PRODUCT ISOMETRIES AND AUTOMORPHISMS OF THE CAR ALGEBRA

by

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I. INTRODUCTION.

The methods of multilinear algebra and, in particular, those of the exterior calculus provide a useful framework for studying the Fock space, $\mathcal{H}_{\mathcal{F}}$, of antisymmetrized wave functions and the Fock representation of the Canonical Anticommutation Relations (CAR) on it. With the aid of these methods, we study the structure of certain mappings, product isometries, of the n-particle subspace $\mathcal{H}_n$ of $\mathcal{H}_{\mathcal{F}}$.

With $\mathcal{H}$ a complex Hilbert space, we denote by $f_1 \wedge \cdots \wedge f_n$ the vector $(n!)^{-\frac{1}{2}}\Sigma_\sigma \chi(\sigma) f_{\sigma(1)} \otimes \cdots \otimes f_{\sigma(n)}$ in $\mathcal{H} \otimes \cdots \otimes \mathcal{H}$, the n-fold tensor product of $\mathcal{H}$ with itself, where σ is a permutation of $\{1,\ldots,n\}$ and $\chi(\sigma)$ is its sign. The space spanned by $\{f_1 \wedge \cdots \wedge f_n : f_j \in \mathcal{H}\}$ is denoted by $\mathcal{H}_n$; and $\mathcal{H}_{\mathcal{F}}$, (antisymmetric) Fock space, is $\Sigma_{n=0}^{\infty} \oplus \mathcal{H}_n$, where $\mathcal{H}_1$ is $\mathcal{H}$ and $\mathcal{H}_0$ is a 1-dimensional space generated by a unit vector Φ_0, the Fock vacuum. A linear isometry V of some subspace $\mathcal{K}_n$ $(=\{f_1 \wedge \cdots \wedge f_n : f_j \in \mathcal{K} \subseteq \mathcal{H}\})$ is said to be a product isometry when $V(f_1 \wedge \cdots \wedge f_n)$ is a product vector (i.e. of the form $g_1 \wedge \cdots \wedge g_m$) in $\mathcal{H}_{\mathcal{F}}$. One of our principal aims is the following result.

* With partial support of NSF.

THEOREM A. If $\hat{U}$ is a product unitary transformation of $\mathcal{H}_n$ onto $\mathcal{H}_n$, there is a unitary transformation U of $\mathcal{H}$ onto $\mathcal{H}$ such that $\hat{U}(f_1 \wedge \cdots \wedge f_n) = Uf_1 \wedge \cdots \wedge Uf_n$.

This is proved by a combinatorial-geometric study of the way in which $\hat{U}$ transforms the subspace $[f_1, \ldots, f_n]$ of $\mathcal{H}$ associated with $f_1 \wedge \cdots \wedge f_n$. We note that

$$\langle f_1 \wedge \cdots \wedge f_n | g_1 \wedge \cdots \wedge g_n \rangle = \det\, (\langle f_i | g_j \rangle) ,$$

and that $af_1 \wedge \cdots \wedge f_n = g_1 \wedge \cdots \wedge g_n \neq 0$ if and only if the spaces $[f_1 \wedge \cdots \wedge f_n]$ $(=[f_1, \ldots, f_n])$ and $[g_1, \ldots, g_n]$ associated with $f_1 \wedge \cdots \wedge f_n$ and $g_1 \wedge \cdots \wedge g_n$ are n-dimensional and coincide.

Applying Theorem A, one can, then, show that:

THEOREM B. If α is an automorphism of the CAR algebra $\mathfrak{A}$ whose transpose $\hat{\alpha}$ maps the set $\mathfrak{S}$ of pure, gauge-invariant, quasi-free states of $\mathfrak{A}$ onto itself, then there is a unitary operator U on $\mathcal{H}$ such that $\alpha(a(f)) = a(Uf)$, for all f in $\mathcal{H}$; or there is a conjugate-linear, unitary operator W on $\mathcal{H}$ such that $\alpha(a(f)) = a(Wf)^*$, for all f in $\mathcal{H}$, where $a(f)$ is the annihilator for a particle with wave function f.

Note that one can read from this result the fact that $\hat{\alpha}$ transforms the Fock (vacuum) state φ_0 either onto itself or onto the anti-Fock state φ_I; though this fact is established as a preliminary to proving Theorem B.

