Note that l is known, but S_κ depends on the co-moving distance between us and the CMB. This requires some knowledge of the subsequent expansion history of the universe, or else there is a degeneracy between Ω_m, Ω_Λ and Ω_κ (Komatsu et al. 2009). An additional constraint, such as a measurement of H_0, or the series of luminosity distance measurements using high-z SNe, allows us to constrain Ω_κ (Komatsu et al. 2009). See Fig. 5.

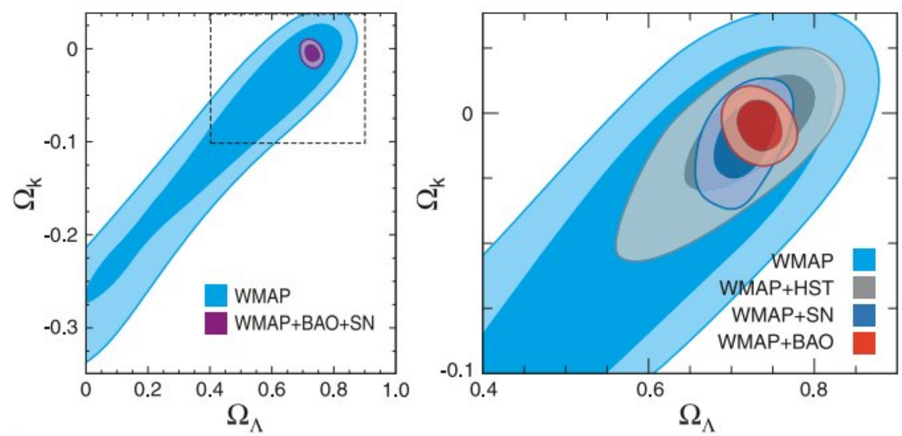


FIG. 5.— Joint two-dimensional marginalized constraint on the dark energy density Ω_Λ, and the spatial curvature parameter, Ω_κ. The contours show the 68% and 95% confidence levels. Additional data is needed to constrain Ω_κ: HST means H_0 from Hubble measurements, SN means luminosity distances from high-z SN, and BAO means baryon acoustic oscillation measurements from galaxy surveys. From Komatsu et al. (2009), their Fig. 6.

1.4. *Question 3*

QUESTION: Outline the development of the Cold Dark Matter spectrum of density fluctuations from the early universe to the current epoch.

Most of this information is from Schneider (2006), Ch. 7.3 - 7.5.

The growth of a single perturbation (described as one of the follow-up questions) in a matter-dominated universe can be described in the following way. We define the relative density contrast $\delta(\vec{r},t) = (\rho(\vec{r},t) - \bar{\rho})/\bar{\rho}$; from this $\delta(\vec{r},t) \leq -1$. At $z \sim 1000$ $|\delta(\vec{r},t)| << 1$. The mean density of the universe $\bar{\rho}(t) = (1+z^3)\bar{\rho}_0 = \bar{\rho}_0/a(t)^3$ from Hubble flow. Like in the classic Newtonian stability argument of an infinite static volume of equally space stars, any overdense region will experience runaway collapse (and any underdense region will become more and more underdense). In the linear perturbative regime, the early stages of this collapse simply make it so that the the expansion of the universe is delayed, so $\delta(\vec{r},t)$ increases. As it turns out, $\delta(\vec{r},t)$ can be written as $D_+(t)\delta_0(\vec{x})$ in the linear growth regime. $D_+(t)$ is normalized to be unity today, and $\delta_0(\vec{x})$ is the linearly-extrapolated (i.e. no non-linear evolution taken into account) density field today.

The two-point correlation function $\xi(r)$ (Sec. 1.18) describes the over-probability of, given a galaxy at $r=0$, there will be another galaxy at r (or x, here). It describes the clustering of galaxies, and is key to understanding the large-scale structure of the universe. We define the matter power spectrum (often shortened to just "the power spectrum") as

$$P(k) = \int_{-\infty}^{-\infty} \xi(r) exp(-ikr) r^2 dr \tag{20}$$

Instead of describing the spatial distribution of clustering, the power spectrum decomposes clustering into characteristic lengths $L \approx 2\pi/k$, and describes to what degree each characteristic contributes to the total overprobability.

