Notes on Smooth Manifolds

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1 Introduction

I am writing these notes to improve my understanding of this beautiful field. These notes are based on "Introduction to Smooth Manifolds" by John Lee and "Topology" by James Munkres.

In one of his blog posts, Terrence Tao claims that to understand mathematics one needs to connect pre-rigorous understanding, i.e., intuitive computational understanding to theoretical or proof based understanding. So, throughout the notes, I will be connecting the two levels.

2 Topology

2.1 Topological Spaces

A topological space is defined as a pair consisting of a set X and a topology T on X. Here, T simply means a collection of open subsets of X.

This collection has the following properties:

1. The sets $\emptyset$ and X are in T so they are open.

2. Given any subcollection of T the union is in T so it is open.

3. Given a finite subcollection in T the intersection is in T so it is open.

Let U be in X. U is an open set of X if it is in T.

2.2 Hausdorff Spaces

A topological space is Hausdorff if there exists neighborhoods U_1, U_2 of any two distinct pairs x_1, x_2 that are disjoint.

If two sets are disjoint then $A \cap B = \emptyset$, i.e., the sets do not share any elements in common.

2.3 Basis and Countability

Let X be a set. Then a basis B of X is a collection of subsets of X with the following properties.

1. Let $x \in X$ and $S \in B$. Then $x \in B$.
2. Let $B_1 \in B$. Let $B_2 \in B$. Then there exists $x \in B_3$ such that $B_3 \subset B_1 \cup B_2$.

We then can use B to generate a topology T as follows.

Let $U \subset X$. Then U is open, i.e., $U \in T$. Then $x \in B$ and $B \subset U$ for any basis element in B.

2.4 Subspaces

We can construct new topological spaces by taking subsets of existing ones, i.e., subspaces.

Let X be a topological space and $S \subseteq X$ then a subspace topology on S is defined by letting $U \subseteq S$. Then the following statements are true.

1. If U is open in S then for a open subset of $V \subseteq X$, $U = V \cup S$.
2. If for an open subset of $V \subseteq X$, $U = V \cup S$ then U is open in S.

3 Topological Manifolds

Topological Manifolds are topological spaces with a given structure that makes behave like Euclidian space i.e., R^n.

Essentially, this structure will allow calculus on smooth manifolds because it gives it a coordinate system.

If a topological space X is a topological manifold then the following statements are true.

1. X is a Hausdorff space.
2. X is second countable.

3. X looks locally like Euclidian.

If a space X is second countable then X has a countable basis. If a space X has a countable basis then there exists a set B of neighborhood basis U_i, U_{i+1}, U_n of x and each U_n contains at least one element of B.

Here, " U is neighborhood of x" means that "U is an open set containing x.

3.1 Coordinate charts

A coordinate chart, on a topological manifold M, is a pair U and φ where U is an open subset of M and $\varphi : U \to \hat{U}$ is a homeomorphism.

If φ is a homeomorphism then $\varphi^{-1} : \hat{U} \to U$. φ is a map from a region of M to $\hat{U} = \varphi(U) \subseteq R^n$ and each point $p \in M$ is in the chart such that $\varphi(p) \to \hat{U}$.

Lemma 3.1. *Let M be any topological manifold. Then M has a countable basis of precompact coordinate balls.*

3.1.1 Connectivity

Proposition 3.2. *Let M be a topological manifold. Then the following statements are true based on lemma 3.1.*

1. *M is locally path connected.*
2. *If M is connected then M is locally path connected and if M is locally path connected then M is connected.*
3. *Let S be the components of M and let T be the path components of M. Then S and T are equivalent.*
4. *Let U be a component of M. Then U is an open subset of M and U is connected topological manifold.*

Definition 3.3 (connected)**.** *Let X_1 and X_2 be any two disjoint, non empty, open subsets of a topological space X. If no such subsets exist whose union is X then X is connected.*

Definition 3.4 (path connected)**.** *Let x_1, x_2 be any pair in X that can be joined in a path in X. Then X is path connected.*

Definition 3.5 (locally path connected)**.** *If a topological space X has basis of path connected open subsets then X is locally path connected.*

3.1.2 Local Compactness and Paracompactness

Proposition 3.6. *Let X be any topological manifold. Then X is locally compact.*

Proof. See proof of lemma 3.1 in "Introduction of Smooth Manifolds". □