AST 301 Problem Set 5

Nathan Spilker

October 19 2019

1

Must prove $F^{\mu\nu}{}_{,\nu} = 4\pi J^\mu$ is equivalent to $c\vec{\nabla}\times\vec{B} = 4\pi\vec{J}+\frac{\partial\vec{E}}{\partial t}$ and $\vec{\nabla}\cdot\vec{E} = 4\pi\rho$, where, taking c = 1:

$$F^{\mu\nu} = \begin{pmatrix} 0 & E^x & E^y & E^z \\ -E^x & 0 & B^z & -B^y \\ -E^y & -B^z & 0 & B^x \\ -E^z & B^y & -B^x & 0 \end{pmatrix}$$

and,

$$J^\mu = (\rho, \vec{J})$$

Gauss's law, $\vec{\nabla}\cdot\vec{E} = 4\pi\rho$, implies one equation:

$$\frac{\partial E^x}{\partial x} + \frac{\partial E^y}{\partial y} + \frac{\partial E^z}{\partial z} = 4\pi\rho$$

and Ampere's law, $c\vec{\nabla}\times\vec{B} = 4\pi\vec{J}+\frac{\partial\vec{E}}{\partial t}$, implies 3 equations:

$$\frac{\partial B^z}{\partial y} - \frac{\partial B^y}{\partial z} - \frac{\partial E^x}{\partial t} = 4\pi J^x$$

$$\frac{\partial B^x}{\partial z} - \frac{\partial B^z}{\partial x} - \frac{\partial E^y}{\partial t} = 4\pi J^y$$

$$\frac{\partial B^y}{\partial x} - \frac{\partial B^x}{\partial y} - \frac{\partial E^z}{\partial t} = 4\pi J^z$$

$F^{\mu\nu}{}_{,\nu} = 4\pi J^\mu$ implies 4 equations, one for each dimension on the free variable μ:

$$\frac{\partial F^{00}}{\partial x^0} + \frac{\partial F^{01}}{\partial x^1} + \frac{\partial F^{02}}{\partial x^2} + \frac{\partial F^{03}}{\partial x^3} = 4\pi J^0$$

$$\frac{\partial F^{10}}{\partial x^0} + \frac{\partial F^{11}}{\partial x^1} + \frac{\partial F^{12}}{\partial x^2} + \frac{\partial F^{13}}{\partial x^3} = 4\pi J^1$$
$$\frac{\partial F^{20}}{\partial x^0} + \frac{\partial F^{21}}{\partial x^1} + \frac{\partial F^{22}}{\partial x^2} + \frac{\partial F^{23}}{\partial x^3} = 4\pi J^2$$
$$\frac{\partial F^{30}}{\partial x^0} + \frac{\partial F^{31}}{\partial x^1} + \frac{\partial F^{32}}{\partial x^2} + \frac{\partial F^{33}}{\partial x^3} = 4\pi J^3$$

Taking $(x^0, x^1, x^2, x^3) = (t, x, y, z)$, and taking the derivatives:

$$\frac{\partial E^x}{\partial x} + \frac{\partial E^y}{\partial y} + \frac{\partial E^z}{\partial z} = 4\pi J^t$$

$$\frac{\partial B^z}{\partial y} - \frac{\partial B^y}{\partial z} - \frac{\partial E^x}{\partial t} = 4\pi J^x$$
$$\frac{\partial B^x}{\partial z} - \frac{\partial B^z}{\partial x} - \frac{\partial E^y}{\partial t} = 4\pi J^y$$
$$\frac{\partial B^y}{\partial x} - \frac{\partial B^x}{\partial y} - \frac{\partial E^z}{\partial t} = 4\pi J^z$$

Which are the equations we derived from Gauss's and Ampere's laws.

Must also prove that $F_{\alpha\beta,\gamma} + F_{\gamma\alpha,\beta} + F_{\beta\gamma,\alpha} = 0$ is equivalent to $\vec{\nabla} \cdot \vec{B} = 0$ and $\frac{\partial \vec{B}}{\partial t} + c\vec{\nabla} \times \vec{E} = 0$.

By definition of lowering indices between orthonormal bases,

$$F_{\alpha\beta} = F^{\mu\nu} g_{\mu\alpha} g_{\nu\beta}$$

where $g_{\mu\alpha}$ is the metric tensor, in orthonormal coordinates, equal to $\eta_{\mu\alpha}$.

$$F^{\mu\nu} g_{\mu 0} g_{\nu 0} = F^{00} g_{00} g_{00} = F_{00}$$
$$F^{\mu\nu} g_{\mu 0} g_{\nu 1} = F^{01} g_{00} g_{11} = -F_{01}$$
$$F^{\mu\nu} g_{\mu 0} g_{\nu 2} = F^{02} g_{00} g_{22} = -F_{02}$$
$$F^{\mu\nu} g_{\mu 0} g_{\nu 3} = F^{03} g_{00} g_{33} = -F_{03}$$

