ISOMORPHISMS OF FACTORS OF INFINITE TYPE

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1. Introduction. One of the striking results of the work done by Murray and von Neumann **(9)** in the analysis of rings of operators[1] on a Hilbert space is the reduction of the unitary equivalence problem for certain types of factors[2] to the problem of algebraic equivalence. Roughly speaking, they associate with each concrete representation of a factor a number (which measures the relative size of the factor and its commutant)—the so-called "coupling constant." Two factors are unitarily equivalent if and only if they are algebraically isomorphic and have the same coupling constant. Somewhat more precisely, Murray and von Neumann show that the given algebraic isomorphism can be implemented by a unitary transformation. Their results do not apply to factors of type III nor does the result concerning the possibility of implementation of an isomorphism by a unitary transformation apply to the case of II_∞ factors with II_1 commutants. Recently[3] Griffin **(4; 7)** pointed out the surprising fact that (at least in the case of a separable Hilbert space) every isomorphism between factors of type III can be implemented by a unitary transformation. By combining the techniques of Nakano **(10)** and Segal **(11)** in their multiplicity theory of abelian rings of operators with the global ring techniques of Dixmier **(1)** and Kaplansky **(6)**, and the Dye-Radon-Nikodym Theorem **(2)**, Griffin **(5)** was able to extend the concept of "coupling constant" from factors in the separable case to "coupling operator" for rings of operators on an arbitrary Hilbert space. He thereby extended the unitary equivalence results of Murray and von Neumann to rings of operators. However, there does not seem to be a description, in the literature, of the possible isomorphisms between II_∞'s with II_1 commutants (Griffin's results are worded so as to exclude this case). One knows, for example, that each $*$-automorphism (adjoint-preserving automorphism) of a factor is implemented by a unitary transformation of the underlying Hilbert space provided the factor is not of type II_∞ with a II_1 commutant. What the situation is, in this last case, seems to be unknown. This note supplies the missing information concerning isomorphisms between rings of type II_∞ with II_1 commutants. In particular, we show that the group of unitarily induced automorphisms of a factor of type II_∞ with a II_1 commutant is a normal subgroup of the group of $*$-automorphisms

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[1]A ring of operators is a weakly closed, self-adjoint algebra of operators on a Hilbert space.

[2]A factor is a ring of operators whose center consists of scalar multiples of its unit element.

[3]The author is indebted to E. L. Griffin for having had this result and proof made available to him in 1952.

and that the quotient group is (canonically) isomorphic to the fundamental group[4] of the II_1 commutant.

The results of the present note will be employed in a forthcoming account of the unitary invariants of representations of arbitrary C*-algebras.

2. The automorphism group. The first question with which we shall deal—the nature of the $*$-automorphisms of a factor of type II_∞ with a II_1 commutant on a separable Hilbert space—is the simplest one, from a technical viewpoint, but, nevertheless, contains all the essential features of the more general investigation of the next section.

THEOREM 1. *If $\mathfrak{M}$ is a factor of type II_∞ with dimension function D and commutant $\mathfrak{M}'$ of type II_1, then the mapping which takes each $*$-automorphism ϕ of $\mathfrak{M}$ into $D[\phi(E)]/D(E)$, with E some fixed, finite, non-zero projection in $\mathfrak{M}$ is a group homomorphism of the group, $\mathfrak{G}$, of $*$-automorphisms of $\mathfrak{M}$ onto the fundamental group of $\mathfrak{M}$ with kernel, $\mathfrak{U}$, consisting of those $*$-automorphisms of $\mathfrak{M}$ which are implemented by unitary transformations of the underlying Hilbert space, $\mathfrak{H}$.*

Proof. Since $D\cdot\phi$ serves as a dimension function on $\mathfrak{M}$ (for each $*$-automorphism, ϕ, of $\mathfrak{M}$), $D\cdot\phi$ is a constant multiple, say $\alpha(\phi)$, of D. If E is chosen to be a projection in $\mathfrak{M}$ with $D(E) = 1$, then clearly, $\alpha(\phi) = D[\phi(E)]$. Thus, if η is another $*$-automorphism of $\mathfrak{M}$, then

$$\alpha(\eta\cdot\phi) = D[\eta\phi(E)] = \alpha(\eta)D[\phi(E)] = \alpha(\eta)\cdot\alpha(\phi);$$

