Deng_Yehong_HW3

February 9, 2020

Conceptual exercises Training/test error for subset selection

1. (5 points) Generate a data set with $p = 20$ features, $n = 1000$ observations, and an associated quantitative response vector generated according to the model

$$Y = X\beta + \epsilon$$

where β has some elements that are exactly equal to zero.

[203]:
```
import numpy as np
import pandas as pd
np.random.seed(1234)
X = []
X_dic = {}
for i in range(20):
    X.append(np.random.normal(0,10,1000))
X = np.array(X)
X_trans = X.transpose()
X_df = pd.DataFrame()
for j in range(20):
    X_df['X{}'.format(j + 1)] = X_trans[:,j]
X_df
```

[203]:
```
            X1         X2         X3        X4         X5         X6  \
0     4.714352 -15.822080 -11.870412 -1.588084  -8.118981  13.777115
1   -11.909757 -16.201902  16.170816 -0.580519 -19.194430   0.188342
2    14.327070   0.465619  -0.426813 -9.282614  -7.877294  -8.008123
3    -3.126519 -16.798289   3.679837  9.123283  25.597029  13.160000
4    -7.205887  13.958923  18.091850  5.383625  -0.045414  13.788188
..         …          …          …         …          …          …
995   2.721640   6.325790   7.030337  2.833529  -5.928772 -13.276657
996   9.315458  20.133818 -10.932684 -1.209014   7.089337  -1.973968
997   3.275323   7.814629   1.864700  7.029969  16.088238   4.938792
998   7.408138  21.471518  26.798537 -3.185988  19.952819   2.146283
999  -8.019050  -9.083505  10.756314  2.326148   0.332693  -5.600972

            X7         X8         X9        X10        X11        X12  \
0    15.073607 -11.755436 -15.214476 -13.876990   7.411468 -12.324561
```

```
1   -16.747214  -0.371710  -3.266494  13.514572  3.267239 -23.198860
2   -10.551765  -0.930554  22.322605  -0.609708  6.662411   0.665484
3    -9.630160  -5.746781  17.077355 -15.209515  0.635578  13.602440
4    13.263866   3.991317   3.777444  15.411191 -1.844548  -1.310472
..         …          …          …          …         …          …
995   1.461406 -14.383922   8.424299 -10.480993 -8.919272   3.020788
996 -15.962519 -25.886965   2.766325  -7.787844  3.334602   1.519641
997  -0.443853 -13.969382  -1.512620 -10.524607 -4.242149 -17.582549
998 -11.980505   7.340459   3.616871  -4.976931  8.151321  -8.316472
999   7.270328  -0.618543  14.205195  -2.560062  9.117900   7.028118

           X13        X14        X15        X16        X17        X18  \
0    -4.445726  14.457552  10.410972   6.874028  -7.736147  -1.642443
1    11.510025  -9.394676 -14.393629   9.727797   5.871652   2.501816
2    -1.354627  -0.525251  -2.410696  12.862934 -12.287901   0.470752
3    10.019327  -2.050978 -18.042717 -22.482470   5.259837   4.149826
4    -4.060836  19.452012  -6.629941 -22.565776   8.458490  -5.159981
..         …          …          …          …         …          …
995   0.354499  -4.020665   9.444323  -2.116472  13.890466  18.364984
996   4.051927  10.212585 -11.600329  -0.507646  -4.021799  -4.252775
997  -4.259394 -15.334754   0.498348  -3.333800   9.773533  -9.742318
998 -34.752990  -9.997654  -0.289133  14.252124 -18.510518  -3.850863
999 -12.907409  13.684463  16.370604  18.513749 -27.822037   9.188391

           X19        X20
0    11.779386 -14.083893
1    -7.256441  -0.178640
2   -11.232715 -24.652302
3    -6.586114   7.794075
4     0.107460  -9.391892
..         …          …
995   5.241202   6.488659
996  -4.664306  11.914236
997   4.049320 -10.248561
998  -4.580651   1.335230
999   8.716782  -4.234142

[1000 rows x 20 columns]
```

[195]:
```python
beta = []
for i in range(20):
    beta.append(np.random.normal(0,1))
zeros_index = np.random.randint(1,5)
beta[:zeros_index] = np.zeros(zeros_index)
beta = np.array(beta)
beta
```

