1. COSMOLOGY (EARLY UNIVERSE, CMB, LARGE-SCALE STRUCTURE)

1.1. *A Very Brief Primer on Cosmology*

Just like in stars far too many question depend on the equations of stellar structure, in cosmology too many questions depend on the basic underpinnings of the FLRW universe. We will summarize the results below. This information comes from Ch. 4 and 5 of Ryden (2003).

1.1.1. *The FLRW Universe*

In accordance with the cosmological principle (that there be a set of observers that see the universe as homogeneous and isotropic), the spatial extent of the universe must have uniform curvature (unless we move to truly non-trivial geometries). This restricts our metric to be of a form known as the Robertson-Walker metric

$$ds^2 = cdt^2 - a(t)^2 \left(\frac{dx^2}{1 - \kappa x^2/R^2} + x^2 d\Omega^2 \right) \tag{1}$$

where $\kappa = -1$, 0 1 and R scales κ. Another way of writing this metric (and making it perhaps more palatable) is

$$ds^2 = cdt^2 - a(t)^2 \left(dr^2 + S_\kappa^2(r) d\Omega^2 \right) \tag{2}$$

where

$$S_\kappa = \begin{cases} R\sin(r/R) & \text{if } \kappa = 1 \\ r & \text{if } \kappa = 0 \\ R\sinh(r/R) & \text{if } \kappa = -1 \end{cases}, \tag{3}$$

Writing the metric in this form shows that if κ is non-zero, angular lengths are either decreased (for positive curvature) or increased (for negative). Just as importantly, this metric indicates that, like in Minkowski space, time is orthogonal to position (meaning we can foliate the spacetime such that each hypersurface slice can be associated with a single time t) and radial distances are independent of curvature S_κ.

The solution to the RW metric is known as the Friedmann-Lemaître equation, and describes how the scale factor $a(t)$ changes with time:

$$\left(\frac{\dot{a}}{a} \right)^2 = H^2 = \frac{8\pi G}{3}\rho - \frac{\kappa c^2}{R^2}\frac{1}{a^2} + \frac{\Lambda c^2}{3} \tag{4}$$

where ρ is the matter-energy density of the universe, $\kappa c^2/R^2$ the curvature term, λ the cosmological constant and a the scale factor in the RW metric. This expression can also be derived (but becomes unscrutable because the terms make little sense) by representing the universe by a homologously expanding sphere (i.e. an explosion at $t = 0$), and considering the dynamics of a test mass within this universe. If $r = r_0 a$ (noting this automatically produces $v = r_0\dot{a}$, so $v/r = \frac{\dot{a}}{a} = H$, reproducing Hubble's law), we can integrate $\frac{d^2r}{dt^2} = \frac{4\pi G r\rho}{3}$[1] to get Eqn. 4 (with Λ subsumed into a constant of integration). Even more easily, we can do the same with energy balance, $K + U = E$.

In a Λ-free universe, if a value of H^2 is given, ρ and κ/R^2 are linked, and there is a critical density

$$\rho_c = \frac{3H^2}{8\pi G}, \tag{5}$$

for which the universe is flat. We define $\Omega \equiv \rho/\rho_c$. We can then rewrite the FL solution as $1 - \Omega = \frac{-\kappa c^2}{R^2 a^2 H^2}$. Note that the right side cannot change sign! This means that if at any time $\rho > \rho_c$, the universe will forever be closed; if $\rho < \rho_c$, the universe will forever be open, and if $\rho = \rho_c$, the equality will hold for all time and the universe will be flat.

1.1.2. *The Fluid and Acceleration Equations*

From the first law of thermodynamics and an assumption of adiabaticity, $dE + PdV = 0$. $\dot{V}/V = 3\dot{a}a$, and $dE/dt = \rho c^2 dV/dt + c^2 V d\rho/dt$. This gives us

$$\dot{\rho} + 3\frac{\dot{a}}{a}(\rho + P/c^2). \tag{6}$$

This can be combined with the FL equation to get

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + 3\frac{P}{c^2} \right) + \frac{\Lambda c^2}{3} \tag{7}$$

[1] Recall that $2\dot{a}\ddot{a} = \frac{d}{dt}\dot{a}^2$.

