Problem Set 3

MACS 30250

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In [11]:

```
import numpy as np
import time
import scipy.optimize as opt
import matplotlib.pyplot as plt
import warnings
warnings.filterwarnings("ignore")

beta = 0.442
nvec = np.array([1, 1, 0.2])
delta = 0.6415
sigma = 3
A = 1
alpha = 0.35
```

In [12]:

```
def get_wage(alpha, A, K, L):
    return (1 -alpha) * A * (K / L) ** alpha

def get_r(alpha, A, L, K, delta):
    return alpha * A * (L / K) ** (1 - alpha) - delta
```

Question 2.1

In [13]:

```
def feasible(f_params, bvec_guess):
        nvec, A, alpha, delta = f_params
        b2, b3 = bvec_guess
        K = sum(bvec_guess)
        K_cnstr = (K <= 0)
        L = sum(nvec)
        wage = get_wage(alpha, A, K, L)
        rate = get_r(alpha, A, L, K, delta)
        c1 = nvec[0] * wage - b2
        c2 = nvec[1] * wage + (1 + rate) * b2 - b3
        c3 = nvec[2] * wage + (1 + rate) * b3
        c_cnstr = [(c1 <= 0), (c2 <= 0), (c3 <= 0)]
        b_cnstr = [c_cnstr[0] or c_cnstr[1], c_cnstr[1] or c_cnstr[2]]
        return b_cnstr, c_cnstr, K_cnstr
```

In [14]:

```
f_params = nvec, A, alpha, delta
bvec_guess = np.array([1.0, 1.2])
print(feasible(f_params, bvec_guess))
```

```
([True, False], [True, False, False], False)
```

Question 2.1 (Part a)
The first-period constraints are violated, as seen in the output above.

In [15]:

```
bvec_guess = np.array([0.06, 0.001])
```

```
bvec_guess = np.array([0.06, -0.001])
feasible(f_params, bvec_guess)
```

Out[15]:

```
([False, False], [False, False, False], False)
```

Question 2.1 (Part b)

None of the constraints are violated

In [16]:

```
bvec_guess = np.array([0.1, 0.1])
feasible(f_params, bvec_guess)
```

Out[16]:

```
([False, False], [False, False, False], False)
```

Question 2.1 (Part c)

None of the constraints are violated

Question 2.2

In [17]:

```
def get_EulErr(c_ss, sigma, beta, r_ss):
    EulErr = np.zeros(2)
    for i in range(2):
        EulErr[i] = (c_ss[i] ** -sigma) - beta * (1 + r_ss) * (c_ss[i + 1] ** -sigma)
    return EulErr
```

In [18]:

```
def get_SS(params, bvec_guess, SS_graphs):
    start_time = time.clock()
    beta, sigma, nvec, L, A, alpha, delta, SS_tol = params

    b_ss = np.array(opt.root(solver, bvec_guess, args=params).x)
    K_ss = sum(b_ss)
    w_ss = get_wage(alpha, A, K_ss, L)
    r_ss = get_r(alpha, A, L, K_ss, delta)

    Y_ss=A * (K_ss ** alpha) * (L ** (1 - alpha))

    c1 = w_ss - b_ss[0]
    c2 = w_ss + (1 + r_ss) * b_ss[0] - b_ss[1]
    c3 = 0.2 * w_ss + (1 + r_ss) * b_ss[1]
    c_ss = np.array([c1, c2, c3])
    C_ss = c1 + c2 + c3
    EulErr_ss = get_EulErr(c_ss, sigma, beta, r_ss)

    RCerr_ss = Y_ss - C_ss - delta * K_ss
    ss_time = time.clock() - start_time

    if SS_graphs:
        %matplotlib notebook
        t1 = np.array([1,2,3])
        t2 = np.array([2,3])
        plt.plot(t1, c_ss, marker='o',label='consumption')
        plt.plot(t2, b_ss, marker='o',label='savings')
        plt.xlabel('Agent\'s age')
        plt.ylabel('Agent\'s comsumption and savings')
        plt.title('Steady-state distribution of consumption and savings by age')
        plt.legend()

    ss_output = {'b_ss': b_ss, 'c_ss': c_ss,
                 'w_ss': w_ss, 'r_ss': r_ss,
                 'K_ss': K_ss, 'Y_ss': Y_ss,
                 'C_ss': C_ss, 'EulErr_ss': EulErr_ss,
```

```
                'RCerr_ss': RCerr_ss, 'ss_time': ss_time}
    return ss_output
```

In [19]:

```
def solver(bvec_guess, *args):
    b2, b3 = bvec_guess
    beta, sigma, nvec, L, A, alpha, delta, SS_tol = args

    K_ss = b2 + b3
    w_ss = (1 - alpha) * A * (K_ss / L) ** alpha
    r_ss = alpha * A * (L / K_ss) ** (1 - alpha) - delta
    C_ss = np.array([w_ss-b2, w_ss + (1 + r_ss) * b2 - b3, 0.2 * w_ss + (1 + r_ss) * b3])
    EulErr_ss = get_EulErr(C_ss, sigma, beta, r_ss)
    return EulErr_ss
```