[195]:
```
array([ 0.        ,  0.        ,  0.        ,  0.        , -1.69566881,
        1.66362859,  0.75788635, -0.80725066,  0.84934261,  0.07161185,
       -1.39349312,  0.84343336, -0.74838974,  0.69696246, -0.02318777,
        0.31478762, -0.3274646 ,  0.22589291, -0.01273708,  0.59368282])
```

[196]:
```
err = np.random.normal(0, 1, 1000)
Y = np.sum(X_df * beta, axis = 1) + err
Y
```

[196]:
```
0       30.880594
1      -20.379907
2       -2.438058
3      -11.006452
4       35.584882
          …
995     22.919576
996      3.526728
997    -38.115009
998    -31.409577
999     35.782003
Length: 1000, dtype: float64
```

2. (10 points) Split your data set into a training set containing 100 observations and a test set containing 900 observations.

[197]:
```
from sklearn.model_selection import train_test_split
X_train, X_test, Y_train, Y_test = train_test_split(X_df, Y, test_size = 0.9)
```

3. (10 points) Perform best subset selection on the training set, and plot the training set MSE associated with the best model of each size. For which model size does the training set MSE take on its minimum value?

[198]:
```
#referenced from http://www.science.smith.edu/~jcrouser/SDS293/labs/lab8-py.html
import statsmodels.api as sm
import itertools
def process(feature_set):
    model = sm.OLS(Y_train, X_train[list(feature_set)])
    regr = model.fit()
    RSS = ((regr.predict(X_train[list(feature_set)]) - Y_train)**2).sum()
    return {"model": regr, "RSS": RSS}

def get_Best(k):
    results = []
    for combo in itertools.combinations(X_train.columns, k):
        results.append(process(combo))
    models = pd.DataFrame(results)
    best_model = models.loc[models['RSS'].idxmin()]
```

```
    return best_model
```

[199]:
```
models_best = pd.DataFrame(columns = ["RSS", "model"])

for i in range(1,21):
    models_best.loc[i] = get_Best(i)
models_best
```

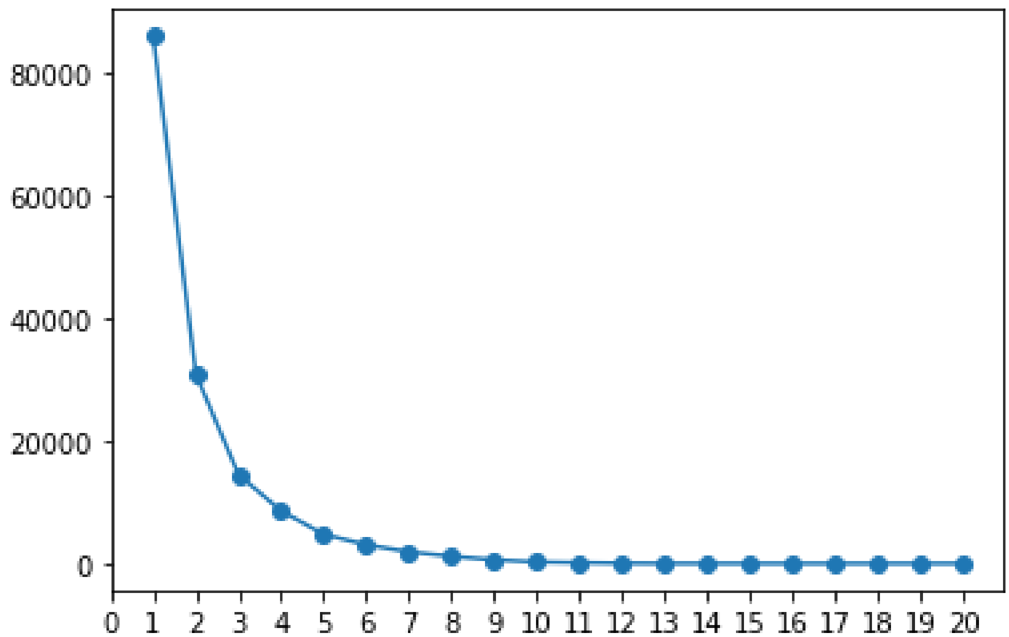
[199]:

	RSS	model
1	86037.446165	<statsmodels.regression.linear_model.Regressio…
2	61701.829039	<statsmodels.regression.linear_model.Regressio…
3	43480.718808	<statsmodels.regression.linear_model.Regressio…
4	34341.392798	<statsmodels.regression.linear_model.Regressio…
5	23760.619900	<statsmodels.regression.linear_model.Regressio…
6	18327.734084	<statsmodels.regression.linear_model.Regressio…
7	13266.209352	<statsmodels.regression.linear_model.Regressio…
8	9305.030454	<statsmodels.regression.linear_model.Regressio…
9	5370.299659	<statsmodels.regression.linear_model.Regressio…
10	2406.913806	<statsmodels.regression.linear_model.Regressio…
11	1493.478101	<statsmodels.regression.linear_model.Regressio…
12	607.218046	<statsmodels.regression.linear_model.Regressio…
13	136.316295	<statsmodels.regression.linear_model.Regressio…
14	74.079620	<statsmodels.regression.linear_model.Regressio…
15	70.235882	<statsmodels.regression.linear_model.Regressio…
16	68.755639	<statsmodels.regression.linear_model.Regressio…
17	67.466786	<statsmodels.regression.linear_model.Regressio…
18	66.505504	<statsmodels.regression.linear_model.Regressio…
19	65.628327	<statsmodels.regression.linear_model.Regressio…
20	65.553434	<statsmodels.regression.linear_model.Regressio…

[200]:
```
import matplotlib.pyplot as plt
mse = []
mse_dic = {}
for i in range(1, 21):
    m = models_best.loc[i,'RSS'] / i
    mse.append(m)
    mse_dic[str(i)] = m
plt.plot(models_best.index, mse, marker = 'o')
plt.xticks(np.arange(0,21))
mse_dic
```

[200]:
```
{'1': 86037.44616521144,
 '2': 30850.91451938499,
 '3': 14493.572935899669,
 '4': 8585.34819948579,
 '5': 4752.123979938449,
 '6': 3054.6223473145255,
```

```
'7': 1895.1727645428007,
'8': 1163.1288067704606,
'9': 596.6999620770449,
'10': 240.69138058482366,
'11': 135.7707364513901,
'12': 50.601503848197986,
'13': 10.485868876468139,
'14': 5.2914014374179486,
'15': 4.68239210554819,
'16': 4.2972274201286895,
'17': 3.9686344479544404,
'18': 3.694750215999465,
'19': 3.4541224745429835,
'20': 3.277671679278184}
```



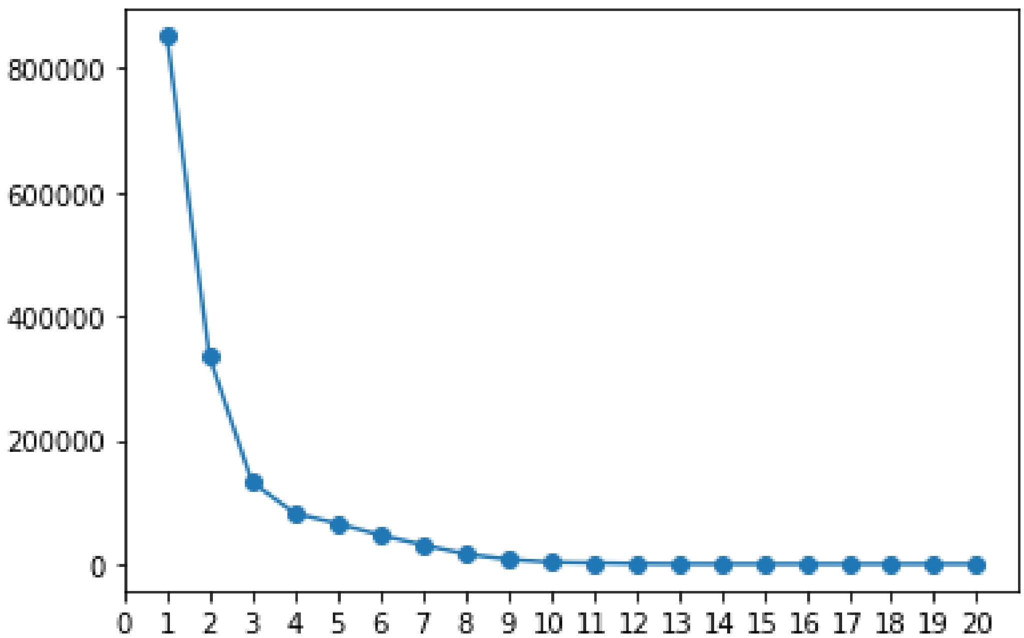
According to the dictionary the dictionary, for size 20, the training set MSE has its minimum 3.277671679278184

4. (5 points) Plot the test set MSE associated with the best model of each size.

