P8104 Homework 3

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Problem 1

(1)

$$f_X(x) = P(X = x) = \frac{10 - x}{63} \text{ for } x \in [-2, -1, 0, 1, 2, 3, 4]$$

$$Y = (X - 1)^2$$

$$f_X(x) = \begin{cases} P(X = -2)[Y = 9] & = 12/63 \\ P(X = -1)[Y = 4] & = 11/63 \\ P(X = 0)[Y = 1] & = 10/63 \\ P(X = 1)[Y = 0] & = 9/63 \\ P(X = 2)[Y = 1] & = 8/63 \\ P(X = 3)[Y = 4] & = 7/63 \\ P(X = 4)[Y = 9] & = 6/63 \end{cases} \Rightarrow f_Y(y) = \begin{cases} P(Y = 0) & = 9/63 \\ P(Y = 1) & = 18/63 \\ P(Y = 4) & = 18/63 \\ P(Y = 9) & = 18/63 \end{cases}$$

(2)

Problem 2

$$f_X(x) = 2xI\{0 < x < 1\}$$

(1)

$$\begin{aligned} Y = g(x) &= 4X - 3 \\ \frac{d}{dx}g(x) &= 4 > 0 \qquad \text{thus } g(x) \text{is an increasing function} \\ g^{-1}(y) &= \frac{y+3}{4} \end{aligned}$$

$$\begin{aligned} f_Y(y) &= f_X(g^{-1}(y))|\frac{d}{dy}g^{-1}(y)| \\ &= f_X(\frac{y+3}{4})|\frac{1}{4}| \qquad\qquad \frac{d}{dy}g^{-1}(y) = \frac{1}{4} \\ &= 2\frac{y+3}{4}(\frac{1}{4}) \\ &= \frac{y+3}{8} \text{ for } -3 < y < 1 \end{aligned}$$

(2)

$$\begin{aligned} Y = g(x) &= 3 - X^2 \\ \frac{d}{dy}g(x) &= -2x < 0 \qquad \text{monotone } \forall\ 0 < x < 1 \\ g^{-1}(y) &= \sqrt{3 - y} \end{aligned}$$

$$
\begin{aligned}
f_Y(y) &= f_X(g^{-1}(y))|\frac{d}{dy}g^{-1}(y)| \\
&= f_x(\sqrt{3-y})\left| -\frac{1}{2\sqrt{3-y}} \right| \quad \frac{d}{dy}g^{-1}(y) = -\frac{1}{2\sqrt{3-y}} \\
&= f_x(\sqrt{3-y})\left(\frac{1}{2\sqrt{3-y}}\right) \\
&= 2(\sqrt{3-y})\left(\frac{1}{2\sqrt{3-y}}\right) \\
&= 1 \text{ for } 2 < y < 3
\end{aligned}
$$

Problem 3

(1)

$$X \sim U(0,1) \quad g(x) = -2\log(x) \quad g^{-1}(y) = e^{-y/2}$$

$$
\begin{aligned}
F_Y(y) &= 1 - F_X(g^{-1}(y)) \\
&= 1 - F_X(e^{-y/2}) \\
&= -e^{-y/2}
\end{aligned}
$$

$\frac{d}{dy}g(x) = -\frac{2}{x} < 0$ for $0 < x < 1$, therefore

$$
\begin{aligned}
f_Y(y) &= f_X(g^{-1}(y))\left|\frac{d}{dy}g^{-1}(y)\right| \\
&= 1(\left| -\frac{e^{-}y/2}{2} \right|) \qquad \frac{d}{dy}e^{-y/2} = e^{-y/2}\frac{d}{dy} - y/2 = -\frac{1}{2}e^{-y/2} \\
&= \frac{e^{-y/2}}{2}
\end{aligned}
$$

(2)

$$
\begin{aligned}
X \sim U(-1,1) \text{ thus } &\quad f_X(x) = \frac{1}{2} \text{ for } -1 \leq x \leq 1 \\
Y = g(x) = e^x \text{ thus } &\quad g^{-1}(y) = log(x)
\end{aligned}
$$

