vec{r}')} + \int d\vec{r}' \int d\vec{k}\, \hat{C}_2\, \eta_j^\dagger(\vec{r}) e^{-i\vec{G}_j\cdot\vec{r}'} e^{i\vec{k}\cdot(\vec{r}-\vec{r}')}$$

$$+\int d\vec{r} \int d\vec{k}\, \hat{C}_2 (\vec{r}_1'-\vec{r}_1)\partial_{r_1}\eta_j^\dagger(\vec{r}) e^{-i\vec{G}_j\cdot\vec{r}'} e^{i\vec{k}\cdot(\vec{r}-\vec{r}')} + \frac{1}{2}\int d\vec{r} \int d\vec{k}\, \hat{C}_2 (\vec{r}_2'-\vec{r}_2)(\vec{r}_1'-\vec{r}_1)\partial_{r_2}\partial_{r_1}\eta_j^\dagger(\vec{r}) e^{-i\vec{G}_j\cdot\vec{r}'} e^{i\vec{k}\cdot(\vec{r}-\vec{r}')} \Big\}$$

① zeroth order terms:

$$\sum_j^3 \int d\vec{r}' \int d\vec{k}\, \hat{C}_2\, \eta_j(\vec{r})\, e^{i(\vec{G}_j-\vec{k})\cdot\vec{r}'} e^{i\vec{k}\cdot\vec{r}} \Rightarrow \sum_j^3 \int d\vec{k}\, \hat{C}_2\, \eta_j(\vec{r})\, e^{i\vec{k}\cdot\vec{r}}\, \delta(\vec{G}_j-\vec{k})$$

$$\Rightarrow \sum_j^3 \hat{C}_2(|\vec{G}_j|)\eta_j(\vec{r}) e^{i\vec{G}_j\cdot\vec{r}}$$, and similarly c.c. gives $\sum_j^3 \hat{C}_2(|-\vec{G}_j|)\eta_j^\dagger(\vec{r}) e^{-i\vec{G}_j\cdot\vec{r}}$

$\hat{C}_2(|-\vec{G}_j|) = \hat{C}_2(|\vec{G}_j|)$

② first order terms:

$$\sum_j \int d\vec{r}' \int d\vec{k}\, \hat{C}_2 (\vec{r}_1'-\vec{r}_1)\partial_{\vec{r}_1}\eta_j(\vec{r}) e^{i\vec{G}_j\cdot\vec{r}'} e^{i\vec{k}(\vec{r}-\vec{r}')}$$

rewrite this term as:

$$\rightarrow -(\vec{r}-\vec{r}_1') e^{i\vec{k}\cdot(\vec{r}-\vec{r}')} = i\partial_{k_1} e^{i\vec{k}\cdot(\vec{r}-\vec{r}')}$$

$$\Rightarrow i\sum_j^3 \int d\vec{r}' \int d\vec{k}\, \hat{C}_2\, \partial_{\vec{k}} e^{i\vec{k}(\vec{r}-\vec{r}')} \partial_{\vec{r}_1}\eta_j(\vec{r}) e^{i\vec{G}_j\cdot\vec{r}'}$$

→ performing a partial integration by parts: i.e. $\int u\,dv = uv - \int v\,du$

$$\int d\vec{k}\, \hat{C}_2\, \partial_{\vec{k}_1} e^{i\vec{k}\cdot(\vec{r}-\vec{r}')} = \hat{C}_2 e^{i\vec{k}\cdot(\vec{r}-\vec{r}')}\Big| - \int d\vec{k}\,(\partial_{\vec{k}_1}\hat{C}_2)\, e^{i\vec{k}\cdot(\vec{r}-\vec{r}')}$$

choose bounds s.t. this term evaluates to zero

$$\Rightarrow -i\sum_j^3 \int d\vec{r}' \int d\vec{k}\, e^{i\vec{k}(\vec{r}-\vec{r}')} \partial_{\vec{k}_1}\hat{C}_2\, \partial_{\vec{r}_1}\eta_j(\vec{r})\, e^{i\vec{G}_j\cdot\vec{r}'} \Rightarrow -i\sum_j^3 \int d\vec{k}\, e^{i\vec{k}\cdot\vec{r}} \partial_{\vec{k}_1}\hat{C}_2\, \partial_{\vec{r}_1}\eta_j(\vec{r})\, \delta(\vec{G}_j-\vec{k})$$

$$\Rightarrow -i\sum_j^3 \partial_{\vec{k}_1}\hat{C}_2(|\vec{G}_j|)\, \partial_{\vec{r}_1}\eta_j(\vec{r})\, e^{i\vec{G}_j\cdot\vec{r}}$$, and cc gives $-i\sum_j^3 \partial_{\vec{k}_1}\hat{C}_2(|\vec{G}_j|)\,\partial_{\vec{r}_1}\eta_j^\dagger(\vec{r})\, e^{-i\vec{G}_j\cdot\vec{r}}$

③ second order terms:

$$\frac{1}{2}\sum_j^3 \int d\vec{r}' \int d\vec{k}\, \hat{C}_2 (\vec{r}_2'-\vec{r}_2)(\vec{r}_1'-\vec{r}_1)\partial_{\vec{r}_2}\partial_{\vec{r}_1}\eta_j(\vec{r}) e^{i\vec{G}_j\cdot\vec{r}'} e^{i\vec{k}\cdot(\vec{r}-\vec{r}')}$$

$$\rightarrow \text{take } (\vec{r}_2-\vec{r}_2')(\vec{r}_1-\vec{r}_1') e^{i\vec{k}\cdot(\vec{r}-\vec{r}')} = -i(\vec{r}_2'-\vec{r}_2)\partial_{\vec{r}_1} e^{i\vec{k}\cdot(\vec{r}-\vec{r}')} = i^2\,\partial_{k_2}\partial_{k_1} e^{i\vec{k}\cdot(\vec{r}-\vec{r}')}$$

$$\Rightarrow \frac{i^2}{2}\sum_j^3 \int d\vec{r}' \int d\vec{k}\, \hat{C}_2\, \partial_{\vec{k}_2}\partial_{\vec{k}_1} e^{i\vec{k}\cdot(\vec{r}-\vec{r}')} \partial_{\vec{r}_2}\partial_{\vec{r}_1}\eta_j(\vec{r})\, e^{i\vec{G}_j\cdot\vec{r}'}$$

and again integrating by parts: $\int d\vec{k}\, \hat{C}_2\, \partial_{\vec{k}_2}\partial_{\vec{k}_1} e^{i\vec{k}\cdot(\vec{r}-\vec{r}')} = \xcancel{\hat{C}_2 \partial_{\vec{k}_1} e^{i\vec{k}(\vec{r}-\vec{r}')}}^{\,\varnothing \text{ as previously}} - \int d\vec{k}\, \partial_{k_2}\hat{C}_2\, \partial_{k_1} e^{i\vec{k}\cdot(\vec{r}-\vec{r}')}$

$$= -\Big[\underbrace{(\partial_{\vec{k}}\hat{C}_2) e^{i\vec{k}(\vec{r}-\vec{r}')}}_{\varnothing} - \int d\vec{k}\,(\partial_{\vec{k}_2}\partial_{\vec{k}_1}\hat{C}_2)\, e^{i\vec{k}\cdot(\vec{r}-\vec{r}')}\Big] = \int d\vec{k}\,(\partial_{\vec{k}_2}\partial_{\vec{k}_1}\hat{C}_2) e^{i\vec{k}(\vec{r}-\vec{r}')}$$

$$\Rightarrow \frac{i^2}{2}\sum_{j}^{3}\int d\bar r' \int d\bar k \left(\partial_{\bar k_2}\partial_{\bar k_1}\hat C_2\right)\partial_{\bar r_2}\partial_{\bar r_1}\eta_j(\bar r)\, e^{i(\bar G_j-\bar k)\cdot\bar r'}\, e^{i\bar k\cdot\bar r}$$

$$\Rightarrow \frac{i^2}{2}\sum_{j}^{3}\partial_{\bar k_2}\partial_{\bar k_1}\hat C_2(|\bar G_j|)\,\partial_{\bar r_2}\partial_{\bar r_1}\eta_j(\bar r)\, e^{i\bar G_j\cdot\bar r}$$, and cc $\frac{i^2}{2}\sum_{j}^{3}\partial_{\bar k_2}\partial_{\bar k_1}\hat C_2(|\bar G_j|)\,\partial_{\bar r_2}\partial_{\bar r_1}\eta_j(\bar r)\, e^{-i\bar G_j\cdot\bar r}$

