Automorphisms of C^*-algebras and von Neumann algebras

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Lemma 1

If α is an automorphisms of a C^*-algebra $\mathfrak{A}$ acting on a Hilbert space and α is weak-operator bicontinuous on the unit ball of $\mathfrak{A}$ (i.e. α is ultra-weakly bicontinuous on $\mathfrak{A}$) then α has an extension $\bar{\alpha}$ which is an automorphism of $\mathfrak{A}^-$, $\bar{\alpha}$ is ultra-weakly bicontinuous on $\mathfrak{A}^-$, and $\|\bar{\alpha} - \iota\| = \|\alpha - \iota\|$.

$*$-representations of self-adjoint operator algebras which have no unitarily equivalent non-zero subrepresentations are called *disjoint representations*

Lemma 2

if $\{\phi_\alpha\}$ are $*$-representations of the self-adjoint operator algebra $\mathfrak{A}$ then $\{\phi_\alpha\}$ consists of mutually disjoint representations if and only if $\phi(\mathfrak{A})^- = \bigoplus(\phi_\alpha(\mathfrak{A})^-)$, where $\phi = \bigoplus \phi_\alpha$.

Lemma 3

If α is an automorphism of a C^*-algebra $\mathfrak{A}$ acting on a Hilbert space, and $\|\alpha - \iota\| < 2$, then α extends to an automorphism $\bar{\alpha}$ of $\mathfrak{A}^-$, leaving each element of the center of $\mathfrak{A}^-$ fixed, such that $\|\bar{\alpha} - \iota\| = \|\alpha - \iota\|$.

With E' a projection in $\mathfrak{A}'$, and ϕ defined by $\phi(A) = \alpha(A)E'$, for $A \in \mathfrak{A}$, $(\phi \bigoplus \iota)(\mathfrak{A})$ acting on $E'(\mathscr{H}) \bigoplus \mathscr{H}$ does not have strong-operator closure $\phi(\mathfrak{A})^- \bigoplus \mathfrak{A}^-$.

Lemma 4

Let α be an inner automorphism of a von Neumann algebra $\mathscr{R}$, for which $\|\alpha - \iota\| < 2$. Then there is a unitary operator U in $\mathscr{R}$, with spectrum $\sigma(U)$ in the half-plane $\{z : \mathrm{Re} z \geq \frac{1}{2}(4 - \|\alpha - \iota\|^2)^{\frac{1}{2}}\}$, such that $\alpha(U) = UAU^*$ for all A in $\mathscr{R}$.

Step 1 : Prove the result when $\mathscr{R} = \mathscr{M}_n$, the space of all operators on an n-dimensional Hilbert space, n being an integer. This involves proving that the convex hull of the eigenvalues of U is at a distance atleast $\frac{1}{2}(4 - \|\alpha - \iota\|^2)^{\frac{1}{2}}$ from the origin.

Step 2 : For any k such that $0 < k < \frac{1}{2}(4 - \|\alpha - \iota\|^2)^{\frac{1}{2}}$, prove that there is a U implementing α such that $\sigma(U)$ is in the half-plane $\{z : \mathrm{Re} z \geq k\}$. This is achieved by suitably approximating U by linear combinations of its spectral projections and appealing to Step 1. (Proof by contradiction)

Step 3 : Use a limiting argument to construct U satisfying the required properties in the theorem.

For $a \in [0, \frac{1}{2}\pi)$, define $S_a = \{\exp it : -a \leq t \leq a\}$. Let b be such that $2\sin b = \|\alpha - \iota\|$. Choose real numbers c, δ such that $b < c < \frac{1}{2}\pi$ and $0 < \delta < \frac{1}{2}\cos c$, and let $\epsilon_n = (c-b)(1-\delta)^{n-1}$.
Let E and F be spectral projections for U_n corresponding to the Borel sets
$\{e^{it} : b + (1-2\delta)\epsilon_n \leq t \leq b + \epsilon_n\}$,
$\{e^{it} : -b - \epsilon_n \leq t \leq -b - (1-2\delta)\epsilon_n\}$.
$C_E C_F = 0$ and thus, there is a central projection $Q \in \mathscr{R}$ such that $E \leq Q$ and $F \leq I - Q$.
The unitary operator $U_{n+1} := \{e^{-i\delta\epsilon_n}Q + e^{i\delta\epsilon_n}(I-Q)\}U_n$ has spectrum in $S_{b+\epsilon_{n+1}}$ and implements the automorphism.

Definition:

An automorphism α of a C^*-algebra $\mathfrak{A}$ acting on a Hilbert space $\mathscr{H}$ is said to be:

(i) **extendable** if there is an automorphism of the weak-operator closure of $\mathfrak{A}$ equal to it on $\mathfrak{A}$.

(ii) **spatial** if there is a unitary operator U on $\mathscr{H}$ such that $\alpha(A) = UAU^*$ for each $A \in \mathfrak{A}$.

