

# P8104 Homework 3

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## Problem 1

(1)

$$f_X(x) = P(X = x) = \frac{10-x}{63} \text{ for } x \in [-2, -1, 0, 1, 2, 3, 4]$$

$$Y = (X - 1)^2$$

$$f_X(x) = \begin{cases} P(X = -2) &= 12/63 \\ P(X = -1) &= 11/63 \\ P(X = 0) &= 10/63 \\ P(X = 1) &= 9/63 \\ P(X = 2) &= 8/63 \\ P(X = 3) &= 7/63 \\ P(X = 4) &= 6/63 \end{cases} \Rightarrow f_Y(y) = \begin{cases} P(Y = 0) &= 9/63 \\ P(Y = 1) &= 18/63 \\ P(Y = 4) &= 18/63 \\ P(Y = 9) &= 18/63 \end{cases}$$

(2)

## Problem 2

$$f_X(x) = 2xI\{0 < x < 1\}$$

(1)

$$Y = g(x) = 4X - 3$$

$$\frac{d}{dx}g(x) = 4 > 0 \quad \text{thus } g(x) \text{ is an increasing function}$$

$$g^{-1}(y) = \frac{y+3}{4}$$

$$\begin{aligned} f_Y(y) &= f_X(g^{-1}(y)) \left| \frac{d}{dy}g^{-1}(y) \right| \\ &= f_X\left(\frac{y+3}{4}\right) \left| \frac{1}{4} \right| \quad \frac{d}{dy}g^{-1}(y) = \frac{1}{4} \\ &= 2 \frac{y+3}{4} \left( \frac{1}{4} \right) \\ &= \frac{y+3}{8} \text{ for } -3 < y < 1 \end{aligned}$$

(2)

$$Y = g(x) = 3 - X^2$$

$$\frac{d}{dy}g(x) = -2x < 0 \quad \text{monotone } \forall 0 < x < 1$$

$$g^{-1}(y) = \sqrt{3-y}$$

$$\begin{aligned}
f_Y(y) &= f_X(g^{-1}(y)) \left| \frac{d}{dy} g^{-1}(y) \right| \\
&= f_X(\sqrt{3-y}) \left| -\frac{1}{2\sqrt{3-y}} \right| \quad \frac{d}{dy} g^{-1}(y) = -\frac{1}{2\sqrt{3-y}} \\
&= f_X(\sqrt{3-y}) \left( \frac{1}{2\sqrt{3-y}} \right) \\
&= 2(\sqrt{3-y}) \left( \frac{1}{2\sqrt{3-y}} \right) \\
&= 1 \text{ for } 2 < y < 3
\end{aligned}$$

### Problem 3

(1)

(2)

$$\begin{aligned}
X &\sim U(-1, 1) \text{ thus } f_X(x) = \frac{1}{2} \text{ for } -1 \leq x \leq 1 \\
Y &= g(x) = e^x \text{ thus } g^{-1}(y) = \log(x)
\end{aligned}$$

Because  $\frac{d}{dx} g(x) = e^x > 0 \forall x \in [-1, 1]$ ,  $g(x)$  is an increasing function and monotone; therefore

$$\begin{aligned}
f_Y(y) &= f_X(g^{-1}(y)) \left| \frac{d}{dy} g^{-1}(y) \right| & F_Y(y) &= F_X(g^{-1}(y)) \\
&= \frac{1}{2} \left| \frac{1}{y} \right| & &= \frac{g^{-1}(y) + 1}{2} \\
&= \frac{1}{2y} & &= \frac{\log(y) + 1}{2}
\end{aligned}$$

$$\begin{aligned}
X &\sim U(-1, 1) \text{ thus } f_X(x) = \frac{1}{2} \text{ for } -1 \leq x \leq 1 \\
Z &= g(x) = -\log(X + 1) \text{ thus } g^{-1}(z) = e^{-z} - 1
\end{aligned}$$

