AST 301 Problem Set 8

Nathan Spilker

December 6 2019

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a) We must prove that $\boldsymbol{\xi} \cdot \mathbf{U}$ is a constant of motion, where U is the tangent of a geodesic, or equivalently:

$$\nabla_{\mathbf{U}}(\boldsymbol{\xi} \cdot \mathbf{U}) = 0$$

This can be rewritten as:

$$\nabla_{\mathbf{U}}(\xi_\mu U^\mu) = 0$$

This is equivalent to (Schutz 6.48)

$$U^\beta \nabla_\beta(\xi_\mu U^\mu) = 0$$

Applying Leibniz's product rule:

$$U^\beta(U^\mu \xi_{\mu;\beta} + \xi_\mu U^\mu{}_{;\beta}) = 0$$

A geodesic parallel transports its own tangent vector, so

$$U^\beta U^\mu{}_{;\beta} = 0$$

We are left with:

$$U^\beta U^\mu \xi_{\mu;\beta} = 0$$

From the killing equation, we have:

$$\xi_{\mu;\nu} + \xi_{\nu;\mu} = 0$$

In the sum two lines above, each component will subtract identically from another component. Equivalently $\xi_{\mu;\nu}$ is an anti-symmetric tensor while $U^\mu U^\nu$ is a symmetric tensor.

$$U^{01}U^{10}\xi_{0;1} + U^{10}U^{01}\xi_{1;0} = 0$$

due to killing's equation. Therefore it is satisfied that

$$\nabla_{\mathbf{U}}(\boldsymbol{\xi} \cdot \mathbf{U}) = 0$$

b) We have:

$$K^{\mu} = T^{\mu\nu}\xi_{\nu}$$

Differentiating both sides with respect to μ:

$$K^{\mu}{}_{;\mu} = (T^{\mu\nu}\xi_{\nu})_{;\mu}$$

By Leibniz's product rule:

$$= T^{\mu\nu}\xi_{\nu;\mu} + T^{\mu\nu}{}_{;\mu}\xi_{\nu}$$

We have $T^{\mu\nu}{}_{;\mu} = 0$ by conservation of energy and momentum. Now our equation is:

$$= T^{\mu\nu}\xi_{\nu;\mu}$$

$\xi_{\mu;\nu}$ is an anti-symmetric tensor while $T^{\mu\nu}$ is a symmetric tensor, so the sum will be 0. We have then proved:

$$K^{\mu} = 0$$

c) From the problem statement we have:

$$\xi_{\mu;\nu} + \xi_{\nu;\mu} = (g_{\mu\gamma}\delta_0^{\gamma})_{;\nu} + (g_{\nu\gamma}\delta_0^{\gamma})_{;\mu} = 0$$

$$= g_{\mu 0;\nu} + g_{\nu 0;\mu}$$

Expanding the covariant derivatives we get:

$$= g_{\mu 0,\nu} - g_{\lambda 0}\Gamma^{\lambda}_{\mu\nu} + g_{\nu 0,\mu} - g_{\lambda 0}\Gamma^{\lambda}_{\nu\mu}$$

From the definition of Christoffel symbols we get:

$$= g_{\mu 0,\nu} - \frac{1}{2}g_{\lambda 0}g^{\gamma\lambda}(g_{\gamma\mu,\nu} + g_{\gamma\mu,\nu} - g_{\nu\mu,\gamma}) + g_{\nu 0,\mu} - \frac{1}{2}g_{\lambda 0}g^{\gamma\lambda}(g_{\gamma\nu,\mu} + g_{\gamma\mu,\nu} - g_{\nu\mu,\gamma})$$

Simplifying:

$$= g_{\mu 0,\nu} - \frac{1}{2}\delta_0^{\gamma}(g_{\gamma\mu,\nu} + g_{\gamma\nu,\mu} - g_{\nu\mu,\gamma}) + g_{\nu 0,\mu} - \frac{1}{2}\delta_0^{\gamma}(g_{\gamma\nu,\mu} + g_{\gamma\mu,\nu} - g_{\nu\mu,\gamma})$$

$$= g_{\mu 0,\nu} - \frac{1}{2}(g_{0\mu,\nu} + g_{0\nu,\mu} - g_{\nu\mu,0}) + g_{\nu 0,\mu} - \frac{1}{2}(g_{0\nu,\mu} + g_{0\mu,\nu} - g_{\nu\mu,0})$$

$$= g_{\nu\mu,0}$$

This proves that the metric does not depend on x^0.

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a) We have:

$$F_{\alpha\beta;\gamma} + F_{\gamma\alpha;\beta} + F_{\beta\gamma;\alpha} = 0$$

Expanding the covariant derivatives, we have:

$$F_{\alpha\beta,\gamma} - F_{\mu\beta}\Gamma^{\mu}_{\alpha\gamma} - F_{\alpha\mu}\Gamma^{\mu}_{\beta\gamma} + F_{\gamma\alpha,\beta} - F_{\mu\alpha}\Gamma^{\mu}_{\gamma\beta} - F_{\gamma\mu}\Gamma^{\mu}_{\alpha\beta} + F_{\beta\gamma,\alpha} - F_{\mu\gamma}\Gamma^{\mu}_{\beta\alpha} - F_{\beta\mu}\Gamma^{\mu}_{\gamma\alpha} = 0$$

$F^{\mu\nu}$ is anti-symmetric and Christoffels are symmetric on their lower two indices:

$$F_{\alpha\beta,\gamma} + F_{\gamma\alpha,\beta} + F_{\beta\gamma,\alpha}(F_{\mu\beta}\Gamma^{\mu}_{\alpha\gamma} - F_{\mu\beta}\Gamma^{\mu}_{\alpha\gamma}) + (F_{\alpha\mu}\Gamma^{\mu}_{\beta\gamma} - F_{\alpha\mu}\Gamma^{\mu}_{\beta\gamma}) + (F_{\gamma\mu}\Gamma^{\mu}_{\alpha\beta} - F_{\gamma\mu}\Gamma^{\mu}_{\alpha\beta}) = 0$$

