# P8104 Homework 3

**Nick Williams**
**ntw2117**

## Problem 1

(1)

(2)

## Problem 2

$$f_X(x) = 2xI\{0 < x < 1\}$$

(1)

$$\begin{aligned}
Y &= g(x) = 4X - 3 \\
\frac{d}{dx}g(x) &= 4 > 0 \qquad \text{thus } g(x)\text{is an increasing function} \\
g^{-1}(y) &= \frac{y+3}{4}
\end{aligned}$$

$$\begin{aligned}
f_Y(y) &= f_X(g^{-1}(y))|\frac{d}{dy}g^{-1}(y)| \\
&= f_X(\frac{y+3}{4})|\frac{1}{4}| \qquad\qquad \frac{d}{dy}g^{-1}(y) = \frac{1}{4} \\
&= 2\frac{y+3}{4}(\frac{1}{4}) \\
&= \frac{y+3}{8} \text{ for } -3 < y < 1
\end{aligned}$$

(2)

$$\begin{aligned}
Y &= g(x) = 3 - X^2 \\
\frac{d}{dy}g(x) &= -2x < 0 \qquad \text{monotone } \forall\ 0 < x < 1 \\
g^{-1}(y) &= \sqrt{3-y}
\end{aligned}$$

$$\begin{aligned}
f_Y(y) &= f_X(g^{-1}(y))|\frac{d}{dy}g^{-1}(y)| \\
&= f_x(\sqrt{3-y})\left| -\frac{1}{2\sqrt{3-y}} \right| \quad \frac{d}{dy}g^{-1}(y) = -\frac{1}{2\sqrt{3-y}} \\
&= f_x(\sqrt{3-y})\left(\frac{1}{2\sqrt{3-y}}\right) \\
&= 2(\sqrt{3-y})\left(\frac{1}{2\sqrt{3-y}}\right) \\
&= 1 \text{ for } 2 < y < 3
\end{aligned}$$

## Problem 3

(1)

(2)

$$X \sim U(-1,1) \text{ thus } \quad f_X(x) = \frac{1}{2} \text{ for } -1 \le x \le 1$$
$$Y = g(x) = e^x \text{ thus } \quad g^{-1}(y) = log(x)$$

Because $\frac{d}{dx}g(x) = e^x > 0 \ \forall x \in [-1,1]$, $g(x)$ is an increasing function and monotone; therefore

$$\begin{aligned} f_Y(y) &= f_X(g^{-1}(y))\left|\frac{d}{dy}g^{-1}(y)\right| & F_Y(y) &= F_X(g^{-1}(y)) \\ &= \frac{1}{2}\left|\frac{1}{y}\right| & &= \frac{g^{-1}(y)+1}{2} \\ &= \frac{1}{2y} & &= \frac{\log(y)+1}{2} \end{aligned}$$

$$X \sim U(-1,1) \text{ thus } \quad f_X(x) = \frac{1}{2} \text{ for } -1 \le x \le 1$$
$$Z = g(x) = -\log(X+1) \text{ thus } \quad g^{-1}(z) = e^{-z} - 1$$

Because $\frac{d}{dx}g(x) = -\frac{1}{x+1} < 0 \ \forall x \in [-1,1]$, $g(x)$ is a decreasing function and monotone; therefore

$$\begin{aligned} F_Z(z) &= 1 - F_X(g^{-1}(z)) & f_Z(z) &= f_X(g^{-1}(z))\left|\frac{d}{dz}g^{-1}(z)\right| \\ &= 1 - F_X(e^{-z}-1) & &= \frac{1}{2}\left| - e^{-z}\right| \\ &= 1 - \frac{(e^{-z}-1)+1}{2} & &= \frac{e^{-z}}{2} \\ &= -\frac{e^{-z}}{2} \end{aligned}$$

## Problem 5

$$f_X(x) = cx^2 I\{-1 < x < 1\}$$

(1) Because $f_X(x) \ge 0 \forall x$, $c > 0$:

$$\begin{aligned} 1 &= \int_{-\infty}^{\infty} f_X(x)dx \\ &= \int_{-\infty}^{\infty} cx^2 I\{-1 < x < 1\}dx \\ &= c\int_{-1}^{1} x^2 dx \\ &= c\frac{1}{3}x^3\Big|_{-1}^{1} \\ &= c\frac{1}{3}(1)^3 - c\frac{1}{3}(-1)^3 \\ c &= \frac{3}{2} > 0 \end{aligned}$$

(2)

$$\begin{aligned}
F_X(x) = P(X \leq x) &= \int f_X(x)dx \\
&= \int \frac{3}{2}x^2 I\{-1 < x < 1\}dx \\
&= \frac{3}{6}x^3 \\
&= \frac{x^3}{2} \forall\{-1 < x < 1\}
\end{aligned}$$

(3)

$$U \sim U(0,1)$$

To find $F^1(U)$, we need to solve $y = \frac{x^3}{2}$ for $x$ in terms of $y \in (0,1)$

$$\begin{aligned}
F_X(F^{-1}(u)) &= u \\
\frac{(F^{-1}(U))^3}{2} &= \\
(F^{-1}(U))^3 &= 2u \\
F^{-1}(U) &= 2u^{1/3} \ \forall U \in (0,1)
\end{aligned}$$