Because  $\frac{d}{dx} g(x) = -\frac{1}{x+1} < 0 \forall x \in [-1, 1]$ ,  $g(x)$  is a decreasing function and monotone; therefore

$$\begin{aligned}
F_Z(z) &= 1 - F_X(g^{-1}(z)) & f_Z(z) &= f_X(g^{-1}(z)) \left| \frac{d}{dz} g^{-1}(z) \right| \\
&= 1 - F_X(e^{-z} - 1) & &= \frac{1}{2} \left| -e^{-z} \right| \\
&= 1 - \frac{(e^{-z} - 1) + 1}{2} & &= \frac{e^{-z}}{2} \\
&= -\frac{e^{-z}}{2}
\end{aligned}$$

### Problem 4

$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \forall x \in (-\infty, \infty) \quad Y = g(x) = \log(|X + 1|)$$

$g(x)$  is not monotone on  $(-\infty, \infty)$ , however it is monotone on  $(-\infty, -1)$  and  $(-1, \infty)$ . We therefore can define:

$$\begin{aligned} A_0 &= -1 & \text{On } A_0 \ y &= \{0\} \\ A_1 &= (-\infty, -1) & \text{On } A_1 \ y &= \log(|x+1|)I\{x < -1\} \quad g^{-1}(y) = -e^y - 1 \\ A_2 &= (-1, \infty) & \text{On } A_2 \ y &= \log(|x+1|)I\{x > -1\} \quad g^{-1}(y) = e^y - 1 \end{aligned}$$

$$f_Y(y) = \sum_{i=1}^k f_X(g_i^{-1}(y)) \left| \frac{d}{dy} g_i^{-1}(y) \right| I\{y \in \Omega_Y\}$$

$$\begin{aligned} f_Y(y) &= f_X(g_1^{-1}(y)) \left| \frac{d}{dy} g_1^{-1}(y) \right| + f_X(g_2^{-1}(y)) \left| \frac{d}{dy} g_2^{-1}(y) \right| \\ &= f_X(-e^y - 1) | -e^y | + f_X(e^y - 1) | e^y | \\ &= f_X(-e^y - 1)(e^y) + f_X(e^y - 1)(e^y) \\ &= e^y \left( f_X(-e^y - 1) + f_X(e^y - 1) \right) \\ &= e^y \left( \frac{1}{\sqrt{2\pi}} e^{\frac{-(-e^y-1)^2}{2}} + \frac{1}{\sqrt{2\pi}} e^{\frac{-(e^y-1)^2}{2}} \right) I\{y > 0\} \end{aligned}$$

## Problem 5

$$f_X(x) = cx^2 I\{-1 < x < 1\}$$

(1) Because  $f_X(x) \geq 0 \forall x$ ,  $c > 0$ :

$$\begin{aligned} 1 &= \int_{-\infty}^{\infty} f_X(x) dx \\ &= \int_{-\infty}^{\infty} cx^2 I\{-1 < x < 1\} dx \\ &= c \int_{-1}^1 x^2 dx \\ &= c \frac{1}{3} x^3 \Big|_{-1}^1 \\ &= c \frac{1}{3} (1)^3 - c \frac{1}{3} (-1)^3 \\ c &= \frac{3}{2} > 0 \end{aligned}$$

(2) By the definition of a cumulative distribution function

$$\begin{aligned} F_X(x) &= P(X \leq x) = \int f_X(x) dx \\ &= \int \frac{3}{2} x^2 I\{-1 < x < 1\} dx \\ &= \frac{3}{6} x^3 \\ &= \frac{x^3}{2} \forall x \in (-1, 1) \end{aligned}$$

(3) If  $U \sim U(0, 1)$ , to find  $F^1(U)$ , we need to solve  $y = \frac{x^3}{2}$  for  $x$  in terms of  $y \in (0, 1)$

$$F_X(F^{-1}(u)) = u$$

$$\frac{(F^{-1}(U))^3}{2} =$$

$$(F^{-1}(U))^3 = 2u$$

$$F^{-1}(U) = 2u^{1/3} \quad \forall u \in (0, 1)$$