It remains to establish the non-normalcy of factors of type II_∞. Each such factor $\mathfrak{M}$ is representable as an infinite matrix ring over a factor $\mathcal{P}$ of type II_1 in the sense of a Kronecker product $\mathcal{P} \otimes \mathcal{B}$ (cf. R.O. IV, Theorem IX, p. 746). Let $\mathfrak{Q}$ be a subfactor of $\mathcal{P}$ such that $(\mathfrak{Q}'_{\mathcal{P}})'_{\mathcal{P}} \neq \mathfrak{Q}$. Then the subfactor, $\mathfrak{Q} \otimes \mathcal{B}$, of $\mathfrak{M}$, obtained by restricting the coefficients of the matrix representation of $\mathfrak{M}$ to $\mathfrak{Q}$, violates the normalcy of $\mathfrak{M}$, as verified by a simple computation. This completes the proof.

In conclusion we note that J. v. Neumann[6] has established the existence of non-normal factors in case III. The techniques applied in the present note do not, however, seem to yield further information in the case III situation.

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[1] Murray, F. J., and Neumann, J. v., "On Rings of Operators," *Ann. Math.*, **37**, 116–229 (1936). We shall refer to this paper as R.O. I, and to the paper "On Rings of Operators IV," *Ibid.*, **44**, 716–808 (1943), by the same authors, as R.O. IV.

[2] "A ring of operators" is a weakly closed, self-adjoint algebra of bounded, linear transformations on a Hilbert space, which contains the identity operator I.

[3] Neumann, J. v., "Zur Algebra der Funktionaloperatoren," *Math. Ann.*, **102**, 370–427 (1929).

[4] We denote by "$T(A)$" the *trace* of the operator A and by "$[[A]]$" the norm, $(T(A^*A))^{1/2}$, of A. Cf., Murray, F. J., and Neumann, J. v., "On Rings of Operators II," *Trans. Am. Math. Soc.*, **41**, 208–248 (1937); see especially pp. 218, 219 and 241 for the properties of the trace and norm.

[5] For a matrix $A = (a_{ij}), i, j = 1, \ldots, n$; $T(A)$ is the normalized trace $(\Sigma a_{ii})/n$, since $T(I) = 1$. Similarly $[[A]]^2 = (\Sigma|a_{ij}|^2)/n$.

[6] Neumann, J. v., "On Rings of Operators III," *Ann. Math.*, **41**, 94–161 (1940). See especially pp. 159–161.

ON DETERMINANTS AND A PROPERTY OF THE TRACE IN FINITE FACTORS

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1. Introduction.—In this note the authors wish to outline a theory of determinants in a finite factor. This theory originated in an attempt to prove that the trace[1] of a generalized nilpotent operator is zero. The properties of the determinant, which we shall derive, will allow us to prove, more generally, that the trace of an arbitrary operator lies in the convex hull of its spectrum.

There is no difficulty in proving that the trace of a *proper* nilpotent is zero or that the trace of a *normal* operator lies in the convex hull of its spectrum. For arbitrary finite matrices, this latter result is proved by passing to a superdiagonal form of the matrix. From this last remark we obtain the general result for operators in an approximately finite factor. These facts did not, however, enable us to deal with arbitrary factors of type II_1.

In §2 we define the determinant on regular operators in a factor of type II_1, and state the properties of this determinant. The proof that the trace lies in the convex hull of the spectrum is given in §3 as an application of the results of §2. The uniqueness of the determinant is noted in §4. The final section, §5, begins with the justification for considering only *positive-valued* determinants. The paper concludes with a study of the possible extensions of the determinant to singular operators in the factor.

The complete details of the theory outlined in this paper will appear in a subsequent publication.

2. *Definition and Properties of the Determinant.*—Let $\mathfrak{M}$ be a factor of type II_1, let T and D be the normalized trace and dimension function, respectively, in $\mathfrak{M}$ (cf. R.O. I and II), and let X be a regular operator (i.e., X has a bounded inverse) in $\mathfrak{M}$. Then X has a unique decomposition of the form $X = UH$, where U is unitary and $H = (X^*X)^{1/2}$ is positive and regular; U and H both belong to $\mathfrak{M}$ (cf. R.O. I, Lemma 4.4.1).