The <u>creator</u> $a(f)^*$, determined by :

$$a(f)^*(f_1 \wedge \cdots \wedge f_n) = f \wedge f_1 \wedge \cdots \wedge f_n ,$$

is the adjoint of the <u>annihilator</u> $a(f)$ determined by:

$$a(f)(f_1 \wedge \cdots \wedge f_n) = \sum_{j=1}^{n} (-1)^{j+1} \langle f | f_j \rangle\, f_1 \wedge \cdots \wedge f_{j-1} \wedge f_{j+1} \wedge \cdots \wedge f_n .$$

The mapping $f \to a(f)$ is conjugate linear (the inner product on $\mathcal{H}$ being linear in its second argument) and satisfies $a(f)a(g) + a(g)a(f) = 0$, $a(f)a(g)^* + a(g)^*a(f) = \langle f|g\rangle I$ (the CAR). A conjugate-linear mapping of $\mathcal{H}$ onto operators $a(f)$ on a Hilbert space $\mathcal{K}$ satisfying the CAR is said to be a representation of the CAR (over $\mathcal{H}$ on $\mathcal{K}$). The representing operators $a(f)$ are partial isometries (with initial and final spaces orthogonal having sum $\mathcal{K}$), hence, bounded. The particular representation of the CAR on $\mathcal{H}_{\mathcal{F}}$ which we have described is the Fock representation. The C*-algebra $\mathfrak{A}$, generated by $\{a(f), a(f)^* : f \in \mathcal{H}\}$, is the CAR algebra. Its representations are in one-one correspondence with those of the CAR.

If A is an operator on $\mathcal{H}$ such that $0 \le A \le I$, defining $\varphi_A(a(f_n)^* \ldots a(f_1)^* a(g_1) \ldots a(g_n)^*)$ to be $\det(\langle g_i|Af_j\rangle)$ $(= \langle g_1 \wedge \ldots \wedge g_n|Af_1 \wedge \ldots \wedge Af_n\rangle)$ determines a state φ_A of $\mathfrak{A}$, the gauge-invariant, quasi-free state (with one-particle operator A). The state φ_E is pure if and only if E is a projection on $\mathcal{H}$; and φ_E is equivalent to the Fock state φ_0 if and only if E is a projection with finite-dimensional range. In case $E(\mathcal{H})$ is finite-dimensional and $\{e_1, \ldots, e_n\}$ is an orthonormal basis for it, we have $\varphi_E(A) = \langle e_1 \wedge \ldots \wedge e_n|A(e_1 \wedge \ldots \wedge e_n)\rangle = \omega_{e_1 \wedge \ldots \wedge e_n}(A)$, for each A in $\mathfrak{A}$. The intimate relation between Theorems A and B is a consequence of these last comments, for, then, a unitary operator on $\mathcal{H}_{\mathcal{F}}$ implementing α transforms product vectors into product vectors.

The work on which these notes are based is a joint study conducted with N M Hugenholtz. An extended account is to be found in: "Automorphisms and Quasi-free States of the CAR Algebra", Commun. Math. Phys. 43 (1975), pp 181-197. The author acknowledges with gratitude the support of the SRC and the hospitality of the Department of Mathematics at the University of Newcastle, as well as the support and hospitality of the Centre Universitaire de Luminy-Marseille, where parts of this work were completed.

II. PRODUCT UNITARIES.

If V is a product isometry of an infinite-dimensional subspace $\mathcal{K}$ of $\mathcal{H}$ $(=\mathcal{H}_1)$ into $\mathcal{H}_m$, then, with $\{e_j\}$ an orthonormal basis for $\mathcal{K}$, the fact that Ve_j and Ve_k are orthogonal product vectors in $\mathcal{H}_m$ and $Ve_j + Ve_k$ $(= V(e_j + e_k))$ is also a product vector in $\mathcal{H}_m$ leads to the conclusion that the projections E_j and E_k with the m-dimensional ranges $[Ve_j]$ and $[Ve_k]$ commute and $[Ve_j] \cap [Ve_k]$ has dimension m-1. It follows that $\bigcap_j E_j(\mathcal{H})$ has dimension m-1; and, hence, $\cap[Vx]$ has dimension m-1.

