

# Lambda calculus

Functional models of computation

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## History

- 1928 — Hilbert's *Entscheidungsproblem* <sup>1</sup>
  - Is there an *algorithm* for deciding whether a proposition in first-order logic is true or false?
- Replacement for set theory as foundation of mathematics
  - 1930 — Combinatory logic (*Curry, Schönfinkel*)
  - 1932 —  $\lambda$ -calculus (*Church*)
  - 1935 — Kleene-Rosser paradox
- Effective computability
  - 1935 — Untyped  $\lambda$ -calculus (*Church, Kleene, Rosser*)
  - 1936 — Turing machine
  - 1936 — Church-Turing thesis
- 1936 — Undecidability of first-order logic
  - Halting problem of Turing machine
  - Equivalence of  $\lambda$ -terms

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## David Hilbert

- Haskell Curry
- Wilhelm Ackermann
- John von Neumann
- Ernst Zermelo
- ...

## Alonzo Church

- Stephen Cole Kleene
- J. Barkley Rosser
- Alan Turing
- Dana Scott
- Michael O. Rabin
- ...

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# Untyped lambda calculus

## Syntax

$$term ::= \underbrace{var}_{\text{Variable}} \mid \underbrace{(term \ term)}_{\text{Application}} \mid \underbrace{(\lambda var. term)}_{\text{Abstraction}}$$

## Examples

$$\lambda x. x \quad (\lambda x. xx)(\lambda y. yy) \quad \lambda f. \lambda x. f(fx)$$

## Conventions

- Application is left associative  
 $abc = (ab)c$
- Abstraction is right associative  
 $\lambda x. \lambda y. x = \lambda x. (\lambda y. x)$
- Consecutive abstractions can be combined  
 $\lambda x. \lambda y. x = \lambda xy. x$

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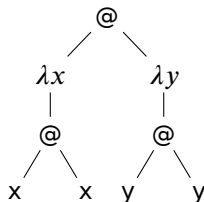
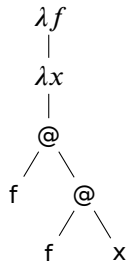
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## Tree representation

$$\begin{array}{c} \lambda x. x \\ | \\ \lambda x \\ | \\ x \end{array}$$
$$(\lambda x. xx)(\lambda y. yy)$$

$$\lambda f. \lambda x. f(fx)$$


# Free and bound variables

## Free variables $FV(t)$

Variable:  $FV(x) = \{x\}$

Application:  $FV(MN) = FV(M) \cup FV(N)$

Abstraction:  $FV(\lambda x. M) = FV(M) \setminus \{x\}$

## Bound variables $BV(t)$

Variable:  $BV(x) = \emptyset$

Application:  $BV(MN) = BV(M) \cup BV(N)$

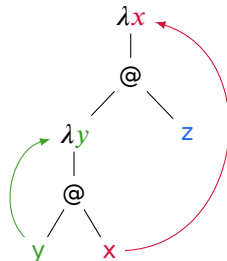
Abstraction:  $BV(\lambda x. M) = BV(M) \cup \{x\}$

## Closed terms

Term  $t$  is called **closed** or **combinator** if  $FV(t) = \emptyset$

## Example

$\lambda x. (\lambda y. yx)z$



# Substitution

Substitution  $t_{[v:=S]}$

$$x_{[v:=S]} = \begin{cases} S & v = x \\ x & v \neq x \end{cases}$$

$$(MN)_{[v:=S]} = (M_{[v:=S]} N_{[v:=S]})$$

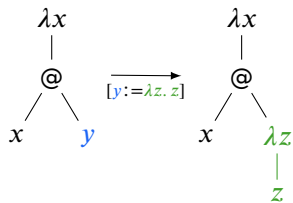
$$(\lambda x. M)_{[v:=S]} = \begin{cases} \lambda x. M & v = x \\ \lambda x. M_{[v:=S]} & v \neq x \end{cases}$$

Safe substitution

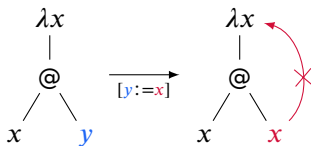
Substitution  $t_{[v:=S]}$  is **safe** if  $BV(t) \cap FV(S) = \emptyset$

