1 The signal model

During each time step k, let s_k^i be the signal at channel i. We assume that this signals are generated by independent Gaussian random processes, with mean denoted by $\mu_i(k \mid u_k, \theta_k, H_k)$, where u_k is the intended velocity of the subject, θ_k is the parameterization of the system, and H_k is the history of the signals s_i and intended velocities u up through time k. The mean $\mu_i(k \mid u_k, \theta_k, H_k)$ is given by

$$\begin{aligned}\mu^i(k \mid u_k, \theta_k, H_k) =& a_0^i(u_0)_x + a_{-1}^i(u_{-1})_x + \ldots + a_{-(L-1)}^i(u_{-(L-1)})_x + \\ & b_0^i(u_0)_y + b_{-1}^i(u_{-1})_y + \ldots + b_{-(L-1)}^i(u_{-(L-1)})_y + \\ & c_0^i(u_0)_z + c_{-1}^i(u_{-1})_z + \ldots + c_{-(L-1)}^i(u_{-(L-1)})_z,\end{aligned} \tag{1.1}$$

where L is the constant number of time lags into the past that the signal at time step k is dependent on, and each a_j^i, b_j^i, and c_j^i is a parameter of the system. The variance $(\sigma^i)^2$ of the signal at each channel is assumed to be known and constant.

1.1 Implementation

For each channel, the variance of the signal is currently estimated by taking the sample variance of the signals received in a 2-second interval before the start of any trials. The mean of this sample is also subtracted from the signal at the channel before being passed to the filter, so that the subject would have the opportunity to drive the value of the signal into either positive or negative voltages, allowing the conribution of the signal at the channel to be in one of two opposite directions.

1.2 Potential modifications

- The variance $(\sigma^i)^2$ could be added to the state, so that if its value evolves over time the filter would be able to track it.

- To attenuate the effect that large signals might have on overpowering the signal from other channels, we could switch from a linear firing model to a sigmoid one.

2 The state vector

The state vector x_k of the system is defined as a $(3NL+6)\times 1$ vector that takes the form

$$\begin{aligned}
&\begin{bmatrix} a_0^0 & a_{-1}^0 & \cdots & a_{-(L-1)}^0 & b_0^0 & \cdots & b_{-(L-1)}^0 & c_0^0 & \cdots & c_{-(L-1)}^0 \end{bmatrix}^T \oplus \\
&\begin{bmatrix} a_0^1 & a_{-1}^1 & \cdots & a_{-(L-1)}^1 & b_0^1 & \cdots & b_{-(L-1)}^1 & c_0^1 & \cdots & c_{-(L-1)}^1 \end{bmatrix}^T \oplus \\
&\qquad\qquad \vdots \\
&\begin{bmatrix} a_0^{N-1} & a_{-1}^{N-1} & \cdots & a_{-(L-1)}^{N-1} & b_0^{N-1} & \cdots & b_{-(L-1)}^{N-1} & c_0^{N-1} & \cdots & c_{-(L-1)}^{N-1} \end{bmatrix}^T \oplus \\
&\begin{bmatrix} p_x & p_y & p_z & v_x & v_y & v_z \end{bmatrix}^T,
\end{aligned} \tag{2.1}$$

where each a_j^i, b_j^i, and c_j^i is a parameter that corresponds to channel i and time step j, as described in "The signal model." p_x, p_y, and p_z comprise the position of the cursor, and v_x, v_y, and v_z comprise its velocity.

3 The filter derivation

We assume that the state x_k of the system evolves according to

$$x_{k+1} = F_k x_k + \nu_k, \tag{3.1}$$

where F_k is the state evolution matrix and ν_k is zero-mean Gaussian noise with covariance matrix Q_k. Both of these matrices are determined from their corresponding free movement matrices by the Joint RSE training method.

