SOME ANALYTIC METHODS IN THE THEORY OF OPERATOR ALGEBRAS

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1. INTRODUCTION

Our aim, in this article, is to illustrate the application of some analytic techniques, drawn from the mathematical theory of quantized fields, to the study of (one-parameter) groups of automorphisms of C*-algebras. These illustrations take the form of three theorems stated and proved in §3, §4 and §5.

Stronger results than the three theorems presented are valid. They apply to many (rather than, one) - parameter groups; and are obtained by applying methods from the theory of several complex variables where we have used those of the theory of functions of one complex variable. The choice of the simpler theorems is prompted by easy accessibility and clarity of method.

These theorems result from adapting more complicated statements arising in the mathematical treatment of certain physical disciplines to simpler circumstances. The physical assumption of 'positive energy' is at the heart of the analytic techniques by which they are proved. The section which follows recalls some basic results and draws the analytic consequence of the positive energy assumption used in their proofs.

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2. PRELIMINARY RESULTS

The Hilbert spaces with which we deal are over the field $\mathbb{C}$ of complex scalars. We denote by '$\mathbb{C}_+$' and '$\mathbb{C}_-$' the (closed) upper and lower half-planes in $\mathbb{C}$, respectively ($\{z : \operatorname{Im} z \geq 0\}$ and $\{z : \operatorname{Im} z \leq 0\}$); and by '$\overset{\circ}{\mathbb{C}}_+$' and '$\overset{\circ}{\mathbb{C}}_-$' the corresponding open half-planes. We denote the inner product of two vectors ϕ and ψ, in a Hilbert space $\mathcal{H}$ by '(ϕ,ψ)' and the bound and adjoint of a bounded operator A on $\mathcal{H}$ by '$\|A\|$' and 'A^*', respectively.

A family $\mathfrak{U}$ of bounded operators on $\mathcal{H}$ which contains AB, $aA + B$ and A^* when it contains A and B and which is closed with respect to the metric (norm) topology induced by $A \to \|A\|$, on the family, $\mathcal{B}(\mathcal{H})$, of all bounded operators on $\mathcal{H}$ is called 'a C^*-algebra'. Those C^*-algebras, $\mathcal{R}$, closed with respect to the strong (operator) topology on $\mathcal{B}(\mathcal{H})$ (the topology of convergence on vectors of $\mathcal{H}$) are called 'von Neumann algebras'.

Denoting by $\mathcal{F}'$ the set of operators in $\mathcal{B}(\mathcal{H})$ commuting with every operator in $\mathcal{F}(\subseteq \mathcal{B}(\mathcal{H}))$ ($\mathcal{F}'$ is called 'the commutant' of $\mathcal{F}$), von Neumann proves, in [9] , his:

Double Commutant Theorem. _If_ $\mathcal{R}$ _is a von Neumann algebra containing the identity operator_ I _then_ $(\mathcal{R}')' = \mathcal{R}$.

The set, $\mathcal{U}(\mathcal{H})$, of unitary operators on $\mathcal{H}$ together with the strong operator topology induced on it from that on $\mathcal{B}(\mathcal{H})$ is a topological group. A continuous homomorphism of another topological group G into the topological group $\mathcal{U}(\mathcal{H})$ is called ' a strongly continuous unitary representation of G on $\mathcal{H}$'. Stone's Theorem [14] classifies all such representations of the additive group, $\mathbb{R}$, of real numbers (with its usual metric topology). If $t \to U_t$ is such a representation (called 'a strongly continuous one-parameter unitary group') on $\mathcal{H}$, this theorem asserts the existence of a (not necessarily bounded) self-adjoint operator H on $\mathcal{H}$ such that $U_t = \exp(-itH)$, for each real t, where $\exp(-itH)$ is understood in the sense of the functional calculus associated with the spectral resolution of H.

We shall have occasion to use this functional calculus in its strong form. The self-adjoint operators are represented as the operations of multiplication by bounded measurable functions on the $\mathcal{L}_2$-space of some measure space, in this version. The process of forming a function of an operator becomes that of composing the function with the 'multiplier' function corresponding to the operator to arrive at the 'multiplier' function corresponding to the desired function of the operator. In more detail, it is proved that if $\mathcal{A}$ is a maximal abelian von Neumann algebra on $\mathcal{H}$, then there are a measure space X, a regular Borel measure m on X and a unitary transformation U of $\mathcal{H}$ onto $\mathcal{L}_2(X)$ such that $U\mathcal{A}U^{-1} = \mathcal{M}$, with $\mathcal{M} = \{M_f : f$ a bounded m-measurable function on $X\}$ and M_f is the (bounded) operator defined on $\mathcal{L}_2(X)$ by: $M_f(g) = f\cdot g$. The unbounded self-adjoint operators whose spectral projections lie in $\mathcal{A}$ are transformed by U onto multiplications by unbounded m-measurable functions. If h is, say, continuous on $\mathbb{R}$ and A is in $\mathcal{A}$, then $U\,h(A)U^{-1} = M_{h\circ f}$, where $UAU^{-1} = M_f$ (cf. [12; Theorem 1, p. 5]).

