Probability: Homework 2

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Problem 1

(1)

By definition, a probability mass function must satisify $\sum_{X \leq x} f(t) = 1$, if $x \in \{0, 1, 2, ...\}$ In addition, k must be positive because $f(t) > 0$ for $f(t)$ to be a pmf.

$$\text{thus, } \sum_{i=0}^{\infty} \frac{k}{2^x} = 1$$

By the common form of the geometric series: $a = \frac{k}{2^0}; r = \frac{1}{2}$

$$1 = \frac{a}{1-r}$$
$$1 = \frac{k}{1-1/2}$$
$$1 = \frac{k}{1/2}$$
$$k = \frac{1}{2}$$

(2)

$$f(x) = \frac{c}{x}$$

If $f(x)$ is a pdf by definition, $\sum_X \leq xf(x) = 1$ and a common ratio should exist, if $c > 0$ then

$$S_n = \frac{c}{x_n}$$
$$S_1 = \frac{c}{1}, S_2 = \frac{c}{2}, S_3 = \frac{c}{3}$$

$$r_1 = S_2/S_1 = \frac{c}{2} \times \frac{1}{c} = \frac{1}{2}$$
$$r_2 = S_3/S_2 = \frac{c}{3} \times \frac{2}{c} = \frac{2}{3}$$
$$r_1 \neq r_2$$

There is no common ratio, r, the series therefore diverges and thus does not sum to 1 meaning $f(x) = \frac{c}{x}$ cannot be a probability density function.

Problem 2

(1) For $f(t)$ to be a pdf, $\int_{-\infty}^{\infty} f(t)dt = 1$ and $f(t) > 0$, thus $c > 0$

$$
\begin{aligned}
1 &= \int_0^\infty ce^{-2t}dt \\
&= c\int_0^\infty e^{-2t}dt \\
&= c\int_0^\infty e^u - \frac{1}{2}du \\
&= c\int_0^\infty \frac{-e^u}{2}du \\
&= \frac{-c}{2}\int_0^\infty e^u du \\
&= \Big[-\frac{1}{2}ce^{-2t}\Big]_0^\infty \\
&= \lim_{t\to\infty}\big(-\frac{1}{2}ce^{-2t}\big) - \big(-\frac{1}{2}c\big) \\
&= 0 + \frac{1}{2}c \\
c &= 2
\end{aligned}
$$

(2) The corresponding cdf of $f(x)$, $F(x)$ is the integral of $f(x)$, thus:

$$F(x) = P(X \leq x) = -e^{-2t}, \text{ where } 0 < t < \infty$$

Problem 3

If $f(t)$ and $g(t)$ are pdfs then:

$$\int_{-\infty}^{\infty} f(t)dt = 1 \text{ and } \int_{-\infty}^{\infty} g(t)dt = 1$$

If $a \geq 0$ and $b \geq 0$ are constants satisfying $a + b = 1$ then, $a = 1 - b$. If $af(t) + bg(t)$ is a pdf then:

$$
\begin{aligned}
1 &= \int_{-\infty}^{\infty} af(t) + \int_{-\infty}^{\infty} bg(t) \\
&= a\int_{-\infty}^{\infty} f(t) + b\int_{-\infty}^{\infty} g(t) \\
&= (1-b)\int_{-\infty}^{\infty} f(t) + b\int_{-\infty}^{\infty} g(t) \\
&= (1-b)(1) + b(1) \\
&= 1 - b + b \\
&= 1
\end{aligned}
$$

Problem 4

(1) For $F(t) = (1 - \frac{1}{1+t})\, I\{t \geq 0\}$ to be a cdf, it must satisfy:

a. $\lim_{t\to\infty} F(t) = 1$ and $\lim_{t\to-\infty} F(t) = 0$:

$$\begin{aligned}
1 &= \lim_{t\to\infty}(1 - \frac{1}{1+t}) & 0 &= \lim_{t\to-\infty}(1 - \frac{1}{1+t})I\{t \geq 0\} \\
&= 1 - \lim_{t\to\infty}\frac{1}{1+t} & &= 1 - \lim_{t\to 0}\frac{1}{1+t} \\
&= 1 - \lim_{t\to\infty}\frac{1}{1+t} & &= 1 - \frac{1}{1+0} \\
&= 1 + 0 & &= 1 - 1 \\
&= 1 & &= 0
\end{aligned}$$

b. $F(t)$ is a non-decreasing function i.e., for any $t_1 < t_2$, $F(t_1) \leq F(t_2)$:

$$\begin{aligned}
F(t_2) - F(t_1) &\geq 0 \\
\int_{-\infty}^{t_2} f(t_2)I\{t \geq 0\}dt - \int_{-\infty}^{t_1} f(t_1)I\{t \geq 0\}dt &\geq \qquad f(t) = \frac{d}{dt}F(t) = \frac{1}{(1+t)^2} \\
\int_{0}^{t_2} f(t_2)dt - \int_{0}^{t_1} f(t_1)dt &\geq \\
\int_{0}^{t_2} \frac{1}{(1+t)^2}dt - \int_{0}^{t_1} \frac{1}{(1+t)^2}dt &\geq \\
\int_{t_1}^{t_2} \frac{1}{(1+t)^2}dt &\geq \\
(1 - \frac{1}{1+t_2}) - (1 - \frac{1}{1+t_1}) &\geq \\
-\frac{1}{1+t_2} + \frac{1}{1+t_1} &\geq 0
\end{aligned}$$

c. $F(t)$ is right continous, for any $t_0 \in (-\infty, \infty), \lim_{t\to+t_0} F(t) = F(t_0)$:

$$\begin{aligned}
\lim_{t\to+t_0}\int_{-\infty}^{t} f(t)dt &= \int_{-\infty}^{t_0} f(t)dt \\
\lim_{t\to+t_0}\int_{0}^{t} f(t)dt &= \int_{0}^{t_0} f(t)dt \\
\int_{0}^{t_0} f(t)dt &= \int_{0}^{t_0} f(t)dt
\end{aligned}$$

