Derivations of C^*-algebras and von Neumann algebras

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Derivations

Definition : Let $\mathscr{A}$ be a Banach algebra, δ a linear mapping on $\mathscr{A}$. Then δ is said to be derivation on $\mathscr{A}$ if $\delta(ab) = \delta(a)b + a\delta(b), a, b, \in \mathscr{A}$. If $\mathfrak{A}$ is a C^*-algebra, δ is said to be a $*$-derivation on $\mathfrak{A}$ if $\delta(a^*) = \delta(a)^*$.

If we define δ^* by $\delta^*(A) = \delta(A^*)^*(A \in \mathfrak{A})$, we may express δ as a linear combination of $*$-derivations in the following manner

$$\delta = \frac{\delta + \delta^*}{2} + i\frac{i\delta^* - i\delta}{2}$$

Theorem 1 (Kleinecke, Sirokov)

Let $\mathscr{A}$ be a Banach algebra, δ a bounded derivation on $\mathscr{A}$. Suppose that $\delta^2(a) = 0$; then $\delta(a)$ is a generalized nilpotent - i.e. $(\|\delta(a)^n\|)^{\frac{1}{n}} \to 0$ $(n \to \infty)$.

$\delta^n(a^n) = n!\delta(a)^n \ \forall n \in \mathbb{N}$

Corollary 2

Let $\mathscr{A}$ be a Banach algebra. Then there are no two elements $a, b \in \mathscr{A}$ such that $ab - ba = 1$.

For $x \in \mathscr{A}$, let $\delta_a(x) = ax - xa = [a, x], (x \in \mathscr{A})$. $\delta_a^2(b) = \delta_a(1) = 0$ if $ab - ba = 1$.

Corollary 3

Let $\mathscr{A}$ be a commutative Banach algebra, δ a bounded derivation on $\mathscr{A}$; then $\delta(\mathscr{A})$ is contained in the radical of $\mathscr{A}$. In particular, if $\mathscr{A}$ is semi-simple, then $\delta \equiv 0$.

$[\delta, L_a] = L_{\delta(a)}$

Corollary 4

Let $C^\infty([0,1])$ be the algebra of all infinitely differentiable functions on the unit interval $[0,1]$. Then there is no norm on $C^\infty([0,1])$ under which $C^\infty([0,1])$ becomes a Banach algebra.

Assume there is one. The evaluation maps are continuous and the derivation, $\frac{d}{dt}$ can be proved to continuous in this norm by using the closed graph theorem. $C^\infty([0,1])$ is semisimple and thus $\frac{d}{dt} \equiv 0$. Contradiction!!

Theorem 5

Let $\mathfrak{A}$ be a C^*-algebra, δ a bounded derivation on $\mathfrak{A}$. Suppose that $\delta(x) = 0$ for a normal element x (i.e. $x^*x = xx^*$) of $\mathfrak{A}$; then $\delta(x^*) = 0$.

$\delta(e^{i\lambda x^*}) = \delta(e^{i\lambda x^*} e^{i\bar{\lambda} x})e^{-i\bar{\lambda} x}$
$f(\lambda) = \delta(e^{i\lambda x^*})e^{-i\lambda x^*}$ is a complex-analytic function on the whole complex plane with $\|f(\lambda)\| \leq \|\delta\|$. Use Liouville's Theorem.

Corollary 6 (Fuglede's Theorem)

Let T be a bounded normal operator on a Hilbert space $\mathscr{H}$ and S a bounded operator on $\mathscr{H}$. If $[S, T] = 0$, then $[S, T^*] = 0$

Theorem 7

Let $\mathfrak{A}$ be a C^*-algebra, δ a derivation on $\mathfrak{A}$. If $[\delta(x), x] = 0$ for a normal element $x \in \mathfrak{A}$, then $\delta(x) = 0$.

As $\delta(x^*x) = \delta(xx^*)$, we have $[\delta(x^*), x] = 0$ and $[\delta(x^*), x^*] = 0$.
x belongs to the center of the C^*-algebra generated by $\{1, x, \delta(x), \delta(x^*)\}$.

Corollary 8 (Singer)

Let $\mathfrak{A}$ be a commutative C^*-algebra and let δ be a derivation on $\mathfrak{A}$; then $\delta = 0$.