[204]:
```
test_mse = []
test_mse_dic = {}
features = []
for i in range(1,21):
    model = models_best.loc[i, 'model']
```

```
    RSS_test = ((model.predict(X_test[list(dict(models_best.loc[i, 'model'].
↪params).keys())]) - Y_test)**2).sum()
    mse_t = RSS_test / i
    test_mse.append(mse_t)
    test_mse_dic[str(i)] = mse_t
plt.plot(np.arange(1,21), test_mse, marker = 'o')
plt.xticks(np.arange(0,21))
test_mse_dic
```

[204]:
```
{'1': 852684.4158056874,
 '2': 334328.6150933516,
 '3': 132234.3191547112,
 '4': 81974.98115865845,
 '5': 65353.836302870885,
 '6': 46878.94080937676,
 '7': 30930.84416089002,
 '8': 16226.520687081165,
 '9': 7482.4674704174995,
 '10': 2854.9687703348054,
 '11': 1752.8718711883462,
 '12': 581.7808013162747,
 '13': 125.72848803562664,
 '14': 74.37567963240333,
 '15': 65.81139805385595,
 '16': 61.114155054945385,
 '17': 58.728981658382224,
 '18': 55.914483787437725,
 '19': 53.25382578953771,
 '20': 53.95246585974844}
```



5. (5 points) For which model size does the test set MSE take on its minimum value? Comment on your results.

[217]:
```python
print(models_best.loc[19,'model'].params)
print(beta)
```

```
X1      0.011549
X3     -0.010122
X4      0.010002
X5     -1.699846
X6      1.670016
X7      0.754349
X8     -0.808657
X9      0.846491
X10     0.076937
X11    -1.391828
X12     0.832116
X13    -0.734209
X14     0.691957
X15    -0.018268
X16     0.305330
X17    -0.332443
X18     0.225389
X19    -0.015089
X20     0.608145
dtype: float64
```

```
[ 0.          0.          0.          0.         -1.69566881  1.66362859
  0.75788635 -0.80725066  0.84934261  0.07161185 -1.39349312  0.84343336
 -0.74838974  0.69696246 -0.02318777  0.31478762 -0.3274646   0.22589291
 -0.01273708  0.59368282]
```

According to the dictionary from 1.4, the model size 19 for the test set MSE takes on its minimum. Comapring the training set and the test set, it shows that there can be a discrapancy between the minimum MSEs for them. The model which size is 19 excludes the X2. As can we see in the beta list, the ture beta I have set for X2 is zero. Hence, it can be a potential reason to explain the difference in minimums for training MSE and test MSE.

6. (10 points) How does the model at which the test set MSE is minimized compare to the true model used to generate the data? Comment on the coefficient sizes.

[219]:
```python
coef = list(models_best.loc[19,'model'].params)
coef.insert(1,0)
difference_dic = {}
for i in range(20):
    difference_dic['x' + str(i + 1)] = coef[i] - beta[i]
difference_dic
```

[219]:
```
{'x1': 0.0115485105455225,
 'x2': 0.0,
 'x3': -0.01012202713623285,
 'x4': 0.010002292535482904,
 'x5': -0.004177031275097054,
 'x6': 0.006387253165412998,
 'x7': -0.003537375611054494,
 'x8': -0.001406836796198685,
 'x9': -0.002851772762482274,
 'x10': 0.005325025250633497,
 'x11': 0.0016649767780980707,
 'x12': -0.011316876739103976,
 'x13': 0.014181010106822711,
 'x14': -0.005005018192484223,
 'x15': 0.004920234971970604,
 'x16': -0.009457246900911154,
 'x17': -0.004978417810654834,
 'x18': -0.0005037017191039384,
 'x19': -0.0023523755286643435,
 'x20': 0.014462404294268483}
```

As we can see in the dictionary, the differences between the coeffcients and the true betas are very small. Moreover, in general, most coefficients are smaller than the true betas.

