C^*-algebras, von Neumann algebras and Comparison Theory of Projections

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Overview

1. Gelfand-Neumark theorem
2. GNS construction
3. Double commutant theorem
4. Kaplansky density theorem
5. Polar decomposition theorem
6. Comparison theorem
7. Type decomposition

C^*-algebras

Definition 1: A C^*-**algebra** is a Banach algebra $\mathfrak{A}$ equipped with an involution i.e. a self-map $A \to A^*$ $(A, A^* \in \mathfrak{A})$, satisfying :

- $(\alpha A + B)^* = \bar{\alpha} A^* + B^*$
- $(AB)^* = B^* A^*$
- $(A^*)^* = A$
- $\|A\|^2 = \|A^* A\|$

$\forall A, B \in \mathfrak{A}, \alpha \in \mathbb{C}$.

Definition 2 : A norm-closed self-adjoint subalgebra of $\mathcal{B}(\mathscr{H})$ (with or without I) is called a C^*-**algebra**.

Continuous functional calculus for self-adjoint elements

If $A \in \mathfrak{A}$ is normal i.e. $AA^* = A^*A$, then $r(A) = \|A\|$.

Theorem 1

If A is a self-adjoint element of a C^*-algebra $\mathfrak{A}$, there is a unique continuous mapping $f \to f(A) : C(sp(A)) \to \mathfrak{A}$ such that $f(A)$ has its elementary meaning when f is a polynomial.

One may deduce from the above theorem that, if $\mathfrak{A}$ is a C^*-algebra, $\mathscr{B}$ is a C^*-subalgebra of $\mathfrak{A}$, and $B \in \mathscr{B}$, then $sp_{\mathfrak{A}}(B) = sp_{\mathscr{B}}(B)$.

Theorem 2

Each element A of a C^*-algebra $\mathfrak{A}$ is a finite linear combination of unitary elements of $\mathfrak{A}$.

Theorem 3

Suppose that $\mathfrak{A}$ and $\mathscr{B}$ are C^*-algebras and ϕ is a $*$-homomorphism from $\mathfrak{A}$ into $\mathscr{B}$.

(i) *For each A in $\mathfrak{A}$, $sp(\phi(A)) \subseteq sp(A)$ and $\|\phi(A)\| \leq \|A\|$; in particular, ϕ is continuous.*

(ii) *If A is a self-adjoint element of $\mathfrak{A}$ and $f \in C(sp(A))$, then $\phi(f(A)) = f(\phi(A))$.*

(iii) *If ϕ is a $*$-isomorphism, then $\|\phi(A)\| = \|A\|$ and $sp(\phi(A)) = sp(A)$ for each A in $\mathfrak{A}$, and $\phi(\mathfrak{A})$ is a C^*-subalgebra of $\mathscr{B}$.*

Definition: A representation of a unital C^*-algebra $\mathfrak{A}$ on a Hilbert space $\mathscr{H}$ is a $*$-homomorphism $\pi : \mathfrak{A} \to \mathcal{B}(\mathscr{H})$.

The above theorem tells us that a representation, in the algebraic sense, is automatically continuous.

Positivity

Theorem 4

If $\mathfrak{A}$ is a C^*-algebra and $A \in \mathfrak{A}$, the following conditions are equivalent :

(i) $A = A^*, sp(A) \subset [0, \infty)]$

(ii) $A = H^2$, *for some* H *such that* $H = H^*$.

(iii) $A = B^*B$, *for some* B *in* $\mathfrak{A}$. *Moreover,* H *is unique and is called the positive square root of* A.

Such A's are said to be *positive*. The set of positive elements $\mathfrak{A}^+$ of $\mathfrak{A}$ forms a positive cone.

Theorem 5

The following conditions on $\rho \in \mathfrak{A}^*$ are equivalent:

(i) $\rho(\mathfrak{A}^+) \subset [0, \infty)$

(ii) $\|\rho\| = \rho(I)$

Such ρ's are called positive linear functionals.

A positive functional ρ is said to be a state if it is normalised so that $\rho(I) = \|\rho\| = 1$. The state space $\mathscr{S}(\mathfrak{A})$ of $\mathfrak{A}$ is convex and weak* compact. By Krein-Milman theorem, it is the convex hull of its extreme points which are called *pure states*. The set of pure states is denoted by $\mathscr{P}(\mathfrak{A})$.