Theorem 9

Let $\mathfrak{A}$ be a C^*-algebra and let δ be a derivation on $\mathfrak{A}$; then δ is bounded.

For $x(= x^*) \in \mathfrak{A}$, let ϕ be a state on $\mathfrak{A}$ such that $|\phi(x)| = \|x\|$. Then $\phi(\delta(x)) = 0$.
Suppose that $x_n(= x_n^*) \to 0$ and $\delta(x_n) \to y(\neq 0)$ and $\delta(x_n) \to y$.
Let ϕ_n be a state on $\mathfrak{A}$ such that $|\phi_n(y + x_n)| = \|y + x_n\|$, and let ϕ_0 be an accumulation point of $\{\phi_n\}$ in the state space of $\mathfrak{A}$.
Then $|\phi_0(y)| = \|y\| = 0$. By the closed graph theorem, δ is bounded.

Theorem 10 (Markov-Kakutani fixed point theorem)

Let X be a locally convex topological vector space. Let C be a compact convex subset of X. Let S be a commuting family of self-mappings T of C which are continuous and affine i.e. $T(tx + (1-t)y)$ $= tT(x) + (1-t)T(y)$ for $t \in [0,1]$ and $x, y \in C$. Then the mappings have a common fixed point in C.

For $x \in C$, define $x(n) := \frac{1}{n+1} \sum_{k=0}^{n} T^k(x)$.
There is a convergent subsequence in C, $x(n_i) \to y$.
$Tx(n) - x(n) = \frac{1}{n+1}(T^{n+1}(x) - x)$.
$f(Ty) = f(y)$ for every $f \in X^*$ and thus, Ty=y by Hahn-Banach Theorem.
If S commutes with T, it takes the fixed-point set of T(which is convex, compact) to itself. Thus the intersection of any finite family of fixed-point sets is non-empty. Use FIP for compact sets.

Polars and the Bipolar Theorem

Definition:
Let X be a locally convex topological vector space. If $A \subseteq X$, then the **polar** of A is defined as $A^{\circ} := \{f \in X^* : |f(x)| \leq 1, \forall x \in A\}$. If $B \subseteq X^*$, then the **prepolar** of B is defined as ${}^{\circ}B := \{x \in X : |f(x)| \leq 1, \forall f \in B\}$. For $A \subseteq X$ the **bipolar** of A is ${}^{\circ}(A^{\circ}) \subseteq X$.

Theorem 11 (The Bipolar Theorem)

Let X be a locally convex TVS and $A \subseteq X$. Then ${}^{\circ}(A^{\circ})$ is the closed, convex and balanced hull of A.

Normal linear functionals

Definition: A state ω of a von Neumann algebra $\mathscr{R}$ is said to be normal when $\omega(H_a) \to \omega(H)$ for each monotone increasing net of operators $\{H_a\}$ in $\mathscr{R}$ with least upper bound H.

A state ω is normal on $\mathscr{R}$ if and only if it is weak-operator continuous on the unit ball of $\mathscr{R}$.

Definition: The **ultraweak topology** on a von Neumann algebra $\mathscr{R}$ acting on a Hilbert space $\mathscr{H}$ is the weakest topology relative to which all functionals of the form $\sum_{n=1}^{\infty} \omega_{x_n,y_n}|\mathscr{R}$, with $\sum(\|x_n\|^2 + \|y_n\|^2) < \infty$, are continuous, where $\omega_{x_n,y_n}(A) = \langle Ax_n, y_n \rangle$ for $A \in \mathcal{B}(\mathscr{H})$.

The Derivation Theorem

Lemma 12

Let $\mathscr{R}$ be a von Neumann algebra and let A be a weakly closed commutative subalgebra of $\mathscr{R}$ containing the identity of $\mathscr{R}$. Let δ be a derivation on $\mathscr{R}$; then there is an element x_0 in $\mathscr{R}$ such that $\delta(a) = [x_0, a]$ for $a \in A$ and $\|x_0\| \leq \|\delta\|$.