DEFINITION: *For X in $\mathfrak{M}$, X regular, and $H = (X^*X)^{1/2}$, we introduce the notion of "determinant" as follows:*[2]

$$\Delta(X) = \Delta(H) = exp[T(log\, H)] = exp[\int log\, \lambda dD(E_\lambda)],$$

with $H = \int \lambda dE_\lambda$, the spectral representation of H.

In particular, we note that $\Delta(I) = 1$.

We state without proof, in the following lemma, the most elementary properties of the determinant.

LEMMA 1. *The determinant satisfies the following relations:*

(1°) $\|X^{-1}\|^{-1} \leq \Delta(X) \leq \|X\|$, *for regular X.*

(2°) $\Delta(\lambda X) = |\lambda|\, \Delta(X)$, *when $\lambda \neq 0$ and X is regular.*

(3°) $\Delta(exp\, A) = |exp\, T(A)| = exp\, \Re e\, T(A)$, *for normal A.*

(4°) $\Delta[f(A)] = exp\, [\int log\, |f(z)|\, dD(E_z)]$, *where A $(= \int z\, dE_z)$ is normal and $f(z)$ is continuous and non-zero on the spectrum of A.*

(5°) $\Delta(AB) = \Delta(A)\Delta(B)$, *for normal, commuting, regular A and B.*

(6°) $\Delta(U_1XU_2) = \Delta(X)$, *for unitary U_1 and U_2 and regular X.*

(7°) $\Delta(X^*) = \Delta(X) = [\Delta(X^*X)]^{1/2}$, *for regular X.*

(8°) $\Delta(X^{-1}) = 1/\Delta(X)$, *for regular X.*

THEOREM 1. *The determinant has the following properties in addition to those listed in Lemma 1.*

(1°) $\Delta(XY) = \Delta(X)\Delta(Y)$, *for arbitrary regular X and Y.*

(2°) $\Delta(exp\ A) = |exp\ T(A)| = exp\ \mathfrak{Re}\ T(A)$, *for arbitrary* A *in* $\mathfrak{M}$.
(3°) *The determinant is continuous on regular elements in the uniform topology.*
(4°) $\Delta(H_1) \geq \Delta(H_2)$ *if* $H_1 \geq H_2 \geq 0$ *and* H_2 *is regular.*
(5°) $\Delta(X)$ *does not exceed the spectral radius of* X.

We shall sketch the proof of this theorem in some detail, but first, however, we state a lemma which is the basic tool for dealing with the non-commutative situation. The proof is based on the fact that $T(AB) = T(BA)$ and may be carried out by the methods of the theory of analytic functions on a Banach algebra.

LEMMA 2. *Let* $f(\lambda)$ *be analytic in a domain*[3] Λ *in the complex* λ*-plane and let* $X(t)$, $0 \leq t \leq 1$, *be a differentiable family of operators in* $\mathfrak{M}$ *such that the spectrum of each* $X(t)$ *lies in* Λ. *Then* $f[X(t)]$ *is differentiable with respect to* t, *and*[4]

$$T\left[\frac{d}{dt}f(X(t))\right] = T[g(X(t))\cdot X'(t)]$$

where $g(\lambda) = df(\lambda)/d\lambda$ *and* $X'(t) = dX(t)/dt$.

We shall derive statements (1°) and (2°) of Theorem 1 from the following lemma:

LEMMA 3. *If* H *is self-adjoint then*

$$\Delta(exp\ A^*\ exp\ H\ exp\ A) = exp\ T(A^* + A)\ exp\ T(H)$$

for arbitrary A *in* $\mathfrak{M}$.

Sketch of Proof:[5] For $0 \leq t \leq 1$ define $X(t) = \exp tA^* \exp H \exp tA$. The function $\log \lambda$ is analytic on a fixed rectangle containing the spectrum of each $X(t)$. Since $X'(t) = A^*X(t) + X(t)A$, we conclude from Lemma 2 that

$$T\left[\frac{d}{dt}\log X(t)\right] = T[X(t)^{-1}A^*X(t) + A] = T(A^* + A)$$

from which the asserted relation follows by integration from 0 to 1.