Putting these terms back into the full expression & expanding the other $n(\bar r)$ term gives:

$$\Rightarrow \frac{B^x}{2}\int d\bar r \sum_{j,l}^{3}\Big\{\eta_l(\bar r)\left[\hat C_2\eta_j(\bar r) - i\partial_{\bar k_1}\hat C_2\,\partial_{\bar r_1}\eta_j(\bar r) + \frac{i^2}{2}\partial_{\bar k_2}\partial_{\bar k_1}\hat C_2\,\partial_{\bar r_2}\partial_{\bar r_1}\eta_j(\bar r)\right]e^{i(\bar G_l+\bar G_j)\cdot\bar r}$$

$$+\,\eta_l(\bar r)\left[\hat C_2\eta_j^{\dagger} - i\partial_{\bar k_1}\hat C_2\,\partial_{\bar r_1}\eta_j^{\dagger} + \frac{i^2}{2}\partial_{\bar k_1}\partial_{\bar k_2}\hat C_2\,\partial_{\bar r_1}\partial_{\bar r_2}\eta_j^{\dagger}\right]e^{i(\bar G_l-\bar G_j)\cdot\bar r}$$

$$+\,\eta_l^{\dagger}(\bar r)\left[\hat C_2\eta_j - i\partial_{\bar k_1}\hat C_2\,\partial_{\bar r_1}\eta_j + \frac{i^2}{2}\partial_{\bar k_1}\partial_{\bar k_2}\hat C_2\,\partial_{\bar r_1}\partial_{\bar r_2}\eta_j\right]e^{i(\bar G_j-\bar G_l)\cdot\bar r}$$

$$+\,\eta_l^{\dagger}(\bar r)\left[\hat C_2\eta_j^{\dagger} - i\partial_{\bar k_1}\hat C_2\,\partial_{\bar r_1}\eta_j^{\dagger} + \frac{i^2}{2}\partial_{\bar k_1}\partial_{\bar k_2}\hat C_2\,\partial_{\bar r_1}\partial_{\bar r_2}\eta_j^{\dagger}\right]e^{-i(\bar G_l+\bar G_j)\cdot\bar r}\Big\}$$

Now the volume averaging function (area in 2D) $\chi_A(\bar r-\bar r')$, can be applied, where $\int d\bar r\,\chi_A(\bar r-\bar r')=1$. We can thus multiply each term by 1, e.g. for first term:

$$\sum_{j,l}^{3}\int d\bar r\,\eta_l(\bar r)\,\hat C_2\,\eta_j(\bar r)\,e^{i(\bar G_l+\bar G_j)\cdot\bar r} \Rightarrow \sum_{j,l}^{3}\int d\bar r \overbrace{\int d\bar r'\,\chi_A(\bar r-\bar r')}^{1}\hat C_2\,\eta_l(\bar r)\eta_j(\bar r)\,e^{i(\bar G_l+\bar G_j)\cdot\bar r}$$

and switching the order of integration to integrate over "$\bar r$" $\Rightarrow$ this is the limiting step (non-invertible):

$$\Rightarrow \sum_{j,l}^{3}\int d\bar r'\int d\bar r\,\hat C_2(|\bar G_j|)\,\eta_l(\bar r)\eta_j(\bar r)\,\chi_A(\bar r-\bar r')\,e^{i(\bar G_l+\bar G_j)\cdot\bar r}$$

Performing the limiting procedure ($\bar k\cdot\bar r \gg 1$) explicitly, expanding $\eta_l(\bar r)\eta_j(\bar r)$ around $\bar r'$ as $\eta_l(\bar r')\eta_j(\bar r') + \left(\eta_l(\bar r')\partial_{\bar r_1}\eta_j(\bar r') + \eta_j(\bar r')\partial_{\bar r_1}\eta_l(\bar r')\right)(\bar r-\bar r') + \ldots$ gives

$$\sum_{j,l}^{3}\hat C_2(|\bar G_j|)\Big[\int d\bar r'\,\eta_l(\bar r')\eta_j(\bar r')\underbrace{\cancelto{\emptyset}{\int d\bar r\,\chi_A(\bar r-\bar r')\,e^{i(\bar G_l+\bar G_j)\cdot\bar r}}}_{\text{The exponential is an oscillatory function on the scale of the lattice spacing; } \chi_A \text{ limits to length scales} \gg a\text{. Therefore coarsegraining this function} \Rightarrow \emptyset} +$$

$$+\int d\bar r'\left(\eta_l(\bar r')\partial_{\bar r}\eta_j(\bar r') + \eta_j(\bar r')\partial_{\bar r}\eta_l(\bar r')\right)\underbrace{\int d\bar r'\,\chi_A(\bar r-\bar r')\,(\bar r-\bar r')\,e^{i(\bar G_l+\bar G_j)\cdot\bar r}}_{\text{Rewrite as } -e^{i(\bar G_l+\bar G_j)\cdot\bar r'}\int du\,\chi_A(u)\,u\,e^{i(\bar G_l+\bar G_j)\cdot u} \text{ where } u=(\bar r-\bar r') \text{ and the same argument applies} \Rightarrow \emptyset} + \ldots\Big]$$

• Thus it is evident that the only terms that will survive are the resonance terms, i.e. terms where $\vec{k}_{total} = 0 \Rightarrow e^{i\vec{k}_{t}\cdot\vec{r}} = e^0 = 1$ in which case we recover $\int d\vec{r}' \chi_A(\vec{r}-\vec{r}') = 1$.

• There are 4 "sets" of terms:

1) $e^{i(\vec{G}_\ell+\vec{G}_j)\cdot\vec{r}} \rightarrow \vec{G}_\ell+\vec{G}_j \neq 0$ for any choice of $j,\ell \Rightarrow 0$
2.) $e^{i(\vec{G}_\ell-\vec{G}_j)\cdot\vec{r}} \rightarrow \vec{G}_\ell-\vec{G}_j = 0$ if $\ell = j$
3.) $e^{i(\vec{G}_j-\vec{G}_\ell)\cdot\vec{r}} \rightarrow \vec{G}_\ell-\vec{G}_j = 0$ if $\ell = j$
4.) $e^{-i(\vec{G}_j+\vec{G}_\ell)\cdot\vec{r}} \rightarrow$ as term 1.)