In its usual form, the observables of a quantum mechanical system are associated with the self-adjoint operators on $\mathcal{H}$. The physical states of the system are associated with the unit vectors in $\mathcal{H}$ (up to a complex multiple of modulus 1). The time development of the system, for a specified set of dynamics, is given by a (strongly continuous) one-parameter unitary group $t \to U_t$ (= exp-itH). The system, in a state corresponding to the unit vector ψ, will evolve, after time t, into the state corresponding to $U_t\psi$ (in a Schrödinger Picture of dynamics). The Hamiltonian H of the system corresponds to the total energy of the system (up to a constant multiple)--its spectral values being the various possible energy levels. The 'positive energy' assumption, we shall make in the statements of the theorems of the succeeding sections takes the form: $H \geq 0$. (For these results, it would do as well to assume that H is bounded below or above--that is, semi-bounded.)

The automorphisms α of the C^*-algebra $\mathfrak{A}$ with which we deal are assumed to preserve the adjoint structure as well as the algebraic structure of $\mathfrak{A}$ (i.e. $\alpha(A^*) = \alpha(A)^*$). With $t \to U_t$ specifying the time development of a physical system whose (bounded) observables are identified with the self-adjoint operators in $\mathfrak{A}$, the assumption we will make, that $(\alpha_t(A) =)\ U_tAU_{-t} \in \mathfrak{A}$ for each real t and each A in $\mathfrak{A}$, is equivalent to assuming that the mappings α_t are automorphisms of $\mathfrak{A}$. It is also the case that $t \to \alpha_t$ is a one-parameter group of automorphisms of $\mathfrak{A}$ (a homomorphism of $\mathbb{R}$ into the group of automorphisms of $\mathfrak{A}$). These automorphisms specify the time-development of the system in a 'Heisenberg Picture' of the dynamics. An observable corresponding to A evolves, after time t, into the observable corresponding to $\alpha_t(A)$. We shall be especially interested in those automorphisms α of $\mathfrak{A}$ such that there is a unitary operater U in $\mathfrak{U}$ for which $\alpha(A) = UAU^{-1}$ for all A in $\mathfrak{U}$. Such α are said to be <u>inner</u>.

With the assumption that $H \geq 0$, $t \to U_t$ (=exp-itH) can be extended to the lower half-plane $\mathbb{C}_-$ with certain continuity and analyticity conditions holding. Define U_z to be $\exp(-izH)$ $(=\exp(-itH) \exp sH)$ for z $(= t + is)$ in $\mathbb{C}_-$. Note that for such z, $\|\exp sH\| \leq 1$ (since $s \leq 0$ and $H \geq 0$) from spectral theory; so that $\|U_z\| = \|U_t \exp sH\| \leq 1$.

For the continuity and analyticity properties of $z \to U_z$, we shall need the following:

Lemma 1. *If (f_n) is a sequence of continuous complex-valued functions on $\mathbb{R}$ which converges pointwise to f, and $|f_n(t)| \leq K$ for all n and all real t, then $f_n(H)$ tends strongly to $f(H)$, for each self-adjoint operator H.*

Proof. From the version of the Spectral Theorem noted, we may assume that H is multiplication by some (possibly, unbounded) measurable function h on $\mathcal{L}_2(X)$ and $f_n(H)$ is multiplication by $f_n \circ h$ (a bounded measurable function; so that there is no question of 'domain'). With g in $\mathcal{L}_2(X)$, we want to show that $(f_n \circ h)\cdot g$ tends to $(f \circ h)\cdot g$ in $\mathcal{L}_2(X)$ (i.e., in the mean of order 2). Now, $|(f_n \circ h)\cdot g| \leq K\cdot g$ and $(f_n \circ h)\cdot g$ tends to $(f \circ h)\cdot g$ almost everywhere (since $|f_n(t)| \leq K$ and f_n tends to f pointwise). From a form of the Lebesque Dominated Convergence Theorem [5; Corollary 16, p. 151], $(f_n \circ h)\cdot g$ tends to $(f \circ h)\cdot g$ in the mean of order 2.