Sketch of Proof of Theorem 1: Ad (1°). Write $X = U_1H_1$, $Y = H_2U_2$ with U_1, U_2 unitary and H_1, H_2 positive. Then

$$\Delta(XY) = \Delta(H_1H_2) = [\Delta(H_2H_1^2H_2)]^{1/2}$$

which reduces to $\Delta(X)\Delta(Y)$ by application of Lemma 3 with $A = A^* = \log H_2$, $H = 2\log H_1$, and Lemma 1, (3°).

Ad (2°). Put $H = 0$ in Lemma 3 and note that, by Lemma 1, (7°),

$$\Delta(\exp A) = [\Delta(\exp A^* \exp A)]^{1/2}.$$

Ad (3°). The continuity of the determinant is implied by the inequality:

$$|\Delta(Y) - \Delta(X)| \leq ||X||\,||X^{-1}||\,||Y - X||.$$

Ad (4°). One has $H_1^{-1/2}H_2H_1^{-1/2} \leq I$, so that, by Lemma 1, (1°), $\Delta(H_1^{-1/2}H_2H_1^{-1/2}) \leq 1$, and thus, by (1°), $\Delta(H_2) \leq \Delta(H_1)$.

Ad (5°). By application of (1°) and Lemma 1, (1°),

$$\Delta(X) = [\Delta(X^n)]^{1/n} \leq ||X^n||^{1/n}$$

for all positive integers n, so that $\Delta(X) \leq \lim ||X^n||^{1/n}$, the spectral radius of X.

3. *Location of the Trace.*—In this section we shall apply the theory sketched above to establish the following result.

THEOREM 2. *The trace $T(A)$ of an arbitrary operator A in $\mathfrak{M}$ is located in the convex hull of the spectrum of A. In particular, $T(A) = 0$ when A is a generalized nilpotent operator in $\mathfrak{M}$.*

Proof: It suffices to prove that $T(A)$ lies in each closed half plane Π which contains the spectrum Σ of A. We may even assume that Π is the left half plane ($\Re\lambda \leq 0$).

In order, now, to prove that $\Re T(A) \leq 0$ when $\Re \Sigma \leq 0$, we introduce the regular operator exp A, whose spectrum is[6] exp Σ, and hence lies in the unit disk. It follows from this remark, by Theorem 1, (2°) and (5°), that $\exp \Re T(A) = \Delta(\exp A) \leq 1$, and hence $\Re T(A) \leq 0$.

As immediate consequences of this result, we have $|T(A)| \leq r$, the spectral radius of A; and $T(A) = 0$ if A is generalized nilpotent.

4. *Uniqueness of the Determinant.*—In this section we shall characterize the determinant in a factor $\mathfrak{M}$ of Type II_1 by means of some of the algebraic properties listed in §2.

THEOREM 3. *A numerically valued function Δ_1, which is defined on the group of regular operators in a factor $\mathfrak{M}$ of Type II_1, and which possesses the properties:*

(1°) $\Delta_1(XY) = \Delta_1(X)\Delta_1(Y)$, *for an arbitrary pair of regular operators X and Y,*

(2°) $\Delta_1(X^*) = \Delta_1(X)$, *for arbitrary regular X,*

(3°) $\Delta_1(\lambda I) = \lambda$, *for some positive $\lambda \neq 1$,*

(4°) $\Delta_1(X) \leq 1$ *if $X \leq I$ and X regular, coincides with the determinant Δ defined in §2.*

The proof of this theorem proceeds by first showing that $\Delta_1(X) > 0$ and that it suffices to prove $\Delta_1(H) = \Delta(H)$ for positive H. This is accomplished by proving that the function $T_1(A) = \log \Delta_1(\exp A)$, defined on self-adjoint A, possesses the properties of the trace. The theorem now follows from the uniqueness of the trace (cf. R.O. I, Theorem XIII, p. 219).

5. *Related Questions.*—In this section we shall indicate the reasons

which compel one to consider *positive-valued* determinants. We shall also discuss the possibilities of extending the determinant to the singular operators in $\mathfrak{M}$.

