AST 301 Problem Set 7

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Must prove $g_{\alpha\beta;\gamma} = 0$. Definition of Christoffel symbols:

$$\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2} g^{\lambda\kappa}(g_{\kappa\nu,\mu} + g_{\mu\kappa,\nu} - g_{\mu\nu,\kappa})$$

From the definition of the covariant derivative, we have:

$$g_{\alpha\beta;\gamma} = g_{\alpha\beta,\gamma} - \Gamma^{\sigma}_{\alpha\gamma} g_{\sigma\beta} - \Gamma^{\omega}_{\beta\gamma} g_{\alpha\omega}$$

using def. of Christoffel symbols:

$$= g_{\alpha\beta,\gamma} - \frac{1}{2} g^{\sigma\kappa}(g_{\kappa\gamma,\alpha} + g_{\kappa\alpha,\gamma} - g_{\alpha\gamma,\kappa}) g_{\sigma\beta} - \frac{1}{2} g^{\omega\epsilon}(g_{\epsilon\beta,\gamma} + g_{\epsilon\gamma,\beta} - g_{\beta\gamma,\epsilon}) g_{\alpha\omega}$$

Relationship between metric tensor and upper metric tensor:

$$g^{\sigma\kappa} g_{\sigma\beta} = \delta^{\kappa}_{\beta}$$

Using this property and the index-canceling property of the kronecker delta, we get:

$$= g_{\alpha\beta,\gamma} - \frac{1}{2}(g_{\beta\gamma,\alpha} + g_{\beta\alpha,\gamma} - g_{\alpha\gamma,\beta}) - \frac{1}{2}(g_{\alpha\beta,\gamma} + g_{\alpha\gamma,\beta} - g_{\beta\gamma,\alpha})$$

$$= (g_{\alpha\beta,\gamma} - \frac{1}{2} g_{\alpha\beta,\gamma} - \frac{1}{2} g_{\alpha\beta,\gamma}) + (\frac{1}{2} g_{\beta\gamma,\alpha} - \frac{1}{2} g_{\beta\gamma,\alpha}) + (\frac{1}{2} g_{\alpha\gamma,\beta} - \frac{1}{2} g_{\alpha\gamma,\beta})$$

$$g_{\alpha\beta;\gamma} = 0$$

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a)

$$\Gamma^{\mu}_{\mu\nu} = \frac{1}{2}(ln|g|)_{,\nu}$$

expanding the derivative on the RHS:

$$= \frac{1}{2g} g_{,\nu}$$

Eqn 6.40 in Schutz is:

$$\Gamma^{\alpha}_{\mu\alpha} = (\sqrt{-g})_{,\mu}/\sqrt{-g}$$

Expanding the derivative on the RHS, we get

$$= \frac{1}{2g} g_{,\mu}$$

Because $\Gamma^{\alpha}_{\mu\alpha} = \Gamma^{\alpha}_{\alpha\mu}$ and we may change the free variable as we choose, we have proved the identity:

$$\Gamma^{\mu}_{\mu\nu} = \frac{1}{2}(ln|g|)_{,\mu}$$

b)

c)

$$F^{\mu\nu}{}_{;\nu} = (\sqrt{-g}F^{\mu\nu})_{,\nu}/\sqrt{-g}$$

$$= F^{\mu\nu}{}_{,\nu} + F^{\nu\mu}\frac{1}{2g}(g)_{,\nu}$$

From the notes, we have the definition:

$$F^{\mu\nu}{}_{;\nu} = g^{-1/2}(g^{1/2}F^{\mu\nu})_{,\nu} + \Gamma^{\mu}_{\lambda\nu}F^{\lambda\nu}$$

expanding the derivative:

$$= F^{\nu\mu}{}_{,\nu} + F^{\nu\mu}\frac{1}{2g}(g)_{,\nu} + \Gamma^{\mu}_{\lambda\nu}F^{\lambda\nu}$$