7. (10 points) Create a plot displaying

$$\sqrt{\sum_{j=1}^{p}(\beta_j - \hat{\beta}_j^r)^2}$$

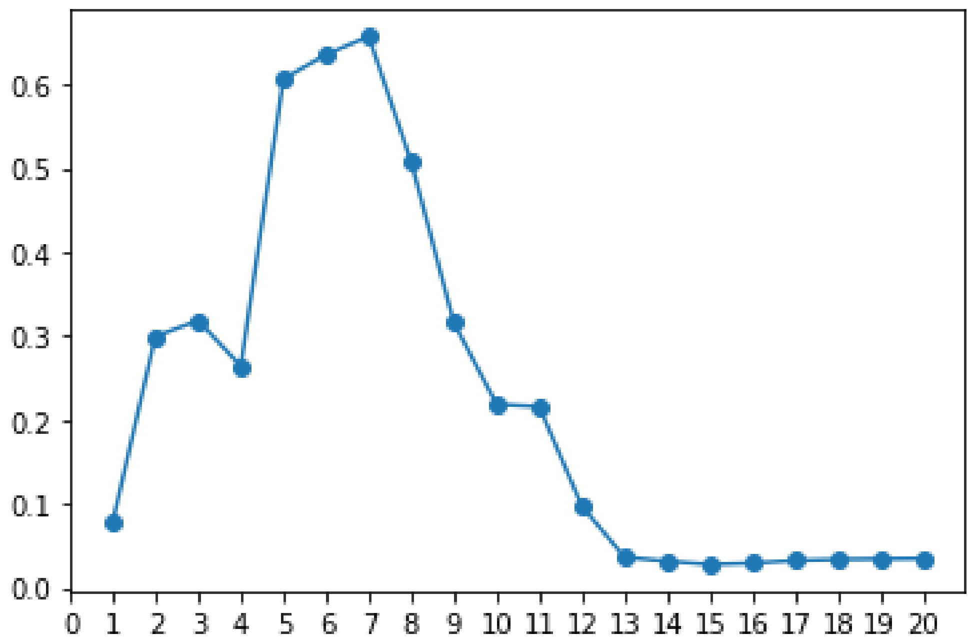
for a range of values of r, where $\hat{\beta}_j^r$ is the jth coefficient estimate for the best model containing r coefficients. Comment on what you observe. How does this compare to the test MSE plot?

[284]:
```
import math
beta_dic = {}
for i in range(20):
    beta_dic['X' + str(i + 1)] = beta[i]

def get_one_res(r):
    res = 0
    for i in list(dict(models_best.loc[r, 'model'].params).items()):
        diff = beta_dic[i[0]] - i[1]
        res += (diff ** 2)
    return (res ** 0.5)

res_ = [get_one_res(k) for k in range(1,21)]
print(res_)
plt.plot(np.arange(1,21), res_, marker = 'o')
plt.xticks(np.arange(0,21))
```

```
[0.07845350456507294, 0.2994134193967109, 0.3183885190260245,
0.2638965214331392, 0.6062287640840333, 0.6352724498744213, 0.6569724119938397,
0.5072234730491388, 0.31556827359214884, 0.218355518574315, 0.2154695920232972,
0.0969516406950457, 0.036155119546721695, 0.03125960036507476,
0.027445055542458074, 0.028979037097520005, 0.03204944072965362,
0.03331533770652615, 0.033992222592772646, 0.03452101219581696]
```

[284]:
```
([<matplotlib.axis.XTick at 0x2a127f68b88>,
  <matplotlib.axis.XTick at 0x2a11e0d52c8>,
  <matplotlib.axis.XTick at 0x2a11e0d5908>,
  <matplotlib.axis.XTick at 0x2a1712a6748>,
  <matplotlib.axis.XTick at 0x2a1712a68c8>,
  <matplotlib.axis.XTick at 0x2a1f00711c8>,
  <matplotlib.axis.XTick at 0x2a1712a6108>,
  <matplotlib.axis.XTick at 0x2a164752208>,
  <matplotlib.axis.XTick at 0x2a164752308>,
  <matplotlib.axis.XTick at 0x2a1a6778708>,
  <matplotlib.axis.XTick at 0x2a11a13cec8>,
  <matplotlib.axis.XTick at 0x2a129b20048>,
  <matplotlib.axis.XTick at 0x2a17d29a788>,
  <matplotlib.axis.XTick at 0x2a1a675ee48>,
  <matplotlib.axis.XTick at 0x2a1641a11c8>,
  <matplotlib.axis.XTick at 0x2a129b201c8>,
```

```
 <matplotlib.axis.XTick at 0x2a1712a6a48>,
 <matplotlib.axis.XTick at 0x2a1a6736508>,
 <matplotlib.axis.XTick at 0x2a1f007de88>,
 <matplotlib.axis.XTick at 0x2a127f3b548>,
 <matplotlib.axis.XTick at 0x2a1a6758748>],
<a list of 21 Text xticklabel objects>)
```



Unlike the test MSE plot, this plot first increases then decrease monotonically. Moreover, this plot start to monotonically decrease when r > 8. This demonstrates while the MSE may decrease constantly as the size of subset increase, the bias may not start to decrease until the subset contain enough predictors. moreover, in this plot, the lowest residials exist when r = 15, but in the MSE graph, the minimum MSE took place when size = 19. This means that we should sometime sacrifice some bias to gat better model.

