AST 301 Problem Set 4

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1

i) By the Schutz equation 3.31, a symmetric tensor $M^{(ab)}$ may be defined from an arbitrary tensor by the equation:

$$M^{(ab)} = \frac{1}{2}(M^{ab} + M^{ba})$$

$$= \begin{pmatrix} 0 & 1 & 1 & \frac{1}{2} \\ 1 & -1 & 0 & \frac{1}{2} \\ 1 & 0 & 0 & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & 0 \end{pmatrix}$$

By the Schutz equation 3.34, an anti-symmetric tensor $M^{[ab]}$ may be defined from an arbitrary tensor by the equation:

$$M^{[ab]} = \frac{1}{2}(M^{ab} - M^{ba})$$

$$= \begin{pmatrix} 0 & 0 & -1 & -\frac{1}{2} \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & \frac{3}{2} \\ \frac{1}{2} & -1 & -\frac{3}{2} & 0 \end{pmatrix}$$

ii) assume bases are orthonormal, so $g^{\mu\nu} = \eta^{\mu\nu}$

$$M^{\alpha}_{\beta} = M^{\alpha\mu} g_{\mu\beta}$$

$$= \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & -1 & 0 & 2 \\ -2 & 0 & 0 & 1 \\ -1 & 0 & -2 & 0 \end{pmatrix}$$

iii)

$$M_\alpha^{\ \beta} = M^{\mu\beta} g_{\mu\alpha}$$

$$= \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & -1 & 0 & 2 \\ -2 & 0 & 0 & 1 \\ -1 & 0 & -2 & 0 \end{pmatrix}$$

iv)

$$M_{\alpha\beta} = M_\alpha^{\ \mu} g_{\mu\beta}$$

$$= \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 2 \\ 2 & 0 & 0 & 1 \\ 1 & 0 & -2 & 0 \end{pmatrix}$$

b) This does not make sense, because its components are a one-form and a vector which are different geometrical objects.

c)

$$\eta^\alpha_{\ \beta} = \eta_{\mu\beta}\eta^{\mu\alpha} = \delta^\alpha_{\ \beta}$$

if $\alpha = \beta$:

$$\eta_{\mu\beta}\eta^{\mu\alpha} = 1$$

if $\alpha \neq \beta$:

$$\eta_{\mu\beta}\eta^{\mu\alpha} = 0$$

2

By the definition of the Lorentz Transformation on a tensor, we have:

$$A^{\bar{\mu}\bar{\nu}} = \wedge^{\bar{\mu}}_{\ \mu} \wedge^{\bar{\nu}}_{\ \nu} A^{\mu\nu}$$

This is equivalent to:

$$A^{\bar{\nu}\bar{\mu}} = \wedge^{\bar{\mu}}_{\ \mu} \wedge^{\bar{\nu}}_{\ \nu} A^{\nu\mu}$$

$A^{\mu\nu}$ is anti-symmetric, so it follows that:

$$A^{\mu\nu} = -A^{\nu\mu}$$

Using these three equations, we can see that:

$$A^{\bar{\mu}\bar{\nu}} = \wedge^{\bar{\mu}}_{\ \mu} \wedge^{\bar{\nu}}_{\ \nu} A^{\mu\nu} = - \wedge^{\bar{\mu}}_{\ \mu} \wedge^{\bar{\nu}}_{\ \nu} A^{\nu\mu} = -A^{\bar{\nu}\bar{\mu}}$$

Which proves that if tensor $A^{\mu\nu}$ is anti-symmetric in one inertial frame, it is anti-symmetric in all others.

3

$g_{\nu\mu} = \eta_{\nu\mu}$ because the bases are orthonormal

$$g_{\nu\mu} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

and because $g^{\nu\mu} g_{\mu\nu} = \delta^{\nu}{}_{\mu}$

$$g^{\nu\mu} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

By definition of lowering indices,

$$\wedge^{\bar{\nu}}{}_{\mu} g_{\bar{\nu}\bar{\mu}} = \wedge_{\bar{\mu}\mu} = \begin{pmatrix} -\gamma & \gamma\beta \\ \gamma\beta & \gamma \end{pmatrix}$$

Then, by the definition of raising indices,

$$\wedge_{\bar{\mu}\nu} g^{\nu\mu} = \wedge_{\bar{\mu}}^{\mu} = \begin{pmatrix} \gamma & \gamma\beta \\ \gamma\beta & \gamma \end{pmatrix}$$

In contrast, the inverse Lorentz transformation (with beta changing sign) is given by:

$$\wedge_{\bar{\mu}}^{\mu} = \begin{pmatrix} \gamma & -\gamma\beta \\ -\gamma\beta & \gamma \end{pmatrix}$$

4

a) Vectorial expansion of $\vec{\bar{E}} \cdot \vec{\bar{B}}$:

$$\vec{\bar{E}} \cdot \vec{\bar{B}} = \bar{E}_x \bar{B}_x + \bar{E}_y \bar{B}_y + \bar{E}_z \bar{B}_z$$

Lorentz Transformation of the E and B field components from frame $\mathcal{O}$ to the frame $\bar{\mathcal{O}}$, which is moving with velocity v along the x axis with respect to $\mathcal{O}$ is given by:

$$\begin{aligned} \bar{E}_x &= E_x & \bar{B}_x &= B_x \\ \bar{E}_y &= \gamma(E_y - vB_z) & \bar{B}_y &= \gamma(B_y + \frac{v}{c^2} E_z) \\ \bar{E}_z &= \gamma(E_z + vB_y) & \bar{B}_z &= \gamma(B_z - \frac{v}{c^2} E_y) \end{aligned}$$