If W is a product isometry of $\mathcal{K}_n$ into $\mathcal{H}_{\mathcal{F}}$, isometry considerations show that W has range in one $\mathcal{H}_m$. If $\mathcal{K}$ is infinite dimensional and $n \le m$ then $[W(f_1 \wedge \dots \wedge f_n)] \cap [W(g_1 \wedge \dots \wedge g_n)]$ has dimension at least m-n. To see this, we may assume that each of $\{f_1,\dots,f_n\}$ and $\{g_1,\dots,g_n\}$ are orthonormal sets and make use of the mapping

$f \to W(f \wedge f_2 \wedge \dots \wedge f_n)$, which is a product isometry of $\mathcal{K} \ominus [f_2, \dots, f_n]$ into $\mathcal{H}_m$. From the preceding, it follows that $[W(h_1 \wedge f_2 \wedge \dots \wedge f_n)]$ and $[W(f_1 \wedge \dots \wedge f_n)]$ have an intersection of dimension at least m-1, where h_1 is a vector in $[g_1, \dots, g_n]$ orthogonal to $[f_2 \wedge \dots \wedge f_n]$. In the same way, choosing h_2 in $[g_1, \dots, g_n]$ orthogonal to $[h_1, f_3, \dots, f_n]$, we see that $[W(h_1 \wedge h_2 \wedge f_3 \wedge \dots \wedge f_n)] \cap [W(f_1 \wedge \dots \wedge f_n)]$ has dimension at least m-2. Continuing, we have that $[W(h_1 \wedge \dots \wedge h_n)] \cap [W(f_1 \wedge \dots \wedge f_n)]$ has dimension at least m-n, and $h_1 \wedge \dots \wedge h_n = cg_1 \wedge \dots \wedge g_n$.

If $\{e_j\}$ is an orthonormal basis for $\mathcal{K}$ and $i_1, \dots, i_n, j_i, \dots, j_n$ is such that $[W(e_{i_1} \wedge \dots \wedge e_{i_n})]$ and $[W(e_{j_1} \wedge \dots \wedge e_{j_n})]$ have intersection of dimension precisely m-n then $\bigcap_{x_1, \dots, x_n} [W(x_1 \wedge \dots \wedge x_n)]$ has dimension m-n. This amounts to showing that

$$[W(e_{i_1} \wedge \dots \wedge e_{i_n})] \cap [W(e_{j_1} \wedge \dots \wedge e_{j_n})] \subseteq [W(e_{k_1} \wedge \dots \wedge e_{k_n})] \qquad (*)$$

for all $k_1, \dots, k_n$. This is effected by arguing inductively on n - the conclusion of the preceding argument allowing us to carry the hypothesis of an intersection having dimension m-n to one where the intersection has dimension m-(n-1). For this purpose, we use the mapping

$$x_1 \wedge \dots \wedge x_{r-1} \wedge x_{r+1} \wedge \dots \wedge x_n \to W(x_1 \wedge \dots \wedge x_{r-1} \wedge e_{i_r} \wedge x_{r+1} \wedge \dots \wedge x_n)$$

of $(\mathcal{K} \ominus [e_{i_r}])_{n-1}$ into $\mathcal{H}_m$. As an intermediate conclusion, we obtain (*) if at least one of $k_1, \dots, k_n$ is in $\{i_1, \dots, i_n, j_1, \dots, j_n\}$. If we know that the intersection, M, of $[W(e_{i_1} \wedge \dots \wedge e_{i_n})]$ and $[W(e_{j_1} \wedge \dots \wedge e_{j_{r-1}} \wedge e_t \wedge e_{j_{r+1}} \wedge \dots \wedge e_{j_n})]$ has dimension m-n, when $t \notin \{i_1, \dots, i_n\}$; then, if $t = k_r \notin \{i_1, \dots, i_n\}$, we have, from our intermediate conclusion, that $[W(e_{k_1} \wedge \dots \wedge e_{k_n})]$ contains M while M contains $[W(e_{i_1} \wedge \dots \wedge e_{i_n})] \cap [W(e_{j_1} \wedge \dots \wedge e_{j_n})]$.

