AST 301 Problem Set 6

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1

a) By definition of the Schwarzschild metric in relativistic units, we have:

$$ds^2 = -(1 - \frac{2M}{r})dt^2 + (1 - \frac{2M}{r})^{-1}dr^2 + r^2(d\theta^2 + sin^2\theta d\phi^2)$$

Because we are looking for a proper "radial" distance, we set $dt = 0$, $d\theta = 0$, and $d\phi = 0$. Therefore, we have:

$$ds^2 = (1 - \frac{2M}{r})^{-1}dr^2$$

$$ds = \sqrt{(1 - \frac{2M}{r})^{-1}}dr$$

Integrating, and setting the lower bound to be the Schwarzschild radius and leaving a variable upper bound, we have:

$$s = D(r) = \int_{2M}^{r} \sqrt{(1 - \frac{2M}{r'})^{-1}}dr' = \int_{2M}^{r} \sqrt{g_{rr}(r')}dr'$$

b) we first make the substitution

$$r = 2M(1 + \epsilon)$$

$$dr = 2Md\epsilon$$

Our integral becomes, with adjusted bounds in epsilon coordinates:

$$D(r) = 2M \int_{0}^{\epsilon} \frac{1}{\sqrt{1 - \frac{1}{1+\epsilon'}}}d\epsilon'$$

rearranging,

$$= 2M \int_0^{\epsilon} \sqrt{1 + \frac{1}{\epsilon'}} d\epsilon'$$

ϵ is less than one, so we must add a factor of ϵ' inside of the square root to use the binomial theorem.

$$= 2M \int_0^{\epsilon} \epsilon'^{-1/2} \sqrt{1 + \epsilon'} d\epsilon'$$

by the binomial theorem,

$$(1 + \epsilon)^{1/2} \approx 1 + \frac{1}{2}\epsilon$$

the integral may now be approximated by:

$$= 2M \int_0^{\epsilon} \epsilon'^{-1/2} + \frac{1}{2}\epsilon'^{1/2} d\epsilon'$$

Integrating and taking the first order term, we get:

$$D(r) = 4M\epsilon'^{\frac{1}{2}}\Big|_0^{\epsilon}$$

we find that

$$\frac{D}{M} = 4\epsilon^{\frac{1}{2}}$$

c) now we may approximate $\sqrt{(1 - \frac{2M}{r})}$ from the original integral with the binomial theorem because $r >> 2M$.

$$\sqrt{(1 - \frac{2M}{r})} \approx 1 - \frac{M}{r}$$

our integral for D becomes:

$$D(r) = \int_{2M}^{r} \frac{r'dr'}{r' - M}$$

Using a substitution $r - m = x$, we write:

$$= \int_{M}^{r-M} \frac{(x + M)dx}{x}$$

integrating,

$$= (x + Mlog(x))\Big|_M^{r-M}$$

we find that

$$D(r) = r - 2M + M(log(\frac{r}{M} - 1))$$

approximated to the leading order.

2

I redefine "curly ν" as β for ease of typing. We define the impact parameter, b, as:

$$sin(\beta) = \frac{b}{D}$$

by small angle approximation,

$$sin(\beta) \approx \beta$$

$$b \approx \beta D$$

by definition of angular momentum, we have:

$$r \times p = L = bpsin(\beta + \frac{\pi}{2}) = bpcos(\beta)$$

Again, by small angle approximation, we have:

$$L \approx bp$$

Subbing in b,

$$L \approx \beta Dp$$

At r_{min}, we know that $\frac{dr}{d\lambda}$ and $\frac{d^2r}{d\lambda^2}$ must vanish. We have these equations for r's derivatives:

$$(\frac{dr}{d\lambda})^2 = E^2 - \frac{L^2}{r^2}(1 - \frac{2M}{r}) = 0$$

$$\frac{d^2r}{d\lambda^2} = \frac{L^2}{r^3}(1 - \frac{3M}{r}) = 0$$

From the second equation, we have:

$$r_{min} = 3M$$

Subbing into the first equation, we get:

$$E^2 = \frac{L^2\beta^2D^2}{3M^2c^2}$$

Using $E_\gamma = h\nu_\infty$, $p_\gamma = \frac{E}{c}$, and $L_\gamma = \beta Dp_\gamma$, we get:

$$h^2\nu_\infty^2 = \frac{h^2\nu_\infty^2\beta^2D^2}{27c^2M^2}$$

reducing,

$$1 = \frac{\beta^2D^2}{27c^2M^2}$$

solving for β, we get:

$$\beta = \frac{3\sqrt{3}cM}{D}$$

3

a) One of our constants of motion for a circular orbit is given by:

$$p_t = -(1 - \frac{2M}{r})m\frac{dt}{d\tau} = -E$$

rearranging,

$$\frac{E}{m} = \frac{dt}{d\tau}(1 - \frac{2M}{r})$$

the energy per unit mass for a circular orbit is given by:

$$\frac{E}{m} = \sqrt{\frac{(r_c - 2M)^2}{r_c(r_c - 3M)}}$$

Subbing the above equation into the one above it, (taking $r = r_c$) we get:

$$\sqrt{\frac{r_c}{r_c - 3M}} = \frac{dt}{d\tau}$$

Separating and integrating, and setting the clocks to start at the same time, we get:

$$\sqrt{\frac{r_c}{r_c - 3M}}\tau = t$$

We now use dimensional analysis to insert our factors of G and c, knowing that the square root term must be dimensionless:

$$\sqrt{\frac{r_c}{r_c - \frac{3GM}{c^2}}}\tau = t$$

multiply the inside of the square root by $\frac{\frac{1}{M}}{\frac{1}{M}}$ because we are interested in the ratio $\mu = \frac{r_c}{M}$

$$\sqrt{\frac{\mu}{\mu - \frac{3G}{c^2}}}\tau = t$$

solving for μ, we get:

$$\mu = \frac{3G}{c^2(1 - (\frac{\tau}{t})^2)}$$

we know that 1 hour on the planet corresponds to 7 years "at infinity", so $\tau = 1$ and $t = 7 * 365 * 24 = 61320$ Solving explicitly for μ,

$$\mu = 2.2278 * 10^{-27}\frac{m}{kg}$$

In relativistic units, we get:

$$\mu = 3.0000000008$$

b) The equation for orbital energy per mass for a circular orbit is given by:

$$\epsilon = \frac{r_c - 2M}{\sqrt{r_c(r_c - 3M)}}$$

dividing num and denom by M, we get:

$$\epsilon = \frac{\mu - 2}{\sqrt{\mu(\mu - 3)}}$$

plugging in our value for μ (relativistic units) we get:

$$\epsilon = 20440$$

c) No, the energy per unit mass for the marginally stable orbit is $\frac{\sqrt{8}}{3}$. Our ϵ is much greater and its orbit is therefore unstable. (We know that $\epsilon \to 1$ as $r_c \to \infty$)

4

We know that the energy input to the system is E_{rad}, so the efficiency of the heat engine may be given by:

$$\frac{E_{rad} - E_{min}}{E_{rad}}$$

The Carnot efficiency is the maximum efficiency of a heat engine:

$$\eta = 1 - \frac{T_{cold}}{T_{hot}}$$

We are given $E_{rad} = aT_{hot}^4 L^3$. E_{min} may be calculated through the metric.