Since the two-point correlation function depends on the square of density, if we switch to co-moving coordinates and stay in the linear regime,

$$\xi(x,t) = D_+^2(t)\xi_0(x,t_0). \tag{21}$$

Likewise,

$$P(k,t) = D_+^2 P(k,t_0) \equiv D_+^2 P_0(k), \tag{22}$$

i.e. everything simply scales with time. This the evolution of the power spectrum is reasonably easily described.

The initial power spectrum $P_0(k)$ was generated by the quantum fluctuations of inflation. It can be argued (pg. 285 of Schneider (2006)) that the primordial power spectrum should be $P(k) = Ak^{n_s}$, where A is a normalization factor that can only be determined empirically. $P(k)$ when $n_s = 1$ is known as the Harrison-Zel'dovich spectrum, which is most commonly used.

An additional correction term needs to be inserted is the transfer function to account for evolution in the radiation-dominated universe, where our previous analysis does not apply. We thus introduce the transfer function $T(k)$, such that $P_0(k) = Ak^{n_s}T(k)^2$. $T(k)$ is dependent on whether or not the universe consists mainly of cold or hot ($k_B T << mc^2$, where T is the temperature at matter-radiation equality) dark matter. If hot dark matter dominate sthe universe, they freely stream out of minor overdensities, leading to a suppression of small-scale perturbations. Since our universe is filled with cold dark matter, this need not be taken into account (and indeed gives results inconsistent with observations). $T(k)$ also accounts for the fact that $a(t) \propto t^{1/2}$ rather than $t^{2/3}$ during radiation domination, and that physical interactions can only take place on scales smaller than $r_{H,c}(t)$ (the co-moving particle horizon) - on scales larger than this GR perturbative theory must be applied.

Growth of a perturbation of length scale L is independent of growths at other length scales. The growth of a density qualitatively goes like this:

1. In the early universe, a perturbation of comoving length L has yet to enter the horizon. According to relativistic perturbation theory, the perturbation grows $\propto a^2$ in a radiation-dominated universe, and $\propto a$ in a matter-dominated universe.

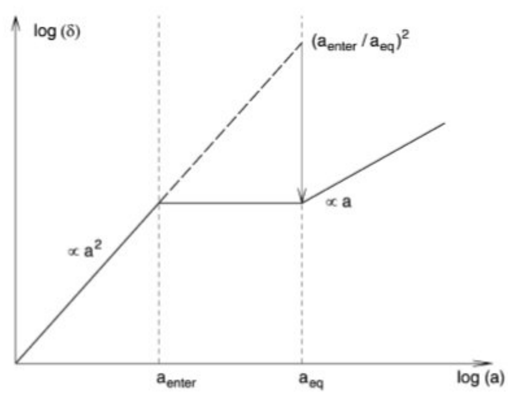
2. At redshift z_e, when $r_{H,c}(z_e) = L$, the perturbation length scale becomes smaller than the horizon. If the universe is still radiation-dominated, the Mészáros effect prevents effective perturbation growth, and the overdensity stalls (Mészáros 2005).

3. Once the universe becomes matter dominated ($z < z_{\rm eq}$), the perturbation continues to grow $\propto a$.

There is therefore a preferred length scale $L_0 = r_{H,c}(z_{\rm eq}) \approx 12(\Omega_m h^2)^{-1}$ Mpc. The transfer function then has two limiting cases: $T(k) \approx 1$ for $k << 1/L_0$, and $T(k) \approx (kL_0)^{-2}$ for $k >> 1/L_0$. This generates a turnover in $P_0(k)$ where $k = 1/L_0$. Note that due to the dependence on the sound horizon on $\Omega_m h^2$ we often define the shape parameter $\Gamma = \Omega_m h$.