So the surviving terms after coarse-graining give (dropping explicit $\vec{r}$-dep. of η's)

$$\Rightarrow \frac{B^x}{2}\sum_j^3 \int d\vec{r}\left\{ \hat{C}_2\eta_j\eta_j^{\dagger} - i\cancel{\partial_{\vec{k}_1}\hat{C}_2\eta_j\partial_{\vec{r}_1}\eta_j^{\dagger}} + \frac{i^2}{2}\partial_{\vec{k}_1}\partial_{\vec{k}_2}\hat{C}_2\eta_j\partial_{\vec{r}_1}\partial_{\vec{r}_2}\eta_j^{\dagger} + \hat{C}_2\eta_j^{\dagger}\eta_j - i\cancel{\partial_{\vec{k}_1}\hat{C}_2\eta_j^{\dagger}\partial_{\vec{r}_1}\eta_j} + \frac{i^2}{2}\partial_{\vec{k}_1}\partial_{\vec{k}_2}\hat{C}_2\eta_j^{\dagger}\partial_{\vec{r}_1}\partial_{\vec{r}_2}\eta_j \right\}$$

0 since $\hat{C}_2(|\vec{G}_j|)$ is a peak (1^{st} deriv $= 0$)

$$\Rightarrow \frac{B^x}{2}\sum_j^3\int d\vec{r}\left\{ 2|\eta_j|^2\hat{C}_2(|\vec{G}_j|) + \frac{i^2}{2}\partial_{\vec{k}_1}\partial_{\vec{k}_2}\hat{C}_2\eta_j\partial_{\vec{r}_1}\partial_{\vec{r}_2}\eta_j^{\dagger} + \frac{i^2}{2}\partial_{\vec{k}_1}\partial_{\vec{k}_2}\hat{C}_2\eta_j^{\dagger}\partial_{\vec{r}_1}\partial_{\vec{r}_2}\eta_j\right\}$$

and evaluating the derivatives of the correlation function:

$$\partial_{\vec{k}_1}\partial_{\vec{k}_2}\hat{C}_2(|\vec{k}|)\Big|_{\vec{k}=\vec{G}_j} = \partial_{\vec{k}_1}\left(\hat{C}_2'(|\vec{k}|)\frac{\partial(|\vec{k}|)}{\partial\vec{k}_2}\right) = \partial_{\vec{k}_1}\left(\hat{C}_2'(|\vec{k}|)\frac{\cancel{2}\vec{k}_2}{\cancel{2}|\vec{k}|}\right)$$

$$= \left(\hat{C}_2''(|\vec{k}|)\frac{\cancel{2}\vec{k}_2}{\cancel{2}|\vec{k}|}\frac{\vec{k}_1}{|\vec{k}|} + \hat{C}_2'(|\vec{k}|)\left(\frac{\partial\vec{k}_1}{\partial\vec{k}_2}\right)\frac{1}{|\vec{k}|} - \hat{C}_2'(|\vec{k}|)\frac{\vec{k}_1\vec{k}_2}{|\vec{k}|^{3/2}}\right)\Bigg|_{\vec{k}=\vec{G}_j}$$

where all integrals are evaluated at $\vec{k}=\vec{G}_j$ and therefore can take the first derivatives $= 0$.

$$= \hat{C}_2''(|\vec{k}|)\frac{\vec{k}_2^2}{|\vec{k}|^2}.$$

$$\Rightarrow \frac{B^x}{2}\sum_j^3\int d\vec{r}\left\{2|\eta_j|^2\hat{C}_2(|\vec{G}_j|) + \frac{i^2}{2}\frac{\hat{C}_2''(|\vec{G}_j|)\vec{G}_j^2}{|\vec{G}_j|^2}\underbrace{\left(\eta_j\partial_{\vec{r}_1}\partial_{\vec{r}_2}\eta_j^{\dagger} + \eta_j^{\dagger}\partial_{\vec{r}_1}\partial_{\vec{r}_2}\eta_j\right)}_{\text{consider now these terms}}\right\}.$$

and write: $\int d\vec{r}\, \eta_j \partial_{\vec{r}_1}\partial_{\vec{r}_2} \eta_j^{\dagger} + \int d\vec{r}\, \eta_j^{\dagger} \partial_{\vec{r}_1}\partial_{\vec{r}_2} \eta_j$

where we can integrate each term by parts to get:

$$= \cancelto{0}{\eta_j \partial_{\vec{r}_2}\eta_j^{\dagger}} - \int d\vec{r}\, \partial_{\vec{r}_1}\eta_j \partial_{\vec{r}_2}\eta_j^{\dagger} + \cancelto{0}{\eta_j^{\dagger}\partial_{\vec{r}_2}\eta_j} - \int d\vec{r}\, \partial_{\vec{r}_1}\eta_j^{\dagger}\partial_{\vec{r}_2}\eta_j$$

(by choice of appropriate boundary conditions as before).

$$= -2\int d\vec{r}\, (\partial_{\vec{r}}\eta_j^{\dagger}\partial_{\vec{r}}\eta_j) = -2\int d\vec{r}\, |\nabla\eta_j|^2$$

$$\Rightarrow \frac{B^x}{2}\sum_j^3 \int d\vec{r} \left\{ 2|\eta_j|^2 \hat{C}_2(|\vec{G}_j|) - \frac{i^2 \hat{C}_2''(|\vec{G}_j|)}{|\vec{G}_j|^2} \vec{G}_j^2 |\nabla\eta_j|^2 \right\}$$

$$\Rightarrow B^x \sum_j^3 \int d\vec{r}\, |\eta_j|^2 \hat{C}_2(|\vec{G}_j|) - \frac{B^x}{2}\sum_j^3 \int d\vec{r}\, \frac{\hat{C}_2''(|\vec{G}_j|)}{|\vec{G}_j|^2} |(i\vec{G}_j\cdot\nabla)\eta_j|^2$$

where $|\vec{G}_j| = 1. \Rightarrow |\vec{G}_j|^2 = 1$

and $\hat{C}_2''(|\vec{k}|) = \frac{d^2}{dk^2}(-2|k|^2 + |k|^4) = \frac{d}{dk}(-4|k| + 4|k|^3) = -4 + 12|k|^2$

so $\hat{C}_2''(|\vec{G}_j|) = -4 + 12|\vec{G}_j|^2 = -4 + 12(1) = 12 - 4 = 8$. ✓

and the coefficients give: $\frac{B^x \hat{C}_2''}{2|\vec{G}_j|^2} = \frac{-B^x \cdot 8}{2\cdot 1} = -4B^x$

* Note here that $\hat{C}_2$ is an approximate form of the correlation! It is obvious from the proper form of correlation peaks that the 2nd derivative of $\hat{C}_2$ should be negative, as the curvature is negative. Thus a (-1) can be inserted above.

And the gradient term can thus be written as:

$$B^x \sum_j^3 \int d\vec{r}\, |(2i\vec{G}_j\cdot\nabla)\eta_j|^2$$

And for the first term, $\hat{C}_2(|\vec{k}|) = -2|k|^2 + |k|^4$

$\hat{C}_2(|\vec{G}_j|) = -2(1)^2 + (1)^4 = -2 + 1 = -1$

$$\Rightarrow \boxed{-B^x \sum_j^3 \int d\vec{r}\, |\eta_j|^2 + B^x \sum_j^3 \int d\vec{r}\, |(2i\vec{G}_j\cdot\nabla_{\vec{r}})\eta_j|^2}$$ from gradient term ✓

Next consider the n^2 term:

$$\frac{B^\ell}{2}\int d\bar{r}\, n(\bar{r})^2 \quad \Rightarrow \text{substitute in form for } n(\bar{r}) = \sum_j^3 \eta_j(\bar{r}) e^{i\bar{G}_j\cdot\bar{r}} + cc$$

$$\Rightarrow \frac{B^\ell}{2}\int d\bar{r} \sum_{j,\ell}^3 \eta_j(\bar{r})\eta_\ell(\bar{r}) e^{i(\bar{G}_j+\bar{G}_\ell)\cdot\bar{r}} + \frac{B^\ell}{2}\int d\bar{r}\sum_{j,\ell}^3 \eta_j(\bar{r})\eta_\ell^\dagger(\bar{r}) e^{i(\bar{G}_j-\bar{G}_\ell)\cdot\bar{r}}$$

$$+ \frac{B^\ell}{2}\int d\bar{r}\sum_{j,\ell}^3 \eta_j^\dagger(\bar{r})\eta_\ell(\bar{r}) e^{i(\bar{G}_\ell-\bar{G}_j)\cdot\bar{r}} + \frac{B^\ell}{2}\int d\bar{r}\sum_{j,\ell}^3 \eta_j^\dagger(\bar{r})\eta_\ell^\dagger(\bar{r}) e^{-i(\bar{G}_\ell+\bar{G}_j)\cdot\bar{r}}$$

and directly apply the volume-averaging function, i.e multiply by 1: where $\int d\bar{r}'\, \chi_A(\bar{r}-\bar{r}') = 1$.