so that α is a group homomorphism of $\mathfrak{G}$ into the group of positive reals. We examine the kernel of α. Suppose then that $\alpha(\phi) = 1$. Since $\mathfrak{M}'$ is of type II_1 and $\mathfrak{M}$ of type II_∞, it is possible (**8**, pp. 178–180) to choose a unit vector x in $\mathfrak{H}$ so that[5] $[\mathfrak{M}x] = \mathfrak{H}$. Let F be the orthogonal projection on the space $[\mathfrak{M}'x]$. Then F lies in $\mathfrak{M}$ and is finite. Moreover $F\mathfrak{M}F$ and $\mathfrak{M}'F$ restricted to the space $[\mathfrak{M}'x]$ are factors of type II_1, one the commutant of the other, with coupling constant 1, since x serves as a cyclic vector for both $F\mathfrak{M}F$ and $\mathfrak{M}'F$ (recall that the total space under consideration, at the moment, is $[\mathfrak{M}'x]$). By assumption on ϕ, however, $D(F) = D[\phi(F)]$, and, since F is finite, one can find a unitary operator, U, in $\mathfrak{M}$ such that $UFU^{-1} = \phi(F)$. Now $[\mathfrak{M}Ux]$ contains $[\mathfrak{M}U^{-1}Ux] = [\mathfrak{M}x] = \mathfrak{H}$, and

$$[\mathfrak{M}'Ux] = [U\mathfrak{M}'x] = U[\mathfrak{M}'x] = U(F(\mathfrak{H})) = \phi(F)(\mathfrak{H}).$$

Thus Ux plays the same role with respect to $\phi(F)$ as x did with respect to F. It follows that $\phi(F)\mathfrak{M}\phi(F)$ and $\mathfrak{M}'\phi(F)$ are factors of type II_1, each the

[4]For the definition of fundamental group of a factor see (**9**). It should be noted that a factor of type II can be viewed as an infinite matrix ring over various factors of type II_1 all belonging to the genus of the II_∞ and, so, all having the same fundamental group which we may call the fundamental group of the given II_∞. This group is also the fundamental group of the commutant.

[5] We denote by $[\mathfrak{M}x]$ the closed subspace spanned by vectors of the form Ax, with A in $\mathfrak{M}$.

commutant of the other, with coupling constant 1. Moreover, ϕ restricted to $F\mathfrak{M}F$ maps this ring isomorphically upon $\phi(F)\mathfrak{M}\phi(F)$. Since $F\mathfrak{M}F$ as represented upon $F(\mathfrak{H})$ and $\phi(F)\mathfrak{M}\phi(F)$ as represented upon $\phi(F)(\mathfrak{H})$ have coupling constant 1, the known theory **(9)** tells us that there is a unitary transformation, U_1, of $F(\mathfrak{H})$ upon $\phi(F)(\mathfrak{H})$ which implements the restricted ϕ. Now choose orthogonal, equivalent projections $F_1, F_2 \ldots$ in $\mathfrak{M}$ with sum I and with $F = F_1$. Let V_i be a partial isometry in $\mathfrak{M}$ with initial space $F_1(\mathfrak{H})$ and final space $F_i(\mathfrak{H})$. (Take $V_1 = F_1$.) The map U_1, discussed above, transforms $F_1(\mathfrak{H})$ onto $\phi(F_1)(\mathfrak{H})$ and implements ϕ restricted to $F_1\mathfrak{M}F_1$. Define U_n to be $\phi(V_n)U_1V_n^*$. Clearly, U_n is a unitary transformation of $F_n(\mathfrak{H})$ onto $\phi(F_n)(\mathfrak{H})$. We assert that U_n implements the isomorphism ϕ restricted to $F_n\mathfrak{M}F_n$. In fact,

$$\begin{aligned} U_nF_nAF_nU_n^{-1} &= \phi(V_n)U_1(V_n^*F_nAF_nV_n)U_1^{-1}\phi(V_n)^* \\ &= \phi(V_n)U_1(F_1V_n^*F_nAF_nV_nF_1)U_1^{-1}\phi(V_n^*) \\ &= \phi(V_n)\phi(F_1V_n^*F_nAF_nV_nF_1)\phi(V_n^*) \\ &= \phi(V_nF_1V_n^*F_nAF_nV_nF_1V_n^*) = \phi(F_nAF_n). \end{aligned}$$

The transformation U defined to be U_n on each of the spaces $F_n(\mathfrak{H})$ is a unitary transformation of $\mathfrak{H}$ onto $\mathfrak{H}$ and certainly implements ϕ on each of the rings $F_n\mathfrak{M}F_n$. Moreover,

$$\begin{aligned} UV_nU^{-1} &= UV_n(\textstyle\sum_k U_k^{-1}) = UV_nU_1^{-1} = (\sum_k U_k)V_nU_1^{-1} = U_nV_nU_1^{-1} \\ &= (\phi(V_n)U_1V_n^*)V_nU_1^{-1} = \phi(V_n)U_1F_1U_1^{-1} = \phi(V_n)\phi(F_1) \\ &= \phi(V_nF_1) = \phi(V_n). \end{aligned}$$

