3-dimensional topology and finite tensor categories

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Joint Meetings

Outline

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Radford's theorem for Hopf algebras

Theorem (Radford 75)

If H is a finite dimensional Hopf algebra and $g \in H$ and $\alpha \in H^*$ are the distinguished grouplike elements, then:

$$S^4(x) = g(\alpha \rightharpoonup x \leftharpoonup \alpha^{-1})g^{-1}.$$

Theorem (Larson-Radford 87-88)

If H is semisimple in characteristic 0, then $S^2 = \mathrm{id}$.

Radford's theorem for tensor categories

Theorem (ENO 04)

If $\mathcal{C}$ is a finite rigid tensor category and $D \in \mathcal{C}$ is the distinguished invertible object, then there's a canonical isomorphism of tensor functors

$$x^{****} \to D \otimes x \otimes D^{-1}.$$

Theorem (ENO 04)

If $\mathcal{C}$ is semisimple in characteristic 0, then $D \cong 1$.

Conjecture (ENO 02)

If $\mathcal{C}$ is semisimple then there's a monoidal isomorphism $x^{**} \to x$.

Goal of talk

Question:

Why is there a nice canonical formula for the quadruple dual? Why is the double dual more difficult?

Answer:

The double dual corresponds to the generator of $\pi_1(\mathrm{SO}_3) = \mathbb{Z}/2$

Technique:

Build a 3-dimensional fully local TFT. Joint work with Chris Douglas and Chris Schommer-Pries.

Outline

Topological Quantum Field Theories

Informal Idea

An *n*-dimensional TQFT is an invariant of *n*-manifolds which can be computed via cutting and pasting.

Formal Definition

A TQFT is a symmetric monoidal functor

$$\mathcal{F} : \mathrm{Bord}_n \to \mathrm{Vec}.$$

In more detail

- Closed $n-1$-manifolds are sent to vector spaces
- *n*-manifolds with boundaries are sent to linear maps
- Gluing goes to composition
- Disjoint union goes to tensor product

Local Topological Field Theories

Informal Idea

It would be even better if we could cut up along lower dimensional pieces, and best of all if we could cut things up all the way down to points.

Formal Definition

An n-dimensional fully local topological field theory with values in a symmetric monoidal n-category $\mathcal{C}$ is a symmetric monoidal functor:

$$\mathcal{F} : \mathrm{Bord}_n \to \mathcal{C}.$$

Topological structures

Flavors of TFT

TFTs come in several flavors based on what topological structures you consider:

- unoriented
- oriented
- spin
- framed (choice of trivialization of the tangent bundle)
- etc.

Lower dimensions

In order to glue structures, we need to pick a structure on small collars of boundaries.

For example, a 3-framing on a 1-manifold is a trivialization of $TM \oplus \mathbb{R}^2$.

Cobordism Hypothesis

Dualizable objects

A symmetric monoidal n-category is fully dualizable if every object has a dual and ever k-morphism for $1 \leq k < n$ has a left adjoint and a right adjoint.

Every symmetric monoidal n-category has a maximal fully dualizable subcategory $\mathcal{C}^{fd}$.

Theorem (Lurie(-Hopkins) 09, Baez-Dolan 95)

$$\mathrm{TFT}^{fr}(\mathcal{C}) \tilde{\to} \mathcal{C}^{fd}$$

as spaces via

$$\mathcal{F} \mapsto \mathcal{F}(\mathrm{pt}_+).$$

Where does the cobordism hypothesis come from?

Why is Bord_n fully dualizable?

- The dual of the positively framed point is the negatively framed point.
- The evaluation and coevaluation are given by 1-handles.
- The adjoints of the 1-handles are other 1-handles.
- The units and counits of these adjunctions are given by 2-handles.

Why framed?

The framing keeps track of the difference between left and right adjoints. Bord_n^{fr} is the universal fully dualizable category. Proof uses Igusa's version of generalized Morse theory.

Our main theorems

Theorem (DSPS)

Finite abelian tensor categories, finite bimodule categories, bimodule functors, and natural transformations form a symmetric monoidal 3-category $\mathcal{TC}^{fin}$ (using a language compatible with Lurie's proof).

Theorem (DSPS)

The rigid fully dualizable objects in $\mathcal{TC}^{fin}$ are exactly the separable (multi-)fusion categories. (Separability is a technical condition that is automatic in characteristic 0.)

Corollary

Any fusion category gives a 3-framed local 3-dimensional TFT. (Note there's no sphericality condition!)

What about finite rigid tensor categories?

What does the proof actually use?

- Objects automatically have duals (just reverse tensor product).
- Adjoints for 1-morphisms uses finiteness, rigidity, and the fact that the Deligne tensor product is a category of module functors (ENO 09).
- Adjoints for 2-morphisms uses the theory of exact module categories (EO 03). With rigidity, but without semisimplicity this shows that all but one kind of 2-handle has an adjoint.

What kind of local TFT do you get?

Finite rigid tensor categories give non-compact 3-framed local 3-dimensional field theories.
In particular, they give 3-framed local 2-dimensional field theory with values in the $(3, 2)$-category $\mathcal{TC}$.

Outline

The Serre automorphism

Framings on the interval

n-framings on the interval correspond to paths in the space of framed points, and hence to $\pi_1(\mathrm{SO}_n)$.

The loop bordism and Serre automorphism

The generator of this group is called the loop bordism. It is denoted by the following picture (the fuzz is a normal framing).

The Serre automorphism

If x is a fully dualizable object the Serre automorphism $\mathcal{S}_x : x \to x$ is the image of the loop bordism.

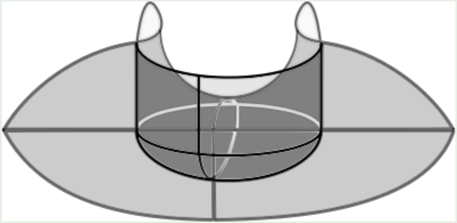
The belt bordism and Radford isomorphism

The square of the Serre

If $n > 2$, we have that $\mathcal{S}_x^2 \cong \mathrm{id}$.

The belt bordism and Radford isomorphism

In fact, we have an explicit distinguished trivialization of the square of the loop bordism, which we call the belt bordism. Its image under a TFT is the Radford isomorphism $\mathcal{R}_x$.



Application to tensor categories

Definition

Let ${}_{\langle \mathcal{F} \rangle}\mathcal{C}$ denote $\mathcal{C}$ as a $\mathcal{C}$–$\mathcal{C}$ bimodule where the action is

$$x \cdot c \cdot y = \mathcal{F}(x) \otimes c \otimes y.$$

Serre for $\mathcal{TC}$

If $\mathcal{C}$ is a finite rigid tensor category then

$$\mathcal{S}_{\mathcal{C}} \cong {}_{\langle {}^{**}(-) \rangle}\mathcal{C}.$$

Radford for $\mathcal{TC}$

We have $\mathcal{R}_{\mathcal{C}}(1) = D$, and the structure of $\mathcal{R}_{\mathcal{C}}$ as a bimodule functor gives a natural isomorphism of tensor functors in Radford's theorem.