Using the notation established preceding the lemma above, the continuity and analyticity properties of $z \to U_z$ we need are contained in:

Lemma 2. *If* $f(z) = ((\exp - izB)AU_{-z}\phi, \psi)$, *for* z *in* $\mathbb{C}_+$, *and* $g(z) = (U_z A(\exp izB)\phi, \psi)$, *for* z *in* $\mathbb{C}_-$, *with* ϕ *and* ψ *vectors in* $\mathcal{H}$ *and* B *a bounded operator on* $\mathcal{H}$; *then* f *and* g *are continuous on* $\mathbb{C}_+$ *and* $\mathbb{C}_-$ *and analytic in* $\mathbb{C}_+^o$ *and* $\mathbb{C}_-^o$, *respectively*.

Proof. We begin by showing that $z \to (U_z\phi, \psi)$ is anylytic in $\mathbb{C}_-^o$. With z_o in $\mathbb{C}_-^o$, we wish to establish that $[(U_z\phi,\psi) - (U_{z_o}\phi,\psi)]\,(z - z_o)^{-1}$ $(=((\exp -izH - \exp -iz_oH)\ (z - z_o)^{-1}\phi,\psi))$ tends to a limit as z tends to z_o. This follows by applying Lemma 1 to the functions f_n defined on $\mathbb{R}$ by: $f_n(t) = (\exp -iz_nt - \exp -iz_ot)(z_n - z_o)^{-1}$ for $t \geq 0$ and $f_n(t) = 0$ for $t \leq 0$, where (z_n) is a sequence of complex numbers, tending to z_o $(= t_o + is_o)$, with $0 < |z_n - z_o| < \frac{-s_o}{2}$. To apply Lemma 1, we note that each f_n is continuous on $\mathbb{R}$, $f_n(t) \to -it \exp -iz_ot$ as $n \to \infty$ for each positive real t (by differentiability of $\exp -izt$ at z_o), and that $|f_n(t)| \leq -2(s_oe)^{-1}$ for all n and t. For this last inequality, with $t > 0$ we have:

$$|f_n(t)| = \frac{|(\exp -z_ot)\ (\exp -i(z_n - z_o)t - 1)|}{|z_n - z_o|}$$

$$\leq \frac{(\exp s_ot)\ (\exp|(z_n - z_o)t| - 1)}{|z_n - z_o|}$$

$$\leq (\exp s_ot)(t + \frac{|z_n - z_o|t^2}{2!} + \frac{|z_n - z_o|^2t^3}{3!} + \cdots)$$

$$\leq (t \exp s_ot)\ \exp|z_n - z_o|t \leq (t \exp s_ot)\ \exp - \frac{s_ot}{2}$$

$$= t \exp \frac{s_ot}{2}.$$

Our inequality follows, now, from the fact that, with $a < 0$, $t \to t \exp at$ has an absolute maximum on the positive t-axis at $t = -a^{-1}$, and that maximum is $-(ae)^{-1}$.

Note, next, that $z \to U_z$ is strong-operator continuous on $\mathbb{C}_-$. To prove this, we apply Lemma 1, again, this time to the functions f_n defined by: $f_n(t) = \exp -iz_n t$ for $t \geq 0$, and $f_n(t) = 1$ for $t < 0$, where (z_n) is a sequence in $\mathbb{C}_-$ tending to z_0. To apply Lemma 1, we observe that $|f_n(t)| \leq 1$ for all n and real t; while $f_n(t) \to \exp -iz_0 t$ as $n \to \infty$ for $t \geq 0$, and $f_n(t) \to 1$ as $n \to \infty$ for $t < 0$. Lemma 1 allows us to conclude that $f_n(H)\phi = U_{z_n}\phi$ tends to $U_{z_0}\phi$ for each ϕ in $\mathcal{H}$, from which the strong-operator continuity of $z \to U_z$ on $\mathbb{C}_-$ follows.