$$\Rightarrow \frac{B^\ell}{2}\sum_{j,\ell}^3 \Big[\int d\bar{r}\int d\bar{r}'\, \chi_A(\bar{r}-\bar{r}')\, \eta_j(\bar{r})\eta_\ell(\bar{r}) e^{i(\bar{G}_j+\bar{G}_\ell)\cdot\bar{r}} + \int d\bar{r}\int d\bar{r}'\, \chi_A(\bar{r}-\bar{r}')\, \eta_j(\bar{r})\eta_\ell^\dagger(\bar{r}) e^{i(\bar{G}_j-\bar{G}_\ell)\cdot\bar{r}}$$

$$+ \int d\bar{r}\int d\bar{r}'\, \chi_A(\bar{r}-\bar{r}')\, \eta_j^\dagger(\bar{r})\eta_\ell(\bar{r}) e^{i(\bar{G}_\ell-\bar{G}_j)\cdot\bar{r}} + \int d\bar{r}\int d\bar{r}'\, \chi_A(\bar{r}-\bar{r}')\, \eta_j^\dagger(\bar{r})\eta_\ell^\dagger(\bar{r}) e^{-i(\bar{G}_\ell+\bar{G}_j)\cdot\bar{r}} \Big]$$

and taking the first term, for example, the limiting procedure is applied by switching the order of integration:

$$\int d\bar{r}'\int d\bar{r}\, \chi_A(\bar{r}-\bar{r}')\, \eta_j(\bar{r})\eta_\ell(\bar{r}) e^{i(\bar{G}_j+\bar{G}_\ell)\cdot\bar{r}}$$

Expanding the amplitudes to 2^{nd} order as before.. (around $\bar{r}'$)

$$\int d\bar{r}'\int d\bar{r}\, \chi_A(\bar{r}-\bar{r}')\Big(\eta_j(\bar{r}') + (\bar{r}-\bar{r}')_1\cdot\partial_{\bar{r}_1'}\eta_j(\bar{r}') + \frac{1}{2}(\bar{r}-\bar{r}')_2(\bar{r}-\bar{r}')_1\partial_{\bar{r}_2'}\partial_{\bar{r}_1'}\eta_j(\bar{r}')\Big)$$

$$\Big(\eta_\ell(\bar{r}') + (\bar{r}-\bar{r}')_1\partial_{\bar{r}_1'}\eta_\ell(\bar{r}') + \frac{1}{2}(\bar{r}-\bar{r}')_2(\bar{r}-\bar{r}')_1\partial_{\bar{r}_2'}\partial_{\bar{r}_1'}\eta_\ell(\bar{r}')\Big) e^{i(\bar{G}_j+\bar{G}_\ell)\cdot\bar{r}}$$

$$= \int d\bar{r}'\int d\bar{r}\, \chi_A(\bar{r}-\bar{r}')\Big\{\eta_j(\bar{r}')\eta_\ell(\bar{r}') + \eta_j(\bar{r}')(\bar{r}-\bar{r}')_1\partial_{\bar{r}_1'}\eta_\ell(\bar{r}') + \frac{1}{2}\eta_j(\bar{r}-\bar{r}')_2(\bar{r}-\bar{r}')_1\partial_{\bar{r}_2'}\partial_{\bar{r}_1'}\eta_\ell(\bar{r}')$$

$$+ \eta_\ell(\bar{r}')(\bar{r}-\bar{r}')_1\cdot\partial_{\bar{r}_1'}\eta_j(\bar{r}') + (\bar{r}-\bar{r}')_1\cdot\partial_{\bar{r}_1}\eta_\ell(\bar{r}')(\bar{r}-\bar{r}')_2\cdot\partial_{\bar{r}_2}\eta_j(\bar{r}') + \frac{1}{2}(\bar{r}-\bar{r}')\cdot\partial_{\bar{r}_1}\eta_j(\bar{r}')(\bar{r}-\bar{r}')(\bar{r}-\bar{r}')\partial_{\bar{r}_2'}\partial_{\bar{r}_1'}\eta_\ell(\bar{r}')$$

$$+\frac{1}{2}\eta_\ell(\bar{r}')(\bar{r}-\bar{r}')_2(\bar{r}-\bar{r}')_1\partial_{\bar{r}_2'}\partial_{\bar{r}_1'}\eta_j(\bar{r}') + \frac{1}{2}(\bar{r}-\bar{r}')\cdot\partial_{\bar{r}_1}\eta_\ell(\bar{r}')(\bar{r}-\bar{r}')(\bar{r}-\bar{r}')\partial_{\bar{r}_2}\partial_{\bar{r}_1}\eta_j(\bar{r}')$$

$$+ \frac{1}{4}(\bar{r}-\bar{r}')(\bar{r}-\bar{r}')\partial_{\bar{r}_2'}\partial_{\bar{r}_1'}\eta_j(\bar{r}')(\bar{r}-\bar{r}')(\bar{r}-\bar{r}')\partial_{\bar{r}_2'}\partial_{\bar{r}_1'}\eta_\ell(\bar{r}')\Big\} e^{i(\bar{G}_j+\bar{G}_\ell)\cdot\bar{r}}$$

we can then take all $\bar{r}'$ terms out of the $\bar{r}$ integral, which will give integrals over $\bar{r}$ of the following forms:

Inserting the Fourier transform of the coarse-graining function

① $\int d\bar{r}\, \chi_A(\bar{r}-\bar{r}')\, e^{i(\bar{G}_j+\bar{G}_\ell)\cdot\bar{r}} = \int d\bar{r}\int d\bar{k}\, \hat{\chi}_A(|\bar{k}|)\, e^{i\bar{k}\cdot(\bar{r}-\bar{r}')}\, e^{i(\bar{G}_j+\bar{G}_\ell)\cdot\bar{r}}$

$= \int d\bar{k}\, \hat{\chi}_A(|\bar{k}|)\, e^{-i\bar{k}\cdot\bar{r}'}\, \delta(\bar{k}+\bar{G}_j+\bar{G}_\ell) = \hat{\chi}_A(|\bar{G}_j+\bar{G}_\ell|)\, e^{i(\bar{G}_j+\bar{G}_\ell)\cdot\bar{r}'}$

② $\int d\bar{r}\, \chi_A(\bar{r}-\bar{r}')(\bar{r}-\bar{r}')\, e^{i(\bar{G}_\ell+\bar{G}_j)\cdot\bar{r}} = \int d\bar{r}\int d\bar{k}\, \hat{\chi}_A(|\bar{k}|)\, e^{i\bar{k}\cdot(\bar{r}-\bar{r}')}(\bar{r}-\bar{r}')\, e^{i(\bar{G}_\ell+\bar{G}_j)\cdot\bar{r}}$

and using partial integration twice...