Concerning the first of these questions, it is natural to ask whether or not a notion of determinant can be developed which, in the finite-dimensional case, reduces to the usual determinant. In this connection, we are faced with the problem of constructing a non-trivial character, viz., the signum of the determinant, on the group of unitary operators in the factor. This character must satisfy certain additional conditions if the determinant theory is to be at all reasonable. The following theorem demonstrates the impossibility of constructing a character satisfying the barest minimum of such conditions.

THEOREM 4. *In a factor $\mathfrak{M}$ of Type II_1 or I_n ($n \geq 2$) there exists on $\mathfrak{M}_u$, the group of unitary operators in $\mathfrak{M}$, no character χ with the property $\chi(\lambda U) = \lambda\chi(U)$, for all λ of modulus 1.*

If $\mathfrak{M}$ if not of Type I_n, n finite, then 1 is the only character in $\mathfrak{M}_u$ which is continuous in the uniform topology.

Indication of Proof: The first statement follows by considering the equivalent unitary operators U and ζU, where $U = \zeta E_1 + \zeta^2 E_2 + \ldots + \zeta^n E_n$, ζ is a primitive nth root of unity, and $E_1, \ldots, E_n$ are orthogonal, equivalent projections with sum I.

The second statement follows by proving first that a uniformly continuous character is 1 on all unitary operators of the form $\lambda E + (I - E)$, where $|\lambda| = 1$ and E is a projection in $\mathfrak{M}$, and then noting that the group generated by such operators is uniformly dense in $\mathfrak{M}_u$.

With regard to the question of extending the notion of determinant to the singular operators in $\mathfrak{M}$, two different possibilities present themselves. On the one hand, guided by classical determinant theory, we can extend the determinant "algebraically" merely by requiring that it be zero on all singular operators in $\mathfrak{M}$. On the other hand, we can extend the determinant in an analytical manner by maintaining the definition in §2 with the understanding that $\Delta(H) = 0$ when $\int \log \lambda dD(E_\lambda) = -\infty$; in particular, $\Delta(H) = 0$ if H has a nullspace.

Except for continuity, both of these extensions preserve all the properties of the determinant noted in §2 (with obvious modifications). The relation $\Delta(XY) = \Delta(X)\Delta(Y)$, for arbitrary X and Y in $\mathfrak{M}$, is proved with the aid of the following lemmas, the first of which refers to the algebraic extension, the second to the analytic extension.

LEMMA 4. *In a factor $\mathfrak{M}$ of Type II_1, the product of two operators is singular unless both operators are regular.*[7]

LEMMA 5. *For the determinant Δ, extended to singular operators by application of the definition in §2, we have the following continuity properties:*

(1°) $\lim_{\epsilon \to 0^+} \Delta (H + \epsilon I) = \Delta(H)$ *for* $H \geq 0$.

(2°) $\Delta(H_1) \geq \Delta(H_2)$ *when* $H_1 \geq H_2 \geq 0$.

(3°) $\overline{\lim_{n \to \infty}} \Delta(X_n) \leq \Delta(X)$ *when* X_n *tends to* X *uniformly.*

(4°) $\lim_{n \to \infty} \Delta(H_n) = \Delta(H)$ *if* $H_n \geq H \geq 0$ *and* H_n *tends to* H *uniformly.*

The inequality

$$[H + \epsilon^2(K + \epsilon I)^{-1}][K + \epsilon I][H + \epsilon^2(K + \epsilon I)^{-1}] \geq HKH$$

where H and K are positive operators, and $\epsilon > 0$, is employed in conjunction with Lemma 5 to prove $\Delta(HKH) = [\Delta(H)]^2\Delta(K)$ for the analytic extension. This implies that $\Delta(XY) = \Delta(X)\Delta(Y)$ for arbitrary regular X and Y.

The two extensions, introduced above, are actually different from one another. In fact, let $H = \int_0^1 \lambda dE_\lambda$, where $D(E_\lambda) = \lambda$. Then $\Delta(H) = 1/e$ for the analytic extension, but $\Delta(H) = 0$ for the algebraic extension.

We state without proof the following additional facts concerning extensions of the determinant.