To prove the analyticity of f in $\overset{\circ}{\mathbb{C}}_+$, note that

$$(f(z)-f(z_0))(z-z_0)^{-1} = (z-z_0)^{-1}[((\exp\text{-}izB-\exp\text{-}iz_0B)AU_{-z}\phi,\psi)$$
$$+ ((\exp\text{-}iz_0B)A(U_{-z}-U_{-z_0})\phi,\psi)] ,$$

and that

$$\|(\exp\text{-}izB-\exp\text{-}iz_0B)(z-z_0)^{-1} + iB\exp\text{-}iz_0B\|$$
$$= \|(-iB\exp -iz_0B)\left(\frac{-iB}{2!}(z-z_0) + \frac{(-iB)^2(z-z_0)^2}{3!} + \cdots\right)\|$$
$$\leq \|B\exp\text{-}iz_0B\| \cdot \|B\| \cdot |z-z_0|\left(\frac{1}{2!} + \frac{\|B\|\cdot|z-z_0|}{3!} + \frac{\|B\|^2|z-z_0|^2}{4!} + \cdots\right) \leq$$
$$\leq \|B\exp -iz_0B\| \cdot \|B\|\,(\exp(\|B\|\cdot|z-z_0|))|z-z_0| \to 0 \text{ as } z \to z_0 .$$

As just proved, $U_{-z}\phi \to U_{-z_0}\phi$ as $z \to z_0$ (z in $\mathbb{C}_+$).
Thus

$(z-z_0)^{-1}((\exp -izB - \exp -iz_0B)AU_{-z}\phi,\psi) \to -i((B\exp -iz_0B)AU_{-z_0}\phi,\psi)$

as $z \to z_0$. Letting ψ' be $A^*(\exp -iz_0B)^*\psi$,

$(z-z_0)^{-1}((\exp -iz_0B)A(U_{-z} - U_{-z_0})\phi,\psi) = (z-z_0)^{-1}((U_{-z} - U_{-z_0})\phi,\psi')$;

and, from the differentiability of $(U_{-z}\phi,\psi)$ at $-z_0$ in $\overset{\circ}{\mathbb{C}}_-$ established at the outset, $(z-z_0)^{-1}((U_{-z}-U_{-z_0})\phi,\psi')$ tends to a limit as $z \to z_0$. Thus $(f(z) - f(z_0))(z-z_0)^{-1}$ tends to a limit as z tends to z_0 in $\overset{\circ}{\mathbb{C}}_+$; and f is anylytic in $\overset{\circ}{\mathbb{C}}_+$.

For the analyticity of g in $\overset{\circ}{\mathbb{C}}_-$, note that

$(z-z_0)^{-1}(g(z)-g(z_0)) = (z-z_0)^{-1}[(U_zA(\exp izB - \exp iz_0B)\phi,\psi)$
$+ ((U_z-U_{z_0})A(\exp iz_0B)\phi,\psi)]$. Again, by analyticity of $z \to (U_z\phi',\psi')$, $(z-z_0)^{-1}((U_z-U_{z_0})A(\exp iz_0B)\phi,\psi)$ tends to a limit as z tends to z_0 in $\overset{\circ}{\mathbb{C}}_-$. As before, $(z-z_0)^{-1}$ $(\exp izB - \exp iz_0B)$ tends, in norm to $iB\exp iz_0B$; while U_z tends strongly to U_{z_0}, as z tends to z_0 in $\mathbb{C}_-$. From the inequality,

$$\|(U_zT_z - U_{z_0}T_{z_0})\psi\| \leq \|U_z(T_z - T_{z_0})\psi\| + \|(U_z - U_{z_0})T_{z_0}\psi\|,$$

U_zT_z tends strongly to $U_{z_0}T_{z_0}$, when U_z tends strongly to U_{z_0}, $\|U_z\| \leq 1$, and T_z tends to T_{z_0} in norm (or just, strongly). Thus $(z-z_0)^{-1}\cdot$ $U_zA(\exp izB - \exp iz_0B)\phi \to iU_{z_0}AB(\exp iz_0B)\phi$ as $z \to z_0$ in $\overset{\circ}{\mathbb{C}}_-$; and g is anylytic in $\overset{\circ}{\mathbb{C}}_-$. The continuity of f on $\mathbb{C}_+$, follows from the (norm, hence) strong continuity of $z \to A^*$ $(\exp -izB)^*$ and that of $z \to U_{-z}$ on $\mathbb{C}_+$; while that of g on $\mathbb{C}_-$ follows from the above inequality with T_z replaced by $\exp izB$.

3. ENERGY AS AN OBSERVABLE IN THE VACUUM THEORY

The first illustrative theorem (related to [1; Prop. 2] and appearing in [6; Theorem 11] essentially with its present proof--th ough, here, with the possibility of an unbounded generator) follows.