Theorem 6 (Gelfand-Neumark 1)

Suppose that $\mathfrak{A}$ is an **abelian** *C^*-algebra and for each A in $\mathfrak{A}$, a complex-valued function $\hat{A}$ is defined throughtout $\mathscr{P}(\mathfrak{A})$ by $\hat{A}(\rho) = \rho(A)$. Then $\mathscr{P}(\mathfrak{A})$ is a compact Hausdorff space, relative to the weak* topology, and the mapping $A \to \hat{A}$ is a $*$-isomorphism onto the C^*-algebra $C(\mathscr{P}(\mathfrak{A}))$.*

Continuous functional calculus for normal elements

Theorem 7

If A is a normal element of a C^*-algebra $\mathfrak{A}$, $C(sp(A))$ is the abelian C^*-algebra of all continuous complex- valued functions on $sp(A)$, and ι in $C(sp(A))$ is defined by $\iota(t) = t$ $(t \in sp(A))$, then there is a unique $*$-isomorphism $\phi : C(sp(A)) \to \mathfrak{A}$ such that $\phi(\iota) = A$. For each $f \in C(sp(A))$, $\phi(f)$ is normal, and is the limit of a sequence of polynomials in I, A, A^*. The set $\{\phi(f) : f \in C(sp(A))\}$ is an abelian C^*-algebra, and is the smallest C^*-algebra of $\mathfrak{A}$ that contains A.

- A is self-adjoint $\Leftrightarrow sp(A) \subset \mathbb{R}$.
- A is positive $\Leftrightarrow sp(A) \subset \mathbb{R}^+$.
- A is unitary $\Leftrightarrow sp(A) \subseteq \mathbb{T}^1$.
- A is a projection $\Leftrightarrow sp(A) \subseteq \{0, 1\}$.

States and representations

Let X be a compact, Hausdorff space.

- States on $C(X)$ may be identified with probability measures μ defined on the σ-algebra $\mathcal{B}_X$ of Borel sets in X, i.e. $\rho_\mu(f) = \int f d\mu$.
- $C(X)$ is 'dense' in $L^2(\mu)$.
- The equation $\pi_\mu(f)(g) = fg$ defines a representation π_μ of $C(X)$.

Definition Let ϕ be a representation of a C^*-algebra $\mathfrak{A}$ on a Hilbert space $\mathscr{H}$. If there is a vector $x \in \mathscr{H}$ for which the linear subspace

$$\phi(\mathfrak{A})x = \{\phi(A)x : A \in \mathfrak{A}\}$$

is everywhere dense in $\mathscr{H}$, ϕ is described as a **cyclic** representation, and x is termed a **cyclic vector** (or **generating vector**) for ϕ.

GNS construction

Theorem 8 (Gelfand-Neumark-Segal)

If ρ is a state of a C^*-algebra $\mathfrak{A}$, there is a cyclic representation π_ρ of $\mathfrak{A}$ on a Hilbert space $\mathscr{H}_\rho$, and a unit cyclic vector x_ρ for π_ρ such that $\rho(A) = \langle \pi_\rho(A)x_\rho, x_\rho \rangle \ \forall A \in \mathfrak{A}$

Cauchy- Schwarz inequality:

If ρ is a state of a C^*-algebra $\mathfrak{A}$, then
$|\rho(B^*A)|^2 \leq \rho(A^*A)\rho(B^*B) \ \forall A, B \in \mathfrak{A}$.
$\mathscr{L}_\rho = \{A \in \mathfrak{A} : \rho(A^*A) = 0\}$ is a closed left ideal in $\mathfrak{A}$
$\mathfrak{A}/\mathscr{L}_\rho$ is the pre-Hilbert space with inner product
$\langle A + \mathscr{L}_\rho, B + \mathscr{L}_\rho \rangle = \rho(B^*A)$. $\pi_\rho(A)(B + \mathscr{L}_\rho) = AB + \mathscr{L}_\rho$

Theorem 9 (Gelfand-Neumark 2)

Each C^*-algebra has a faithful representation.

von Neumann algebras

Definition: Let $\mathscr{F} \subseteq \mathcal{B}(\mathscr{H})$. The commutant of $\mathscr{F}$ is defined to be $\{A' \in \mathcal{B}(\mathscr{H}) : AA' = A'A,\ \forall A \in \mathscr{F}\}$

Theorem 10 (Double commutant theorem)

Let $\mathfrak{A}$ be a unital self-adjoint algebra of operators acting on the Hilbert space $\mathscr{H}$. Then the following are equivalent :

(i) $\mathfrak{A}$ *is weak operator closed.*

(ii) $\mathfrak{A}$ *is strong-operator closed.*

(iii) $(\mathfrak{A}')' = \mathfrak{A}$.