One last modification must be made to this picture: at a certain point growth becomes non-linear, and our analysis must be modified.

Fig. 6 shows the schematic growth of a perturbation, as well as both the primordial Harrison-Zel'dovich spectrum and the modern-day power spectrum for a series of different cosmological parameters.



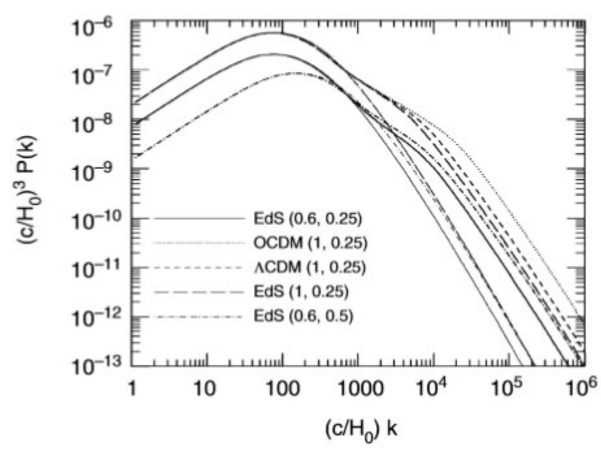


FIG. 6.— Left: evolution of a density perturbation. The $(a_{\rm enter}/a_{\rm eq})^2$ line indicates the degree of suppression during radiation domination. Right: the current power spectrum of density fluctuations for CDM models. The various curves have different cosmological models (EdS, $\Omega_m = 1$, $\Omega_\Lambda = 0$, OCDM, $\Omega_m = 0.3$, $\Omega_\Lambda = 0$, ΛCDM, $\Omega_m = 0.3$, $\Omega_\Lambda = 0.7$). Values in paretheses specify (σ_8, Γ). The thin curves correspond to power spectra linearly extrapolated, while the thick curves include non-linear corrections. From Schneider (2006), his Figs. 7.5 and 7.6.

1.4.1. *How do baryon and photon density perturbations grow?*

This information is from Schneider (2006), pg. 288 - 289.

Baryon and photon density perturbations grew alongside dark matter perturbations until z_e, at which point baryon acoustic oscillations began, stimying any growth until recombination, $z_r < z_{\rm eq}$. Following this, the photon overdensities escaped while the baryon overdensities began to track the dark matter overdensities.

1.4.2. *How does an individual density perturbation grow?*

This is described in much greater detail in Schneider (2006), Ch. 7.2.

If we assume a pressure-free ideal fluid, we can write the Euler's and continuity equations in comoving coordinates and linearize them to obtain $\frac{\partial^2\delta}{\partial t^2} + \frac{2\dot{a}}{a}\frac{\partial\delta}{\partial t} = 4\pi G\bar{\rho}\delta$. This means we can separate $\delta(\vec{x}, t)$ into $D(t)\delta_0(\vec{x})$; i.e. at all comoving points $\vec{x}$ the overdensity rises in the exact same manner over time. Our equation of motion then becomes

$$\ddot{D} + \frac{2\dot{a}}{a}\dot{D} - 4\pi G\bar{\rho}(t)D = 0. \tag{23}$$

There are two solutions to this equation, and we call the increasing one the growth factor, $D_+(a)$. In general $D_+(a)$ is normalized so that $D_+(1) = 1$, so that $\delta_0(\vec{x})$ is the density distribution we would have today if no non-linear effects take hold. We can show that the general increasing solution to this equation, when we switch from time to a, is

$$D_+(a) \propto \frac{H(a)}{H_0}\int_0^a \frac{da'}{(\Omega_m/a' + \Omega_\Lambda a'^2 + \Omega_k)^{3/2}} \tag{24}$$

For an Einstein-de Sitter universe, we can show, through an ansatz that $D \propto t^q$ and Eqn. 23 that $D_+(t) = (t/t_0)^{2/3}$. During matter domination, overdensities grew with the scale length.