$= \int d\bar{r}\int d\bar{k}\, \hat{\chi}_A(|\bar{k}|)\left(-i\partial_{\bar{k}_1} e^{i\bar{k}\cdot(\bar{r}-\bar{r}')}\right) e^{i(\bar{G}_\ell+\bar{G}_j)\cdot\bar{r}} = i\int d\bar{r}\int d\bar{k}\left(\partial_{\bar{k}_1}\hat{\chi}_A(|\bar{k}|)\right) e^{i\bar{k}\cdot(\bar{r}-\bar{r}')}\, e^{i(\bar{G}_\ell+\bar{G}_j)\cdot\bar{r}}$

$= i\left(\partial_{\bar{k}_1}\hat{\chi}_A(|\bar{G}_\ell+\bar{G}_j|)\right) e^{i(\bar{G}_\ell+\bar{G}_j)\cdot\bar{r}'}$

③ $\int d\bar{r}\, \chi_A(\bar{r}-\bar{r}')(\bar{r}-\bar{r}')_2(\bar{r}-\bar{r}')_1\, e^{i(\bar{G}_\ell+\bar{G}_j)\cdot\bar{r}} = \int d\bar{r}\int d\bar{k}\, \hat{\chi}_A(|\bar{k}|)\left(i^2\partial_{\bar{k}_2}\partial_{\bar{k}_1} e^{i\bar{k}\cdot(\bar{r}-\bar{r}')}\right) e^{i(\bar{G}_\ell+\bar{G}_j)\cdot\bar{r}}$

$= \int d\bar{r}\int d\bar{k}\,(-i^2)\left(\partial_{\bar{k}_1}\partial_{\bar{k}_2}\hat{\chi}_A(|\bar{k}|)\right) e^{i\bar{k}\cdot(\bar{r}-\bar{r}')}\, e^{i(\bar{G}_\ell+\bar{G}_j)\cdot\bar{r}} = -i^2\left(\partial_{\bar{k}_1}\partial_{\bar{k}_2}\hat{\chi}_A(|\bar{G}_\ell+\bar{G}_j|)\right) e^{i(\bar{G}_\ell+\bar{G}_j)\cdot\bar{r}'}$

④ $\int d\bar{r}\, \chi_A(\bar{r}-\bar{r}')(\bar{r}-\bar{r}')_3(\bar{r}-\bar{r}')_2(\bar{r}-\bar{r}')_1\, e^{i(\bar{G}_\ell+\bar{G}_j)\cdot\bar{r}} = \int d\bar{r}\int d\bar{k}\, \hat{\chi}_A(|\bar{k}|)\left(-i^3\partial_{\bar{k}_3}\partial_{\bar{k}_2}\partial_{\bar{k}_1} e^{i\bar{k}\cdot(\bar{r}-\bar{r}')}\right) e^{i(\bar{G}_\ell+\bar{G}_j)\cdot\bar{r}}$

$= i^3\int d\bar{r}\int d\bar{k}\left(\partial_{\bar{k}_1}\partial_{\bar{k}_2}\partial_{\bar{k}_3}\hat{\chi}_A(|\bar{k}|)\right) e^{i\bar{k}\cdot(\bar{r}-\bar{r}')}\, e^{i(\bar{G}_\ell+\bar{G}_j)\cdot\bar{r}} = i^3\left(\partial_{\bar{k}_1}\partial_{\bar{k}_2}\partial_{\bar{k}_3}\hat{\chi}_A(|\bar{G}_\ell+\bar{G}_j|)\right) e^{i(\bar{G}_\ell+\bar{G}_j)\cdot\bar{r}'}$

⑤ $\int d\bar{r}\, \chi_A(\bar{r}-\bar{r}')(\bar{r}-\bar{r}')_4(\bar{r}-\bar{r}')_3(\bar{r}-\bar{r}')_2(\bar{r}-\bar{r}')_1\, e^{i(\bar{G}_\ell+\bar{G}_j)\cdot\bar{r}}$

$= \int d\bar{r}\int d\bar{k}\, \hat{\chi}_A(|\bar{k}|)\left(i^4\partial_{\bar{k}_4}\partial_{\bar{k}_3}\partial_{\bar{k}_2}\partial_{\bar{k}_1} e^{i\bar{k}\cdot(\bar{r}-\bar{r}')}\right) e^{i(\bar{G}_\ell+\bar{G}_j)\cdot\bar{r}} = -i^4\int d\bar{r}\int d\bar{k}\left(\partial_{\bar{k}_1}\partial_{\bar{k}_2}\partial_{\bar{k}_3}\partial_{\bar{k}_4}\hat{\chi}_A(|\bar{k}|)\right) e^{i\bar{k}\cdot(\bar{r}-\bar{r}')}\, e^{i(\bar{G}_\ell+\bar{G}_j)\cdot\bar{r}}$

$= -i^4\left(\partial_{\bar{k}_1}\partial_{\bar{k}_2}\partial_{\bar{k}_3}\partial_{\bar{k}_4}\hat{\chi}_A(|\bar{G}_\ell+\bar{G}_j|)\right) e^{i(\bar{G}_\ell+\bar{G}_j)\cdot\bar{r}'}$

Inserting each of these back into the integral over $\bar{r}'$ and considering the coarse-graining limiting factor, it is evident that the integrals over the rapid oscillating terms $e^{i(\bar{G}_\ell+\bar{G}_j)\cdot\bar{r}'}$ will be negligible; thus all of these terms will $\Rightarrow 0$.

The only terms that will survive are those at resonance (as with the gradient term) i.e. with $e^{i(\bar{G}_\ell-\bar{G}_j)\cdot\bar{r}'}$ where $\bar{G}_\ell=\bar{G}_j$ ie where $\ell=j$.
Thus from ⊛ we can see that we only need to retain the following terms:

$$\Rightarrow \frac{B^\ell}{2}\sum_j^3\left[\int d\bar{r}\int d\bar{r}'\,\chi_A(\bar{r}-\bar{r}')\,\eta_j(\bar{r})\eta_j^\dagger(\bar{r}) + \int d\bar{r}\int d\bar{r}'\,\chi_A(\bar{r}-\bar{r}')\,\eta_j^\dagger(\bar{r})\eta_j(\bar{r})\right]$$

where the integral over $\bar{r}'$ is just $\int d\bar{r}'\, \chi_p(\bar{r}-\bar{r}') = 1$ and what remains is:

$$\Rightarrow \frac{B^\ell}{2}\sum_j^3\left[\int d\bar{r}\, \eta_j(\bar{r})\eta_j^\dagger(\bar{r}) + \int d\bar{r}\, \eta_j^\dagger(\bar{r})\eta_j(\bar{r})\right] \Rightarrow \boxed{B^\ell \sum_j^3 \int d\bar{r}\, |\eta_j(\bar{r})|^2}$$

← from n^2 term

For the n^3 term the same procedure is applied:

$$-\frac{t}{3}\int d\bar{r}\, n(\bar{r})^3 \quad \text{(dropping explicit } \bar{r}\text{-dep. of amplitudes for convenience)}$$

$$\Rightarrow -\frac{t}{3}\int d\bar{r}\left(\sum_j^3 \eta_j e^{i\bar{G}_j\cdot\bar{r}} + \sum_j^3 \eta_j^\dagger e^{-i\bar{G}_j\cdot\bar{r}}\right)\left(\sum_\ell^3 \eta_\ell e^{i\bar{G}_\ell\cdot\bar{r}} + \sum_j^3 \eta_\ell^\dagger e^{-i\bar{G}_\ell\cdot\bar{r}}\right)\left(\sum_m^3 \eta_m e^{i\bar{G}_m\bar{r}} + \sum_m \eta_m^\dagger e^{-i\bar{G}_m\bar{r}}\right)$$

$$\Rightarrow -\frac{t}{3}\sum_{j,\ell,m}^3\int d\bar{r}\left\{\eta_j\eta_\ell e^{i(\bar{G}_j+\bar{G}_\ell)\cdot\bar{r}} + \eta_j\eta_\ell^\dagger e^{i(\bar{G}_j-\bar{G}_\ell)\cdot\bar{r}} + \eta_j^\dagger\eta_\ell e^{i(\bar{G}_\ell-\bar{G}_j)\cdot\bar{r}} + \eta_j^\dagger\eta_\ell^\dagger e^{-i(\bar{G}_j+\bar{G}_\ell)\cdot\bar{r}}\right\}\left(\eta_m e^{i\bar{G}_m\bar{r}} + \eta_m^\dagger e^{-i\bar{G}_m\bar{r}}\right)$$

