Semigroups and monoids

Functional abstractions

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Algebraic structures

Functional abstractions start with algebra

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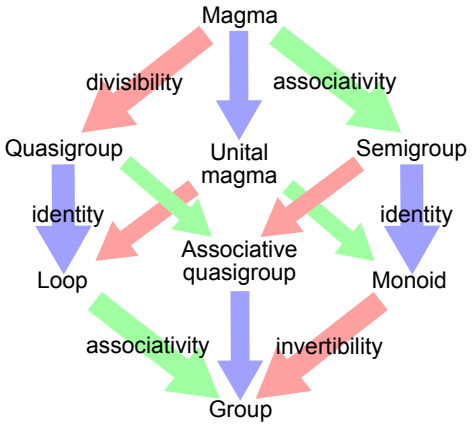


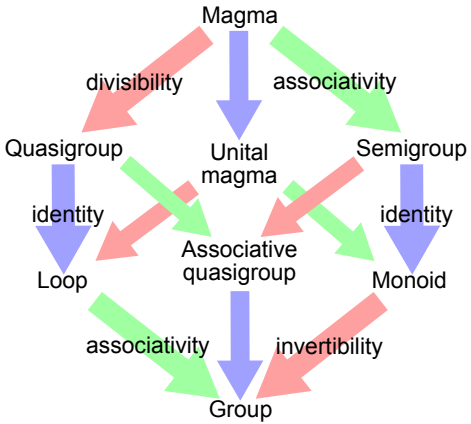
https://en.wikipedia.org/wiki/Semigroup

Algebraic structures

Functional abstractions start with algebra

Semigroup $\langle S, \cdot \rangle$

- Set S
- Binary operation
  $\cdot : (S \times S) \to S$
- *Associativity*
  $\forall a, b, c \in S : (a \cdot b) \cdot c = a \cdot (b \cdot c)$



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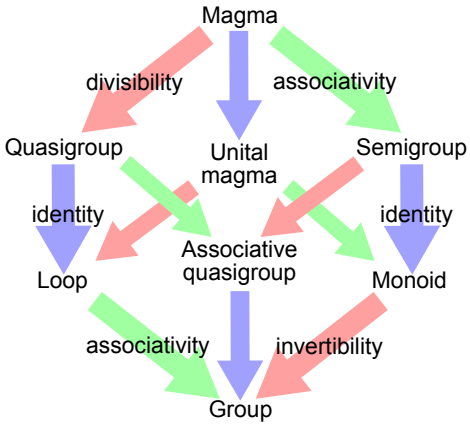

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class Semigroup a where
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  stimes  :: Integral b => b -> a -> a
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Minimal complete definition

`(<>)` | `sconcat`

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Data.Monoid

```
class Semigroup a => Monoid a where
  mempty  :: a
  mappend :: a -> a -> a
  mconcat :: [a] -> a
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Q&A