Definition 1 : A von Neumann algebra $\mathscr{R}$ is the commutant of a unitary group representation (say π of G) on a Hilbert space $\mathscr{H}$ i.e.

$$\mathscr{R} = \{A \in \mathcal{B}(\mathscr{H}) : A\pi(g) = \pi(g)A, \ \forall g \in G\}$$

Definition 2 : A von Neumann algebra $\mathscr{R}$ is a unital self-adjoint algebra which satisfies any of the equivalent conditions described in the double commutant theorem.

Theorem 11

A von Neumann algebra $\mathscr{R}$ is a C^*-algebra which is also a dual Banach space.

Projections

von Neumann algebras contain plenty of projections. Let $\mathscr{R}$ be a von Neumann algebra over a Hilbert space $\mathscr{H}$.

- If $A \in \mathscr{R}$, the range projection $R(A) \in \mathscr{R}$.
- The union and intersection of each family of projections in $\mathscr{R}$ lie in $\mathscr{R}$.
- $\mathscr{R}$ is generated by its projections in the sense that it is the norm closure of the subspace spanned by its projections.

Spectral resolution

Theorem 12

If A is a self-adjoint operator acting on a Hilbert space $\mathscr{H}$ and $\mathscr{A}$ is an abelian von Neumann algebra containing A, there is a family $\{E_\lambda\}$ of projections, indexed by $\mathbb{R}$, in $\mathscr{A}$ such that

(i) $E_\lambda = 0$ *if* $\lambda < -\|A\|$, *and* $E_\lambda = I$ *if* $\|A\| \leq \lambda$;

(ii) $E_\lambda \leq E_{\lambda'}$, *if* $\lambda \leq \lambda'$;

(iii) $E_\lambda = \bigwedge_{\lambda' > \lambda} E_{\lambda'}$;

(iv) $AE_\lambda \leq \lambda E_\lambda$ *and* $\lambda(I - E_\lambda) \leq A(I - E_\lambda)$ *for each* λ;

(v) $A = \int_{-\|A\|}^{\|A\|} \lambda dE_\lambda$, *in the sense of norm convergence of approximating Riemann sums; and A is the norm limit of finite linear combinations with coefficients in $sp(A)$ of orthogonal projections* $E_{\lambda'} - E_\lambda$.

Suppose $\mathscr{R} = \pi(G)'$ as in definition 1. Then $E \leftrightarrow ran(E)$ establishes a bijection $\mathcal{P}(\mathscr{R}) \leftrightarrow G$-stable subspaces of $\mathscr{H}$. In other words, the G-stable subspaces of $\mathscr{H}$ are precisely the ranges of projection operators in $\mathscr{R}$.

The notion of unitary equivalence of subrepresentations of π translates to the so-called *Murray-von Neumann equivalence* of projections relative to $\mathscr{R}$. $P \sim_{\mathscr{R}} Q$ if and only if there exists an operator $V \in \mathscr{R}$ such that $P = V^*V, Q = VV^*$ (V is said to be a *partial isometry* with *initial space* $ran(P)$ and *final space* $ran(Q)$).

One naturally defines : If $P, Q \in \mathcal{P}(\mathscr{R})$ say $P \precsim_{\mathscr{R}} Q$ if there exists $P_0 \in \mathcal{P}(\mathscr{R})$ such that $P \sim_{\mathscr{R}} P_0 \leq Q$ (From now on , we will use $\sim$ instead of $\sim_{\mathscr{R}}$ when its unambiguous which von Neumann algebra is under consideration.)