$$\Rightarrow -\frac{t}{3}\sum_{j,\ell,m}^3\int d\bar{r}\left\{\eta_j\eta_\ell\eta_m e^{i(\bar{G}_j+\bar{G}_\ell+\bar{G}_m)\cdot\bar{r}} + \eta_j\eta_\ell^\dagger\eta_m e^{i(\bar{G}_j-\bar{G}_\ell+\bar{G}_m)\cdot\bar{r}} + \eta_j^\dagger\eta_\ell\eta_m e^{i(-\bar{G}_j+\bar{G}_\ell+\bar{G}_m)\cdot\bar{r}}\right.$$
$$+ \eta_j^\dagger\eta_\ell^\dagger\eta_m e^{-i(\bar{G}_j+\bar{G}_\ell-\bar{G}_m)\cdot\bar{r}} + \eta_j\eta_\ell\eta_m^\dagger e^{i(\bar{G}_j+\bar{G}_\ell-\bar{G}_m)\cdot\bar{r}} + \eta_j\eta_\ell^\dagger\eta_m^\dagger e^{i(\bar{G}_j-\bar{G}_\ell-\bar{G}_m)\cdot\bar{r}}$$
$$\left. + \eta_j^\dagger\eta_\ell\eta_m^\dagger e^{i(\bar{G}_\ell-\bar{G}_j-\bar{G}_m)\cdot\bar{r}} + \eta_j^\dagger\eta_\ell^\dagger\eta_m^\dagger e^{-i(\bar{G}_j+\bar{G}_\ell+\bar{G}_m)\cdot\bar{r}}\right\}$$

where we can see that the only terms that can give us resonance are:

$$e^{\pm i(\bar{G}_j+\bar{G}_\ell+\bar{G}_m)\cdot\bar{r}} \quad \text{where } j\neq\ell\neq m, \quad \text{that is:}$$

$$\bar{G}_1+\bar{G}_2+\bar{G}_3 = \left(-\tfrac{\sqrt{3}}{2}\hat{x} - \tfrac{1}{2}\hat{y}\right) + (\hat{y}) + \left(\tfrac{\sqrt{3}}{2}\hat{x} - \tfrac{1}{2}\hat{y}\right)$$
$$= \left(-\tfrac{\sqrt{3}}{2}+0+\tfrac{\sqrt{3}}{2}\right)\hat{x} + \left(-\tfrac{1}{2}+1-\tfrac{1}{2}\right)\hat{y} = 0$$

(This can be seen visually by tip-to-tail addition of the lattice vectors, also)
Thus following the procedure from the n^2 term all other terms are discarded and the result is:

$$\Rightarrow -\frac{t}{3}\sum_{j,\ell,m}^3\int d\bar{r}\left\{\eta_j\eta_\ell\eta_m e^{i(\bar{G}_j+\bar{G}_\ell+\bar{G}_m)\cdot\bar{r}} + \eta_j^\dagger\eta_\ell^\dagger\eta_m^\dagger e^{-i(\bar{G}_j+\bar{G}_\ell+\bar{G}_m)\cdot\bar{r}}\right\}, \text{ for } j\neq\ell\neq m$$

where there are 6 choices of j,ℓ,m that satisfy this, thus:

$$\Rightarrow -\frac{6t}{3}\int d\bar{r}\left(\eta_1\eta_2\eta_3 + \eta_1^\dagger\eta_2^\dagger\eta_3^\dagger\right) = \boxed{-2t\int d\bar{r}\left(\prod_j^3\eta_j + \prod_j^3\eta_j^\dagger\right)}$$

← from n^3 term

And finally, for the n^4 term:

$$\frac{v}{4}\int d\vec{r}\; n(\vec{r})^4$$

$$\Rightarrow \frac{v}{4}\sum_{j,l,m,n}^{3}\int d\vec{r}\left\{\eta_j\eta_l\eta_m\eta_n e^{i(\vec{G}_j+\vec{G}_l+\vec{G}_m+\vec{G}_n)\cdot\vec{r}} + \eta_j\eta_l^\dagger\eta_m\eta_n e^{i(\vec{G}_j-\vec{G}_l+\vec{G}_m+\vec{G}_n)\cdot\vec{r}} + \eta_j^\dagger\eta_l^\dagger\eta_m^\dagger\eta_n e^{i(-\vec{G}_j-\vec{G}_l-\vec{G}_m+\vec{G}_n)\cdot\vec{r}}\right.$$

$$\left. + \eta_j^\dagger\eta_l^\dagger\eta_m\eta_n e^{i(-\vec{G}_j-\vec{G}_l+\vec{G}_m+\vec{G}_n)\cdot\vec{r}} + \eta_j\eta_l\eta_m^\dagger\eta_n e^{i(\vec{G}_j+\vec{G}_l-\vec{G}_m+\vec{G}_n)\cdot\vec{r}} + \dots\right.$$

Without writing out all the terms, we know that we are looking for resonances. We will require pairs of complex conjugates for this to occur ie.

$\vec{G}_1+\vec{G}_2-\vec{G}_1-\vec{G}_2$ and so $\eta_j\eta_j^\dagger\eta_l\eta_l^\dagger$

Remaining terms will thus be:

$$\Rightarrow \frac{v}{4}\sum_{j,l,m,n}^{3}\int d\vec{r}\left\{\eta_j^\dagger\eta_l^\dagger\eta_m\eta_n e^{i(-\vec{G}_j-\vec{G}_l+\vec{G}_m+\vec{G}_n)\cdot\vec{r}} + \eta_j\eta_l^\dagger\eta_m^\dagger\eta_n e^{i(\vec{G}_j-\vec{G}_l-\vec{G}_m+\vec{G}_n)\cdot\vec{r}}\right.$$

$$+ \eta_j^\dagger\eta_l\eta_m^\dagger\eta_n e^{i(-\vec{G}_j+\vec{G}_l-\vec{G}_m+\vec{G}_n)\cdot\vec{r}} + \eta_j\eta_l^\dagger\eta_m\eta_n^\dagger e^{i(\vec{G}_j-\vec{G}_l+\vec{G}_m-\vec{G}_n)\cdot\vec{r}} + \eta_j^\dagger\eta_l\eta_m\eta_n^\dagger e^{i(-\vec{G}_j+\vec{G}_l+\vec{G}_m-\vec{G}_n)\cdot\vec{r}}$$

$$\left. + \eta_j\eta_l\eta_m^\dagger\eta_n^\dagger e^{i(\vec{G}_j+\vec{G}_l-\vec{G}_m-\vec{G}_n)\cdot\vec{r}}\right\}$$

and enforcing pairs of cc's, where each set of 4 gives 2 choices (ie $j=m$ or $j=n$ for $\eta_j^\dagger\eta_l^\dagger\eta_m\eta_n$) so

$$\Rightarrow \frac{v}{4}\int d\vec{r}\; 6\cdot\left\{2\sum_{j\neq l}^{3}\eta_j^\dagger\eta_j\eta_m^\dagger\eta_m + \sum_{j}^{3}\eta_j^\dagger\eta_j\eta_j^\dagger\eta_j\right\}$$

which can also be written as

(since there is only one choice $j=l=m=n$ for each term)

$$\Rightarrow \frac{3v}{2}\left[2\sum_{j,l}|\eta_j(\vec{r})|^2|\eta_l(\vec{r})|^2 - \sum_{j}|\eta_j(\vec{r})|^4\right]$$

$\longrightarrow$ from n^4 term.

