Probability: Homework 2

Nick Williams
ntw2117

Problem 1:

(1)

By definition, a probability mass function must satisify $\sum_{X \leq x} f(t) = 1$, if $x \in \{0, 1, 2, ...\}$ In addition, k must be positive because $f(t) > 0$ for $f(t)$ to be a pmf.

$$\text{thus,} \sum_{i=0}^{\infty} \frac{k}{2^x} = 1$$

By the common form of the geometric series: $a = \frac{k}{2^0}; r = \frac{1}{2}$

$$1 = \frac{a}{1 - r}$$
$$1 = \frac{k}{1 - 1/2}$$
$$1 = \frac{k}{1/2}$$
$$k = \frac{1}{2}$$

(2)

$$f(x) = \frac{c}{x}$$

If $f(x)$ is a pdf by definition, $\sum_X \leq xf(x) = 1$ and a common ratio should exist, if $c > 0$ then

$$S_n = \frac{c}{x_n}$$
$$S_1 = \frac{c}{1}, S_2 = \frac{c}{2}, S_3 = \frac{c}{3}$$

$$r_1 = S_2/S_1 = \frac{c}{2} \times \frac{1}{c} = \frac{1}{2}$$
$$r_2 = S_3/S_2 = \frac{c}{3} \times \frac{2}{c} = \frac{2}{3}$$
$$r_1 \neq r_2$$

There is no common ratio, r, the series therefore diverges and thus does not sum to 1 meaning $f(x) = \frac{c}{x}$ cannot be a probability density function.

Problem 2:

(1) For $f(t)$ to be a pdf, $\int_{-\infty}^{\infty} f(t)dt = 1$ and $f(t) > 0$, thus $c > 0$

$$1 = \int_0^{\infty} ce^{-2t}dt$$
$$= c\int_0^{\infty} e^{-2t}dt$$
$$= c\int_0^{\infty} e^u - \frac{1}{2}du$$
$$= c\int_0^{\infty} \frac{-e^u}{2}du$$
$$= \frac{-c}{2}\int_0^{\infty} e^u du$$
$$= \left[-\frac{1}{2}ce^{-2t}\right]_0^{\infty}$$
$$= \lim_{t\to\infty}\left(-\frac{1}{2}ce^{-2t}\right) - \left(-\frac{1}{2}c\right)$$
$$= 0 + \frac{1}{2}c$$
$$c = 2$$

(2) The corresponding cdf of $f(x)$, $F(x)$ is the integral of $f(x)$, thus:

$$F(x) = P(X \le x) = -e^{-2t}, \text{ where } 0 < t < \infty$$

Problem 3:

If $f(t)$ and $g(t)$ are pdfs then:

$$\int_{-\infty}^{\infty} f(t)dt = 1 \text{ and } \int_{-\infty}^{\infty} g(t)dt = 1$$

If $a \ge 0$ and $b \ge 0$ are constants satisfying $a + b = 1$ then, $a = 1 - b$. If $af(t) + bg(t)$ is a pdf then:

$$1 = \int_{-\infty}^{\infty} af(t) + \int_{-\infty}^{\infty} bg(t)$$
$$= a\int_{-\infty}^{\infty} f(t) + b\int_{-\infty}^{\infty} g(t)$$
$$= (1-b)\int_{-\infty}^{\infty} f(t) + b\int_{-\infty}^{\infty} g(t)$$
$$= (1-b)(1) + b(1)$$
$$= 1 - b + b$$
$$= 1$$

Problem 4:

(1) For $F(t) = (1 - \frac{1}{1+t})\ I\{t \ge 0\}$ to be a cdf, it must satisfy:

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a. $\lim_{t\to\infty} F(t) = 1$ and $\lim_{t\to-\infty} F(t) = 0$:

$$1 = \lim_{t\to\infty}(1 - \frac{1}{1+t}) \qquad 0 = \lim_{t\to-\infty}(1 - \frac{1}{1+t})I\{t \geq 0\}$$
$$= 1 - \lim_{t\to\infty}\frac{1}{1+t} \qquad = 1 - \lim_{t\to 0}\frac{1}{1+t}$$
$$= 1 - \lim_{t\to\infty}\frac{1}{1+t} \qquad = 1 - \frac{1}{1+0}$$
$$= 1 + 0 \qquad = 1 - 1$$
$$= 1 \qquad = 0$$

b. $F(t)$ is a non-decreasing function i.e., for any $t_1 < t_2$, $F(t_1) \leq F(t_2)$:

$$F(t_2) - F(t_1) \geq 0$$
$$\int_{-\infty}^{t_2} f(t_2)I\{t \geq 0\}dt - \int_{-\infty}^{t_1} f(t_1)I\{t \geq 0\}dt \geq \qquad f(t) = \frac{d}{dt}F(t) = \frac{1}{(1+t)^2}$$
$$\int_0^{t_2} f(t_2)dt - \int_0^{t_1} f(t_1)dt \geq$$
$$\int_0^{t_2} \frac{1}{(1+t)^2}dt - \int_0^{t_1} \frac{1}{(1+t)^2}dt \geq$$
$$\int_{t_1}^{t_2} \frac{1}{(1+t)^2}dt \geq$$
$$(1 - \frac{1}{1+t_2}) - (1 - \frac{1}{1+t_1}) \geq$$
$$-\frac{1}{1+t_2} + \frac{1}{1+t_1} \geq 0$$

c. $F(t)$ is right continous, for any $t_0 \in (-\infty, \infty)$, $\lim_{t\to+t_0} F(t) = F(t_0)$:

$$\lim_{t\to+t_0} \int_{-\infty}^t f(t)dt = \int_{-\infty}^{t_0} f(t)dt$$
$$\lim_{t\to+t_0} \int_0^t f(t)dt = \int_0^{t_0} f(t)dt$$
$$\int_0^{t_0} f(t)dt = \int_0^{t_0} f(t)dt$$