Thus

$$\begin{aligned} UAU^{-1} &= U(\textstyle\sum_n F_n)A(\sum_m F_m)U^{-1} = U(\sum_{n,m} F_nAF_m)U^{-1} \\ &= \textstyle\sum_{n,m} UF_nA\,F_mU^{-1} \\ &= \textstyle\sum_{n,m} UV_nV_n^*A\,V_mV_m^*U^{-1} \\ &= \textstyle\sum_{n,m} (UV_nU^{-1})(UF_1V_n^*A\,V_mF_1U^{-1})(UV_m^*U^{-1}) \\ &= \textstyle\sum_{n,m} \phi(V_n)\phi(V_n^*A\,V_m)\phi(V_m^*) = \sum_n \phi(F_n)\phi(A)\phi(F_m) \\ &= (\textstyle\sum_n \phi(F_n))\phi(A)(\sum_m \phi(F_m)) = \phi(A), \end{aligned}$$

since each $*$-automorphismis countably additive We have established that each automorphism, ϕ, in the kernel of α is induced by a unitary transformation of $\mathfrak{H}$. Suppose, on the other hand, that ϕ is an automorphism of $\mathfrak{M}$ induced by the unitary transformation U of $\mathfrak{H}$. Once again, choosing x a unit vector in $\mathfrak{H}$ such that $[\mathfrak{M}x] = \mathfrak{H}$ and defining F to be the orthogonal projection with range $[\mathfrak{M}'x]$, we have that F lies in $\mathfrak{M}$ and is finite. In addition,

$$[\mathfrak{M}Ux] = [UU^{-1}\,\mathfrak{M}Ux] = U[\mathfrak{M}x] = U(\mathfrak{H}) = \mathfrak{H},$$

and

$$[\mathfrak{M}'\,Ux] = U[U^{-1}\,\mathfrak{M}'\,Ux] = U[\mathfrak{M}'x] = UFU^{-1}\,U(\mathfrak{H}) = \phi(F)(\mathfrak{H}).$$