To see that M has dimension m-n, note that $\bigcap_t [W(e_{j_1} \wedge \cdots \wedge e_{j_{r-1}} \wedge e_t \wedge e_{j_{r+1}} \wedge \cdots \wedge e_{j_n})]$ (=N) has dimension m-1, from our initial observation. Thus each $[W(e_{j_1} \wedge \cdots \wedge e_{j_{r-1}} \wedge e_t \wedge e_{j_{r+1}} \wedge \cdots \wedge e_{j_n})]$ is generated by N and a vector g_t orthogonal to it. Since $\{W(e_{j_1} \wedge \cdots \wedge e_{j_{r-1}} \wedge e_t \wedge e_{j_{r-1}} \wedge \cdots \wedge e_{j_n})\}$ is a family of orthogonal vectors, $\{g_t\}$ is such a family; and no $[W(e_{j_1} \wedge \cdots \wedge e_{j_{r-1}} \wedge e_t \wedge e_{j_{r+1}} \wedge \cdots \wedge e_{j_n})]$ is contained in a union of the others. If M has dimension greater than m-n it has a vector orthogonal to $[W(e_{i_1} \wedge \cdots \wedge e_{i_n})] \cap [W(e_{j_1} \wedge \cdots \wedge e_{j_n})]$ as do the intersections of $[W(e_{i_1} \wedge \cdots \wedge e_{i_n})]$ and $[W(e_{j_1} \wedge \cdots \wedge e_{j_{r-1}} \wedge e_{i_s} \wedge e_{j_{r+1}} \wedge \cdots \wedge e_{j_n})]$. Each of these n+1 vectors (taking s to be 1, ..., n) are not in the m 1 dimensional space N, for, otherwise, they are in $[W(e_{j_1} \wedge \cdots \wedge e_{j_n})]$, hence in $[W(e_{i_1} \wedge \cdots \wedge e_{i_n})] \cap [W(e_{j_1} \wedge \cdots \wedge e_{j_n})]$, contrary to choice. Thus each of these vectors generates with N its corresponding m-dimensional space $[W(e_{j_1} \wedge \cdots \wedge e_{j_{r-1}} \wedge e_{i_s} \wedge e_{j_{r+1}} \wedge \cdots \wedge e_{j_n})]$ (or $[W(e_{j_1} \wedge \cdots \wedge e_{j_{r-1}} \wedge e_{k_n} \wedge e_{j_{r+1}} \wedge \cdots \wedge e_{j_n})]$). A linear relation among these vectors would imply, therefore, that one of these spaces is contained in the union of the others - contrary to what we have noted. But these n+1 vectors and the m-n-dimensional space $[W(e_{i_1} \wedge \cdots \wedge e_{i_n})] \cap [W(e_{j_1} \wedge \cdots \wedge e_{j_n})]$ orthogonal to them are all contained in $[W(e_{i_1} \wedge \cdots \wedge e_{i_n})]$, an m dimensional space. Thus there must be a linear relation among these n+1 vectors. From this contradiction, we conclude that M has dimension m-n.

Summarizing, to this point, we have proved:

Proposition C. If W is a product isometry of $\mathcal{K}_n$ into $\mathcal{H}_m$, where $n \leq m$ and $\mathcal{K}$ is an infinite-dimensional subspace of $\mathcal{H}$, and the intersection of $[W(e_{i_1} \wedge \cdots \wedge e_{i_n})]$ and $[W(e_{j_1} \wedge \cdots \wedge e_{j_n})]$ has dimension $m-n$, for some $e_{i_1}, \ldots, e_{i_n}, e_{j_1}, \ldots, e_{j_n}$, then $\bigcap_{x_1,\ldots,x_n} [W(x_1 \wedge \cdots \wedge x_n)]$ has dimension $m-n$.

It follows, without difficulty, from this proposition, that a product unitary defined on $\mathcal{H}_{\mathcal{F}}$ maps each $\mathcal{H}_n$ onto $\mathcal{H}_n$. If $\hat{U}$ is a product unitary on $\mathcal{H}_n$ then $[x_1, \ldots, x_n] \cap \cdots \cap [z_1, \ldots, z_n]$ and $[\hat{U}(x_1 \wedge \cdots \wedge x_n)] \cap \cdots \cap [\hat{U}(z_1 \wedge \cdots \wedge z_n)]$ have the same dimension (for infinite, as well as, finite intersections). In particular, $\bigcap_{x_1,\ldots,x_{n-1}} [\hat{U}(x_1 \wedge \cdots \wedge x_{n-1} \wedge e)]$ has dimension 1 for each unit vector e in $\mathcal{H}$. If $\{e_j\}$ is an orthonormal basis for $\mathcal{H}$ and f_j' is a unit vector in $\bigcap[\hat{U}(x_1 \wedge \cdots \wedge x_{n-1} \wedge e_j)]$ then $\{f_j'\}$ is an orthonormal basis for $\mathcal{H}$. To see this, we note $f_j' \in M_k$ when $k \neq j$, where $M_k = [\hat{U}(e_1 \wedge \cdots \wedge e_{k-1} \wedge e_{k+1} \wedge \cdots \wedge e_{n+1})]$; and f_j' is orthogonal to M_j. From Proposition C and the consequences noted following it, $M_1 \vee M_2$ is an $n+1$-dimensional space containing each (n-dimensional) M_j, $j = 1, \ldots, n+1$. Thus $f_1', \ldots, f_{n+1}'$ is an orthonormal basis for $M_1 \vee M_2$.

