DUALIZABLE TENSOR CATEGORIES

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Abstract.

Contents

1. Introduction

The story: "Fusion categories provided local field theories, and the structure (pivotality) of the fusion category corresponds to the structure (spinness) of the local field theory."

We are aiming to keep this paper to 30 pages.

1.1. **Background and motivation.**

1.2. **Results.** .

The first half of the paper, sections 3 and **??**, focuses on [local field theory in dimension three and the dualizability of fusion categories].

The main theorem:

Theorem 1.1. *Fusion categories are dualizable.*

A key application of this theorem is the construction of a plethora of local field theories:

Corollary 1.2. *For any fusion category there is a local topological quantum field theory whose value on a point is that fusion category.*

In particular, the theorem provides localizations of Turaev-Viro field theories:

Corollary 1.3. *There is a local field theory whose value on a circle is the center of the fusion category of representations of a loop group at (any nondegenerate?) level.*

Of course, the fusion category of representations of a loop group is merely an example, and can be replaced by any fusion category here. This result is related to recent work of Kirillov and Balsam [**?**], which constructs a semi-local (that is, $1+1+1$-dimensional) version of Turaev-Viro theory. In particular, our $0+1+1+1$-dimensional theory has the same value on a circle as the Kirillov-Balsam theory.

The second half of the paper, sections 5 and 6, focuses on [Serre/doubledual/pivotality].

The main theorem:

Theorem 1.4. *The Serre automorphism of a fusion category $\mathcal{C}$ is the bimodule associated to the double dual functor $** : \mathcal{C} \to \mathcal{C}$.*

Because the Serre automorphism is necessarily order 2, this theorem provides a simple topological proof of the following generalization (?) of a theorem of ENO:

Corollary 1.5. *The quadruple dual functor $* * ** : \mathcal{C} \to \mathcal{C}$ on a fusion category is naturally isomorphic to the identity functor.*

A key insight resulting from the field-theoretic perspective on fusion categories is that the ENO conjecture, namely that fusion categories are pivotal, is equivalent to the spin-independence of the topological field theories associated to fusion categories:

Theorem 1.6. *A fusion category $\mathcal{C}$ is pivotal if and only if the local field theory associated to $\mathcal{C}$ is independent of spin structure.*

A precise formulation of this result, in terms of a descent condition for the bordism structure group of the local field theory, is given in section **??**.

Acknowledgments. André Henriques, Scott Morrison, Kevin Walker.

CD's comment color

CSP's comment color

NS's comment color

The organization of this section might well change as we decide what exactly we should include.

2. Tensor categories

Our goal is to define $TC(3)$, ie [...]. We are not trying to include a huge discussion of all the different variations $TC(i)$, which can occur in DTCII.

2.1. **Linear categories.** .

[Fix the ground ring to be $\mathbb{C}$? Linear categories will mean linear over $\mathbb{C}$?]

2.2. **Tensor products and colimits of linear categories.**

2.3. **Tensor category bimodules and bimodule composition.**

2.4. **The 3-category of tensor categories.** .

Formalism for 3-categories.

Formalism for symmetric monoidal 3-categories. A symmetric monoidal category is the same as a functor $Fin_* \to$ Cat which sends coproducts to products. A symmetric monoidal 3-category is a functor $Fin_* \to$ 3-Cat which sends coproducts to products.

Definition of TC. [Objects of TC are *idempotent complete linear categories*.]

CD continues to vote for using 'fusion' rather than 'multifusion'

2.5. **Multi-Fusion categories.**

3. Local Field Theory in Dimension Three

3.1. **Dualizability in 3-categories.** .

[Adjunction convention: ${}_C M_D \dashv {}_D N_C$ if there are $M \boxtimes_D N \to C$ and $D \to N \boxtimes_C M$ satisfying the S relations.]

Set up terminology for evaluation and coevaluation.

This proposition is stated as a remark, without proof in Jacob's paper.