Let A^u be the group of all unitary elements in A.
For $u \in A^u$, define $Tu(x) = (ux + \delta(u))u^{-1}(x \in \mathscr{R}).TuTv = Tuv$.
Let K be the σ-closed convex subset of $\mathscr{R}$ generated by $\{Tu(0) : u \in A^u\}$; then $Tu(K) \subseteq K$. $\{Tu : u \in A^u\}$ is commutative and Tu is σ-continuous.
Thus, there is an element x_0 such that $Tu(x_0) = x_0$ for $u \in A^u$.

Lemma 13

Let δ be a $*$-derivation on $\mathscr{R}$ and let ϕ be a normal state on $\mathscr{R}$, then there is a self-adjoint element h in $\mathscr{R}$ such that $\phi(\delta(x)) = \phi(\delta_{ih}(x))$ for $x \in \mathscr{R}$, where $\delta_{ih} = [ih, x](x \in \mathscr{R})$ and $\|h\| \leq \|\delta\|$.

$\mathscr{L} := \{\delta_{ih}^*(\phi) : h^* = h, \|h\| \leq \|\delta\|, h \in \mathscr{R}\}$
Since the mapping $h \to \delta_{ih}^*\phi$ of $\mathscr{R}^s$ with $\sigma(\mathscr{R}, \mathscr{R}_*)$ (ultraweak topology) onto $\mathscr{R}_*^s$ with $\sigma(\mathscr{R}_*, \mathscr{R})$ is continuous, $\mathscr{L}$ is $\sigma(\mathscr{R}_*, \mathscr{R})$-compact in $\mathscr{R}_*^s$.
By the bipolar theorem, there exists $a \in \mathscr{R}^s$ such that
$\delta(a) \notin \{\delta_{ih}(a) : h^* = h, \|h\| \leq \|\delta\|, h \in \mathscr{R}\}$ if we assume $\delta_{ih}^*(\phi) \notin \mathscr{L}$.
But since $(\delta(u)u^{-1})^* = -\delta(u)u^{-1}$ (as δ is a *-derivation),
$\delta(a) = [x_0, a] = [\frac{x_0 - x_0^*}{2}, a] = [ih, a]$ for some self-adjoint element $h \in \mathscr{R}$.

1-parameter groups

Theorem 14 (Borchers)

Let $t \to \alpha_t$ be a σ-weakly continuous one-parameter group of $*$-automorphisms of a von Neumann algebra $\mathscr{R}$ acting on the Hilbert space $\mathscr{H}$ containing the identity operator. Then the following two conditions are equivalent:

(1) *There is a strongly continuous one-parameter unitary group $t \to U_t \in \mathcal{B}(\mathscr{H})$ with non-negative spectrum (namely, $U_t = \exp(itH), H \geq 0$) such that $\alpha_t(a) = U_t a U_t^* (a \in \mathscr{R}, t \in \mathbb{R})$.*

(2) *There is a strongly continuous one-parameter unitary group $t \to V_t \in \mathscr{R}$ with non-negative spectrum such that $\alpha_t(a) = V_t a V_t^* (a \in \mathscr{R}, t \in \mathbb{R})$*

Theorem 15 (Stone's Theorem)

If H is a (possibly unbounded) self-adjoint operator on the Hilbert space $\mathscr{H}$, then $t \to \exp itH$ is a strongly continuous one-parameter unitary group on $\mathscr{H}$. Conversely, if $t \to U_t$ is a strongly continuous one-parameter unitary group on $\mathscr{H}$, there is a (possibly unbounded) self-adjoint operator H on $\mathscr{H}$ such that $U_t = \exp itH$ for each real t. The domain of H consists of precisely those vectors x in $\mathscr{H}$ for which $t^{-1}(U_t x - x)$ tends to a limit as t tends to 0, in which case this limit is iHx.

Theorem 16

Let $\{T(t)\}$ be a uniformly continuous semi-group. Then there exists a bounded operator A such that $T(t) = \exp tA$ for $t \geq 0$. The operator A is given by the formula $A = \lim_{h\to 0}(T(h) - I)/h$.