Now it follows **(8**, pp. 178–180**)** that $[\mathfrak{M}'x]$ is equivalent to $[\mathfrak{M}'\,Ux]$ modulo $\mathfrak{M}$, i.e., $D(F) = D[\phi(F)]$, since $[\mathfrak{M}x] = \mathfrak{H}$ is equivalent to $[\mathfrak{M}\,Ux] = \mathfrak{H}$ modulo $\mathfrak{M}'$, so that $\alpha(\phi) = 1$; and ϕ lies in the kernel of α. Thus we have identified the kernel of α with the group $\mathfrak{U}$ of unitarily induced automorphisms of $\mathfrak{M}$. Finally we show that the image of α is precisely the fundamental group of $\mathfrak{M}$. In fact, with F a projection of relative dimension 1 in $\mathfrak{M}$, $F\mathfrak{M}F$ is a factor of type II_1, and $\phi(F)\mathfrak{M}\phi(F)$ is its $D[\phi(F)] = \alpha(\phi)$th power[4] (for, if, say $\alpha(\phi) \leqslant 1$, then $\phi(F)\mathfrak{M}\phi(F)$ is unitarily equivalent to the restriction of $F\mathfrak{M}F$ to any projection in $F\mathfrak{M}$ of relative dimension $\alpha(\phi)$). However, ϕ induces an isomorphism of $F\mathfrak{M}F$ upon $\phi(F)\mathfrak{M}\phi(F)$, so that $\alpha(\phi)$ lies in the fundamental group of $F\mathfrak{M}F$ (which is the fundamental group of $\mathfrak{M}$). Suppose that a is in the fundamental group of $\mathfrak{M}$. Choose **(9)** two (infinite, sets of matrix units $[E_{ij}]_{i,j=1,2,\ldots}$ and $[F_{ij}]_{i,j=1,2,\ldots}$ in $\mathfrak{M}$, with E_{nn}, F_{nn} projections of relative dimensions 1 and a, respectively, for each n. If we denote by $\mathfrak{N}_1$ and $\mathfrak{N}_2$ the sets of elements in $\mathfrak{M}$ which commute with $[E_{ij}]$ and $[F_{ij}]$, respectively, then $\mathfrak{N}_1$ and $\mathfrak{N}_2$ are subfactors of $\mathfrak{M}$ of type II_1, and are isomorphic to $E_{11}\mathfrak{M}E_{11}$ and $F_{11}\mathfrak{M}F_{11}$, respectively. Thus $\mathfrak{N}_2$ is the ath power of $\mathfrak{N}_1$, and, since a lies in the fundamental group of $\mathfrak{M}$ (hence, of $\mathfrak{N}_1$), there is an isomorphism η of $\mathfrak{N}_1$ onto $\mathfrak{N}_2$. Now $\mathfrak{M}$ is isomorphic to the denumerably infinite matrix rings over $\mathfrak{N}_1$ and over $\mathfrak{N}_2$ in which only those matrices occur which yield bounded operators on the denumerably infinite direct sum of $\mathfrak{H}$ with itself. Let ϕ_1, ϕ_2 be these isomorphisms and $\mathfrak{N}_1^\circ$, $\mathfrak{N}_2^\circ$ the matrix rings, respectively. If A° is a denumerably infinite matrix over $\mathfrak{N}_1$ (or $\mathfrak{N}_2$) it will act as a bounded operator if and only if the bounds of the operators obtained from A° by replacing the entries whose row or column index exceeds n by 0, forms a bounded set of numbers. Now η extends, in the obvious way, to a $*$-isomorphism η_n of the $n \times n$ matrix ring over $\mathfrak{N}_1$ onto the $n \times n$ matrix ring over $\mathfrak{N}_2$. Then η_n is norm preserving and it follows from the foregoing characterization of the operators in $\mathfrak{N}_1^\circ$, $\mathfrak{N}_2^\circ$ that η°, the extension of η to $\mathfrak{N}_1^\circ$, is a $*$-isomorphism of $\mathfrak{N}_1^\circ$ onto $\mathfrak{N}_2^\circ$. Under ϕ_1 and ϕ_2, respectively, E_{11} and F_{11}, respectively, map onto the matrices in $\mathfrak{N}_1^\circ$ and $\mathfrak{N}_2^\circ$, respectively, whose entry in the first column and row is I and whose other entries are 0. Thus the $*$-automorphism, $\phi_2^{-1}\eta^\circ\phi_1$, of $\mathfrak{M}$ carries E_{11} onto F_{11}. It follows that $\alpha(\phi_2^{-1}\eta^\circ\phi_1) = a$, so that the homomorphism α maps onto the fundamental group of $\mathfrak{M}$.

COROLLARY. *There exist factors of type* II_∞ *with* II_1 *commutants which admit non-unitarily induced automorphisms.*

Proof. Since the fundamental group of the approximately finite II_1 is the multiplicative group of positive reals, the automorphism group of the approximately finite II_∞ (bounded, denumerably infinite matrices over the approximately finite II_1) contains distinct cosets modulo the group of unitarily induced automorphisms corresponding to each positive real number. The denumerably infinite matrix representation, acting in the usual way on the direct sum of Hilbert space with itself a denumerably infinite number of times has a II_1 commutant and provides the desired example.

It is a rather surprising observation that, in a certain sense, the more complicated factors of type II_∞ have the less complicated automorphism groups.

Indeed, one tends to think of a factor of type II_∞ whose fundamental group consists of 1 alone as being quite complicated structurally (the approximately finite factors of type II_∞ would appear to be the least complicated), while, in this case, all $*$-automorphisms are unitarily induced. On the other hand, the only information we have about the fundamental group of any factor is that the fundamental group of an approximately finite factor of type II_1 is the group of positive reals. It may well be that this is the fundamental group of all factors of type II_1. It would be quite interesting to know, for example, the fundamental group of the factor of type II_1 which is the group algebra of the free group on two generators.

3. The linking operator of an isomorphism. In this section, we shall deal with the more general situation of $*$-isomorphisms between rings of type II_∞ with II_1 commutants on arbitrary Hilbert spaces.

DEFINITION. *If ϕ is a $*$-isomorphism between two rings of operators $\mathfrak{M}_1$ and $\mathfrak{M}_2$ of type II_∞ with II_1 commutants acting on the Hilbert spaces $\mathfrak{H}_1$ and $\mathfrak{H}_2$, respectively, we shall call the operator $D[\phi(E)]$ in the center of $\mathfrak{M}_2$, the* linking operator *for ϕ, where E is the projection in $\mathfrak{M}_1$ with range $[\mathfrak{M}_1'x]$, where x is a unit vector such that $[\mathfrak{M}_1 x] = \mathfrak{H}_1$, and where D is the center-valued dimension function on $\mathfrak{M}_2$ normalized so that $D(F) = \mathrm{I}$. (F defined for $\mathfrak{M}_2$ in the same way as E is defined for $\mathfrak{M}_1$; assuming $\mathfrak{M}_1$, $\mathfrak{M}_2$ have countably decomposable centers.*