Proposition 3.1 (Lurie, Remark 3.4.22)**.** *Let C be a symmetric monoidal 3-category. Let $f : x \to y$ be a 1-morphism in C, and suppose that f admits a right adjoint f^R, so we have unit and counit maps $u : id_x \to f^R \circ f$ and $v : f \circ f^R \to id_y$. If u and v admit left adjoints u^L and v^L, the u^L and v^L exhibit f^R also as a left adjoint to f.*

Proof. .... □

3.2. **Structure groups of 3-manifolds.**

4. Dualizability and fusion categories

4.1. **Fusion categories are dualizable.**

4.1.1. *Functors of finite semisimple module categories have adjoints.* cf ENOPartII "Solution to item (1)"

Lemma 4.1. *Let $F : {}_{\mathcal{C}}\mathcal{M}_{\mathcal{D}} \to {}_{\mathcal{C}}\mathcal{N}_{\mathcal{D}}$ be a functor of bimodule categories, and suppose the underlying functor $\tilde{F} : \mathcal{M} \to \mathcal{N}$ of linear categories has an ambidextrous adjoint $\tilde{G} : \mathcal{N} \to \mathcal{M}$. Then F has an ambidextrous adjoint $G : {}_{\mathcal{C}}\mathcal{N}_{\mathcal{D}} \to {}_{\mathcal{C}}\mathcal{M}_{\mathcal{D}}$.*

Proof. Look at the wave □

Lemma 4.2. *Let $F : {}_{\mathcal{C}}\mathcal{M}_{\mathcal{D}} \to {}_{\mathcal{C}}\mathcal{N}_{\mathcal{D}}$ be a functor of bimodule categories. If $\mathcal{M}$ and $\mathcal{N}$ are semisimple categories with finitely many simple objects, then the functor $\tilde{F} : \mathcal{M} \to \mathcal{N}$ of linear categories underlying F has an ambidextrous adjoint.*

Proof. The ambidextrous adjoint F^* is given by the transpose of the entrywise dual of F. That is, write F as a matrix of vector spaces in terms of a chosen basis of simple objects for M and N, then take the matrix of dual vector spaces, and take the transpose. ... □

Proposition 4.3. *A functor $F : {}_{\mathcal{C}}\mathcal{M}_{\mathcal{D}} \to {}_{\mathcal{C}}\mathcal{N}_{\mathcal{D}}$ of bimodules categories has an ambidextrous adjoint if the linear categories $\mathcal{M}$ and $\mathcal{N}$ are semisimple with finitely many simple objects.*

This proposition follows from the above two lemmas.

4.1.2. *Indecomposable modules with braided fusion commutant have adjoints.*

Lemma 4.4. *Let $\mathcal{C}$ be a fusion category and $\mathcal{M}$ an indecomposable $\mathcal{C}$-module. Let $\mathcal{C}'$ denote the commutant of $\mathcal{C}$ acting on $\mathcal{M}$. In this case the bimodule ${}_{\mathcal{C}}\mathcal{M}_{\mathcal{C}'}$ is invertible.*

Proof. See ENOPartIII Apr 7 "Lemma: When C is fusion" for a sketch. Can this be shown only using that C is semisimple? □

In the situation of this lemma we will denote the inverse of ${}_{\mathcal{C}}\mathcal{M}_{\mathcal{C}'}$ by ${}_{\mathcal{C}'}\mathcal{N}_{\mathcal{C}}$.

Lemma 4.5. *Let $\mathcal{C}$ be a fusion category and $\mathcal{M}$ an indecomposable $\mathcal{C}$-module such that the commutant $\mathcal{C}'$ of the $\mathcal{C}$ action on $\mathcal{M}$ is bradied fusion. In this case there exist maps*

$$\lambda : {}_{\mathrm{Vect}}\mathcal{C}' \boxtimes_{\mathcal{C}'} \mathcal{C}'_{\mathrm{Vect}} \Rightarrow {}_{\mathrm{Vect}}\mathrm{Vect}_{\mathrm{Vect}}$$
$$\mu : {}_{\mathcal{C}'}\mathcal{C}'_{\mathcal{C}'} \Rightarrow {}_{\mathcal{C}'}\mathcal{C}' \boxtimes \mathcal{C}'_{\mathcal{C}'}$$

that form an adjunction

$$ {}_{\mathrm{Vect}}\mathcal{C}'_{\mathcal{C}'} \dashv {}_{\mathcal{C}'}\mathcal{C}'_{\mathrm{Vect}}. $$

We refer to λ and μ as "conditional expectation" maps.

If we have anything to say about the existence of conditional expectations when neither tensor category is Vect, then this proposition can be generalized to that case. In that case, the theorem below might also be able to be generalized.

Do you need to assume fusion for the commutant, or is something weaker enough?

Do you need to assume $\mathcal{C}$ is fusion?

cf ENOPartIII "Proposition3".