Eventually $D(t)\delta(\vec{x})$ approaches 1, and the linear approximation fails. Growth increases dramatically (Fig. 7).

In the case of a uniform homogeneous sphere with density $\rho = \bar{\rho}(1+\delta)$, where δ is the average density perturbation in the sphere, and we have switched back to proper distances rather than comoving. The total mass within the sphere is $M \approx \frac{4\pi}{3}R_{\rm com}^3\rho_0(1+\delta_i)$, where $R_{\rm com} = a(t_i)R$ is the initial comoving sphere radius (R is the physical radius of the sphere), and $\rho_0 = \bar{\rho}/a^3$ is the present average density of the universe. We may then model the mass and radius of the sphere as a miniature universe governed by the Friedmann equations. If the initial density of the system is greater than critical, the sphere collapses. Because of the time-reversibility of the Friedmann equations, if we know the time $t_{\rm max}$ where $R_{\rm com}$ is maximum, we know the time $t_{\rm coll}$ when the universe collapses back into a singularity.

This collapse is unphysical. In reality violent relaxation will occur - density perturbations in the infalling cloud will create knots due to local gravitational collapse; these knots then scatter particles, which create more perturbations, creating more knots. This creates an effective translation of gravitational potential energy to kinetic (thermal) energy, within one dynamical time. It turns out a virialized cloud has density

$$\langle\rho\rangle = (1+\delta_{\rm vir})\bar{\rho}(t_{\rm coll}), \tag{25}$$

where $1 + \delta_{\rm vir} \approx 178\Omega_m^{-0.6}$.

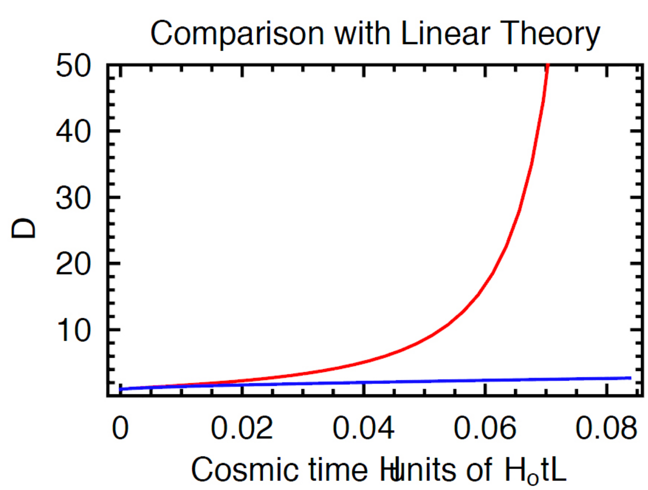


FIG. 7.— Growth of a density fluctuation taking into account non-linear evolution, versus the equivalent linear evolution. The singularity that eventually forms in the non-linear case is not physical, as the dust approximation eventually fails and virialization occurs. In baryonic material dissipative cooling also occurs. From Abraham (2011b).

1.4.3. *What is violent relaxation?*

This information is from Schneider (2006), pg. 235 and 290.

Violent relaxation is a process that very quickly establishes a virial equilibrium in the course of a gravitational collapse of a mass concentration. The reason for it are the small-scale density inhomogeneities within the collapsing matter distribution which generate, via Poisson's equation, corresponding fluctuations in the gravitational field. These then scatter the infalling particles and, by this, the density inhomogeneities are further amplified.

This virialization occurs on a dynamical time, and once virialization is complete, the average density of the perturbation becomes, as noted earlier, $\langle\rho\rangle \approx 178\Omega_m^{-0.6}\bar{\rho}(t_{\mathrm{collapse}})$

1.4.4. *What are top-down and bottom-up growth?*

This information is from Schneider (2006), pg. 286.