Now we must prove that

$$\Gamma^{\mu}_{\lambda\nu}F^{\lambda\nu} = 0$$

Both ν and λ are dummy indices. There are 4^2 components for each value of μ. 4 are null, when $\lambda = \nu$ ($F^{\lambda\nu}$ is anti-symmetric). Because $\Gamma^{\mu}_{\lambda\nu} = \Gamma^{\mu}_{\nu\lambda}$

all components subtract each other out, for example: $\Gamma^{\mu}_{01}F^{01} = -\Gamma^{\mu}_{10}F^{10}$ Therefore,

$$F^{\mu\nu}{}_{;\nu} = (\sqrt{-g}F^{\mu\nu})_{,\nu}/\sqrt{-g}$$

d)

$$g^{\alpha\beta}g_{\beta\mu,\nu} + g^{\alpha\beta}{}_{,\nu}g_{\beta\mu} = 0$$

By the product rule, we get:

$$(g^{\alpha\beta}g_{\beta\mu})_{,\nu} = 0$$

Using $g^{\alpha\beta}g_{\beta\mu} = \delta^{\alpha}_{\mu}$, we can see that:

$$(\delta^{\alpha}_{\mu})_{,\nu} = 0$$

proving the identity:

$$g^{\alpha\beta}g_{\beta\mu,\nu} = -g^{\alpha\beta}{}_{,\nu}g_{\beta\mu}$$

e)

$$g^{\mu\nu}{}_{,\alpha} = -\Gamma^{\mu}_{\beta\alpha}g^{\beta\nu} - \Gamma^{\nu}_{\beta\alpha}g^{\mu\beta}$$

From definition of the covariant derivative of the metric tensor, we have:

$$g^{\mu\nu}{}_{;\alpha} = g^{\mu\nu}{}_{,\alpha} + \Gamma^{\mu}_{\beta\alpha}g^{\beta\nu} + \Gamma^{\nu}_{\beta\alpha}g^{\mu\beta} = 0$$

It follows directly that:

$$g^{\mu\nu}{}_{,\alpha} = -\Gamma^{\mu}_{\beta\alpha}g^{\beta\nu} - \Gamma^{\nu}_{\beta\alpha}g^{\mu\beta}$$

3

from the metric, we have:

$$g_{tt} = -e^{2\Phi(r)}$$

$$g_{rr} = e^{2\Lambda(r)}$$

$$g_{\theta\theta} = r^2$$

$$g_{\phi\phi} = r^2 sin^2\theta$$

The $g^{\mu\mu} = 1/g_{\mu\mu}$, because $g_{\mu\nu}$ is diagonal.
Definition of Christoffel symbols:

$$\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2}g^{\lambda\kappa}(g_{\kappa\nu,\mu} + g_{\mu\kappa,\nu} - g_{\mu\nu,\kappa})$$

We have the relation $\lambda = \kappa$ because the metric is a diagonal matrix. We can only have nonzero components when $\lambda = \nu$, $\lambda = \mu$, or $\mu = \nu$ also due to the metric's diagonality. Using the equation for the Christoffel symbols:

$$\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2} g^{\lambda\kappa}(g_{\kappa\nu,\mu} + g_{\mu\kappa,\nu} - g_{\mu\nu,\kappa})$$

We find:

$$\Gamma^{t}_{tr} = \Gamma^{t}_{rt} = \Phi'$$

$$\Gamma^{\theta}_{\theta r} = \Gamma^{\theta}_{r\theta} = \frac{1}{r}$$

$$\Gamma^{\phi}_{\phi r} = \Gamma^{\phi}_{r\phi} = \frac{1}{r}$$

$$\Gamma^{\phi}_{\phi\theta} = \Gamma^{\phi}_{\theta\phi} = cot(\theta)$$

$$\Gamma^{r}_{rr} = \Lambda'$$

$$\Gamma^{r}_{tt} = -e^{2(\Phi - \Lambda)}$$

$$\Gamma^{r}_{\theta\theta} = -re^{-2\Lambda}$$

$$\Gamma^{r}_{\phi\phi} = -rsin^2(\theta)e^{-2\Lambda}$$

All other Christoffel symbols are zero.