(iii) **weakly-inner** if it is spatial and U can be chosen in the weak-operator closure of $\mathfrak{A}$.

If ϕ is a faithful representation of $\mathfrak{A}$ on a Hilbert space, $\epsilon_\phi(\mathfrak{A}), \sigma_\phi(\mathfrak{A})$, and $\iota_\phi(\mathfrak{A})$, denote the groups of those elements α of the automorphism group of $\mathfrak{A}$ for which $\phi\alpha\phi^{-1}$ is *extendable*, *spatial*, and *weakly-inner*, respectively.

$\pi(\mathfrak{A})$ denotes the intersection of all the subgroups $\iota_\phi(\mathfrak{A})$ and refer to its elements as *permanently weakly* (for brevity, $\pi-$) *inner* automorphisms of $\mathfrak{A}$.

Lemma 5

If $t \to \alpha(t)$ is a norm-continuous one-parameter group of automorphisms of a C^*-algebra $\mathfrak{A}$ acting on a Hilbert space $\mathscr{H}$ then each $\alpha(t)$ is weakly-inner.

$\alpha(t)[AB] = AB + t\delta(AB) + O(t^2) = \alpha(t)[A]\beta(t)[B] = AB + t(A\delta(B) + \delta(A)B) + O(t^2)$. Thus δ, the generator of the one-parameter group is a derivation.

By the Derivation Theorem, we have that $\delta = \mathrm{ad}\, iA|\mathfrak{A}$ with $A \in \mathfrak{A}^-$ (and $A = A^*$ as $\delta(B^*) = \delta(B)^*$).

$\alpha(t)[B] = U_t B U_{-t}$, with $U_t (= \exp itA)$ a unitary operator in $\mathfrak{A}^-$.

Lemma 6

If $\mathfrak{A}$ is a C^*-algebra and U a unitary operator acting on a Hilbert space $\mathscr{H}$ such that $\alpha(A) = UAU^*$ lies in $\mathfrak{A}$ for all A in $\mathfrak{A}$ and $\mathrm{Re}\, a > 0$ for each a in $\sigma(U)$, then α lies on a norm-continuous one-parameter subgroup of $\alpha(\mathfrak{A})$ and is π-inner.

Let $\bar{\alpha}$ be the extension of α to $\mathcal{B}(\mathscr{H})$ defined by $\bar{\alpha}(B) = UBU^*$. We may choose H self-adjoint with $\sigma(H)$ in $(-\frac{\pi}{2}, \frac{\pi}{2})$ such that $U = \exp iH$. Then, $\bar{\alpha} = \exp(i\mathrm{ad}H)$.
$\mathrm{ad}\, iH$ has spectrum in $\{it : |t| \leq r\}$, where $2\|H\| = r < \pi$, by choice of H.
Thus, $\bar{\alpha}$ has spectrum in $\{\exp it : |t| \leq r\}$.
$\bar{\alpha}^s = \exp(\mathrm{ad}\, isH)$ is an automorphism of $\mathcal{B}(\mathscr{H})$, for all real s.

$$\bar{\alpha}^s = \frac{1}{2\pi i} \int_C g_s(z)(z - \bar{\alpha})^{-1}\, dz$$

so that $\bar{\alpha}^s$ leaves $\mathfrak{A}$ invariant. (use Runge's theorem to approximate $(z_0 - z)^{-1}$ by polynomials where $z_0 \in C$)

Theorem 7

If α is an automorphism of a C^*-algebra $\mathfrak{A}$ and $\|\alpha - \iota\| < 2$, then α lies on a norm-continuous one-parameter subgroup of $\alpha(\mathfrak{A})$. Such subgroups generate $\gamma(\mathfrak{A})$, the connected component of ι in $\alpha(\mathfrak{A})$ with its norm topology, as a group; and $\gamma(\mathfrak{A})$ is an open subgroup of $\alpha(\mathfrak{A})$. Each element of $\gamma(\mathfrak{A})$ is π-inner.

Pass to the reduced atomic representation of $\mathfrak{A}$. $\mathfrak{A}^-$ is a type I von Neumann algebra. $\bar{\alpha}$ is a $*$-automorphism which preserves the center and hence is implemented by a unitary in $\mathfrak{A}^-$. Use previous theorems to conclude that α lies on a norm-continuous one-parameter subgroup of $\alpha(\mathfrak{A})$.

Corollary 8

Each norm-continuous representation of a connected topological group by automorphisms of a C^*-algebra has range consisting of π-inner automorphisms.

Corollary 9

If $\mathfrak{A}$ is a C^*-algebra which has a faithful representation φ as a von Neumann algebra then $\iota_0(\mathfrak{A}) = \gamma(\mathfrak{A}) = \pi(\mathfrak{A}) = \iota_\varphi(\mathfrak{A})$; and each element of $\gamma(\mathfrak{A})$ lies on some norm-continuous one-parameterr subgroup of $\alpha(\mathfrak{A})$.