Example

$$(\lambda x. xy)_{[y:=\lambda z. z]} = \lambda x. x(\lambda z. z)$$

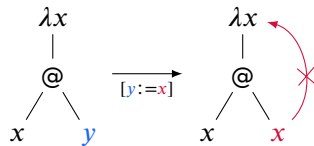


$$(\lambda x. xy)_{[y:=x]} = \lambda x. xx$$



## Problem

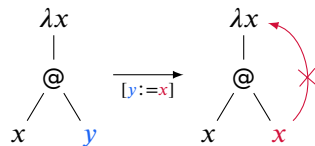
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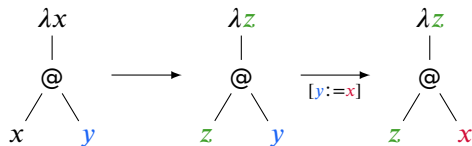
## Problem

$$\lambda x. x y_{[y:=x]} = \lambda x. x x$$



## Solution

$$(\lambda x. x y)_{[y:=x]} \equiv (\lambda z. z y)_{[y:=x]} = \lambda z. z x$$



## $\alpha$ -equivalence

$$\lambda x. M \equiv_{\alpha} \lambda y. M_{[x:=y]} \quad \text{if } x \notin FV(M)$$

$$\lambda x. M \equiv_{\alpha} \lambda x. N \quad \text{if } M \equiv_{\alpha} N$$

$$MP \equiv_{\alpha} NP \quad \text{if } M \equiv_{\alpha} N$$

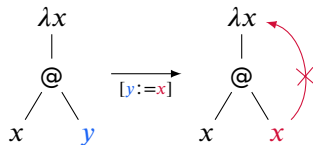
$$PM \equiv_{\alpha} PN \quad \text{if } M \equiv_{\alpha} N$$

## Conventions

- $\lambda$ -terms are considered **identical** up to  $\alpha$ -equivalence
- Appropriate renaming happens **implicitly** if required during substitution

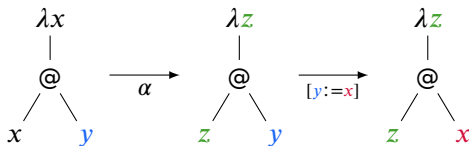
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$$\lambda x. xy_{[y:=x]} = \lambda x. xx$$



## Solution

$$(\lambda x. xy)_{[y:=x]} \equiv_{\alpha} (\lambda z. zy)_{[y:=x]} = \lambda z. zx$$



## Definitions

- Subterm of form  $(\lambda x. M)N$  is called  $\beta$ -redex
- Redex  $(\lambda x. M)N$  can be reduced to  $M_{[x:=N]}$
- $M \rightarrow_{\beta} M'$  denotes single  $\beta$ -reduction
- $M \twoheadrightarrow_{\beta} M'$  denotes several  $\beta$ -reductions
- $M \leftrightarrow_{\beta} M'$  denotes  $\beta$ -conversion as smallest equivalence relation containing  $\rightarrow_{\beta}$

## $\beta$ -reduction

$$(\lambda x. M)N \rightarrow_{\beta} M_{[x:=N]}$$

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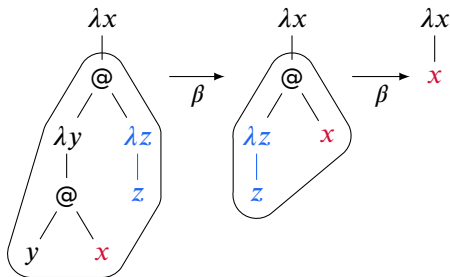
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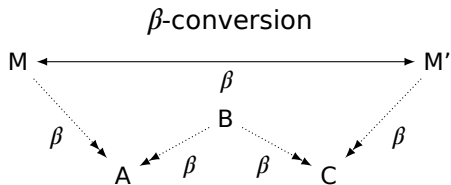
$$(\lambda x. (\lambda y. yx)(\lambda z. z)) \rightarrow_{\beta}$$

$$(\lambda x. (\lambda z. z)x) \rightarrow_{\beta} \lambda x. x$$



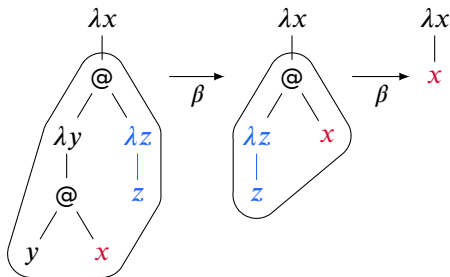
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## Example

$$(\lambda x. (\lambda y. yx)(\lambda z. z)) \rightarrow_{\beta} (\lambda x. (\lambda z. z)x) \rightarrow_{\beta} \lambda x. x$$