1.1.3. *Equations of State*

Cosmologists generally use equations of state that look like

$$P = \omega c^2 \rho. \tag{8}$$

The ideal gas law, for example, has an ω that is dependent on temperature, and therefore time. "Dust", which is pressure-free, has $\omega = 0$ - stars exert little enough pressure to be considered a dust. A relativistic equation of state always has $P = \frac{1}{3}\rho c^2$, including photons. Dark energy has $\omega = -1$. In substances with positive ω, $\sqrt{\frac{P}{\rho}} = c_s \leq c$, which restricts $w \leq 1$.

Combining Eqn. 6 with $P = \omega c^2 \rho$ gives us $\rho = \rho_0 a^{-3(1+\omega)}$. From this we determine that matter density $\rho = \rho_0 a^{-3}$ and radiation density $\rho_r = \rho_{r,0} a^{-4}$. We can compare the densities of any component to the critical density to obtain Ω. For example, $\rho/\rho_c = \rho/(3H^2/8\pi G) = \frac{8\pi G}{3}\rho/H^2$. We then note that $\frac{8\pi G}{3}\rho = \frac{8\pi G}{3}\rho\frac{\rho_0}{\rho_0} = H_0^2 \Omega_{m,0} a^{-3}$ - conversions such as this will be useful in the following section. Note that $\rho_\Lambda = \frac{\Lambda c^2}{8\pi G}$, giving $\Omega_\Lambda = \frac{\Lambda c^2}{3H^2}$.

Taking ratios of Ωs gives us the energy component that dominates (ex. radiation to matter is $\Omega_r/\Omega_m = \rho_{r,0}/\rho_{m,0}\frac{1}{a} \approx \frac{1}{3600a}$ if $a_0 = 1$, indicating there was a time when radiation dominanted the energy of the universe.

A related question is how the radiation field temperature scales with time. Assuming adiabaticity, $dQ = dE + PdV$ (the work is being done by the radiation field on the universe). Since $P = \frac{1}{3}U = \frac{1}{3}aT^4$ we obtain $\frac{1}{T}\frac{dT}{dt} = -\frac{1}{3V}dVdT$, which implies (through integration and the fact that V scales like a^{-3}) that $T \propto a^{-1}$.

1.1.4. *History of Expansion*

Let us consider several possibilities:

- An empty flat universe is static. An empty, open universe goes like $a = ct/R$. An empty, closed universe is impossible.
- A flat universe with a single component would have $\dot{a}^2 = \frac{8\pi G}{3}\rho_0 a^{-(1+3\omega)}$. This gives $a \propto t^{2/(3+3\omega)}$.
- A Λ-dominated universe would have $\dot{a}^2 = \frac{\Lambda c^2}{3}a^2$, which gives $a \propto \exp(\sqrt{\Lambda c^2/3}t)$. We note we could have snuck Λ into the energy density of the universe if we set $\omega = -1$ and $\rho_0 = c^2/8\pi G$.

We may now consider a universe with radiation, stars, and a cosmological constant. Since $\kappa/R^2 = \frac{H_0^2}{a^2}(\Omega_0 - 1)$ (so that R may be written as $\frac{c}{H_0}\sqrt{|\Omega_\kappa|}$), we can actually write the FL equation as $H^2 = \frac{8\pi G}{3}\rho - \frac{H_0^2}{a^2}(\Omega_0 - 1)$, where ρ includes radiation, matter and dark energy, and if we divided by H_0^2, we get

$$\frac{H^2}{H_0^2} = \frac{\Omega_{r,0}}{a^4} + \frac{\Omega_{m,0}}{a^3} + \Omega_{\Lambda,0} + \frac{1-\Omega_0}{a^2} \tag{9}$$

where $\Omega_0 = \Omega_{r,0} + \Omega_{m,0} + \Omega_{\Lambda,0}$. Note how the curvature still responds to the total matter-energy density in the universe, but the expansion history may now be altered by Λ. Assuming that radiation is negligible, Fig. 16 describes the possible fates and curvatures of the universe.

Using measured values of the Ωs, we find the universe to have the expansion history given in Fig. 2.

1.1.5. *Distance and Size Measurements*

The redshift z is given by

$$1 + z = \frac{\lambda_0}{\lambda_e} = \frac{a_0}{a_e} \tag{10}$$

where subscript e stands for "emitted".