LEMMA 6. *If* Δ_1 *is an arbitrary extension of the determinant from regular operators to all operators in* $\mathfrak{M}$, *and* X *is an arbitrary operator with a nullspace, then* $\Delta_1(X) = 0$ *(in fact, we need only use that* $\Delta_1(XY) = \Delta_1(X)\Delta_1(Y)$, *and that* $\Delta_1 \not\equiv 1$*).*

THEOREM 5. *No extension of the determinant* Δ *from the regular operators to all operators in* $\mathfrak{M}$, *is continuous in the uniform topology.*

We observe, finally, that no determinant with properties (1°) – (4°) of Theorem 3 exists in an infinite factor.

* The second named author is a National Research Fellow.

[1] The term "trace" refers, throughout this note, to the normalized trace which takes the value 1 at the identity operator. For a complete account of the theory of factors, the reader is referred to the original papers on this subject:

Murray, F. J., and Neumann, J. v., "On Rings of Operators," *Ann. Math.*, **37**, 116–229 (1936).

Murray, F. J., and Neumann, J. v., "On Rings of Operators, II," *Trans. Am. Math. Soc.*, **41**, 208–248 (1937).

Neumann, J. v., "On Rings of Operators, III," *Ann. Math.*, **41**, 94–161 (1940).

Murray, F. J., and Neumann, J. v., "On Rings of Operators, IV," *Ibid.*, **44**, 716–808 (1943).

In making reference to these papers, we shall use the abbreviation R.O. I, II, III, and IV.

[2] Throughout this note, "log" refers to the principal value of the logarithm.

[3] For our purposes, we need only deal with the simplest types of domains Λ, for example, rectangles; our considerations are valid, however, in much more general circumstances.

[4] The formula obtained from that of this lemma by omitting the trace, T, is not, in general, valid, as illustrated by the example $f(\lambda) = \lambda^2$.

[5] The formal mechanism behind this lemma, and, in fact, behind the multiplicativity of the determinant is contained in the Campbell-Baker-Hausdorff formula (cf., for example, Baker, H. F., "Alternants and Continuous Groups," *Proc. London Math. Soc.* (2), **3**, 24–47 (1905)) $\exp x \exp y = \exp z$, where $z = x + y +$ commutators (note that the trace of a commutator is zero). The convergence precaution necessary to apply this formula to our situation makes the approach we indicate as short, however, and preferable in the sense that it does not rely upon this algebraic result.

[6] Cf. Dunford, N., "Spectral Theory I. Convergence to Projections," *Trans. Am. Math. Soc.*, **54**, 185–217 (1943). (See especially p. 195.)

[7] In an infinite factor, there are singular operators whose product is regular; for example, one can find U such that $U^*U = I$ but $UU^* = E$, a projection different from I.

A GENERALIZATION OF THE FROBENIUS RECIPROCITY THEOREM

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Let G be a locally compact topological group whose left-invariant Haar measure is also right invariant and K an arbitrary (but fixed) compact subgroup of G. Suppose we are given a continuous unitary representation u: $k \to u(k)(k \in K)$ of the subgroup K in a Hilbert (or finite dimensional) space $\mathfrak{h}$ over the complex numbers. Then we can construct a unitary representation U of G as follows:

Consider those functions $X(g)$ defined on G and with values in $\mathfrak{h}$ for which the inner products $(X(g), \varphi_n)$ are Haar measurable functions of g for each fixed $n = 1, \ldots, \dim \mathfrak{h}$, and for some fixed complete orthonormal system $\varphi_1, \varphi_2, \ldots$ in the space $\mathfrak{h}$. Among such functions $X(g)$ we consider only those for which

$$\|X\|^2 = \int_G \|X(g)\|^2 \, dg \qquad (1)$$

is finite, where $\|X(g)\|$ denotes the norm of $X(g)$ as an element of $\mathfrak{h}$ for each fixed $g \in G$. We restrict ourselves to only those functions $X(g)$ which satisfy also

$$X(kg) = u(k)X(g) \text{ for all } k \in K \text{ and } g \in G. \qquad (2)$$

Clearly all such "Haar-measurable" vector valued functions $X(g)$ which satisfy (1) and (2) form a linear space over the complex numbers, from which we obtain a Hilbert (or finite dimensional space) $\mathfrak{H}$ if we identify