In a universe dominated by hot dark matter, all small perturbations cease to exist, and therefore the largest structures in the universe must for first, with galaxies fragmenting during the formation of larger structures. This top-down growth is incompatible with the fact that galaxies appear to have already collapsed, while superclusters are still in the linear overdensity regime. In a universe dominated by cold dark matter, small overdensities collapse first, and this bottom-up growth is consistent with observations.

1.4.5. *How can the power spectrum be observed?*

The matter power spectrum, if one assumes that baryons track dark matter, can be determined observationally recovering the two-point correlation function from galaxy surveys (Sec. 1.18).

1.4.6. *How can the power spectrum constrain cosmological parameters?*

This information is from Schneider (2006), Ch. 8.1.

The turnover of the power spectrum is determined by the wavenumber corresponding to the sound horizon at matter-radiation equality. This allows us to determine the shape parameter $\Omega_m h$, which can be combined with measurements of H_0 to obtain Ω_m. Detailed modelling of the power spectrum shows that the transfer function depends on Ω_b as well as Ω_m. As a result, this modelling can also derive the baryon to total matter ratio.

One important use of the dark matter power spectrum is to determine the shape and frequency of the baryon acoustic oscillations (Sec. 1.14).

1.4.7. *How can we determine the dark matter mass function from perturbation analysis?*

This information is from Schneider (2006), pg. 291 - 292.

As noted previously, a spherical region with an average density δ greater than some critical density will collapse. We can therefore back-calculate $\delta(\vec{x}, t)$ from the power spectrum[3], smooth it out over some comoving radius R to determine the average density, and determine using the critical density (given the redshift and cosmological model), the normalized number density of relaxed dark matter halos. Since the power spectrum has a normalization factor that must be determined empirically, this normalized number density can then be scaled to the true number density using observations. The result is the Press-Schechter function $n(M, z)$ which describes the number density of halos of mass $> M$ at redshift z. See Fig. 8.

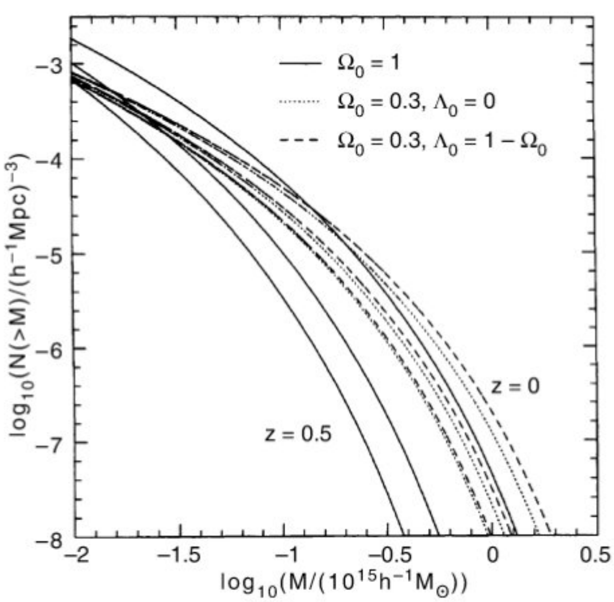


Fig. 8.— Number density of dark matter halos with mass $> M$ (i.e. a reverse cumulative model), computed from the PressSchechter model. The comoving number density is shown for three different redshifts and for three different cosmological models. The normalization of the density fluctuation field has been chosen such that the number density of halos with $M > 10^{14}h^1$ M$_\odot$ at z = 0 in all models agrees with the local number density of galaxy clusters. From Schneider (2006), his Fig. 7.7.

[3] This only works if $P(k)$ alone is sufficient to describe δ, but because this distribution is Gaussian (for complicated reasons), $P(k)$ completely constrains δ.

Since there is only one normalization, a survey of galaxy cluster total masses at different redshifts (using hot gas and a mass-to-light conversion, gravitational microlensing, etc) can be used to determine cosmological parameters. This is because the minimum overdensity for collapse δ_{min} is dependent on both the growth rate of overdensities and the expansion of the universe. Increasing Ω_m, for example, decreases $n(M,z)/n(M,0)$, since massive halo growth is more extreme the higher Ω_m is. A large Ω_Λ dampens massive halo growth.