Kaplansky density theorem

Theorem 13 (Kaplansky)

Let $\mathscr{A}$ and $\mathscr{B}$ be $*$-subalgebras of $\mathcal{B}(\mathscr{H})$ and denote by $\mathscr{A}_1$ and $\mathscr{B}_1$ respectively their unit balls. Suppose that $\mathscr{A} \subseteq \mathscr{B}$ and $\mathscr{A}$ is strong -operator dense in $\mathscr{B}$. Then $\mathscr{A}_1$ is strong operator dense in $\mathscr{B}_1$.

A family $\mathscr{F}$ of bounded operators on a Hilbert space $\mathscr{H}$ is said to act *topologically irreducibly* when (0) and $\mathscr{H}$ are the only (closed) stable spaces under $\mathscr{F}$. If (0) and $\mathscr{H}$ are the only linear manifolds (not necessarily closed) in $\mathscr{H}$ stable under $\mathscr{F}$, we say that $\mathscr{F}$ acts *algebraically irreducibly*.

Theorem 14

If the C^*-algebra $\mathfrak{A}$ acts topologically irreducibly on the Hilbert space $\mathscr{H}$, then it acts algebraically irreducibly.

Polar decomposition theorem

Theorem 15 (Polar decomposition theorem)

If T is a bounded operator on the Hilbert space $\mathscr{H}$, there is a partial isometry V with initial space the closure $ran(T^*)$ of the range of T^* and final space $ran(T)$ such that $T = V(T^*T)^{\frac{1}{2}} = (TT^*)^{\frac{1}{2}}V$. If $T = WH$ with H positive and W a partial isometry whose initial space is $ran(H)$, then $H = (T^*T)^{\frac{1}{2}}$ and $W = V$. If neither T nor T^* annihilates a non-zero vector, then V is a unitary operator.

If $\mathscr{R}$ is a von Neumann algebra and $T \in \mathscr{R}$, then $R(T) \sim R(T^*)$.

Definition : The **central carrier** of an element A of a von Neumann algebra $\mathscr{R}$, denoted by C_A, is the smallest projection Q in $\mathscr{C} = \mathscr{R} \cap \mathscr{R}'$ (center of $\mathscr{R}$) for which $QA = A$.

$$C_A = [\{RAx : R \in \mathscr{R}, x \in \mathscr{H}\}]$$

Let E, F be projections in $\mathscr{R}$.

(i) If $E \precsim F$, then $C_E \leq C_F$.

(ii) If $E \sim F$, then $C_E = C_F$.

(iii) Two projections E and F in a von Neumann algebra $\mathscr{R}$ have non-zero equivalent subprojections if and only if $C_E C_F \neq 0$.

$\precsim$ is a partial ordering on the equivalence classes of $\mathcal{P}(\mathscr{R})$. (Schroeder-Bernstein)

Theorem 16 (Comparison theorem)

Suppose that E and F are projections in $\mathscr{R}$. Then there exist unique projections P, Q, R in the center $\mathscr{C}$ such that $P + Q + R = I$, $PE \sim PF, Q_0E \prec Q_0F$ for each projection Q_0 in $\mathscr{C}$ such that Q_0 is a nonzero central subprojection of Q and $R_0F \prec R_0E$ for each projection R_0 in $\mathscr{C}$ such that R_0 is a nonzero central subprojection of R.

Definition : A <u>factor</u> $\mathscr{M}$ is a von Neumann algebra whose center is $\{\lambda I : \lambda \in \mathbb{C}\}$.
$\precsim$ is a total ordering on the equivalence classes of $\mathcal{P}(\mathscr{R})$ if $\mathscr{R}$ is a factor.

A projection E in $\mathscr{R}$ is said to be

(i) <u>finite</u> if there is no projection E_0 in $\mathscr{R}$ such that $E \sim E_0 < E$;

(ii) <u>infinite</u> if it is not finite;

(iii) <u>properly infinite</u> if QE is infinite for each projection Q in $\mathscr{C}$ such that $0 < Q \leq C_E$, and $E \neq 0$.

Definition : A projection E in a von Neumann algebra $\mathscr{R}$ is said to be an **abelian projection** in $\mathscr{R}$ when $E\mathscr{R}E$ is abelian. A projection in $\mathscr{R}$ is abelian if and only if it is minimal in the class of projections in $\mathscr{R}$ with the same central carrier.