# Reduction order

## Normal order

- Term without any redex is in  $\beta$ -normal form
- Reduction sequence that always reduces leftmost outermost redex is called  $\beta$ -normal order reduction

## Example

$$\underline{(\lambda z. ((\lambda x. xz)f) ((\lambda y. yz)g))a} \rightarrow_{\beta}$$

$$\underline{((\lambda x. xa)f) ((\lambda y. ya)g)} \rightarrow_{\beta}$$

$$(fa) \underline{((\lambda y. ya)g)} \rightarrow_{\beta} (fa) (ga)$$

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$$\Omega = \underline{(\lambda x. xx)(\lambda x. xx)} \rightarrow_{\beta}$$

$$(xx)_{[x := (\lambda x. xx)]} \rightarrow_{\beta}$$

$$\underline{(\lambda x. xx)(\lambda x. xx)} \rightarrow_{\beta} \dots$$



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## Strictness

$$\underline{(\lambda x. y)\Omega} \rightarrow_{\beta} y$$

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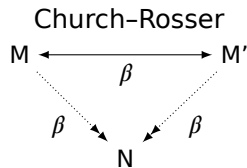
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Normal order reduction models  
**non-strict** or **lazy** evaluation



## Church-Rosser theorem

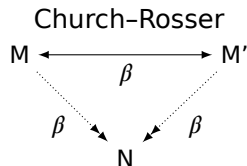
If  $M \leftrightarrow_{\beta} M'$  then there exists  $N$  such that  $M \twoheadrightarrow_{\beta} N$   
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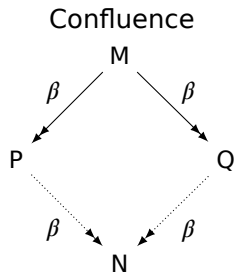
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## Confluence property

If  $M \twoheadrightarrow_{\beta} P$  and  $M \twoheadrightarrow_{\beta} Q$  then there exists  $N$  such  
that  $P \twoheadrightarrow_{\beta} N$  and  $Q \twoheadrightarrow_{\beta} N$



# Reduction order

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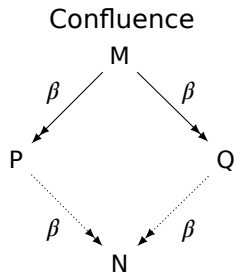
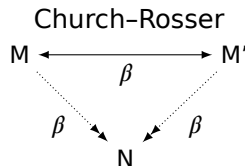
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## Corollary

$\beta$ -normal form is **unique** up to  $\alpha$ -equivalence

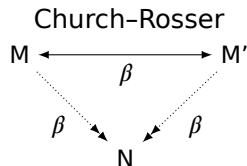




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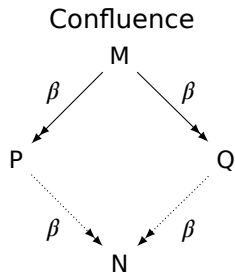
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## Normal order theorem

If  $M \twoheadrightarrow_{\beta} N$  and  $N$  is in  $\beta$ -normal form then there exists **normal order reduction** sequence from  $M$  to  $N$

Fixed-point combinator

$$Yf \leftrightarrow_{\beta} f(Yf)$$

Curry's  $Y$ -combinator

$$Y = \lambda f. (\lambda x. f(xx)) (\lambda x. f(xx))$$

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Turing's  $\Theta$ -combinator

$$\Theta = (\lambda xy. x(xxy)) (\lambda xy. x(xxy))$$

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Note:  $\Theta f \rightarrow_{\beta} f(\Theta f)$

Undecidability

Church numerals

Relation to folds

Algebraic data types

Predecessor

$\eta$ -conversion

$$(\lambda x. Mx) \underset{\eta}{\longleftrightarrow} M \text{ if } x \notin FV(M)$$



Q&A