P8104 Homework 3

Nick Williams
ntw2117

Problem 1

(1)

(2)

Problem 2

$$f_X(x) = 2xI\{0 < x < 1\}$$

(1)

$$\begin{aligned} Y &= g(x) = 4X - 3 \\ \frac{d}{dx}g(x) &= 4 > 0 \qquad \text{thus } g(x) \text{is an increasing function} \\ g^{-1}(y) &= \frac{y+3}{4} \end{aligned}$$

$$\begin{aligned} f_Y(y) &= f_X(g^{-1}(y))|\frac{d}{dy}g^{-1}(y)| \\ &= f_X(\frac{y+3}{4})|\frac{1}{4}| \qquad\qquad \frac{d}{dy}g^{-1}(y) = \frac{1}{4} \\ &= 2\frac{y+3}{4}(\frac{1}{4}) \\ &= \frac{y+3}{8} \text{ for } -3 < y < 1 \end{aligned}$$

(2)

$$\begin{aligned} Y &= g(x) = 3 - X^2 \\ \frac{d}{dy}g(x) &= -2x < 0 \qquad \text{monotone } \forall\ 0 < x < 1 \\ g^{-1}(y) &= \sqrt{3-y} \end{aligned}$$

$$\begin{aligned} f_Y(y) &= f_X(g^{-1}(y))|\frac{d}{dy}g^{-1}(y)| \\ &= f_x(\sqrt{3-y})\left| -\frac{1}{2\sqrt{3-y}} \right| \quad \frac{d}{dy}g^{-1}(y) = -\frac{1}{2\sqrt{3-y}} \\ &= f_x(\sqrt{3-y})\left(\frac{1}{2\sqrt{3-y}}\right) \\ &= 2(\sqrt{3-y})\left(\frac{1}{2\sqrt{3-y}}\right) \\ &= 1 \text{ for } 2 < y < 3 \end{aligned}$$

Problem 3

(1)

(2)

$$
\begin{aligned}
X \sim U(-1,1) \quad f_X(x) &= \frac{1}{2} \text{ for } -1 \le x \le 1 \\
Y = g(x) = e^x \qquad\qquad & g^{-1}(y) = log(x)
\end{aligned}
$$

Because $\frac{d}{dx}g(x) = e^x > 0 \ \forall x \in [-1,1]$, $g(x)$ is an increasing function and monotone; therefore

$$
\begin{aligned}
f_Y(y) &= f_X(g^{-1}(y))\left|\frac{d}{dy}g^{-1}(y)\right| & F_Y(y) &= F_X(g^{-1}(y)) \\
&= \frac{1}{2}\left|\frac{1}{y}\right| & &= \frac{g^{-1}(y)+1}{2} \\
&= \frac{1}{2y} & &= \frac{\log(y)+1}{2}
\end{aligned}
$$

$$
\begin{aligned}
X \sim U(-1,1) \quad f_X(x) &= \frac{1}{2} \text{ for } -1 \le x \le 1 \\
Z = g(x) = -\log(X+1) \qquad\qquad & g^{-1}(z) = e^{-z} - 1
\end{aligned}
$$

Because $\frac{d}{dx}g(z) = -\frac{1}{x+1} < 0 \ \forall x \in [-1,1]$, $g(x)$ is a decreasing function and monotone; therefore

$$
\begin{aligned}
F_Z(z) &= 1 - F_X(g^{-1}(z)) & f_Z(z) &= f_X(g^{-1}(z))\left|\frac{d}{dz}g^{-1}(z)\right| \\
&= 1 - F_X(e^{-z}-1) & &= \frac{1}{2}\left| - e^{-z}\right| \\
&= 1 - \frac{(e^{-z}-1)+1}{2} & &= \frac{e^{-z}}{2} \\
&= -\frac{e^{-z}}{2}
\end{aligned}
$$