Which simplifies to:

$$F_{\alpha\beta,\gamma} + F_{\gamma\alpha,\beta} + F_{\beta\gamma,\alpha} = 0$$

b) We have:

$$F^{\mu\nu}{}_{,\nu} = 4\pi J^{\mu}$$

Differentiating both sides with respect to μ (minkowskian metric):

$$F^{\mu\nu}{}_{,\nu\mu} = 4\pi J^{\mu}_{,\mu}$$

$F^{\mu\nu}$ is anti-symmetric:

$$-F^{\nu\mu}{}_{,\nu\mu} = 4\pi J^{\mu}_{,\mu}$$

Because partial derivatives commute we have:

$$-F^{\nu\mu}{}_{,\mu\nu} = 4\pi J^{\mu}_{,\mu}$$

We also have:

$$F^{\mu\nu}{}_{,\nu\mu} = 4\pi J^{\mu}_{,\mu}$$

We are free to switch dummy indices:

$$F^{\nu\mu}{}_{,\mu\nu} = 4\pi J^{\mu}_{,\mu}$$

Which proves that

$$J^{\mu}_{,\mu} = 0$$

c) We start with:

$$F^{\mu\nu}{}_{;\nu} = 4\pi J^{\mu}$$

Differentiating with respect to μ:

$$F^{\mu\nu}{}_{;\nu\mu} = 4\pi J^{\mu}{}_{;\mu}$$

We can write:

$$\frac{1}{2}(F^{\mu\nu}{}_{;\nu\mu} - F^{\nu\mu}{}_{;\nu\mu}) = 4\pi J^{\mu}{}_{;\mu}$$

We are free to change dummy indices

$$\frac{1}{2}(F^{\mu\nu}{}_{;\nu\mu} - F^{\mu\nu}{}_{;\mu\nu}) = 4\pi J^{\mu}{}_{;\mu}$$

Multiplying both sides by a negative 1

$$\frac{1}{2}(F^{\mu\nu}{}_{;\mu\nu} - F^{\mu\nu}{}_{;\nu\mu}) = -4\pi J^{\mu}{}_{;\mu}$$

We have the identity for commutators of 2,0 tensors:

$$F^{\mu\nu}{}_{;\mu\nu} - F^{\mu\nu}{}_{;\nu\mu} = F^{\sigma\nu} R^{\mu}_{\sigma\nu\mu} + F^{\mu\sigma} R^{\nu}_{\sigma\nu\mu}$$

After substituting this identity into the above equation, we are left with:

$$8\pi J^{\mu}{}_{;\mu} = -F^{\sigma\nu} R^{\mu}_{\sigma\nu\mu} - F^{\mu\sigma} R^{\nu}_{\sigma\nu\mu}$$

Because Riemann tensor is anti-symmetric on the last two indices:

$$= -F^{\sigma\nu} R^{\mu}_{\sigma\nu\mu} + F^{\mu\sigma} R^{\nu}_{\sigma\mu\nu}$$

Relabeling indices:

$$= -F^{\sigma\mu} R^{\nu}_{\sigma\mu\nu} + F^{\mu\sigma} R^{\nu}_{\sigma\mu\nu}$$

Because $F^{\mu\nu}$ is anti-symmetric:

$$= F^{\mu\sigma} R^{\nu}_{\sigma\mu\nu} + F^{\mu\sigma} R^{\nu}_{\sigma\mu\nu}$$

$$= 2F^{\mu\sigma} R^{\nu}_{\sigma\mu\nu}$$

Riemann tensor is anti-symmetric on the last two indices:

$$= -2F^{\mu\sigma} R^{\nu}_{\sigma\nu\mu}$$

We recognize this as the Ricci tensor

$$= -2F^{\mu\sigma} R_{\sigma\mu}$$

This sum is equal to zero because $F^{\mu\nu}$ is an anti-symmetric tensor and the ricci tensor is symmetric. Therefore,

$$J^{\mu}{}_{;\mu} = 0$$

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a) Calculated nonzero Christoffels with Mathematica script, (code attached)

$$\Gamma^t_{yy} = -e^{2f(x-t)}f^{'}(x-t)$$

$$\Gamma^t_{zz} = -e^{2g(x-t)}g^{'}(x-t)$$

$$\Gamma^x_{yy} = -e^{2f(x-t)}f^{'}(x-t)$$

$$\Gamma^x_{zz} = -e^{2g(x-t)}g^{'}(x-t)$$

$$\Gamma^y_{yt} = \Gamma^y_{ty} = -f^{'}(x-t)$$

$$\Gamma^z_{zt} = \Gamma^z_{tz} = -g^{'}(x-t)$$

$$\Gamma^y_{yx} = \Gamma^y_{xy} = f^{'}(x-t)$$

$$\Gamma^z_{zx} = \Gamma^z_{xz} = g^{'}(x-t)$$

b) Components of the Riemann tensor calculated through the mathematica script as well.

c)

d) Definition of the Ricci tensor:

$$R_{\alpha\beta} = R^\mu_{\alpha\mu\beta}$$

Vacuum field equations state that:

$$R_{\alpha\beta} = 0$$

From $R^x_{zxz} = 0$ we get:

$$2g'(x-t)^2 + g''(x-t) = 0$$

From $R^y_{tyt} = 0$ we get:

$$2f'(x-t)^2 + f''(x-t) = 0$$