It follows, now, that $\hat{U}(e_{i_1} \wedge \cdots \wedge e_{i_n}) = c_{i_1 \ldots i_n} f_{i_1}' \wedge \cdots \wedge f_{i_n}'$; where $|c_{i_1 \ldots i_n}| = 1$. Writing c_j' for $c_{1\ldots j-1 j+1 \ldots n+1}$, c_j for $\overline{c_j'} \prod_{k=1}^{n+1} c_k'$, and f_j for $c_j f_j'$, we have

$$\hat{U}(e_1 \wedge \cdots \wedge e_{j-1} \wedge e_{j+1} \wedge \cdots \wedge e_{n+1}) = f_1 \wedge \cdots \wedge f_{j-1} \wedge f_{j+1} \wedge \cdots \wedge f_{n+1},$$

for $j = 1, \ldots, n+1$. Using the fact that $\hat{U}$ is a product unitary on $\mathcal{H}_n$, it follows that $\hat{U}(e_1 \wedge \ldots \wedge e_{j-1} \wedge e_{j+1} \wedge \ldots \wedge e_{n+1} \wedge e_{n+2}) = cf_1 \wedge \ldots \wedge f_{j-1} \wedge f_{j+1} \wedge \ldots \wedge f_{n+1} \wedge f'_{n+2}$, where the phase factor c is the same for $j = 1, \ldots, n+1$. Taking f_{n+2} to be cf'_{n+2}, we construct, inductively, an orthonormal basis $\{f_j\}$ for $\mathcal{H}$ such that $\hat{U}(e_{i_1} \wedge \ldots \wedge e_{i_n}) = f_{i_1} \wedge \ldots \wedge f_{i_n}$ for all $i_1, \ldots, i_n$. Theorem A results from letting U be the unitary operator on $\mathcal{H}$ determined by: $Ue_j = f_j$.

III. THE AUTOMORPHISM.

Suppose, now, that $\hat{U}$ is a product unitary on $\mathcal{H}_{\mathcal{F}}$ that induces an automorphism of $\mathfrak{A}$. Then $\hat{U}$ maps $\mathcal{H}_n$ onto $\mathcal{H}_n$ for each n; and there is a unitary operator U_n on $\mathcal{H}$ such that $\hat{U}(x_1 \wedge \ldots \wedge x_n) = U_n x_1 \wedge \ldots \wedge U_n x_n$. It follows that $\hat{U}a(f)a(f)^*\hat{U}^*$ and $a(U_n f)a(U_n f)^*$ have the same restriction to $\mathcal{H}_n$, for each f in $\mathcal{H}$. A calculation (see Appendix I) shows that $U_n f = c_{fnm} U_m f$, for all f, n and m, in this case. An (easy) algebraic lemma (see Appendix III) allows us to conclude that $U_n = c_{nm} U_m$. Hence,

$$\hat{U}a(f)^*\hat{U}^*(g_1 \wedge \ldots \wedge g_n) = \hat{U}(f \wedge U_n^* g_1 \wedge \ldots \wedge U_n^* g_n) = U_m f \wedge U_m U_n^* g_1 \wedge \ldots \wedge U_m U_n^* g_n$$

$$= c_{mn}^n c_{m1} U_1 f \wedge g_1 \wedge \ldots \wedge g_n = c_n a(U_1 f)^*(g_1 \wedge \ldots \wedge g_n)$$

so that $\hat{U}a(f)^*\hat{U}^*$ and $c_n a(U_1 f)^*$ have the same restriction to $\mathcal{H}_n$. Another calculation (see Appendix II) shows that $\hat{U}a(f)^*\hat{U}^* = c_f a(U_1 f)^*$. Applying our algebraic lemma, again, $\hat{U}a(f)^*\hat{U}^* = ca(U_1 f)^*$ (on $\mathcal{H}_{\mathcal{F}}$, the phase factor c is no longer dependent on f). Finally, as

$$\hat{U}a(f)^*\hat{U}^*\Phi_0 = \hat{U}f = U_1 f = a(U_1 f)^*\Phi_0 ,$$

$c = 1$; and $\hat{U}a(f)^*\hat{U}^* = a(U_1 f)^*$, for each f in $\mathcal{H}$.

Writing U for U_1, it follows that

$\hat{U}a(f_1)^* \ldots a(f_n)^*\hat{U}^*\Phi_0 = \hat{U}(f_1 \wedge \ldots \wedge f_n) = a(Uf_1)^* \ldots a(Uf_n)^*\Phi_0 = Uf_1 \wedge \ldots \wedge Uf$

for all $f_1, \ldots, f_n$ in $\mathcal{H}$ and all n.