$$\Rightarrow \boxed{3v\left(\sum_{j}^{3}|\eta_j(\vec{r})|^2\right)^2 - \frac{3v}{2}\sum_{j}^{3}|\eta_j(\vec{r})|^4}$$

and combining all of these we get:

$$\hat{F}(n)_{\text{course grained}} = \int d\vec{r}\left\{B^l\sum_{j}^{3}|\eta_j(\vec{r})|^2 - 2t\left(\prod_{j}^{3}\eta_j + \prod_{j}^{3}\eta_j^\dagger\right) + 3v\left(\sum_{j}^{3}|\eta_j(\vec{r})|^2\right)^2 - \frac{3v}{2}\sum_{j}^{3}|\eta_j(\vec{r})|^4\right.$$

$$\left. - B^x\sum_{j}^{3}|\eta_j(\vec{r})|^2 + B^x\sum_{j}^{3}|(2i\vec{G}_j\cdot\nabla_{\vec{r}})\eta_j(\vec{r})|^2\right\}$$

Nik: in his 2010 PRE Ken Elder defines A^2 with a factor of 2. This is the only way I can get the necessary coefficients. (this differs from the definition in the slides)

Gathering like terms and using the following notation: $A^2 = 2\sum_{j=1}^{3} |\eta_j(\vec{r})|^2$

$$\Delta B = B^l - B^x$$

$$\hat{F}(\eta)_{\text{coarse grained}} = \int d\vec{r} \left\{ \frac{\Delta B}{2} A^2 + \frac{3v}{4} A^4 + \sum_{j=1}^{3} \left(B^x |(2i\vec{G}_j \cdot \vec{\nabla}_{\vec{r}})\eta_j(\vec{r})|^2 - \frac{3v}{2} |\eta_j(\vec{r})|^4 \right) \right.$$

$$\left. - 2t \left(\prod_j^3 \eta_j(\vec{r}) + \prod_j^3 \eta_j^*(\vec{r}) \right) \right\}$$

Bravo!

50/50

To calculate the elastic tensor we start with

$$\hat{F}_{\text{coarse grained}}(n) = \int d\bar{r}\left\{ \frac{\Delta B}{2} A^2 + \frac{3v}{4}A^4 + \sum_j^3 \left(B^x |(2i\bar{G}_j\cdot\nabla_{\bar{r}})\eta_j(\bar{r})|^2 - \frac{3v}{2}|\eta_j(\bar{r})|^4\right) - 2t\left(\prod_j^3 \eta_j(\bar{r}) + \prod_j^3 \eta_j^*(\bar{r})\right)\right\}$$

To represent a small displacement $n(\bar{x}) \Rightarrow n(\bar{x}+\bar{u})$ the amplitudes in the coarse-grained model above are replaced: $\eta_j \Rightarrow \phi_j e^{i\bar{G}_j\cdot\bar{u}}$

since $n = \sum_{j=1}^3 (\phi_j e^{i\bar{G}_j\cdot\bar{u}})e^{i\bar{G}\cdot\bar{x}} + c.c.$; and taking ϕ both real & the same for all j
$(\phi = \phi^* ;\ \phi_1 = \phi_2 = \phi_3)$

$$\Rightarrow \hat{F}[n(\bar{x}+\bar{u})] \Rightarrow \int d\bar{r}\left\{ \Delta B \sum_j^3 |\phi e^{i\bar{G}_j\bar{u}}|^2 + 3v\sum_{j\neq l}^3 |\phi e^{i\bar{G}_j\bar{u}}|^2|\phi e^{i\bar{G}_l\cdot\bar{u}}|^2 + \sum_j^3 B^x|(2iG_j\cdot\bar{\nabla})(\phi e^{i\bar{G}_j\bar{u}})|^2 - \frac{3v}{2}\sum_j^3|\phi e^{i\bar{G}_j\bar{u}}|^4 - 2t\prod_j^3 \phi e^{i\bar{G}_j\cdot\bar{u}} - 2t\prod_j^3 \phi e^{-i\bar{G}_j\bar{u}}\right\}$$

as seen previously, $\bar{G}_1+\bar{G}_2+\bar{G}_3=\emptyset$

$$\Rightarrow \int d\bar{r}\left\{ 3\Delta B\phi^2 + 3v\phi^4 \overbrace{\sum_{j\neq l}^3}^{=9} - \frac{3}{2}v\phi^4\overbrace{\sum_j^3}^{=3} - 2t\phi^3 - 2t\phi^3 + \sum_j^3 B^x|(2i\bar{G}_j\cdot\bar{\nabla})(\phi e^{i\bar{G}_j\bar{u}})|^2\right\}$$

$$\Rightarrow \int d\bar{r}\left\{ 3\Delta B\phi^2 - 4t\phi^3 + \frac{45}{2}v\phi^4 + \underbrace{\sum_j^3 B^x|(2i\bar{G}_j\cdot\bar{\nabla})(\phi e^{i\bar{G}_j\bar{u}})|^2}_{*}\right\}$$

Looking just at the remaining gradient term & rewriting the expression:

$$* \sum_j^3 B^x 4i^2\left\{(\bar{G}_j\cdot(\bar{\nabla}\phi e^{i\bar{G}_j\cdot\bar{u}} + \phi\nabla e^{i\bar{G}_j\cdot\bar{u}}))(\bar{G}_j\cdot(\bar{\nabla}\phi e^{-i\bar{G}_j\cdot\bar{u}} + \phi\bar{\nabla}e^{-i\bar{G}_j\cdot\bar{u}}))\right\}$$

$$= \sum_j^3 B^x 4i^2\left\{ \overset{①}{(\bar{G}_j\cdot\bar{\nabla}\phi)^2} + \overset{②}{(\bar{G}_j\cdot(\bar{\nabla}\phi e^{i\bar{G}_j\bar{u}}))(\bar{G}_j\cdot\phi\nabla e^{-i\bar{G}_j\bar{u}})} + \overset{③}{(\bar{G}_j\cdot\phi\bar{\nabla}e^{i\bar{G}_j\cdot\bar{u}})(\bar{G}_j\cdot(\bar{\nabla}\phi e^{-i\bar{G}_j\bar{u}}))} + \overset{④}{\phi^2(\bar{G}_j\cdot\nabla e^{i\bar{G}_j\cdot\bar{u}})(\bar{G}_j\cdot\nabla e^{-i\bar{G}_j\cdot\bar{u}})}\right\}$$

term ①: summing over $\bar{G}_j$'s.

$$\sum_j^3(\bar{G}_j\cdot\bar{\nabla}\phi)^2 = \left\{\left[\left(\frac{-\sqrt{3}}{2}\hat{x} - \frac{1}{2}\hat{y}\right)\cdot(\partial_x\phi\hat{x}+\partial_y\phi\hat{y})\right]^2 + \left[\hat{y}\cdot(\partial_x\phi\hat{x} - \partial_y\phi\hat{y})\right]^2 + \left[\left(\frac{\sqrt{3}}{2}\hat{x} - \frac{1}{2}\hat{y}\right)\cdot(\partial_x\phi\hat{x}+\partial_y\phi\hat{y})\right]^2\right\} = \left[\frac{-\sqrt{3}}{2}\partial_x\phi - \frac{1}{2}\partial_y\phi\right]^2 + (\partial_y\phi)^2 + \left[\frac{\sqrt{3}}{2}\partial_x\phi - \frac{1}{2}\partial_y\phi\right]^2$$

$$= \frac{3}{4}(\partial_x\phi)^2 + \frac{\sqrt{3}}{4}\partial_x\phi\partial_y\phi + \frac{1}{4}(\partial_y\phi)^2 + (\partial_y\phi)^2 + \frac{3}{4}(\partial_x\phi)^2 - \frac{\sqrt{3}}{4}\partial_x\phi\partial_y\phi + \frac{1}{4}(\partial_y\phi)^2$$

$$= \frac{3}{2}(\partial_x\phi)^2 + \frac{3}{2}(\partial_y\phi)^2 = \frac{3}{2}|\bar{\nabla}\phi|^2 \quad \boxed{\Rightarrow B^x 4i^2\sum_j^3(\bar{G}_j\cdot\bar{\nabla}\phi)^2 = 6B^x i^2|\bar{\nabla}\phi|^2}$$

terms ② & ③ become:

$$(\bar{G}_j \cdot (\nabla\phi) e^{i\bar{G}_j \cdot \bar{u}})(\bar{G}_j \cdot \phi(-i\bar{G}_j) e^{-i\bar{G}_j \cdot \bar{u}} \nabla \bar{u}) + (\bar{G}_j \cdot \phi(i\bar{G}_j) e^{i\bar{G}_j \cdot \bar{u}} \bar{\nabla} u)(\bar{G}_j \cdot (\nabla\phi) e$$

$$= -i\phi(\bar{G}_j \cdot (\bar{\nabla}\phi))(\bar{G}_j \cdot \bar{G}_j \nabla \bar{u}) + i\phi(\bar{G}_j \cdot \bar{G}_j \bar{\nabla} u)(\bar{G}_j \cdot (\bar{\nabla}\phi)) = \emptyset$$