Proof. The first map $\lambda : \mathcal{C}' \to \mathrm{Vect}$ is defined by $\lambda(x) = \mathrm{Hom}_{\mathcal{C}'}(1, x)$. The second map $\mu : \mathcal{C}' \to \mathcal{C}' \boxtimes \mathcal{C}'$ is determined, using the left $\mathcal{C}'$-module structure, by the condition that

$$ \mu(1) = \sum_{\sigma \in \mathcal{I}} {}^*\sigma \boxtimes \sigma. $$

Here $\mathcal{I}$ is a basis of simple objects of $\mathcal{C}'$.

By construction μ is a left module map, but we also need to give μ the structure of a right module map. It is sufficient to check that for $\tau \in \mathcal{C}'$ simple, there is an isomorphism

$$ \sum_{\sigma \in \mathcal{I}} {}^*\sigma \boxtimes \sigma\tau \cong \sum_{\sigma \in \mathcal{I}} \tau({}^*\sigma) \boxtimes \sigma $$

Is that right that there is no further condition, ie you pick the module structure to be whatever iso you want on each simple, and that's it?

Let N^a_{bc} denote the vector spaces defining the tensor structure on the tensor category $\mathcal{C}'$. We have a series of isomorphisms

$$ \begin{aligned} \sum_{\sigma \in \mathcal{I}} {}^*\sigma \boxtimes \sigma\tau &= \sum_{\sigma,\rho \in \mathcal{I}} N^{\rho}_{\sigma\tau}\, {}^*\sigma \boxtimes \rho \\ &\cong (1) \sum_{\sigma,\rho \in \mathcal{I}} N^{{}^*\sigma}_{\tau({}^*\rho)}\, {}^*\sigma \boxtimes \rho \\ &\cong (2) \sum_{\sigma,\rho \in \mathcal{I}} N^{{}^*\sigma}_{({}^*\rho)\tau}\, {}^*\sigma \boxtimes \rho \\ &\cong (3) \sum_{\sigma,\rho \in \mathcal{I}} N^{\rho}_{\sigma\tau}\, \rho \boxtimes \sigma^* \\ &= \sum_{\sigma \in \mathcal{I}} \sigma\tau \boxtimes \sigma^* \\ &\cong (4) \sum_{\sigma \in \mathcal{I}} \tau({}^*\sigma) \boxtimes \sigma \end{aligned} $$

The first isomorphism $N^{\rho}_{\sigma\tau} \cong N^{{}^*\sigma}_{\tau({}^*\rho)}$ exists by the standard properties of structure constants for fusion categories. say more? Because $\mathcal{C}'$ is braided, the constant $N^{{}^*\sigma}_{\tau({}^*\rho)}$ is isomorphic to $N^{{}^*\sigma}_{({}^*\rho)\tau}$, giving the second isomorphism. Reindexing the sum by substituting ρ^* for σ and σ^* for ρ provides the third isomorphism. Braiding σ and τ and then substituting ${}^*\sigma$ for σ produces the fourth isomorphism.

Was *that* the argument? Using the braiding twice?

Finally we need to know that the maps λ and μ do indeed satisfy the adjunction S-relations. The first relation is the composite

$$ {}_{\mathrm{Vect}}\mathcal{C}'_{\mathcal{C}'} = \mathcal{C}' \boxtimes_{\mathcal{C}'} \mathcal{C}' \to \mathcal{C}' \boxtimes_{\mathcal{C}'} \mathcal{C}' \boxtimes \mathcal{C}' \to \mathrm{Vect} \boxtimes \mathcal{C}' = \mathcal{C}' $$

sending 1 to $\sum_{\sigma \in \mathcal{I}} \mathrm{Hom}(1, {}^*\sigma)\sigma = Hom(1, {}^*1)1 = Hom(1 \cdot 1, 1)1 = 1$. The second relation is the composite

$$ {}_{\mathcal{C}'}\mathcal{C}'_{\mathrm{Vect}} = \mathcal{C}' \boxtimes_{\mathcal{C}'} \mathcal{C}' \to \mathcal{C}' \boxtimes \mathcal{C}' \boxtimes_{\mathcal{C}'} \mathcal{C}' \to \mathcal{C}' \boxtimes \mathrm{Vect} = \mathcal{C}' $$

sending 1 to $\sum_{\sigma \in \mathcal{I}} {}^*\sigma\, \mathrm{Hom}(1, \sigma) = {}^*1 = 1$. Both maps are indeed equivalent to the identity. $\square$

Theorem 4.6. *Let $\mathcal{C}$ be a tensor category and $\mathcal{M}$ an indecomposable $\mathcal{C}$-module. If the commutant $\mathcal{C}'^{\mathcal{M}}$ is a braided fusion category, then the bimodule ${}_{\mathcal{C}}\mathcal{M}_{\mathrm{Vect}}$ has an ambidextrous adjoint.*

cf ENOPartIII "Proof details" etc.