2 Application exercises

1. (10 points) Fit a least squares linear model on the training set, and report the test MSE.

[282]:
```
gss_train = pd.read_csv('gss_train.csv')
gss_test = pd.read_csv('gss_test.csv')
y_train = gss_train['egalit_scale']
y_test = gss_test['egalit_scale']
x_train = gss_train.drop(['egalit_scale'], axis = 1)
x_test = gss_test.drop(['egalit_scale'], axis = 1)
```

[290]:
```
#referenced from https://scikit-learn.org/stable/supervised_learning.
↪html#supervised-learning
```

```
from sklearn.linear_model import LinearRegression
from sklearn.linear_model import RidgeCV
from sklearn.linear_model import LassoCV
from sklearn.linear_model import ElasticNetCV
from sklearn.metrics import mean_squared_error as mse
```

[291]:
```
OLS = LinearRegression().fit(x_train, y_train)
MSE = mse(OLS.predict(x_test), y_test)
print('The test MSE for the least squares linear model is', MSE)
```

```
The test MSE for the least squares linear model is 63.21362962301501
```

2. (10 points) Fit a ridge regression model on the training set, with λ chosen by 10-fold cross-validation. Report the test MSE.

[293]:
```
ridge = RidgeCV(alphas = (0.1, 1, 10), cv = 10).fit(x_train, y_train)
MSE = mse(ridge.predict(x_test), y_test)
print('The test MSE for the ridge regression model is', MSE)
```

```
The test MSE for the ridge regression model is 62.49920243957809
```

3. (10 points) Fit a lasso regression on the training set, with λ chosen by 10-fold cross-validation. Report the test MSE, along with the number of non-zero coefficient estimates.

[294]:
```
lasso = LassoCV(alphas = (0.1, 1, 10), cv = 10).fit(x_train, y_train)
MSE = mse(lasso.predict(x_test), y_test)

print('The test MSE for the lasso regression model is', MSE)
```

```
The test MSE for the lasso regression model is 62.7784155547739
```

[306]:
```
print("the number of non-zero coefficients is",(lasso.coef_ != 0).sum())
```

```
the number of non-zero coefficients is 24
```

4. (10 points) Fit an elastic net regression model on the training set, with α and λ chosen by 10-fold cross-validation. That is, estimate models with $\alpha = 0, 0.1, 0.2, \ldots, 1$ using the same values for λ across each model. Select the combination of α and λ with the lowest cross-validation MSE. For that combination, report the test MSE along with the number of non-zero coefficient estimates.

[303]:
```
#hide warnings referenced from https://stackoverflow.com/questions/9031783/
↪hide-all-warnings-in-ipython
import warnings
warnings.filterwarnings('ignore')
elastic_net = ElasticNetCV(l1_ratio = .1, alphas = np.arange(0,1,0.1), cv = 10).
↪fit(x_train, y_train)
MSE =  mse(elastic_net.predict(x_test), y_test)
print("The test MSE for the elastic net regression model is", MSE)
```

```
The test MSE for the elastic net regression model is 62.467338809713716
```

[307]:
```python
print("the number of non-zero coefficients is",(elastic_net.coef_ != 0).sum())
```

```
the number of non-zero coefficients is 68
```

5. (5 points) Comment on the results obtained. How accurately can we predict an individual's egalitarianism? Is there much difference among the test errors resulting from these approaches?

In general, we can predict an individual's egalitarianism at a MSE that close to 62.5.In other words, there is not much difference among the test errors from different approaches. More specifically, though, among different approaches, the least squares linear model has the highest MSE at 63.21362962301501, and the eastic net regression model has the lowest 62.4673388097l3716. Moreover, when we compare lasso regression with elastic net regression, the lasso model has fewer non-zero coefficients, 24, than the elastic net model, 68.