Correction des exercices du TD3

21 mars 2020

Exercice 1

2B, 4N.

$X(\Omega) = \{2, \ldots, 5\}$

Soient :

B_i : "la ième boule tiré est blanche",

N_i : "la ième boule tiré est noir".

$$\begin{aligned}
\mathbb{P}(X = 2) &= \mathbb{P}(B_1 \cap B_2) \\
&= \mathbb{P}(B_1) \cdot \mathbb{P}(B_2|B_1) \\
&= \frac{2}{6} \cdot \frac{1}{5} = \frac{1}{15}
\end{aligned}$$

$$\begin{aligned}
\mathbb{P}(X = 3) &= \mathbb{P}(B_1 \cap N_2 \cap B_3) + \mathbb{P}(N_1 \cap B_2 \cap B_3) \\
&= \frac{8}{15}
\end{aligned}$$

On continue ainsi de suite et on obtient finalement

k	2	3	4	5
$\mathbb{P}(X = k)$	$\frac{1}{15}$	$\frac{2}{15}$	$\frac{4}{15}$	$\frac{8}{15}$

Exercice 2

$X(\Omega) = \{1, \ 2, \ n\}$

$\mathbb{P}(X = 3) = \mathbb{P}(A \cap B \cap C) = \frac{1}{4}$

$\mathbb{P}(X = 0) = \frac{1}{3}$
$\mathbb{P}(X = 1) = \frac{1}{4}$
$\mathbb{P}(X = 2) = \frac{1}{6}$
$\mathbb{E}(X) = \sum_{i=0}^{3} i \cdot \mathbb{P}(X = i) = \frac{4}{3}$

Exercice 3

X prend ses valeurs dans $\{0, \ldots, n\}$
Calcul de l'ésperance

$$\mathbb{E}(X) = \sum_{k=0}^{n} k\mathbb{P}(X = k)$$

$$= \sum_{i=1}^{n} \mathbb{E}(\mathbb{1}_{A_i}) \quad \text{par linéarité}$$

Posons $Y = \mathbb{1}_{A_i}$, on a alors $Y(\Omega) = \{0, \ 1\}$
$\mathbb{P}(Y = 1) = \mathbb{P}(A_i) = p_i$
$\mathbb{P}(Y = 0) = 1 - p_i$
$\mathbb{E}(\mathbb{1}_{\mathbb{A}}) = p_i$ Ainsi, $\mathbb{E}(X) = \sum_{i=1}^{n} p_i$.

Calcul de la variance
$\text{Var}(X) = (E(X^2)) - \mathbb{E}(X)^2$

$$\mathbb{E}(X^2) = \mathbb{E}[(\sum_{i=1}^{n} \mathbb{1}_{A_i})^2]$$

$$= \mathbb{E}[\sum_{i=1}^{n} \mathbb{1}_{A_i}^2 + 2 \sum_{1 \leq i < k \leq n} \mathbb{E}(\mathbb{1}_{A_i}\mathbb{1}_{A_j})]$$

$\mathbb{E}(\mathbb{1}_{A_i}^2) = 1^2\mathbb{P}(\mathbb{1}_{A_i} = 1) = p_i$
Posons $W = \mathbb{1}_{A_i}\mathbb{1}_{A_j}$, on a,

$$\mathbb{P}(W = 1) = P(\{\mathbb{1}_{A_i} = 1\} \cap \{\mathbb{1}_{A_j} = 1\})$$
$$= \mathbb{P}(A_i \cap A_j) = q_i j$$

Et $\mathbb{P}(W = 0) = 1 - q_i j$
Ainsi, $\mathbb{E}(W) = q_i j$, donc

$$\text{Var}(X) = \sum_{i=1}^{n} p_i + 2 \sum_{1 \leq i < j \leq n} q_i j - (\sum_{j=1}^{n} p_i)^2$$

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