Proof. Let ${}_{\mathcal{C}'}\mathcal{N}_{\mathcal{C}}$ denote the inverse of the bimodule ${}_{\mathcal{C}}\mathcal{M}_{\mathcal{C}'}$ provided by Lemma 4.4. We will construct an ambidextrous adjunction

$$ {}_{\mathcal{C}}\mathcal{M}_{\mathrm{Vect}} \vdash\dashv {}_{\mathrm{Vect}}\mathcal{N}_{\mathcal{C}}. $$

First we build the adjunction

$$ {}_{\mathcal{C}}\mathcal{M}_{\mathrm{Vect}} \dashv {}_{\mathrm{Vect}}\mathcal{N}_{\mathcal{C}} $$

as follows. Write the bimodules ${}_{\mathcal{C}}\mathcal{M}_{\mathrm{Vect}}$ and ${}_{\mathrm{Vect}}\mathcal{N}_{\mathcal{C}}$ as tensor products:

$$ \mathcal{M} = \mathcal{M} \boxtimes_{\mathcal{C}'} \mathcal{C}' $$

$$ \mathcal{N} = \mathcal{C}' \boxtimes_{\mathcal{C}'} \mathcal{N} $$

The desired adjunction is the composite of the following two adjunctions:

(1) ${}_{\mathcal{C}'}\mathcal{C}'_{\mathrm{Vect}} \dashv {}_{\mathrm{Vect}}\mathcal{C}'_{\mathcal{C}'}$
(2) ${}_{\mathcal{C}}\mathcal{M}_{\mathcal{C}'} \dashv {}_{\mathcal{C}'}\mathcal{N}_{\mathcal{C}}$

The bimodules ${}_{\mathcal{C}}\mathcal{M}_{\mathcal{C}'}$ and ${}_{\mathcal{C}'}\mathcal{N}_{\mathcal{C}}$ are inverse by construction, therefore adjoint as needed. The unit and counit for the first adjunction are given by

$\phi : \mathcal{C}' \boxtimes \mathcal{C}' \to \mathcal{C}' \boxtimes_{\mathcal{C}'} \mathcal{C}' = \mathcal{C}'$
$\psi : \mathrm{Vect} \to \mathcal{C}' = \mathcal{C}' \boxtimes_{\mathcal{C}'} \mathcal{C}'$

The S-relations for this unit and counit can be checked as follows:

$${}_{\mathcal{C}'}\mathcal{C}'_{\mathrm{Vect}} = \mathcal{C}' \boxtimes \mathrm{Vect} \to \mathcal{C}' \boxtimes \mathcal{C}' = \mathcal{C}' \boxtimes \mathcal{C}' \boxtimes_{\mathcal{C}'} \mathcal{C}' \to \mathcal{C}' \boxtimes_{\mathcal{C}'} \mathcal{C}' \boxtimes_{\mathcal{C}'} \mathcal{C}' = \mathcal{C}' \boxtimes_{\mathcal{C}'} \mathcal{C}' = \mathcal{C}'$$
$${}_{\mathrm{Vect}}\mathcal{C}'_{\mathcal{C}'} = \mathrm{Vect} \boxtimes \mathcal{C}' \to \mathcal{C}' \boxtimes \mathcal{C}' = \mathcal{C}' \boxtimes_{\mathcal{C}'} \mathcal{C}' \boxtimes \mathcal{C}' \to \mathcal{C}' \boxtimes_{\mathcal{C}'} \mathcal{C}' \boxtimes_{\mathcal{C}'} \mathcal{C}' \to \mathcal{C}' \boxtimes_{\mathcal{C}'} \mathcal{C}' = \mathcal{C}'$$

Explicitly, the unit and counit for the adjunction ${}_{\mathcal{C}}\mathcal{M}_{\mathrm{Vect}} \dashv {}_{\mathrm{Vect}}\mathcal{N}_{\mathcal{C}}$ are respectively the composites:

The above adjunction could be generalized to $\mathcal{D}$ instead of Vect.

