Notes on Smooth Manifolds

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1 Introduction

I am writing these notes to improve my understanding of this beautiful field. These notes are based on "Introduction to Smooth Manifolds" by John Lee and "Topology" by James Munkres.

In one of his blog posts, Terrence Tao claims that to understand mathematics one needs to connect pre-rigorous understanding, i.e., intuitive computational understanding to theoretical or proof based understanding. So, throughout the notes, I will be connecting the two levels.

2 Topology

2.1 Topological Spaces

A topological space is defined as a pair consisting of a set X and a topology T on X. Here, T simply means a collection of open subsets of X.

This collection has the following properties:

1. The sets $\emptyset$ and X are in T so they are open.
2. Given any subcollection of T the union is in T so it is open.
3. Given a finite subcollection in T the intersection is in T so it is open.

Let U be in X. U is an open set of X if it is in T.

2.2 Hausdorff Spaces

A topological space is Hausdorff if there exists neighborhoods U_1, U_2 of any two distinct pairs x_1, x_2 that are disjoint.

If two sets are disjoint then $A \cap B = \emptyset$, i.e., the sets do not share any elements in common.

2.3 Basis and Countability

Let X be a set. Then a basis B of X is a collection of subsets of X with the following properties.

1. Let $x \in X$ and $S \in B$. Then $x \in B$.
2. Let $B_1 \in B$. Let $B_2 \in B$. Then there exists $x \in B_3$ such that $B_3 \subset B_1 \cup B_2$.

We then can use B to generate a topology T as follows.

Let $U \subset X$. Then U is open, i.e., $U \in T$. Then $x \in B$ and $B \subset U$ for any basis element in B.

2.4 Subspaces

We can construct new topological spaces by taking subsets of existing ones, i.e., subspaces.

Let X be a topological space and $S \subseteq X$ then a subspace topology on S is defined by letting $U \subseteq S$. Then the following statements are true.

1. If U is open in S then for a open subset of $V \subseteq X$, $U = V \cup S$.
2. If for an open subset of $V \subseteq X$, $U = V \cup S$ then U is open in S.