Let $\mathfrak{A}$ be a C^*-algebra, φ a faithful representation of $\mathfrak{A}$.
Then $\gamma(\mathfrak{A}) \subseteq \pi(\mathfrak{A}) \subseteq \iota_\varphi(\mathfrak{A}) \subseteq \sigma_\varphi(\mathfrak{A}) \subseteq \epsilon_\varphi(\mathfrak{A}) \subseteq \alpha(\mathfrak{A})$. Thus each of the above groups contains the open ball, with center ι and radius 2, in $\alpha(\mathfrak{A})$.
Each of these groups is open, hence closed, and the quotient of any of them by a smaller one is discrete.
The subgroups $\gamma(\mathfrak{A}), \pi(\mathfrak{A}), \iota_0(\mathfrak{A})$ of $\alpha(\mathfrak{A})$ are normal.

Let $\mathfrak{A} := C(X) \otimes \mathscr{M}_n$, where X is a compact Hausdorff space and $\mathscr{M}_n$ is the algebra of $n \times n$ complex matrices. The center $\mathscr{C}$ of $\mathfrak{A}$ is the set of matrices whose only non-zero entries consist of a single A in $C(X)$ and as continuous functions on X with values in $\mathscr{M}_n$.

> $\pi(\mathfrak{A})$ consists of precisely those automorphisms of $\mathfrak{A}$ which leave each element of $\mathscr{C}$ fixed.

With $\alpha \in \pi(\mathfrak{A})$ and ρ a point of X, a homomorphism φ_ρ of $C(X) \otimes \mathscr{M}_n$ onto $\mathscr{M}_n$ is determined by $\varphi_\rho(A \otimes B) = \rho(A)B$.
Define $\alpha(\rho)(B) := \varphi_\rho(\alpha(I \otimes B))$. Then $\alpha(\rho)$ is an isomorphism of $\mathscr{M}_n$ into $\mathscr{M}_n$. $\rho \to \alpha(\rho)$ is norm-continuous. Conversely, a norm-continuous map $\rho \to \alpha(\rho)$ from X to $\alpha(\mathscr{M}_n)$ gives rise to an element of $\pi(\mathfrak{A})$.

> The correspondence between elements of $\pi(\mathfrak{A})$ and continuous mappings of X into $\alpha(\mathscr{M}_n)$ is a group isomorphism when this second set is provided with pointwise multiplication through the group structure of $\alpha(\mathscr{M}_n)$.

$\alpha(\mathscr{M}_n) \approx U(n)/T_1$, where $U(n)$ is the group of unitary operators in $\mathscr{M}_n$ and T_1 is the circle group.

Theorem 10 (Covering Homotopy Theorem)

Let $\mathscr{B}'$ be a bundle over X'. Let X be a C_σ space (any covering has a countable subcovering), let $f_0 : X \to B'$ be a map, and let $\bar{f} : X \times I \to X'$ be a homotopy of $p' f_0 = \bar{f}_0$. Then there is a homotopy $f : X \times I \to B'$ of f_0 covering $\bar{f}$ (i.e. $p' f = \bar{f}_0$), and f is stationary with f.

$\gamma(\mathfrak{A}) \subseteq \iota_0(\mathfrak{A}) \subseteq \pi(\mathfrak{A})$.
$\pi(\mathfrak{A})/\gamma(\mathfrak{A})$ is the group of homotopy classes of mappings of X into $U(n)/T_1$.
$\iota_0(\mathfrak{A})/\gamma(\mathfrak{A})$ is the group of homotopy of classes of mappings of X into $U(n)/T_1$ which can be lifted to $U(n)$.

If X is contractible, then each continuous map of X is homotopic to a constant mapping. Thus $\gamma(\mathfrak{A}) = \iota_0(\mathfrak{A}) = \pi(\mathfrak{A})$.
If $X = U(n)/T_1$, then $\gamma(\mathfrak{A}) \subsetneq \iota_0(\mathfrak{A}) \subsetneq \pi(\mathfrak{A})$.
$\iota_0(\mathfrak{A}) \subsetneq \pi(\mathfrak{A})$ as $\pi_1(U(n)/T_1) \approx \mathbb{Z}_n$ has torsion but $\pi_1(U(n)) \approx \mathbb{Z}$ has no torsion. Thus, the identity map from $U(n)/T_1$ to itself has no lifting to $U(n)$.
$\gamma(\mathfrak{A}) \subsetneq \iota_0(\mathfrak{A})$ as $p \circ i \circ r$ is not nulhomotopic where $U(n)/T_1 \approx SU(n)/\mathbb{Z}_n \xrightarrow{r} SU(n) \xrightarrow{i} U(n) \xrightarrow{p} U(n)/T_1$, r being the map that takes $U\mathbb{Z}_n$ to U^n, i being the inclusion map and p being the projection map.

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