Because $\frac{d}{dx}g(x) = e^x > 0 \; \forall x \in (-1,1)$, $g(x)$ is an increasing function and monotone; therefore

$$
\begin{aligned}
f_Y(y) &= f_X(g^{-1}(y))\left|\frac{d}{dy}g^{-1}(y)\right| & F_Y(y) &= F_X(g^{-1}(y)) \\
&= \frac{1}{2}\left|\frac{1}{y}\right| & &= \frac{g^{-1}(y)+1}{2} \\
&= \frac{1}{2y} & &= \frac{\log(y)+1}{2}
\end{aligned}
$$

$$
\begin{aligned}
X \sim U(-1,1) \text{ thus } &\quad f_X(x) = \frac{1}{2} \text{ for } -1 \leq x \leq 1 \\
Z = g(x) = -\log(X+1) \text{ thus } &\quad g^{-1}(z) = e^{-z} - 1
\end{aligned}
$$

Because $\frac{d}{dx}g(x) = -\frac{1}{x+1} < 0 \; \forall x \in (-1,1)$, $g(x)$ is a decreasing function and monotone; therefore

$$\begin{aligned} F_Z(z) &= 1 - F_X(g^{-1}(z)) & f_Z(z) &= f_X(g^{-1}(z))\left|\frac{d}{dz}g^{-1}(z)\right| \\ &= 1 - F_X(e^{-z}-1) & &= \frac{1}{2}\left|-e^{-z}\right| \\ &= 1 - \frac{(e^{-z}-1)+1}{2} & &= \frac{e^{-z}}{2} \\ &= -\frac{e^{-z}}{2} \end{aligned}$$

Problem 4

$$f_X(x) = \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}} \forall x \in (-\infty,\infty) \quad Y = g(x) = log(|X+1|)$$

$g(x)$ is not monotone on $(-\infty,\infty)$, however it is monotone on $(-\infty,-1)$ and $(-1,\infty)$. We therefore can define:

$$\begin{array}{lll} A_0 = -1 & \text{On } A_0 \; y = \{0\} & \\ A_1 = (-\infty,-1) & \text{On } A_1 \; y = \log(|x+1|)I\{x < -1\} & g^{-1}(y) = -e^y - 1 \\ A_2 = (-1,\infty) & \text{On } A_2 \; y = \log(|x+1|)I\{x > -1\} & g^{-1}(y) = e^y - 1 \end{array}$$

$$f_Y(y) = \sum_{i=1}^{k} f_X(g_i^{-1}(y))\left|\frac{d}{dy}g_i^{-1}(y)\right|I\{y \in \Omega_Y\}$$

$$\begin{aligned} f_Y(y) &= f_X(g_1^{-1}(y))\left|\frac{d}{dy}g_1^{-1}(y)\right| + f_X(g_2^{-1}(y))\left|\frac{d}{dy}g_2^{-1}(y)\right| \\ &= f_X(-e^y-1)|-e^y| + f_X(e^y-1)|e^y| \\ &= f_X(-e^y-1)(e^y) + f_X(e^y-1)(e^y) \\ &= e^y\Big(f_X(-e^y-1) + f_X(e^y-1)\Big) \\ &= e^y\left(\frac{1}{\sqrt{2\pi}}e^{\frac{-(-e^y-1)^2}{2}} + \frac{1}{\sqrt{2\pi}}e^{\frac{-(e^y-1)^2}{2}}\right)I\{y > 0\} \end{aligned}$$

Problem 5

$$f_X(x) = cx^2 I\{-1 < x < 1\}$$

(1) Because $f_X(x) \geq 0 \forall x$, $c > 0$:

$$
\begin{aligned}
1 &= \int_{-\infty}^{\infty} f_X(x)dx \\
&= \int_{-\infty}^{\infty} cx^2 I\{-1 < x < 1\}dx \\
&= c\int_{-1}^{1} x^2 dx \\
&= c\frac{1}{3}x^3\Big|_{-1}^{1} \\
&= c\frac{1}{3}(1)^3 - c\frac{1}{3}(-1)^3 \\
c &= \frac{3}{2} > 0
\end{aligned}
$$

(2) By the definition of a cumulative distribution function

$$
\begin{aligned}
F_X(x) = P(X \leq x) &= \int f_X(x)dx \\
&= \int \frac{3}{2}x^2 I\{-1 < x < 1\}dx \\
&= \frac{3}{6}x^3 \\
&= \frac{x^3}{2} \forall x \in (-1, 1)
\end{aligned}
$$

(3) If $U \sim U(0,1)$, to find $F^1(U)$, we need to solve $y = \frac{x^3}{2}$ for x in terms of $y \in (0,1)$

$$
\begin{aligned}
F_X(F^{-1}(u)) &= u \\
\frac{(F^{-1}(U))^3}{2} &= \\
(F^{-1}(U))^3 &= 2u \\
F^{-1}(U) &= 2u^{1/3} \ \forall u \in (0,1)
\end{aligned}
$$