(2) The pdf, $f(t)$, of $F(t)$ is the derivative of $F(t)$, $\frac{d}{dt}F(t)$:

$$\begin{aligned}
f(t) &= \frac{d}{dt}F(t) \\
&= \frac{d}{dt}(1 - \frac{1}{1+t}) \\
&= \frac{d}{dt}(1) - \frac{d}{dt}(1+t)^{-1} \\
&= -(-(1+t)^{-2}) \\
&= \frac{1}{(1+t)^2}
\end{aligned}$$

Problem 5

(1)

$$f(x) = P(X = x) = \begin{cases} c(x+1) & \text{for } 0, 1, 2, 3, 4, 5 \\ 0 & \text{otherwise} \end{cases}$$

Because $f(x) > 0$ then $c > 0$. In addition, by the definition of a probability mass function:

$$\begin{aligned} 1 &= \sum_x f(x) \\ &= \sum_x c(x+1) \\ &= \sum_{x=0}^{5} = c(x+1) \\ &= c + 2c + 3c + 4c + 5c + 6c \\ &= 21c \\ c &= \frac{1}{21} \end{aligned}$$

(2)

$$F(x) = P(X \le x) = \sum_{X \le x} \frac{x+1}{21}$$

(3)

$$\begin{aligned} P(X > 2) = P(X \ge 3) &= 1 - P(X \le 2) \\ &= 1 - \sum_{x=0}^{2} \frac{x+1}{21} \\ &= 1 - \frac{1}{21} + \frac{2}{21} + \frac{3}{21} \\ &= 0.7143 \end{aligned}$$

Problem 6

(1) Because $f(x) \geq 0$, $c > 0$. In addition $\int_{-\infty}^{\infty} f(x)dx = 1$, thus

$$\begin{aligned}
1 &= \int_{-\infty}^{\infty} f(x)dx \\
&= \int_{-\infty}^{\infty} c(1+x)^{-5} I\{x \geq 0\} \\
&= \int_{0}^{\infty} c(1+x)^{-5} \\
&= c\int_{0}^{\infty} (1+x)^{-5} \\
&= \left[c \times \frac{(1+x)^{-4}}{-4}\right]_0^{\infty} \\
&= \lim_{x\to\infty} \left(c \times \frac{1}{-4(1+x)^4}\right) - \left(c \times \frac{1}{-4(1+0)^4}\right) \\
&= 0 + \frac{1}{4}c \\
c &= 4
\end{aligned}$$

(2) $F(x)$ is the integral of $f(x)$

$$F(x) = P(X \leq x) = 4\left(\frac{1}{-4(1+x)^4}\right) = -\frac{1}{(x+1)^4}$$

(3)

$$\begin{aligned}
P(0.4 < x < 0.45) &= \int_{\infty}^{0.45} f(x)dx - \int_{-\infty}^{0,40} f(x)dx \\
&= \int_{0.40}^{0.45} f(x)dx \\
&= \big[F(x)\big]_{0.4}^{0.45} \\
&= -\frac{1}{(0.45+1)^4} - \left(-\frac{1}{(0.4+1)^4}\right) \\
&= -0.2262 + 0.2603 \\
&= 0.0341
\end{aligned}$$

The probability of a flaw occuring between 0.4 and 0.45 meters is 0.0341

Problem 7

If $g(x)$ is a pdf then it must satisfy:

(1) $g(x) \geq 0$ for all $x \in (-\infty, \infty)$

$$g(x) = \frac{f(x)I\{x \geq x_0\}}{1 - F(x_0)}$$

If $f(x)$ and $F(x)$ are the pdf and cdf of random variable X then $f(x)I\{x \geq x_0\} \geq 0$ and $1 > F(x_0) > 0$, then

$$g(x) \geq 0 \text{ for all } x$$

(2) $\int_{-\infty}^{\infty} g(x)dx = 1$ where x_0 is fixed

$$\begin{aligned}
1 &= \int_{-\infty}^{\infty} g(x)dx \\
&= \int_{-\infty}^{\infty} \frac{f(x)I\{x \geq x_0\}}{1 - F(x_0)} dx \\
&= \frac{1}{1 - F(x_0)} \int_{-\infty}^{\infty} f(x)I\{x \geq x_0\}dx \\
&= \frac{1}{1 - F(x_0)} \int_{x_0}^{\infty} f(x)dx \\
&= \left[\frac{F(x)}{1 - F(x_0)}\right]_{x_0}^{\infty} \\
&= \frac{1}{1 - 0} \\
&= 1
\end{aligned}$$

The cdf of $g(x) = G(x)$

$$\begin{aligned}
G(x) &= \int_{-\infty}^{\infty} g(x)dx \\
&= \frac{F(x)}{1 - F(x_0)}
\end{aligned}$$