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From the metric, we have:

$$g_{tt} = -e^{2f(t,x)}$$

$$g_{xx} = e^{2f(t,x)}$$

$g^{xx} = \frac{1}{g_{xx}}$ and $g^{tt} = \frac{1}{g_{tt}}$ because the metric is diagonal. From the definition of the Riemann tensor

$$R^{t}_{xtx} = \Gamma^{t}_{xx,t} - \Gamma^{t}_{xt,x} + \Gamma^{t}_{\sigma t}\Gamma^{\sigma}_{xx} - \Gamma^{t}_{\sigma x}\Gamma^{\sigma}_{xt}$$

Calculating the Christoffel symbols:

$$\Gamma^t_{xx} = \frac{1}{2}g^{tt}(g_{tx,x} + g_{xt,x} - g_{xx,t})$$
$$= \frac{1}{2}e^{-2f}(2\frac{\partial f}{\partial t}e^{2f})$$
$$= \frac{\partial f}{\partial t}$$

$$\Gamma^t_{xt} = \frac{1}{2}g^{tt}(g_{tx,t} + g_{tt,x} - g_{xt,t})$$
$$= \frac{-1}{2}e^{-2f}(-2\frac{\partial f}{\partial x}e^{2f})$$
$$= \frac{\partial f}{\partial x}$$

$$\Gamma^t_{tt} = \frac{1}{2}g^{tt}(g_{tt,t} + g_{tt,t} - g_{tt,t})$$
$$= \frac{-1}{2}e^{-2f}(-2\frac{\partial f}{\partial t}e^{2f})$$
$$= \frac{\partial f}{\partial t}$$

$$\Gamma^x_{xx} = \frac{1}{2}g^{xx}(g_{xx,x} + g_{xx,x} - g_{xx,x})$$
$$= \frac{1}{2}e^{-2f}(2\frac{\partial f}{\partial x}e^{2f})$$
$$= \frac{\partial f}{\partial x}$$

$$\Gamma^x_{xt} = \frac{1}{2}g^{xx}(g_{xx,t} + g_{xt,x} - g_{xt,x})$$
$$= \frac{1}{2}e^{-2f}(2\frac{\partial f}{\partial t}e^{2f})$$
$$= \frac{\partial f}{\partial t}$$

Plugging these into our equation for the Riemann tensor, we get:

$$R^t_{xtx} = \frac{\partial^2 f}{\partial t^2} - \frac{\partial^2 f}{\partial x^2} - (\frac{\partial f}{\partial t})^2 + (\frac{\partial f}{\partial x})^2 + (\frac{\partial f}{\partial t})^2 - (\frac{\partial f}{\partial x})^2$$

$$= \frac{\partial^2 f}{\partial t^2} - \frac{\partial^2 f}{\partial x^2}$$

We are also looking for, using null coordinates:

$$R^v_{vuv} = \Gamma^v_{vv,u} - \Gamma^v_{vu,v} + \Gamma^v_{\sigma u}\Gamma^\sigma_{vu} - \Gamma^v_{\sigma v}\Gamma^\sigma_{vu}$$

Metric:

$$g_{uv} = -e^{2f}$$

upper metric obtained from taking inverse of the matrix g_{uv},

$$g^{uv} = -e^{-2f}$$

Finding Christoffels:

$$\Gamma^v_{vv} = \frac{1}{2} g^{vu}(g_{uv,v} + g_{uv,v} - g_{vv,v})$$

$$= \frac{-1}{2} e^{-2f}(-4\frac{\partial f}{\partial v} e^{2f})$$

$$= 2\frac{\partial f}{\partial v}$$

$$\Gamma^v_{vu} = \frac{1}{2} g^{vu}(g_{uv,u} + g_{uu,v} - g_{vu,u})$$

$$= 0$$

$$\Gamma^v_{uu} = \frac{1}{2} g^{vu}(g_{uu,u} + g_{uu,u} - g_{uu,u})$$

$$= 0$$

$$\Gamma^u_{vu} = \frac{1}{2} g^{vu}(g_{vv,u} + g_{vu,v} - g_{vu,v})$$

$$= 0$$

Therefore,

$$R^v_{vuv} = 2\frac{\partial^2 f}{\partial u \partial v}$$