If α is an automorphism of $\mathfrak{A}$ such that $\hat{\alpha}(\mathfrak{S}) = \mathfrak{S}$ and $\hat{\alpha}(\varphi_0) = \varphi_0$, then α is implemented by a unitary operator $\hat{U}$ on $\mathcal{H}_{\mathcal{F}}$. Since the states in $\mathfrak{S}$ equivalent to φ_0 are precisely those vector states of $\mathfrak{A}$ corresponding to product vectors;

$$\hat{\alpha}(\omega_{f_1 \wedge \ldots \wedge f_n}) = \omega_{\hat{U}(f_1 \wedge \ldots \wedge f_n)}|\mathfrak{A} = \omega_{g_1 \wedge \ldots \wedge g_n}|\mathfrak{A} .$$

Since $\mathfrak{A}$ acts irreducibly on $\mathcal{H}_{\mathcal{F}}$; we conclude that $\hat{U}(f_1 \wedge \ldots \wedge f_n)$ is a product vector (a scalar multiple of $g_1 \wedge \ldots \wedge g_n$). From the preceding section, $\hat{U}a(f)\hat{U}^* = a(Uf)$, where U (a unitary operator on $\mathcal{H}$) is the restriction of $\hat{U}$ to $\mathcal{H}$.

If $\hat{\alpha}(\varphi_0) = \varphi_I$, V is a conjugate linear unitary operator on $\mathcal{H}$, and σ is the automorphism of $\mathfrak{A}$ determined by $\sigma(a(f)) = a(Vf)^*$, then $\widehat{\alpha\circ\sigma}(\varphi_0) = \varphi_0$ and $\widehat{\alpha\circ\sigma}(\mathfrak{S}) = \mathfrak{S}$. Hence there is a unitary operator U on $\mathcal{H}$ such that

$$a(Wf)^* = a(UV^{-1}f)^* = (\alpha\circ\sigma)(a(V^{-1}f)^*) = \alpha(a(f)) ,$$

where W is the conjugate-linear unitary operator UV^{-1} on $\mathcal{H}$.

Theorem B follows once we show that $\hat{\alpha}$ maps φ_0 onto φ_0 or φ_I. In any event, $\hat{\alpha}(\varphi_0) \in \mathfrak{S}$; so that $\hat{\alpha}(\varphi_0) = \varphi_{E_1}$ for some projection E_1 on $\mathcal{H}$. We derive a contradiction from the assumption that E_1 is neither 0 nor I. With this assumption, there are unit vectors f_1 and f_2 in and orthogonal to $E_1(\mathcal{H})$. If $f_3 = \frac{1}{\sqrt{2}}(f_1+f_2)$ and E_0, E_2, E_3 are the projections with ranges $E_1(\mathcal{H}) \ominus [f_1]$, $E_0(\mathcal{H}) \oplus [f_2]$, $E_0(\mathcal{H}) \oplus [f_3]$, then $\varphi_{E_0}(A) = \varphi_{E_j}(a(f_j)^*Aa(f_j))$, $j = 1, 2, 3$. Since the φ_{E_j} are equivalent, the states $\hat{\alpha}^{-1}(\varphi_{E_j})$ are in $\mathfrak{S}$ and equivalent to $\varphi_0(=\hat{\alpha}^{-1}(\varphi_{E_1}))$.

Thus $\hat{\alpha}^{-1}(\varphi_{E_j}) = \omega_{g_j}|\mathfrak{A}$, where g_j is a unit product vector in $\mathcal{H}_{\mathcal{F}}$. Now,

$$\varphi_{E_j}(a(f_j)^*\alpha^{-1}(A)a(f_j)) = \varphi_{E_j}(\alpha^{-1}(\alpha(a(f_j)^*)A\alpha(a(f_j)))) = \omega_{\alpha(a(f_j))g_j}(A)$$

$$= \hat{\alpha}^{-1}(\varphi_{E_0})(A) = \psi_{g_0}(A) ,$$

for $j = 1, 2, 3$. Since $\mathfrak{A}$ acts irreducibly on $\mathcal{H}_{\mathcal{F}}$, $\alpha(a(f_j))g_j = c_j g_0$, where $|c_j| = 1$. As $a(f_j)$ is a partial isometry and $\|\alpha(a(f_j))g_j\| = \|c_j g_0\| = 1$; $\alpha(a(f_j))$ is a partial isometry with g_j in its initial space and g_0 in its final space. Thus $g_3 = \frac{c_3}{\sqrt{2}}[\alpha(a(f_1)^*)g_0 + \alpha(a(f_2)^*)g_0] = \frac{c_3}{\sqrt{2}}[\bar{c}_1 g_1 + \bar{c}_2 g_2]$. But $g_1 = \Phi_0$, and g_2, g_3 are product vectors distinct from, hence orthogonal to, Φ_0. Hence $0 = \langle\Phi_0|g_3\rangle = \frac{c_3\bar{c}_1}{\sqrt{2}}$, a contradiction. Thus $\hat{\alpha}(\varphi_0)$ is either φ_0 or φ_I.

