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On the Early Work of I. E. Segal

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Some of Irving Segal's earliest work was in the area of operator algebras and representations of locally compact groups. We sample these early publications of Segal's in the pages that follow.

A note [S1] communicated to the Proceedings of the National Academy of Sciences, USA (PNAS), in June of 1941, contains a definition of a "group ring" for a general locally compact group and the basic properties of this group ring. It is defined in terms of left convolution of L_1-functions on the group relative to right invariant Haar measure. Segal notes the semi-simplicity of the group ring for the case of a compact or locally compact abelian group and studies the maximal ideal structure of the group ring.

Of course, we know much more about these matters today (we did even at the end of the 1950s), but the technical aspects of the general measure theory on locally compact groups were not sufficiently understood at that time for *honest* proofs of the more general results to be given. Segal, himself, discovered and proved many of those technical results (see, for example, [S3]), as well as many of the more general results about the group ring (now called "group algebra").

One of Segal's main goals in the study of his group ring was to provide an appropriately general setting for the Wiener Tauberian theory, the theory of almost periodic functions, and harmonic analysis on locally compact groups. In a second PNAS note [S2], communicated in May of 1944, Segal develops some of the Tauberian theory, studying the span of the translates of an L_p-function in a general Lebesgue space and its relation to the vanishing of the (L_p) Fourier transform.

A later paper [S4], appearing in early 1947, gives a full account of the group ring, now the (complex) "group algebra," and its applications to Tauberian theory and the theory of almost periodic functions on a group. Again, in the applications, the main force appears in the case of locally compact abelian and compact groups. The connections with operator

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algebras begin to appear in this paper, as well as an awareness of the relation of these matters to the work of Murray and von Neumann (to quote footnote 15 on p. 77, "we have incidentally proved, clearly, that any self-adjoint algebra of bounded operators on a Hilbert space is weakly semi-simple").

In [S5], appearing in early 1947, Segal defines the C^*-algebra formally. He notes that "Our interest in operator algebras grew out of our investigation of group algebras of locally compact groups, a group algebra being isomorphic with a certain self-adjoint operator algebra." At the same time, his strong interest in quantum mechanics and its mathematical underpinnings begins to make itself evident. He notes, "Our results make possible a broader and more rigorous treatment of certain parts of quantum mechanics..." In this article, Segal defines the concepts of "state" and "pure state" on a C^*-algebra and describes his fundamental construction of the representation of a C^*-algebra associated with a state of the algebra. He points out that "a state ω of $\mathscr{A}$ is a normalizing function for some normal representation of $\mathscr{A}$, using a procedure due in part to Gelfand and Neumark." (See [G–N]. Some of this terminology, in the way Segal used it—*normalizing* and *normal*—has disappeared and changed.) This is the all-important "GNS construction" (Gelfand, Neumark, Segal), so basic to the subject of operator algebras.

In this same article, Segal defines and establishes the existence of *approximate identities* in C^*-algebras (without a unit—equally, in ideals of C^*-algebras). He also defines and establishes the existence of *pure states* on C^*-algebras and shows that the irreducible representations of C^*-algebras are precisely those coming from pure states. Using the Krein–Millman theorem and the (weak*) compactness of the set of states, Segal shows that there is a *separating* family of pure states of a C^*-algebra (only 0 is annihilated by all pure states). By associating a C^*-algebra with a locally compact group, a *C^*-group algebra of the group* (leaving aside some technical distinctions to be made here), Segal constructs a unitary representation of the group on a Hilbert space from each state of that group algebra—irreducible unitary representations from the pure states. With this construction, Segal then gives the most natural proof of the Gelfand–Raikov theorem [G–R] on the existence of a separating family of irreducible unitary representations of locally compact groups. The importance of what Segal achieved in this short (16-page) article is difficult to overstate! Some small indication of the stage of the subject of C^*-algebras at this point (its infancy) is contained in a footnote thanking "G. W. Mackey for the observation that a representation of a C^*-algebra is necessarily continuous."

In [S6], Segal presents "a set of postulates for a physical system and deduce[s] from these the main general features of the quantum theory of stationary states." He notes that "An aspect of our theory which is significant for general physics is the fact that a general indeterminancy principle follows from the postulates." In essence, Segal introduces and studies real Jordan Banach algebras, developing a spectral theory for the elements in the process. At the same time, he introduces the theory of states for such algebras. That area has grown into an active and interesting aspect of (mathematical) quantum measurement theory. (See, for example, [A–S, H–S, U].) Of course, the von Neumann article [N1] was an important precursor to Segal's paper (as Segal notes). Although there are clever technical devices developed and used by Segal in this article, it is more the focus that the investigation brings to quantum measurement theory that is important. The metric (norm), * algebra aspect of the structure, and, eventually, the Hilbert space representations are vital to the study.

The main result of [S7] settled a basic open question in the foundations of the theory of C^*-algebras. Segal proves that each norm-closed, two-sided ideal in such an algebra is closed under the * operation (that is, contains A^* when it contains A). It follows that the quotient of a C^*-algebra by such an ideal is a C^*-algebra. Though this is not explicitly noted, Segal's arguments lead, at once, to the fact that a left (or right) norm-closed ideal is generated (as a norm-closed ideal) by the cone of positive elements it contains. Those arguments involve some of the first and most basic techniques of non-commutative analysis. They provide the key technique for establishing the existence of approximate identities and approximate polar decompositions.

In [S8], Segal extends the non-commutative, measure-theoretic aspects of von Neumann's work in [N2] from factors to von Neumann algebras with arbitrary center. In some sense, it was work that had to be done, but contributed little in the way of ideas to what von Neumann had accomplished in [N2]. That project needed some-one with well-developed (general) measure-theoretic and operator-algebra skills to carry it out. Segal could supply these in abundance.

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