Theorem 3. <u>If</u> $\mathcal{R}$ <u>is a von Neumann algebra</u>, $t \to U_t(= \exp -itH)$ <u>a strongly continuous, one-parameter, unitary group, both acting on the Hilbert space</u> $\mathcal{H}$; <u>and</u> ψ_o <u>is a unit vector in</u> $\mathcal{H}$ <u>such that</u>:

(i) $U_t A U_{-t} \in \mathcal{R}$, <u>for each</u> A <u>in</u> $\mathcal{R}$ <u>and all real</u> t,

(ii) $H \geq 0$,

(iii) $H\psi_o = 0$

(iv) $\mathcal{R}\psi_o$ <u>is dense in</u> $\mathcal{H}$;

<u>then</u> $U_t \in \mathcal{R}$ <u>for all</u> t.

Before beginning the proof, we note that, aside from the general physical interpretation described in §2, condition (iii) is the assumption that the system has a state of (lowest) 0-energy--a type of "vacuum state". It is a crucial assumption for the argument we give; though, the result of the next section (Borcher's Theorem) removes this assumption (in essence). Condition (iv) is a normalizing assumption which permits us to draw the strong (and unphysical) conclusion that $U_t \in \mathcal{R}$. Without this assumption (though replaced by the condition that ψ_o has "contact" with all of $\mathcal{R}$--mathematically, that ψ_o has "central support" I relative to $\mathcal{R}$), we could conclude the more physical result: each α_t is inner. As far as the conclusion is concerned, it amounts to the fact that H (identified with the "total energy" of the system) is observable. Since H is (generally) unbounded, we do not expect to establish that H itself lies in $\mathcal{R}$. Knowing that each U_t is in $\mathcal{R}$, we can conclude (from Spectral Theory) that all the bounded functions of H lie in $\mathcal{R}$.

Proof of Theorem 3. From Lemma 2 and the observations preceding it, g defined on $\mathbb{C}_-$ by: $g(z) = (U_z A\psi_o, A'\psi_o)$, with A and A' self-adjoint operators in $\mathcal{R}$ and $\mathcal{R}'$, respectively, is continuous on $\mathbb{C}_-$ and analytic in $\mathbb{C}_-^o$ (taking B to be 0 in the definition of g in Lemma 2). Moreover, for real t,

$$g(t) = (U_t A\psi_o, A'\psi_o) = (A'U_t AU_{-t}\psi_o, \psi_o) = (U_t AU_{-t}A'\psi_o, \psi_o)$$
$$= (A'\psi_o, U_t A\psi_o) = \overline{g(t)},$$

noting that $U_t\psi_o = \psi_o$ for all t, since $H\psi_o = 0$ (and applying the Spectral Theorem). Thus g is real on the real axis and the Schwarz Reflection Principle (cf. [15; §4.5, pp. 155-157]) applies to guarantee the existence of an entire function G such that $G(z) = g(z) = \overline{G(\bar{z})}$ for z in $\mathbb{C}_-$. We have

$$|G(z)| = |G(\bar{z})| = |g(z)| \leq \|U_z\| \cdot \|A\psi_o\| \cdot \|A'\psi_o\| \leq \|A\psi_o\| \cdot \|A'\psi_o\|,$$

for z in $\mathbb{C}_-$. Liouville's Theorem applies; and we conclude that G and g are constant. In particular,

$$g(0) = (A\psi_o, A'\psi_o) = g(t) = (U_t A\psi_o, A'\psi_o) = (A\psi_o, U_{-t}A'U_t\psi_o).$$

Thus $(A\psi_o, (A' - U_{-t}A'U_t)\psi_o) = 0$, for all self-adjoint A in $\mathcal{R}$, self-adjoint A' in $\mathcal{R}'$ and real t. Since $\mathcal{R}$ contains the adjoint of each operator in it, using the decomposition of an operator as the sum of a self-adjoint and a skew-adjoint operator, we see that the same equality holds for each A in $\mathcal{R}$. From the fact that $\mathcal{R}\psi_o$ is dense in $\mathcal{H}$ (condition (iv)), we conclude that $(A' - U_{-t}A'U_t)\psi_o = 0$. With T in $\mathcal{R}$,

$$0 = T(A' - U_{-t}A'U_t)\psi_o = (A'T - U_{-t}U_tTU_{-t}A'U_t)\psi_o$$

$$= (A'T - U_{-t}A'U_tT)\psi_o = (A' - U_{-t}A'U_t)T\psi_o \ .$$

Again, since $\mathcal{R}\psi_o$ is dense in $\mathcal{H}$, $A' - U_{-t}A'U_t = 0$. Thus $U_tA' = A'U_t$ for each self-adjoint A' in $\mathcal{R}'$; and $U_t \in (\mathcal{R}')'$ $(= \mathcal{R})$ for each real t.