Correction des exercices du TD5

20 mars 2020

Exercice 4

Question 1

Convergence simple :
$\forall n \in \mathbb{N},\ f_n(1) = 0$
$\forall x \neq 1$ clairement $\lim\limits_{n \to +\infty} f_n(x) = 0$
Convergence uniforme :
On suppose $\varepsilon < 1$
Si $1 - \varepsilon < x$, on a $\forall n \in \mathbb{N}$:

$$x^n(1-x) \leq 1 - x < \varepsilon$$

Si $x \leq 1 - \varepsilon$, on a :

$$x^n(1-x) \leq (1-\varepsilon)^n(1-x) \leq (1-\varepsilon)^n$$

Si on prend donc n_0 tel que $(1-\varepsilon)^{n_0} < \varepsilon$ on a : $x^n(1-x) < \varepsilon,\ \forall n \geq n_0$
Ainsi $\forall n \geq n_0,\ \forall x \in [0,1],\ f_n(x) < \varepsilon$

Question 2

$g_n(x) = x^n sin(\pi x) = x^n sin(\pi - \pi x) = x^n sin(\pi(1-x))$
De plus $\forall x \geq 0,\ sin(x) \leq x$ Donc :

$$\begin{aligned} & sin(\pi(1-x)) \leq \pi(1-x) \\ \Rightarrow\ & x^n sin(\pi(1-x)) \leq \pi x^n(1-x),\ \forall x \in [0,1] \\ \Rightarrow\ & 0 \leq g_n(x) \leq f_n(x) \forall x \in [0,1] \end{aligned}$$

On sait que $f_n \xrightarrow{u} 0$, alors
$\forall \varepsilon > 0 \exists n_0$ tel que $\forall n \geq n_0,\ \forall x |f_n(x)| < \varepsilon$ Ainsi $\forall x,\ \forall n \geq n_0$
$|g_n(x)| \leq |f_n(x)| < \varepsilon$ Donc $g_n \xrightarrow{u} 0$

Exercice 14

Question 1

Montrons que $\lim_{k \to +\infty} f_k(x_k) = f(x)$. Fixons $\varepsilon > 0$.

$$\begin{aligned}\forall k \in \mathbb{N}\ |f_k(x_k) - f(x)| &= |f_k(x_k) - f(x_k) + f(x_k) - f(x)| \\ &\leq |f_k(x_k) - f(x_k)| + |f(x_k) - f(x)|\end{aligned}$$