Taylor expanding the current a, we obtain $a(t) \approx a(t_0) + \dot{a}|_{t=t_0}(t - t_0) + \frac{1}{2}\ddot{a}|_{t=t_0}(t - t_0)^2$. Dividing both sides by a_0 (which is equal to 1) we get $1 + H_0(t - t_0) - \frac{1}{2}q_0 H_0^2 (t - t_0)^2$. $q_0 = -\ddot{a}/aH^2|_{t=t_0}$ is known as the deceleration parameter, and helps constrain the makeup of the universe, since $q_0 = \frac{1}{2}\sum_\omega \Omega_\omega(1 + 3\omega)$.

The comoving distance (interpretable as how distant the object would be today) to an object whose light we are seeing is given by

$$d_c(t_0) = c\int_{t_e}^{t_0} \frac{dt}{a}, \tag{11}$$

which can be converted into $d_c = c\int_0^z \frac{dz}{H}$. Since radial distances are not curvature-dependent, the proper (physical) distance is simply given by (Davis & Lineweaver 2004)

$$d_p(t) = a(t)d_c \tag{12}$$

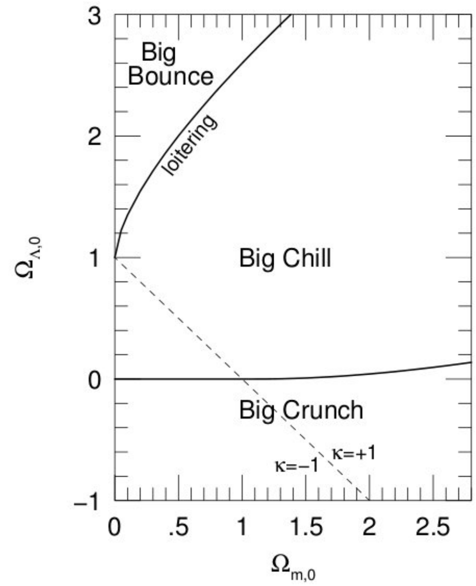

FIG. 1.— Fate of the universe as a function of Ω_m and Ω_Λ. From Ryden (2003), her Fig. 6.3.

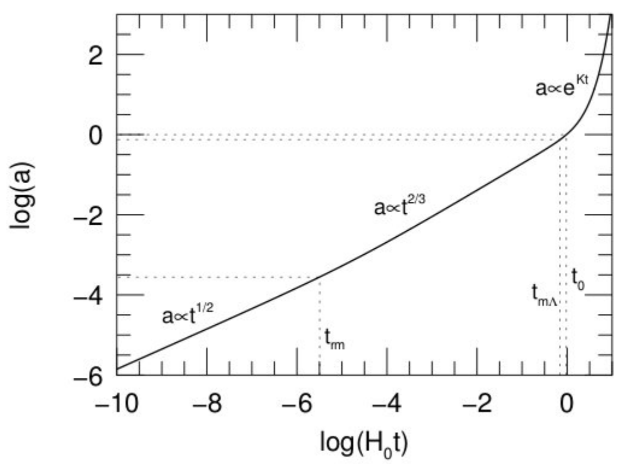

FIG. 2.— Fate of the universe, using measured values of Ω_i. From Ryden (2003), her Fig. 6.5.

where $a_0 = 1$ is assumed, and t represents time since the beginning of the universe. The luminosity distance is defined as

$$d_L = \sqrt{\frac{L}{4\pi F}} = S_\kappa(r)(1+z). \tag{13}$$

The second expression is due to two factors - first, the expansion of space drops energy with $1+z$, and increases the thickness of any photon shell dr by $1+z$ as well. The area covered by the wave of radiation is $4\pi S_\kappa^2(r)$ ($S_\kappa = r$ for a flat universe), where r should be interpreted as the comoving distance d_c. The angular diameter distance $d_A = \frac{l}{d\theta}$ (l is the *physical* length of an object at the time the light being observed was emitted) is given by the fact that $ds = a(t_e)S_\kappa(r)d\theta$. If the length l is known, then $ds = l$ and we obtain

$$d_A = \frac{S_\kappa}{1+z}. \tag{14}$$

Note that the angular diameter distance is related to the luminosity distance by $d_A = d_L/(1+z)^2$. For $z \to 0$, all these distances are equal to cz/H_0, but at large distances they begin to differ significantly.

1.2. *Question 1*