(2) The pdf, $f(t)$, of $F(t)$ is the derivative of $F(t)$, $\frac{d}{dt}F(t)$:

$$f(t) = \frac{d}{dt}F(t)$$
$$= \frac{d}{dt}(1 - \frac{1}{1+t})$$
$$= \frac{d}{dt}(1) - \frac{d}{dt}(1+t)^{-1}$$
$$= -(-(1+t)^{-2})$$
$$= \frac{1}{(1+t)^2}$$

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Problem 5:

(1)

$$f(x) = P(X = x) = \begin{cases} c(x+1) & \text{for } 0, 1, 2, 3, 4, 5 \\ 0 & \text{otherwise} \end{cases}$$

Because $f(x) > 0$ then $c > 0$. In addition, by the definition of a probability mass function:

$$1 = \sum_x f(x)$$
$$= \sum_x c(x+1)$$
$$= \sum_{x=0}^{5} = c(x+1)$$
$$= c + 2c + 3c + 4c + 5c + 6c$$
$$= 21c$$
$$c = \frac{1}{21}$$

(2)

$$F(x) = P(X \le x) = \sum_{X \le x} \frac{x+1}{21}$$

(3)

$$P(X > 2) = P(X \ge 3) = 1 - P(X \le 2)$$
$$= 1 - \sum_{x=0}^{2} \frac{x+1}{21}$$
$$= 1 - \frac{1}{21} + \frac{2}{21} + \frac{3}{21}$$
$$= 0.7143$$

Problem 6

(1) Because $f(x) \geq 0$, $c > 0$. In addition $\int_{-\infty}^{\infty} f(x)dx = 1$, thus

$$
\begin{aligned}
1 &= \int_{-\infty}^{\infty} f(x)dx \\
&= \int_{-\infty}^{\infty} c(1+x)^{-5} I\{x \geq 0\} \\
&= \int_{0}^{\infty} c(1+x)^{-5} \\
&= c \int_{0}^{\infty} (1+x)^{-5} \\
&= \left[c \times \frac{(1+x)^{-4}}{-4} \right]_{0}^{\infty} \\
&= \lim_{x \to \infty} \left(c \times \frac{1}{-4(1+x)^4} \right) - \left(c \times \frac{1}{-4(1+0)^4} \right) \\
&= 0 + \frac{1}{4}c \\
c &= 4
\end{aligned}
$$

(2) $F(x)$ is the integral of $f(x)$

$$
F(x) = P(X \leq x) = 4\left(\frac{1}{-4(1+x)^4} \right) = -\frac{1}{(x+1)^4}
$$

(3)

$$
\begin{aligned}
P(0.4 < x < 0.45) &= \int_{\infty}^{0.45} f(x)dx - \int_{-\infty}^{0,40} f(x)dx \\
&= \int_{0.40}^{0.45} f(x)dx \\
&= \left[F(x) \right]_{0.4}^{0.45} \\
&= -\frac{1}{(0.45+1)^4} - \left(-\frac{1}{(0.4+1)^4} \right) \\
&= -0.2262 + 0.2603 \\
&= 0.0341
\end{aligned}
$$

The probability of a flaw occuring between 0.4 and 0.45 meters is 0.0341

Problem 7

If $g(x)$ is a pdf then it must satisfy:

(1) $g(x) \geq 0$ for all $x \in (-\infty, \infty)$

$$g(x) = \frac{f(x)I\{x \geq x_0\}}{1 - F(x_0)}$$

If $f(x)$ and $F(x)$ are the pdf and cdf of random variable X then $f(x)I\{x \geq x_0\} \geq 0$ and $1 > F(x_0) > 0$, then

$$g(x) \geq 0 \text{ for all } x$$

(2) $\int_{-\infty}^{\infty} g(x)dx = 1$ where x_0 is fixed

$$
\begin{aligned}
1 &= \int_{-\infty}^{\infty} g(x)dx \\
&= \int_{-\infty}^{\infty} \frac{f(x)I\{x \geq x_0\}}{1 - F(x_0)} dx \\
&= \frac{1}{1 - F(x_0)} \int_{-\infty}^{\infty} f(x)I\{x \geq x_0\}dx \\
&= \frac{1}{1 - F(x_0)} \int_{x_0}^{\infty} f(x)dx \\
&= \left[\frac{F(x)}{1 - F(x_0)} \right]_{x_0}^{\infty} \\
&= \frac{1}{1 - 0} \\
&= 1
\end{aligned}
$$

The cdf of $g(x) = G(x)$

$$
\begin{aligned}
G(x) &= \int_{-\infty}^{\infty} g(x)dx \\
&= \frac{F(x)}{1 - F(x_0)}
\end{aligned}
$$