$$\mathcal{M} \boxtimes \mathcal{N} = (\mathcal{M} \boxtimes_{\mathcal{C}'} \mathcal{C}') \boxtimes (\mathcal{C}' \boxtimes_{\mathcal{C}'} \mathcal{N}) \xrightarrow{\phi} \mathcal{M} \boxtimes_{\mathcal{C}'} \mathcal{C}' \boxtimes_{\mathcal{C}'} \mathcal{N} = \mathcal{M} \boxtimes_{\mathcal{C}'} \mathcal{N} \cong \mathcal{C}$$
$$\mathrm{Vect} \xrightarrow{\psi} \mathcal{C}' \boxtimes_{\mathcal{C}'} \mathcal{C}' = \mathcal{C}' \boxtimes_{\mathcal{C}'} \mathcal{C}' \boxtimes_{\mathcal{C}'} \mathcal{C}' \cong \mathcal{C}' \boxtimes_{\mathcal{C}'} \mathcal{N} \boxtimes_{\mathcal{C}} \mathcal{M} \boxtimes_{\mathcal{C}'} \mathcal{C}' = \mathcal{N} \boxtimes_{\mathcal{C}} \mathcal{M}$$

Second we construct an adjunction

$$ {}_{\mathrm{Vect}}\mathcal{N}_{\mathcal{C}} \dashv {}_{\mathcal{C}}\mathcal{M}_{\mathrm{Vect}}. $$

Again this adjunction is constructed as the composite of two adjunctions, namely

(1) ${}_{\mathrm{Vect}}\mathcal{C}'_{\mathcal{C}'} \dashv {}_{\mathcal{C}'}\mathcal{C}'_{\mathrm{Vect}}$
(2) ${}_{\mathcal{C}'}\mathcal{N}_{\mathcal{C}} \dashv {}_{\mathcal{C}}\mathcal{M}_{\mathcal{C}'}$

The bimodules ${}_{\mathcal{C}'}\mathcal{N}_{\mathcal{C}}$ and ${}_{\mathcal{C}}\mathcal{M}_{\mathcal{C}'}$ are inverse, so again adjoint as needed. The unit and counit for the first adjunction, namely

$\lambda : {}_{\mathrm{Vect}}\mathcal{C}' \boxtimes_{\mathcal{C}'} \mathcal{C}'_{\mathrm{Vect}} \Rightarrow {}_{\mathrm{Vect}}\mathrm{Vect}_{\mathrm{Vect}}$
$\mu : {}_{\mathcal{C}'}\mathcal{C}'_{\mathcal{C}'} \Rightarrow {}_{\mathcal{C}'}\mathcal{C}' \boxtimes \mathcal{C}'_{\mathcal{C}'}$,

are provided by Lemma 4.5. □

Since this adjunction depends on the lemma, which depends on $\mathcal{D} =$ Vect, we don't know how to generalize this part at the moment.

4.1.3. *Fusion categories have duals.*

Theorem 4.7. *[... fusion categories have duals ...]*

Remark 4.8. In non-zero characteristic, it is not the case that all fusion categories are dualizable. In particular, if the global dimension of a fusion category is zero, then the category cannot be dualizable.

Recalling the discussion of dualizability in 3-categories from Section 3.1, the theorem follows from the following three propositions.

Proposition 4.9. *Every tensor category $\mathcal{C} \in \mathrm{TC}$ has a dual in the homotopy category of* TC, *namely the monoidal opposite category $\mathcal{C}^{mp}$.*

Proof. The unit of the duality is $\mathcal{C}$ as a $\mathcal{C} \boxtimes \mathcal{C}^{mp}$–Vect bimodule. The counit of the duality is $\mathcal{C}$ as a Vect–$\mathcal{C}^{mp} \boxtimes \mathcal{C}$ bimodule. □

Proposition 4.10. *Let $\mathcal{C}$ be a fusion category. The unit ${}_{\mathcal{C}\boxtimes\mathcal{C}^{mp}}\mathcal{C}_{\mathrm{Vect}}$ and counit ${}_{\mathrm{Vect}}\mathcal{C}_{\mathcal{C}^{mp}\boxtimes\mathcal{C}}$ of the duality between $\mathcal{C}$ and $\mathcal{C}^{mp}$ both have ambidextrous adjoints.*