Finally, term ④:

$$\sum_j^3 4 i^2 B^x (\bar{G}_j \cdot \phi \bar{\nabla} e^{i\bar{G}_j \cdot \bar{u}})(\bar{G}_j \cdot \phi \bar{\nabla} e^{-i\bar{G}_j \cdot \bar{u}})$$

where $\nabla e^{i\bar{G}_j \cdot \bar{u}} = \partial_x e^{i(G_x u_x + G_y u_y)} \hat{x} + \partial_y e^{i(G_x u_x + G_y u_y)} \hat{y}$

$$= i\left(G_x \frac{\partial u_x}{\partial x} + G_y \frac{\partial u_y}{\partial x}\right) e^{i\bar{G}\cdot\bar{u}} \hat{x} + i\left(G_x \frac{\partial u_x}{\partial y} + G_y \frac{\partial u_y}{\partial y}\right) e^{i\bar{G}\cdot\bar{u}} \hat{y}$$

so term ④ becomes:

$$\sum_j^3 -i^4 4 B^x \phi^2 \left\{ \bar{G}_j \cdot \left[(G_x \partial_x u_x + G_y \partial_x u_y)\hat{x} + (G_x \partial_y u_x + G_y \partial_y u_y)\hat{y} \right]^2 \right\}$$

$$\Rightarrow -4 i^4 \phi^2 B^x \sum_j^3 (G_x^2 \partial_x u_x + G_x G_y \partial_x u_y + G_y G_x \partial_y u_x + G_y^2 \partial_y u_y)^2$$

where by definition $U_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$ so

$$\Rightarrow -4 i^4 \phi^2 B^x \underbrace{\sum_j^3 (G_x^2 U_{xx} + 2 G_x G_y U_{xy} + G_y^2 U_{yy})^2}_{f_{amp}}$$

f_{amp}. (let * represent expression inside brackets).

This can be expanded out to match the form found in the slides.
To calculate the elastic tensor, each term $K_{ijkl} = \frac{\partial^2 f_{amp}}{\partial U_{ij} \partial U_{kl}}$.

$$\frac{\partial^2 f_{amp}}{\partial U_{xx} \partial U_{xx}} = \frac{\partial}{\partial U_{xx}} \left[2(\text{---}*\text{---}) . G_x^2 \right] = 2 G_x^4 \; ; \; 2\sum_j^3 G_x^4 = 2\left(\frac{9}{16} + \emptyset + \frac{9}{16}\right) = \frac{9}{4}$$

$$\frac{\partial^2 f_{amp}}{\partial U_{xy} \partial U_{xx}} = \frac{\partial^2 f_{amp}}{\partial U_{xx} \partial U_{xy}} = \frac{\partial}{\partial U_{xy}} \left[2(\text{---}*\text{---}) G_x^2 \right] = 4 G_x^3 G_y \; ; \; \sum_j^3 4 G_x^3 G_y = \emptyset$$

$$\frac{\partial^2 f_{amp}}{\partial U_{xx} \partial U_{yy}} = \frac{\partial^2 f_{amp}}{\partial U_{yy} \partial U_{xx}} = 2 G_x^2 G_y^2 \; ; \; \sum_j^3 2 G_x^2 G_y^2 = \frac{3}{4}$$

$$\frac{\partial^2 f_{amp}}{\partial U_{yy} \partial U_{yy}} = 2 G_y^4 \; ; \; \sum_j^3 2 G_y^4 = \frac{9}{4}$$

$$\frac{\partial^2 f_{amp}}{\partial U_{xy} \partial U_{yy}} = \frac{\partial^2 f_{amp}}{\partial U_{yy} \partial U_{xy}} = 4 G_y^3 G_x \; ; \; \sum_j^3 4 G_y^3 G_x = \emptyset$$

and lastly:

$$\frac{\partial^2 f_{amp}}{\partial U_{xy}\partial U_{xy}} = 8G_x^2 G_y^2 \; ; \; \sum_j^3 8G_x^2 G_y^2 = 3$$

so the values for the elastic tensor K_{ijkl} are:

$$K_{xxxx} = K_{yyyy} = \frac{9}{4}$$

$$K_{xxxy} = K_{xyxx} = K_{yyxy} = K_{xyyy} = \emptyset$$

$$K_{xxyy} = K_{yyxx} = \frac{3}{4}$$

$$K_{xyxy} = 3$$

and going back to term ④ to expand,

$$\Rightarrow -4i^4\phi^2 B^x \sum_j^3 (G_x^2 U_{xx} + 2G_xG_yU_{xy} + G_y^2 U_{yy})(G_x^2 U_{xx} + 2G_xG_yU_{xy} + G_y^2 U_{yy})$$

$$= -4i^4\phi^2B^x\sum_j^3 [G_x^4 U_{xx}^2 + \underbrace{\cancel{2G_x^3 G_y}}_{\emptyset \text{ over sum}} U_{xx}U_{xy} + G_x^2G_y^2 U_{xx}U_{yy} + \underbrace{\cancel{2G_x^3G_y}}_{\emptyset} U_{xy}U_{xx}$$

$$+ 4G_x^2G_y^2 U_{xy}^2 + \underbrace{\cancel{2G_xG_y^3}}_{\emptyset} U_{xy}U_{yy} + G_x^2G_y^2 U_{xx}U_{yy} + \underbrace{\cancel{2G_xG_y^3}}_{\emptyset} U_{yy}U_{xy}$$

$$+ G_y^4 U_{yy}^2]$$

$$= -4i^4\phi^2B^x\sum_j^3 [G_x^4(U_{xx}^2 + U_{yy}^2) + 2G_x^2G_y^2U_{xx}U_{yy} + 4G_x^2G_y^2U_{xy}^2]$$

$$= -4i^4\phi^2B^x\left\{\frac{9}{8}\sum_{i=1}^2 U_{ii}^2 + \frac{3}{4}U_{xx}U_{yy} + \frac{3}{2}U_{xy}^2\right]$$

$$= -3i^4\phi^2B^x\left\{\left(\frac{3}{2}\sum_{i=1}^2 U_{ii}^2\right) + U_{xx}U_{yy} + 2U_{xy}^2\right].$$

So the full expression yields:

$$F(n(x,\omega)) = \int dF\Big\{3\Delta B\phi^2 - 4t\phi^3 + \frac{45}{2}v\phi^4 + 6i^2B^x|\vec\nabla\phi|^2 + 3i^2B^x\underbrace{\left[\sum_{i=1}^2 \frac{3}{2}U_{ii}^2 + U_{xx}U_{yy} + 2U_{xy}^2\right]}\phi^2$$

This can also be the f_{amp} → will yield slightly different values for K_{ijkl} but the ratios will be the same.