1.5. *Question 4*

QUESTION: State and explain three key pieces of evidence for the Big Bang theory of the origin of the Universe.

This information is cribbed from Emberson (2012).

The Big Bang theory is the theory that the universe started off in an extremely hot, dense state, which then rapidly expanded, cooled, and became more tenuous over time. The Big Bang theory requires that at some point in the past a). the universe was born, b). the universe was extremely hot and c). objects were much closer together. The three key pieces of evidence are:

1. **Hubble's Law**: galaxies isotropically recede from our position with the relationship

$$\vec{v} = H_0 \vec{r} \tag{26}$$

   known as Hubble's Law. As it turns out, moving into the frame of another galaxy ($\vec{r}' = \vec{r} - \vec{k}$, $\vec{v}' = \vec{v} - H_0\vec{k} = H_0(\vec{r} - \vec{k}) = H_0\vec{r}'$) does not change any observations. At larger distances, Hubble's Law breaks down (see Sec. 1.7), but the rate of expansion only increases with distance. Because of this isotropic radial motion outward, we can back-calculate a time when all the galaxies ought to be together at one point. This time is $t_0 = r/v = 1/H_0 \approx 14$ Gyr, the Hubble Time. This gives an age to the universe, and indicates that in the distant past everything was closer together.

2. **The Cosmic Microwave Background**: the cosmic microwave background (CMB) is a near perfect isotropic blackbody with a (current) $T_0 \approx 2.73$ K. For a blackbody, $\lambda_{\mathrm{peak}} = 0.0029\,\mathrm{mK}/T$, $U = aT^4$ and $n = \beta T^3$, which gives us $n \approx 400$ cm^{-3}, $\epsilon \approx 0.25$ eV cm^{-3}, and $\lambda \approx 2$ mm. In Big Bang cosmology, this microwave background is the redshifted ($T \propto a^{-1}$) vestige of the surface of last scattering, when $T \approx 3000$ K and the universe became neutral enough for photons to travel unimpeded. This is evidence that the universe used to be hot.

3. **Big Bang Nucleosynthesis**: in the Big Bang theory, the lightest elements were created out of subatomic particles when the temperature dropped enough that the average photon was significantly below the binding energy of light elements. A detailed calculation of nucleosynthetic rates of H, D, He and Li during the first few minutes of the universe is consistent with the current abundances of light elements in the universe. See Sec. 1.8.

Additionally, no object has been found to be older than the currently accepted age of the universe, 13.7 Gyr. As we look back in time, we notice that the average galaxy in the universe looked considerably different - this evolution is consistent with ΛCDM cosmology, which has small, dense cores of dark matter forming due to gravitational instability, and then merging to form larger cores.

1.5.1. *What is Olbers's Paradox?*

Olbers's paradox is the apparent contradiction one has when an infinitely old, infinitely large universe with a fixed stellar density is assumed. In such a universe every single line of sight would eventually reach a star's photosphere. Since a typical photospheric temperature is ~ 5000 K and surface brightness is independent of distance, we would expect the entire sky to be at ~ 5000 K, roasting the Earth. Setting a finite age to the universe is one solution to the paradox; another would be that stars only formed in the last several billion years, and light from more distant stars have yet to reach us.

1.5.2. *Are there Big Bang-less cosmologies?*

It is impossible to generate a matter dominated universe for which there is no Big Bang. It is possible, however, for a Λ-dominated universe to be infinitely old, since an exponential (see Sec 1.1.4) never goes to zero. This is consistent with the steady state theory (Sec. 1.6).

1.6. *Question 5*

QUESTION: Define and describe the "tired light hypothesis" and the "steady state universe" as alternatives to the Big Bang. How have they been disproved observationally?