Further,

$$F^{10} = -F^{10}$$
$$F^{20} = -F^{20}$$
$$F^{30} = -F^{30}$$

The bottom right nine entries remain unchanged. Therefore,

$$F_{\alpha\beta} = \begin{pmatrix} 0 & -E^x & -E^y & -E^z \\ E^x & 0 & B^z & -B^y \\ E^y & -B^z & 0 & B^x \\ E^z & B^y & -B^x & 0 \end{pmatrix}$$

Faraday's law implies three equations:

$$\frac{\partial E^z}{\partial y} - \frac{\partial E^y}{\partial z} + \frac{\partial B^x}{\partial t} = 0$$
$$\frac{\partial E^x}{\partial z} - \frac{\partial E^z}{\partial x} + \frac{\partial B^y}{\partial t} = 0$$
$$\frac{\partial E^y}{\partial x} - \frac{\partial E^x}{\partial y} + \frac{\partial B^z}{\partial t} = 0$$

And no magnetic monopoles implies:

$$\frac{\partial B^x}{\partial x} + \frac{\partial B^y}{\partial y} + \frac{\partial B^z}{\partial z} = 0$$

$F_{\alpha\beta,\gamma} + F_{\gamma\alpha,\beta} + F_{\beta\gamma,\alpha} = 0$ implies 4^3 equations, but many are trivial. If all three indices are equal, the equation is $0 = 0$. This is due to the zeros on the diagonal of $F_{\alpha\beta}$. If two of the indices are equal, the equation becomes of the form $\frac{\partial E^\mu}{\partial x^\nu} = \frac{\partial E^\mu}{\partial x^\nu}$, because of the anti-symmetric nature of $F_{\alpha\beta}$ and the zero diagonal. An example is $F_{22,1} + F_{12,2} + F_{21,2} = \frac{\partial B^z}{\partial y} - \frac{\partial B^z}{\partial y} = 0$. Cyclic permutations of unique α, β, γ also return the same equation due to the commutativity of addition. Further, swapping of two indices (a non-cyclic permutation) returns an identical negative equation, due to the anti-symmetry of $F_{\alpha\beta}$. An example is $F_{01,2}+F_{20,1}+F_{12,0} = -F_{10,2}-F_{21,0}-F_{02,1} = 0$ Therefore, these are the nontrivial equations:

$$F_{01,2} + F_{20,1} + F_{12,0} = -\frac{\partial E^x}{\partial y} + \frac{\partial E^y}{\partial x} + \frac{\partial B^z}{\partial t}$$
$$F_{02,3} + F_{30,2} + F_{23,0} = -\frac{\partial E^y}{\partial z} + \frac{\partial E^z}{\partial y} + \frac{\partial B^x}{\partial t}$$
$$F_{10,3} + F_{31,0} + F_{03,1} = \frac{\partial E^x}{\partial z} + \frac{\partial E^z}{\partial x} + \frac{\partial B^y}{\partial t}$$
$$F_{12,3} + F_{31,2} + F_{23,1} = \frac{\partial B^z}{\partial z} + \frac{\partial B^y}{\partial y} + \frac{\partial B^x}{\partial x}$$

Which are the equations derived from the other two of Maxwell's equations.

2

call the tangent vector $\vec{r} = (r^x, r^y)$.

a)

$$r^x = \frac{dx}{d\lambda}\bigg|_{\lambda=0} = cos(\lambda)\bigg|_{\lambda=0} = 1$$

$$r^y = \frac{dy}{d\lambda}\bigg|_{\lambda=0} = -sin(\lambda)\bigg|_{\lambda=0} = 0$$

b)

$$r^x = -4\pi t sin(2\pi t^2)\bigg|_{t=0} = 0$$

$$r^y = 4\pi t cos(2\pi t^2 + \pi)\bigg|_{t=0} = 0$$

c)

$$r^x = 1\bigg|_{s=0} = 1$$

$$r^y = 1\bigg|_{s=0} = 1$$

d)

$$r^x = 2s\bigg|_{s=0} = 0$$

$$r^y = -2s\bigg|_{s=0} = 0$$

e)

$$r^x = 1\bigg|_{\mu=0} = 1$$

$$r^y = 0\bigg|_{t=0} = 0$$

3

a)

$$D(u) = \mathbb{R}$$

and

$$D(v) = \mathbb{R}$$

c) taking the differential of the definitions of x and y, we get:

$$dx = -asinh(u)cos(v)du - acosh(u)sin(v)dv$$

$$dy = acosh(u)sin(v)du + asinh(u)cos(v)dv$$

Therefore a line element is defined as:

$$ds^2 = dx^2 + dy^2 = a^2(sinh^2(u)cos^2(v) + cosh^2(u)sin^2(v))(du^2 + dv^2)$$

Simplifying,

$$ds^2 = \frac{a^2}{2}(cosh(2u) - cos(2v))(du^2 + dv^2)$$

d)Using the equation for transformation of bases:

$$\vec{e_k} = \frac{\partial x^{\bar{k}}}{\partial x^k}\vec{e_{\bar{k}}}$$

In our coordinates:

$$\vec{e_v} = \frac{\partial x}{\partial v}\vec{e_x} + \frac{\partial y}{\partial v}\vec{e_y} = -acosh(u)sin(v)\vec{e_x} + asinh(u)cos(v)\vec{e_y}$$

$$\vec{e_u} = \frac{\partial x}{\partial u}\vec{e_x} + \frac{\partial y}{\partial u}\vec{e_y} = -asinh(u)cos(v)\vec{e_x} + acosh(u)sin(v)\vec{e_y}$$

e) Definition of Christoffel symbol:

$$\Gamma^{\gamma}{}_{\beta\mu} = \frac{1}{2}g^{\alpha\nu}(g_{\alpha\beta,\mu} + g_{\alpha\mu,\beta} - g_{\beta\mu,\alpha})$$

In our case:

$$\Gamma^{v}{}_{vv} = \frac{1}{2}g^{\alpha v}(g_{\alpha v,v} + g_{\alpha v,v} - g_{vv,\alpha}) = \frac{sin(2v)}{cosh(2u) - cos(2v)}$$

$$\Gamma^{v}{}_{uv} = \Gamma^{v}{}_{vu} = \frac{1}{2}g^{\alpha v}(g_{\alpha v,u} + g_{\alpha u,v} - g_{vu,\alpha}) = \frac{-sinh(2u)}{cosh(2u) - cos(2v)}$$

$$\Gamma^{v}{}_{uu} = \frac{1}{2}g^{\alpha v}(g_{\alpha u,u} + g_{\alpha u,u} - g_{uu,\alpha}) = \frac{-sin(2v)}{cosh(2u) - cos(2v)}$$

$$\Gamma^{u}{}_{uu} = \frac{1}{2}g^{\alpha u}(g_{\alpha u,u} + g_{\alpha u,u} - g_{uu,\alpha}) = \frac{-sinh(2v)}{cosh(2u) - cos(2v)}$$

$$\Gamma^{u}{}_{vu} = \Gamma^{u}{}_{uv} = \frac{1}{2}g^{\alpha u}(g_{\alpha u,v} + g_{\alpha v,u} - g_{uv,\alpha}) = \frac{sin(2v)}{cosh(2u) - cos(2v)}$$

$$\Gamma^{u}{}_{vv} = \frac{1}{2}g^{\alpha u}(g_{\alpha v,v} + g_{\alpha v,v} - g_{vv,\alpha}) = \frac{sinh(2u)}{cosh(2u) - cos(2v)}$$

4

a) by equation of the geodesic in a non-constant metric, choosing the parameter to be arc length:

$$\frac{d}{ds}(g_{mk}\frac{dx^k}{ds}) = \frac{1}{2}\frac{\partial g_{jk}}{\partial x^m}\frac{dx^j}{ds}\frac{dx^k}{ds}$$

In our case, we get two equations:

$$\frac{d}{ds}(g_{xx}\frac{dx}{ds} + g_{xy}\frac{dy}{ds}) = \frac{1}{2}\frac{\partial g_{jk}}{\partial x}\frac{dx^j}{ds}\frac{dx^k}{ds}$$

$$\frac{d}{ds}(g_{yx}\frac{dx}{ds} + g_{yy}\frac{dy}{ds}) = \frac{1}{2}\frac{\partial g_{jk}}{\partial y}\frac{dx^j}{ds}\frac{dx^k}{ds}$$

The RHS of the first equation becomes zero because $\frac{\partial g_{jk}}{\partial x} = 0$
Integrating once, the first equation becomes:

$$g_{xy}\frac{dy}{ds} + g_{xx}\frac{dx}{ds} = constant$$

$g_{xy} = 0$ and $g_{xx} = y^{-2}$ from the definition of a line element. Therefore, taking $\frac{dx}{ds} = \dot{x}$,

$$y^{-2}\dot{x} = constant = \frac{1}{R}$$

from definition of the line element, we have:

$$y^{-2}(\dot{x}^2 + \dot{y}^2) = 1$$

Eliminating $\dot{x}$, we get:

$$y^{-2}(\frac{y^4}{R^2} + \dot{y}^2) = 1$$

$$\dot{y} = \sqrt{y^2 - \frac{y^4}{R^2}}$$

Further,

$$\frac{\dot{y}}{\dot{x}} = \frac{dy}{dx} = \sqrt{y^2 - \frac{y^4}{R^2}}\frac{R}{y^2} = \sqrt{\frac{R^2}{y^2} - 1}$$

Integrating this function, we get:

$$\int dx = \int \frac{y dy}{\sqrt{R^2 - y^2}}$$

$$x = -\sqrt{R^2 - y^2} - c$$

Rearranging, we find that geodesics are semicircles with diameters on the x axis:

$$(x - c)^2 + y^2 = R^2$$