A von Neumann algebra $\mathscr{R}$ is said to be :

(i) **type I** if it has an abelian projection with central carrier I - of **type** I_n if I is the sum of n equivalent abelian projections.

(ii) **type II** if it has no non-zero abelian projection but has a finite projection with central carrier I - of type II_1 if I is finite - of type II_∞ if I is properly infinite.

(iii) **type III** if it has no non-zero finite projections.

Theorem 17 (Type decomposition)

If $\mathscr{R}$ is a von Neumann algebra acting on a Hilbert space $\mathscr{H}$, there are (mutually orthogonal) central projections P_n, n not exceeding $\dim \mathscr{H}$, P_{c_1}, P_{c_∞} and P_∞, with sum I, maximal with respect to the properties that $\mathscr{R}P_n$ is of type I_n or $P_n = 0$, $\mathscr{R}P_{c_1}$ is of type II_1 or $P_{c_1} = 0$, $\mathscr{R}P_{c_\infty}$ is of type II_∞ or $P_{c_\infty} = 0$, and $\mathscr{R}P_\infty$ is of type III or $P_\infty = 0$.

A factor $\mathscr{M}$ is either of type I_n, or II_1, or II_∞, or III.

Theorem 18

If $\mathscr{M}$ is a factor of type I_n, then $\mathscr{M}$ is $*$-isomorphic to $\mathcal{B}(\mathscr{H})$, where $\mathscr{H}$ has dimension n.

Examples of type II_1 factors :

Let G be a discrete group with unit element e, and $\mathscr{H}$ denote the Hilbert space $l_2(G)$.
Define $x * y(g_0) := \sum_{g \in G} x(g_0 g^{-1}) y(g)$.
Denote by x_g the function on G that takes the value 1 at g and 0 at other elements of G.
$(x * x_g)(g_0) = x(g_0 g^{-1})$ and $(x_g * x)(g_0) = x(g^{-1} g_0)$.

Lemma 19

If $T \in \mathcal{B}(\mathscr{H}), x \in \mathscr{H}$, and $\langle T x_g, x_h \rangle = \langle x * x_g, x_h \rangle$ for all g and h in G, then $T = L_x$.

Theorem 20

If x and y in $l_2(G)$ are such that L_x and L_y are bounded operators on $l_2(G)$, then

(i) $L_x + L_y = L_{x+y}, aL_x = L_{ax}, L_x L_y = L_{x*y}, L_x^* = L_{x^*}$ *where* $x^*(g) = \overline{x(g^{-1})}, L_{x_e} = I$ *and* $x = y$ *if* $L_x = L_y$*;*

(ii) *the sets* $\mathscr{L}_G := \{L_x : x \in l_2(G), L_x \in \mathcal{B}(l_2(G))\}$ *and* $\mathscr{R}_G := \{R_x : x \in l_2(G), R_x \in \mathcal{B}(l_2(G))\}$ *are von Neumann algebras such that* $\mathscr{L}_G' = \mathscr{R}_G$*;*

(iii) $\{L_{x_g} : g \in G\}$ *generates* $\mathscr{L}_G$ *and* $\{R_{x_g} : g \in G\}$ *generates* $\mathscr{R}_G$ *as von Neumann algebras; and* L_{x_g}, R_{x_g} *are unitary operators.*

$$\langle L_x L_y x_g, x_g \rangle = \langle L_y L_x x_g, x_g \rangle$$

Proposition 21

The von Neumann algebras $\mathscr{L}_G$ *and* $\mathscr{R}_G$ *are finite.*

Theorem 22

If G is a group with unit e and the conjugacy class (g) of each element g different from e is infinite, then $\mathscr{L}_G$ and $\mathscr{R}_G$ are factors of type II_1 when $G \neq e$.

$\mathscr{F}_n$, the free (non-abelian group on n generators ($n \geq 2$) and an infinity of generators is permissible) and Π, the group of permutations of the integers that leave fixed all but a finite set of integers, are examples of groups that satisfy the infinite-conjugacy-class condition above (*i.c.c. groups*).

References :

- *Fundamentals of the Theory of Operator Algebras - Vol. I* - Richard Kadison, John Ringrose
- *Fundamentals of the Theory of Operator Algebras - Vol. II* - Richard Kadison, John Ringrose
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