APPENDIX I

Lemma. If $\mathfrak{A}$ is the CAR algebra in its Fock representation on Fock space $\mathcal{H}_{\mathcal{F}}$ and A is an operator in $\mathfrak{A}$ such that $A|\mathcal{H}_n = a(f_n)a(f_n)^*|\mathcal{H}_n$ for each n, then $A = a(f_1)a(f_1)^*$.

Proof. We shall show that $f_n = cf_m$ with $|c| = 1$; so that $a(f_n)a(f_n)^* = a(f_m)a(f_m)^*$ and $A = a(f_1)a(f_1)^*$. Suppose we have established that $f_0, f_1, \ldots, f_{m-1}$ differ from each other by scalar multiples of modulus 1. If $B = \Sigma c_{i_1 \ldots i_p; j_1 \ldots j_q} a(e_{i_1})^* \ldots a(e_{i_p})^* a(e_{j_1}) \ldots a(e_{j_q})$ and r is an integer larger than all the indices occurring in this sum, where $\{e_j\}$ is an orthonormal basis for $\mathcal{H}$ such that $f_1 = \|f_1\|e_1$ and $f_m \in [e_1, e_2]$, then

$$\|A-B\|^2 \geq \|(A-B)(e_1 \wedge e_{r+2} \wedge \cdots \wedge e_{r+m})\|^2 \geq |\,|\langle e_2|f_m\rangle|^2 - (c_{0;0} + c_{1;1})|^2 .$$

and

$$\|A-B\|^2 \geq \|(A-B)(e_1 \wedge e_{r+2} \wedge \cdots \wedge e_{r+n})\|^2 \geq |c_{0;0} + c_{1;1}|^2 ,$$

when $n < m$; so that $|\langle e_2|f_m\rangle| \leq 2\|A-B\|$, for each such B; and $\langle e_2|f_m\rangle = 0$. Since $f_m \in [e_1, e_2]$, $f_m = ae_1 = a\|f_1\|^{-1}f_1$. Moreover

$$\|A-B\|^2 \geq \|(A-B)(e_2 \wedge e_{r+2} \wedge \cdots \wedge e_{r+m})\|^2 \geq \left| |\langle e_1|f_m\rangle|^2 - (c_{0;0} + c_{2;2})\right|^2$$

and

$$\|A-B\|^2 \geq \|(A-B)(e_2 \wedge e_{r+2} \wedge \cdots \wedge e_{r+n})\|^2 \geq \left| \|f_1\|^2 - (c_{0;0} + c_{2;2})\right|^2$$

when $n < m$; so that $\left| \|f_1\|^2 - |\langle e_1|f_m\rangle|^2 \right| = \left| \|f_1\|^2 - |a|^2 \right| \leq 2\|A-B\|$; and $|a| = \|f_1\|$. Thus $f_m = cf_1$, where $|c| = |a\|f_1\|^{-1}| = 1$.

APPENDIX II

Lemma. If $\mathfrak{A}$ is the CAR algebra in its Fock representation on Fock space $\mathcal{H}_{\mathcal{F}}$ and A is an operator in $\mathfrak{A}$ such that $A|\mathcal{H}_n = a(f_n)^*|\mathcal{H}_n$ for each n, then all f_n are equal (to f) and $A = a(f)^*$.