Several remarks are appropriate with regard to this definition. In the first place, the cyclic vector x exists since $\mathfrak{M}_1$ is of type II_∞ and $\mathfrak{M}_1'$ is of type II_1. Secondly, E is finite with central carrier the identity, and any other projection in $\mathfrak{M}_1$ arising from a cyclic vector such as x is equivalent to E. Since ϕ maps finite projections into finite projections and equivalent projections into equivalent projections, $D[\phi(E)]$ is independent of the choice of x and is a positive operator in the center of $\mathfrak{M}_2$ (observe that $\phi(E)$ is finite and has central carrier I). Using direct sums, we assume our rings have countably decomposable centres.

THEOREM 2. *If ϕ is a $*$-isomorphism of the ring of operators $\mathfrak{M}_1$ of type II_∞ onto the ring of operators $\mathfrak{M}_2$ and the commutants of $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are of type II_1, then ϕ is implemented by a unitary transformation of $\mathfrak{H}_1$ onto $\mathfrak{H}_2$, the Hilbert spaces upon which $\mathfrak{M}_1$ and $\mathfrak{M}_2$, respectively, act, if and only if the linking operator for ϕ is the identity operator.*

Proof. Suppose first that ϕ is implemented by a unitary transformation U of $\mathfrak{H}_1$ onto $\mathfrak{H}_2$. In this case, with the notation of the above definition,

$$[\mathfrak{M}_2\, Ux] = U[U^{-1}\,\mathfrak{M}_2\, Ux] = U[\mathfrak{M}_1\, x] = U(\mathfrak{H}_1) = \mathfrak{H}_2,$$

and

$$[\mathfrak{M}_2'\, Ux] = U[U^{-1}\,\mathfrak{M}_2'\, Ux] = U[\mathfrak{M}_1'\, x] = U(E(\mathfrak{H}_1)) = \phi(E)(\mathfrak{H}_2).$$

Thus $D[\phi(E)] = \mathrm{I}$, for $\phi(E)$ arises from a cyclic vector Ux, and is therefore equivalent to the normalizing projection F for D, in $\mathfrak{M}_2$.

We assume now that $D[\phi(E)] = I$, so that $\phi(E)$ is equivalent to the normalizing projection F for D and arises from a cyclic vector for $\mathfrak{M}_2$. Thus $\phi(E)\mathfrak{M}_2\phi(E)$ and $\mathfrak{M}_2' \phi(E)$ are of type II_1 with a joint cyclic vector, as are $E\,\mathfrak{M}_1 E$ and $\mathfrak{M}_1' E$. Moreover, ϕ yields a $*$-isomorphism of $E\,\mathfrak{M}_1 E$ onto $\phi(E)\,\mathfrak{M}_2\,\phi(E)$. This last $*$-isomorphism is implemented **(3; 5)** by a unitary transformation U_1 of $E(\mathfrak{H}_1)$ onto $\phi(E)(\mathfrak{H}_2)$. Again, as in Theorem 1, one can find a (possibly uncountable) family of projections $[E_\alpha]$ in $\mathfrak{M}_1$, mutually orthogonal and equivalent to E, with sum I, and with E as one of the projections of the family. Let V_α be a partial isometry in $\mathfrak{M}_1$ with initial space $E(\mathfrak{H}_1)$ and final space $E_\alpha(\mathfrak{H}_1)$ (let E be the partial isometry with initial and final space $E(\mathfrak{H}_1)$). Define U_α to be $\phi(V_\alpha)U_1\,V_\alpha{}^*$. As in Theorem 1, the unitary operator U defined as U_α on $E_\alpha(\mathfrak{H}_1)$, for each α, implements ϕ on each of the rings $E_\alpha\,\mathfrak{M}_1\,E_\alpha$ and $UV_\alpha\,U^{-1} = \phi(V_\alpha)$. Thus the isomorphisms $A \rightarrow UAU^{-1}$ and $A \rightarrow \phi(A)$ agree on a subset of $\mathfrak{M}_1$ dense in the strongest topology. Since both mappings are strongestly continuous **(3; 5)**, they agree on $\mathfrak{M}_1$ and ϕ is implemented by the unitary transformation U.

It is now a simple matter to incorporate the above result into the statements of Griffin **(3; 5)** to completely answer the question of when $*$-isomorphisms between arbitrary rings of operators on arbitrary Hilbert spaces are induced by unitary transformations.

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