To derive the desired update equations for the mean and covariance of the state, we must find expressions describing the distribution $p(x_k \mid s_k, H_k)$. By Bayes's rule, we have that

$$p(x_k \mid s_k, H_k) = \frac{p(s_k \mid x_k, H_k)p(x_k \mid H_k)}{p(s_k \mid H_k)}. \tag{3.2}$$

Since we assumed a Gaussian distribution on the signal at each channel i, the first term in the numerator $p(s_k \mid x_k, H_k)$ is proportional to

$$\prod_{i=0}^{N-1} \exp\left(-\frac{1}{2}[(s_k^i - \mu_k^i)/\sigma^i]^2\right). \tag{3.3}$$

The second term in the numerator of the Bayesian expression is given by

$$p(x_k \mid H_k) = \int p(x_k \mid x_{k-1}, H_k)p(x_{k-1} \mid s_{k-1}, H_{k-1})\, dx_{k-1}. \tag{3.4}$$

Letting $x_{k|k-1} = \mathrm{E}[x_k \mid H_k]$, $W_{k|k-1} = \mathrm{var}[\theta_k \mid H_k]$, $x_{k|k} = \mathrm{E}[x_k \mid s_k, H_k]$ and $W_{k|k} = \mathrm{var}[x_k \mid s_k, H_k]$, and using the formula for convolution of Gaussians we get that

$$p(x_k \mid H_k) = \exp\left(-\frac{1}{2}(x_k - x_{k|k-1})^T(W_{k|k-1})^{-1}(x_k - x_{k|k-1})\right). \tag{3.5}$$

Since the denominator in 3.2 does not depend on x_k, it serves only as a normalizing factor, so plugging equations 3.3 and 3.5 into 3.2, we get that

$$\begin{aligned} p(x_k \mid s_k, H_k) \propto & \left[\prod_{i=0}^{N-1} \exp\left(-\frac{1}{2}[(s_k^i - \hat{s}_k^i)/\sigma^i]^2\right)\right] \cdot \\ & \exp\left(-\frac{1}{2}(x_k - x_{k|k-1})^T(W_{k|k-1})^{-1}(x_k - x_{k|k-1})\right). \end{aligned} \tag{3.6}$$

Assuming that x_k has a Gaussian distribution, we also have that

$$p(x_k \mid s_k, H_k) \propto \exp\left(-\frac{1}{2}(x_k - x_{k|k})^T(W_{k|k})^{-1}(x_k - x_{k|k})\right), \tag{3.7}$$

so the right-hand expressions of both 3.6 and 3.7 are proportional to each other. Taking the log of both these expressions, we get that

$$-\frac{1}{2}(x_k - x_{k|k})^T (W_{k|k})^{-1}(x_k - x_{k|k}) = -\frac{1}{2}(x_k - x_{k|k-1})^T (W_{k|k-1})^{-1}(x_k - x_{k|k-1}) - \sum_{i=0}^{N-1} \frac{1}{2}[(s_k^i - \mu_k^i)/\sigma^i]^2 + R, \tag{3.8}$$

where R is a catch-all constant with no dependence on x_k. Differentiating this with respect to x_k, we get that

$$(W_{k|k})^{-1}(x_k - x_{k|k}) = (W_{k|k-1})^{-1}(x_k - x_{k|k-1}) + \sum_{i=0}^{N-1} \frac{1}{(\sigma^i)^2}(-s_k^i + \mu_k^i)\frac{\partial \mu_k^i}{\partial x_k}. \tag{3.9}$$

Since this equation should hold true for all x_k, if we let $x_k = x_{k|k-1}$, we get that

$$(W_{k|k})^{-1}(x_{k|k-1} - x_{k|k}) = \sum_{i=0}^{N-1} \left[\frac{1}{(\sigma^i)^2}(-s_k^i + \mu_k^i)\frac{\partial \mu_k^i}{\partial x_k}\right]_{x_{k|k-1}} \tag{3.10}$$

$$\iff x_{k|k} = x_{k|k-1} + W_{k|k} \cdot \sum_{i=0}^{N-1} \left[\frac{1}{(\sigma^i)^2}(s_k^i - \mu_k^i)\frac{\partial \mu_k^i}{\partial x_k}\right]_{x_{k|k-1}}, \tag{3.11}$$

giving us the update equation for the state. Taking the derivative of 3.9 again, we obtain

$$(W_{k|k})^{-1} = (W_{k|k-1})^{-1} + \sum_{i=0}^{N-1} \frac{1}{(\sigma^i)^2}\left[\left(\frac{\partial \mu_k^i}{\partial x_k}\right)\left(\frac{\partial \mu_k^i}{\partial x_k}\right)^T + (-s_k^i + \mu_k^i)\frac{\partial \mu_k^i}{\partial x_k\,\partial x_k^T}\right]_{x_{k|k-1}}, \tag{3.12}$$

giving us the update equation for the covariance, as desired.