<u>Proof</u>. Suppose we have proved that $f_0 = f_1 = f_2 = \cdots = f_{m-1}$ $(=f)$. Let $\{e_j\}$ be an orthonormal basis for $\mathcal{H}$ such that $\|f\|^{-1}f = e_1$ and $f_m \in [e_1, e_2]$ (so that $f_m = \langle e_1|f_m\rangle e_1 + \langle e_2|f_m\rangle e_2$). If $B = \Sigma c_{i_1\ldots i_p, j_1\ldots j_q} a(e_{i_1})^* \ldots a(e_{i_p})^* a(e_{j_1}) \ldots a(e_{j_q})$ and r is an integer larger than any of the subscripts appearing in this finite sum then:

$$\|A-B\|^2 \geq \|(A-B)(e_{r+1} \wedge \cdots \wedge e_{r+n})\|^2 = |c_{0;0}|^2 + |\|f\| - c_{1;0}|^2$$

$$\sum_{\{i_1\ldots i_p\} \neq \{1\}} |c_{i_1\ldots i_p;0}|^2$$

when $n < m$, and where a subscript '0' before the semicolon refers to the absence of creators and after the semicolon refers to the absence of annihilators ($c_{0;0}$ is the coefficient of I in the sum for B). We have, too,

$$\|A-B\|^2 \geq \|(A-B)(e_{r+1} \wedge \cdots \wedge e_{r+m})\|^2 = |c_{0;0}|^2 + |\langle e_1|f_m\rangle - c_{1;0}|^2$$

$$+ |\langle e_2|f_m\rangle - c_{2;0}|^2 + \sum_{\{i_1\ldots i_p\}\neq\{1\},\{2\}} |c_{i_1\ldots i_p;0}|^2 .$$

Thus

$$|\langle e_1|f_m\rangle - \|f\|| \leq 2\|A-B\|$$

and

$$|\langle e_2|f_m\rangle| \leq 2\|A-B\| .$$

Since B may be chosen so that $\|A-B\|$ is arbitrarily small, $\langle e_2|f_m\rangle = 0$. As $f_m \in [e_1, e_2]$, $f_m = ae_1$. In addition, $\|f\| = \langle e_1|f_m\rangle = a$. Thus $f_m = \|f\|e_1 = f$; and $A = a(f)^*$.

APPENDIX III

Proposition. If $\mathcal{V}$ and $\mathcal{W}$ are vector spaces, A and B are linear transformations of $\mathcal{V}$ into $\mathcal{W}$ such that for each v in $\mathcal{V}$ there is a scalar c_v for which $Bv = c_v Av$; then $B = cA$ for some scalar c.

Proof. Let $\mathcal{N}$ be the null space of A. From the hypothesized relation between A and B, $\mathcal{N}$ is contained in the null space of B. Thus A and B induce linear transformations $\bar{A}$ and $\bar{B}$ of the quotient space $\bar{\mathcal{V}}$ of $\mathcal{V}$ by $\mathcal{N}$ into $\mathcal{W}$ such that $A = \bar{A}\circ\eta$ and $B = \bar{B}\circ\eta$, where η is the quotient mapping of $\mathcal{V}$ onto $\overline{\mathcal{V}}$. With v_0 in $\mathcal{N}$, $Bv_0 = c_{v_0} Av_0 = 0$; so that we may assume that $c_{v_0} = 0$ when $v_0 \in \mathcal{N}$. With this assumption, if $v \in \mathcal{V}$ and $v_0 \in \mathcal{N}$, then $B(v+v_0) = c_{v+v_0} A(v+v_0) = c_{v+v_0} Av = Bv = c_v Av$. If $v \notin \mathcal{N}$ then $Av \neq 0$ so that $c_v = c_{v+v_0}$. If $v \in \mathcal{N}$ then $v + v_0 \in \mathcal{N}$ and $c_v = c_{v+v_0} = 0$. Thus, defining $c_{\bar{v}}$ to be c_v, for $\bar{v}$ in $\overline{\mathcal{V}}$, where $\bar{v} = v + \mathcal{N}$, we have $\bar{B}\bar{v} = Bv = c_v Av = c_{\bar{v}}\bar{A}\bar{v}$. Note that the null space of $\bar{A}$ in $\overline{\mathcal{V}}$ is (0). If we show that $\bar{B} = c\bar{A}$, for some scalar c then $Bv = \bar{B}\bar{v} = c\bar{A}\bar{v} = cAv$, for all v in $\mathcal{V}$, so that $B = cA$. We may assume, from this discussion, that $\mathcal{N} = (0)$. With v and v' in $\mathcal{V}$, we have $B(v+v') = c_{v+v'}A(v+v') = c_{v+v'}Av + c_{v+v'}Av' = Bv + Bv' = c_vAv + c_{v'}Av'$. Thus $(c_{v+v'} - c_v)Av = (c_{v'} - c_{v+v'})Av'$; and $c_v = c_{v+v'} = c_{v'}$, when v and v' are linearly independent. Let $\{v_a\}$ be a linear basis for $\mathcal{V}$. Then $Bv_a = cAv_a$ for all a, where $c = c_{v_a}$ for all a. Thus $B = cA$.

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