Covariant representation of $\{\mathfrak{A}, \mathbb{R}, \alpha\}$

Let $\alpha : \mathbb{R} \to \mathfrak{A}$ be a strongly continuous one-parameter group in the C^*-algebra $\mathfrak{A}$. Let ϕ be an invariant state on $\mathfrak{A}$ i.e. $\phi(\alpha_t(x)) = \phi(x), \forall x \in \mathfrak{A}$ and let $\{\pi_\phi, \mathscr{H}_\phi\}$ be the associated GNS representation. Define $u_\phi(t)a_\phi = (\alpha_t(a))_\phi$ for $t \in \mathbb{R}$ and $a \in \mathfrak{A}$; then $\|u_\phi(t)a_\phi\|^2 = \|a_\phi\|^2$.
$t \to u_\phi(t)$ is a strongly continuous representation of $\mathbb{R}$ and $u_\phi(t)1_\phi = 1_\phi(t \in \mathbb{R})$.
$\pi_\phi(\alpha_t(a)) = u_\phi(t)\pi_\phi(a)u_\phi(t)^*$. The above representation is said to be covariant representation of the C^*-dynamical system $\{\mathfrak{A}, \mathbb{R}, \alpha\}$, and is denoted by $\{\pi_\phi, u_\phi, \mathscr{H}_\phi\}$

Theorem 17 (The Derivation Theorem)

Let $\mathscr{R}$ be a von Neumann algebra and let δ be a derivation on $\mathscr{R}$; then there is an element a in $\mathscr{R}$ such that $\delta(x) = [a, x] (x \in \mathscr{R})$ and $\|a\| \leq \|\delta\|$.

Choose any normal state for $\mathscr{M}$. As in a previous lemma (13), pick $h \in \mathscr{M}$ such that $\phi(\delta(x) - \delta_{ih}(x)) = 0$. $\delta_1 := \delta - \delta_{ih}$ generates $\alpha_t = \exp(t\delta_1)$ a norm-continuous one-parameter group of $*$-automorphisms on $\mathscr{M}$ and $\phi(\alpha_t(x)) = \phi(x), (x \in \mathscr{M}, t \in \mathbb{R})$. Use Stone's representation theorem for $\alpha_t(x)$ in the covariant representation of the dynamical system $\{\mathscr{M}, \alpha\}$. Deduce that δ is inner on $\mathscr{M}z$ where z is a central projection such that $\mathscr{M}(1 - z) =$ kernel of the covariant representation. Dixmier's approximation theorem may be used to perturb the element implementing the derivation by a central element so as to restrict its norm $\leq \|\delta\|$.
Use Zorn's lemma on a maximal family of orthogonal projections such that δ is inner on $\mathscr{M}z_\alpha$.

Corollary 18

Let $\mathfrak{A}$ be a C^*-algebra on a Hilbert space $\mathscr{H}$ and let δ be a derivation on $\mathfrak{A}$, then there is an element a in the weak closure of $\mathfrak{A}$ in $\mathcal{B}(\mathscr{H})$ with $\|a\| \leq \|\delta\|$ such that $\delta(x) = [a, x](a \in \mathfrak{A})$.

For $A \geq 0$, $\langle \delta(A)x, y\rangle = \langle \delta(A^{1/2})A^{1/2}x, y\rangle + \langle A^{1/2}\delta(A^{1/2})x, y\rangle$. Thus δ satisfies the assumptions of the lemma mentioned below. Thus it has an extension to the weak-operator closure of $\mathfrak{A}$ which is a von Neumann algebra. Appeal to the Derivation Theorem now.

Lemma 19

If $\mathfrak{A}$ is a C^*-algebras acting on the Hilbert space $\mathscr{H}$ and η is a linear mapping of $\mathfrak{A}$ into $\mathcal{B}(\mathscr{K})$ that is continuous at 0 on $(\mathfrak{A})_r^+$, the set of positive operators in $(\mathfrak{A})_r$, the ball of radius r with center 0 in $\mathfrak{A}$, from $\mathfrak{A}$ in the strong-operator topology to $\mathcal{B}(\mathscr{K})$ in the weak-operator topology, then η is continuous on $(\mathfrak{A})_s$ from $\mathfrak{A}$ in the weak-operator topology to $\mathcal{B}(\mathscr{K})$ in the weak-operator topology.

References :

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- *Derivations and Automorphisms of C^*-algebras* - Richard V. Kadison, John R. Ringrose , Comm. Math. Phys. Volume 4, Number 1 (1967), 1-76.