Proof. The unit category $\mathcal{C}$ is indecomposable as a $\mathcal{C} \boxtimes \mathcal{C}^{mp}$-module, and the commutant of this module structure is the Drinfeld center $Z(\mathcal{C})$ which is braided fusion, by [...]. Theorem 4.6 therefore ensures that this module has an ambidextrous adjoint. The argument for the counit is analogous. □

Let ${}_{\mathcal{C}\boxtimes\mathcal{C}^{mp}}\mathcal{C}_{\mathrm{Vect}} \vdash\dashv {}_{\mathrm{Vect}}\mathcal{D}_{1\mathcal{C}\boxtimes\mathcal{C}^{mp}}$ and ${}_{\mathrm{Vect}}\mathcal{C}_{\mathcal{C}^{mp}\boxtimes\mathcal{C}} \vdash\dashv {}_{\mathcal{C}^{mp}\boxtimes\mathcal{C}}\mathcal{D}_{2\mathrm{Vect}}$ denote the ambidextrous adjoints provided by Proposition 4.10.

Proposition 4.11. *For $\mathcal{C}$ a fusion category, the units and counits of the four adjunctions ${}_{\mathcal{C}\boxtimes\mathcal{C}^{mp}}\mathcal{C}_{\mathrm{Vect}} \dashv \mathcal{D}_1$, $\mathcal{D}_1 \dashv {}_{\mathcal{C}\boxtimes\mathcal{C}^{mp}}\mathcal{C}_{\mathrm{Vect}}$, ${}_{\mathrm{Vect}}\mathcal{C}_{\mathcal{C}^{mp}\boxtimes\mathcal{C}} \dashv \mathcal{D}_2$, and $\mathcal{D}_2 \dashv {}_{\mathrm{Vect}}\mathcal{C}_{\mathcal{C}^{mp}\boxtimes\mathcal{C}}$ all have ambidextrous adjoints.*

Proof. Need to know that the inverse bimodule provided by Lemma 4.4 is finite semisimple. Given that, this follows from Proposition 4.3. (This uses the fact that $A \boxtimes_{\mathcal{C}} B$ is finite semisimple if $A, B, \mathcal{C}$ all are.) ... □

4.2. **Examples of dualization structures.** For a variety of fusion categories, we explicitly describe the dualization structure provided by Theorem 4.7, namely the dual categories and the adjunctions and higher adjunctions for the adjunctions, and so on. Throughout we use implicitly that the adjunctions are all ambidextrous.

Example 4.12. $Rep(Z/2) = \{\mathbb{C}[x]/(x^2 - 1)\} -$ mod (Symmetric) [Start this by copying the content from Wave "Rep(Z/2)", ENO Part III, on Jun 22.]

Example 4.13. Here we give an example of a tensor category that is not fusion, namely $\{\mathbb{C}[x]/x^2\}-$ mod and highlight the failure of dualizability. Prove that this category is not dualizable. Cf Wave ENO Part III on May 4 search for "An example to consider", and Wave ENO Part III on Jun 17, search for "is still not fully dualizable".

Example 4.14. $Vect[G, \lambda]$ for some simple G and λ?

Example 4.15. Fibonacci category (Modular)

Example 4.16. D_4 (or even part of E_6) as small non-braided category. $Z(D_4) = A_5[x]Z/3$.

5. The Serre automorphism of a fusion category

5.1. The double dual is the Serre automorphism.

5.1.1. *3-framed 1-manifolds and the Serre automorphism.*

5.1.2. *Computing the Serre automorphism.*

Theorem 5.1. *Serre(C) = [**].*

5.2. The quadruple dual is trivial.

Lemma 5.2 (Bimodulification Lemma)**.**

Theorem 5.3. *If C is dualizable, that is fusion, then $* * ** = 1$.*

6. Pivotality as a descent condition

6.1. Fusion category TFTs are string.

6.2. **Pivotal fusion category TFTs are orpo.**

6.3. **Structure groups of fusion category TFTs.**

Theorem 6.1. *A fusion category is pivotal if and only if the associated TFT is orpo.*

In particular, the ENO conjecture can be reformulated as follows:

Conjecture 6.2. Every local framed topological field theory with values in tensor categories descends to an oriented p_1 field theory.

Conjecture 6.3. All TC-valued orpo TFTs are oriented.

Do we think we can prove this or should we leave it as a conjecture?

Sketch. Drinfeld centers of pivotal fusion categories are anomaly free modular (ref Mueger), therefore oriented 123; pushout to show oriented as 0123. □

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