The dot product may be written, then, as:

$$\vec{\bar{E}} \cdot \vec{\bar{B}} = E_x B_x + \gamma^2 (E_y - vB_z)(B_y + \frac{v}{c^2} E_z) + \gamma^2 (E_z + vB_y)(B_z - \frac{v}{c^2} E_y)$$

Expanding,

$$= E_x B_x + \gamma^2 (E_y B_y - vB_z By + \frac{v}{c^2} E_y E_z - \frac{v^2}{c^2} B_z E_z +$$

$$E_z B_z + vB_y B_z - \frac{v}{c^2} E_y E_z - \frac{v^2}{c^2} B_y E_y)$$

Reducing,

$$= E_x B_x + \gamma^2 (E_y B_y - \frac{v^2}{c^2} B_z E_z + E_z B_z - \frac{v^2}{c^2} B_y E_y)$$

Grouping terms and using $\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$,

$$= E_x B_x + \gamma^2 (\frac{E_y B_y}{\gamma^2} + \frac{E_z B_z}{\gamma^2}) = E_x B_x + E_y B_y + E_z B_z$$

This proves that $\vec{\bar{E}} \cdot \vec{\bar{B}}$ is invariant under the Lorentz Transformation.

$$\vec{\bar{E}}^2 - \vec{\bar{B}}^2 = \vec{\bar{E}} \cdot \vec{\bar{E}} - c^2 \vec{\bar{B}} \cdot \vec{\bar{B}}$$

Subbing in Lorentz transformation from $\mathcal{O}$,

$$= E_x E_x + \gamma^2 (E_y - vB_z)^2 + \gamma^2 (E_z + vB_y)^2 - c^2 B_x B_x -$$

$$\gamma^2 c^2 (B_y + \frac{v}{c^2} E_z)^2 - \gamma^2 c^2 (B_z - \frac{v}{c^2} E_y)^2$$

Expanding,

$$= E_x E_x - c^2 B_x B_x + \gamma^2 (E_y E_y - 2vEyBz + v^2 B_z^2 + E_z^2 + 2vE_z B_y + v^2 B_y^2)$$

$$-\gamma^2 c^2 (B_y^2 + \frac{2v}{c^2} E_z B_y + \frac{v^2}{c^4} E_z^2 + B_z^2 - \frac{2v}{c^2} E_y B_z + \frac{v^2}{c^4} E_y^2)$$

Moving around and canceling terms,

$$= E_x E_x + \gamma^2 (1 - \frac{v^2}{c^2}) E_y E_y + \gamma^2 (1 - \frac{v^2}{c^2}) E_z E_z - B_x B_x -$$

$$\gamma^2 c^2 (1 - \frac{v^2}{c^2}) B_y B_y - \gamma^2 c^2 (1 - \frac{v^2}{c^2}) B_z B_z$$

Substituting in γ, we get:

$$= E_x E_x + E_y E_y + E_z E_z - B_x B_x - B_y B_y - B_z B_z$$

Proving that $\vec{\vec{E}}^2 - \vec{\vec{B}}^2$ is Lorentz invariant.

b) Take $\vec{E} = (0, E_y, 0)$ and $\vec{B} = (0, 0, B_z)$ measured in frame $\mathcal{O}$ such that $\vec{E} \cdot \vec{B} = 0$ and $B_z > E_y$. In a frame $\mathcal{O}'$ moving with velocity v w/r to $\mathcal{O}$, the E-field components are, set equal to zero:

$$E'_x = E_x = 0$$

$$E'_y = \gamma(E_y - vB_z) = 0$$
$$E'_z = \gamma(E_z - vB_y) = 0$$

Using the equation for E'_y:

$$E_y = vB_z$$

$$v = fracE_y B_z = \frac{\vec{E}}{\vec{B}}$$

5

The stress-energy tensor in the rope's MCRF is given by:

$$T^{\mu\nu} = \begin{pmatrix} \mu & 0 & 0 & 0 \\ 0 & -\tau & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

In the frame $\bar{\mathcal{O}}$ moving upwards, along the rope's (x) axis at a velocity v relative to the rope, the four-velocity of the rope is (taking the Lorentz transformation):

$$\underset{\sim}{U} \rightarrow_{\bar{\mathcal{O}}} (\gamma, \gamma v_x, 0, 0)$$

The space-like unit vector along the cord is given by (also taking the Lorentz transformation):

$$\underset{\sim}{S} \rightarrow_{\bar{\mathcal{O}}} (-\gamma\beta, \gamma, 0, 0)$$

$T^{\bar{0}\bar{0}}$ is given by:

$$T^{\bar{0}\bar{0}} = \mu U^{\bar{0}} U^{\bar{0}} - \tau S^{\bar{0}} U^{\bar{0}} = \mu\gamma^2 - \beta\ \gamma^2$$

Stipulating that $T^{\bar{0}\bar{0}} > 0$, we have:

$$\mu > \tau\beta^2$$

6

a) The i-th component of force on a cube of side length l on the surface which has normal vector $\vec{x}$ is given by:

$$F^i = T^{ix} l^2$$

Calculating torque (approximating the force as applied at the center of each face):

$$\tau = \sum (r \times F^i) = l^3 T^{yx} - l^3 T^{xy}$$

b) The moment of inertia about the z-axis for a cube is:

$$I = \frac{\rho l^5}{6}$$

Where α is some constant. Solving for angular acceleration, we get:

$$\frac{\tau}{I} = \ddot{\theta} = \frac{6(T^{yx} - T^{xy})}{\rho l^2}$$

If the cube side length goes to 0, the angular acceleration goes to infinity. This proves that the stress-energy tensor must be symmetric.