Question 2.2 (Part a)

In [20]:

```
L = 2.2
SS_tol = 1e-9
params = (beta, sigma, nvec, L, A, alpha, delta, SS_tol)
res = get_SS(params, bvec_guess, False)
res
```

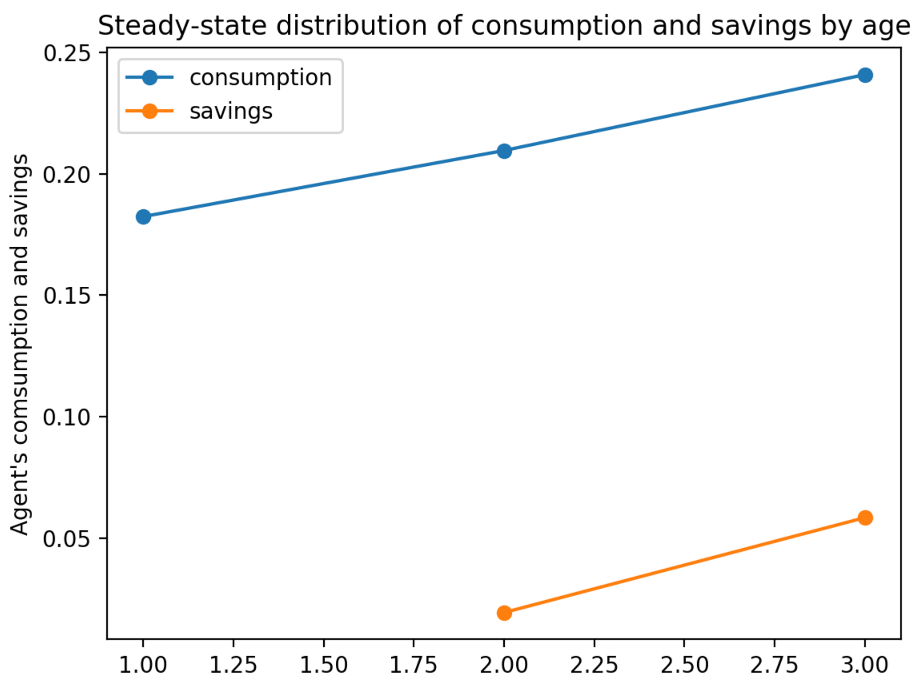
Out[20]:

```
{'b_ss': array([0.01931253, 0.0584111 ]),
 'c_ss': array([0.18241213, 0.20961468, 0.24087387]),
 'w_ss': 0.20172465739052575,
 'r_ss': 2.433062339127069,
 'K_ss': 0.07772362575316386,
 'Y_ss': 0.6827603788602411,
 'C_ss': 0.6329006729395864,
 'EulErr_ss': array([-1.39266376e-12,  2.62900812e-12]),
 'RCerr_ss': 8.326672684688674e-17,
 'ss_time': 0.0003869999999999152}
```

Question 2.2 (Part b)

In [21]:

```
get_SS(params, bvec_guess, True)
```



Agent's age

Out[21]:

```
{'b_ss': array([0.01931253, 0.0584111 ]),
 'c_ss': array([0.18241213, 0.20961468, 0.24087387]),
 'w_ss': 0.20172465739052575,
 'r_ss': 2.433062339127069,
 'K_ss': 0.07772362575316386,
 'Y_ss': 0.6827603788602411,
 'C_ss': 0.6329006729395864,
 'EulErr_ss': array([-1.39266376e-12,  2.62900812e-12]),
 'RCerr_ss': 8.326672684688674e-17,
 'ss_time': 0.00042100000000000047}
```

Question 2.2 (Part c)

In [22]:

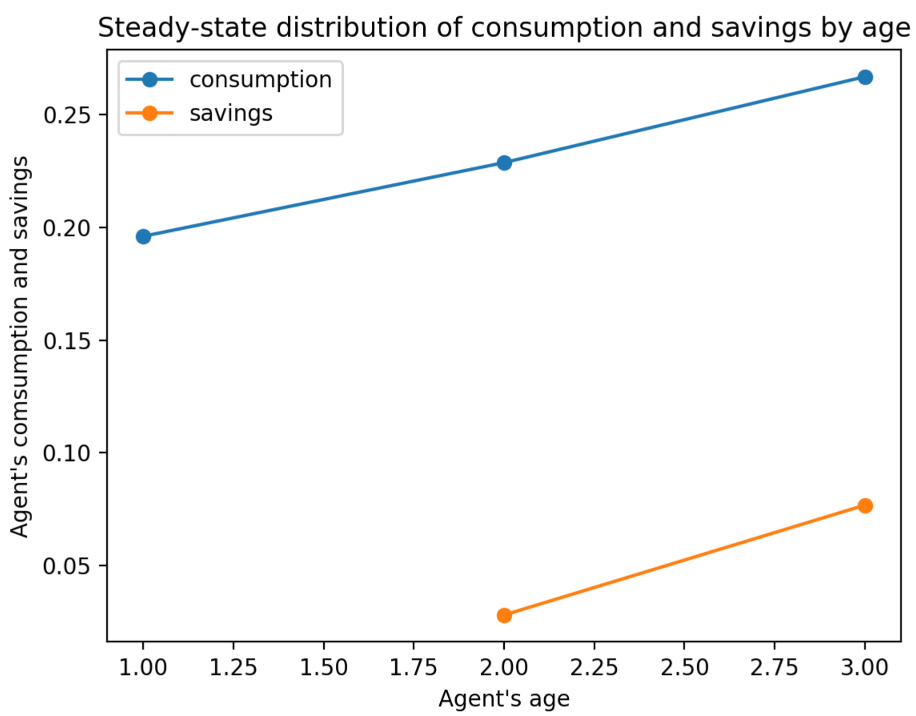
```python
beta = 0.55
params2 = (beta, sigma, nvec, L, A, alpha, delta, SS_tol)
res2 = get_SS(params2, bvec_guess, False)
res2
```

Out[22]:

```
{'b_ss': array([0.02817692, 0.07686545]),
 'c_ss': array([0.19597528, 0.22861594, 0.26669307]),
 'w_ss': 0.22415219593446706,
 'r_ss': 1.8863765057189819,
 'K_ss': 0.1050423702259807,
 'Y_ss': 0.7586689708551193,
 'C_ss': 0.6912842903551526,
 'EulErr_ss': array([6.87805368e-12, 2.52953214e-12]),
 'RCerr_ss': 9.71445146547012e-17,
 'ss_time': 0.00033400000000000565}
```

In [23]:

```python
get_SS(params2, bvec_guess, True)
```



Out[23]:

```
{'b_ss': array([0.02817692, 0.07686545]),
 'c_ss': array([0.19597528, 0.22861594, 0.26669307]),
 'w_ss': 0.22415219593446706,
 'r_ss': 1.8863765057189819,
 'K_ss': 0.105042370225980 7,
 'Y_ss': 0.7586689708551193,
 'C_ss': 0.6912842903551526,
 'EulErr_ss': array([6.87805368e-12, 2.52953214e-12]),
 'RCerr_ss': 9.71445146547012e-17,
 'ss_time': 0.0004610000000000447}
```

As beta value increases (here, from 0.442 to 0.55) consumption, savings and wage also increases. However, the interest rate decreases with an increase in the value of beta.
Hence, people start saving more in the current periods which results in higher savings which results in an increase in firm capital.
As a consequence of increase in firm capital, the consumption increases due to higher wage rates.

Question 2.3

In [24]:

```
beta = 0.442
nvec = np.array([1, 1, 0.2])
delta = 0.6415
sigma = 3
A = 1
alpha = 0.35
```

In [25]:

```
T = 30
max_iter = 300
TPI_tol = 1e-9
xi = 0.2
b_ss = res['b_ss']
K_ss = res['K_ss']
w_ss = res['w_ss']
r_ss = res['r_ss']
b1vec = np.array([0.8 * b_ss[0], 1.1 * b_ss[1]])
K1 = b1vec.sum()
Kpath = np.zeros(T + 1)
Kpath[:-1] = np.linspace(K1, K_ss, T)
Kpath[-1] = K_ss
```

In [26]:

```
def get_wage(alpha, A, K, L):
    return (1 -alpha) * A * (K / L) ** alpha

def get_r(alpha, A, L, K, delta):
    return alpha * A * (L / K) ** (1 - alpha) - delta

def get_euler_err1(b3, *args):
    nvec, beta, sigma, b2, w_path, r_path = args
    n2, n3 = nvec
    w1, w2 = w_path
    r1, r2 = r_path
    c2 = (1 + r1) * b2 + w1 * n2 - b3
    c3 = (1 + r2) * b3 + w2 * n3
    MU2 = c2 ** (-sigma)
    MU3 = c3 ** (-sigma)
    err1 = MU2 - beta * (1 + r2) * MU3
    return err1

def get_euler_err2(bvec, *args):
    nvec, beta, sigma, w_path, r_path = args
    b = np.append([0], bvec)
    b1 = np.append(bvec, [0])
    cvec = (1 + np.append([0], r_path)) * b + w_path * nvec - b1
    MU = cvec ** (-sigma)
    err2 = MU[:-1] - beta * (1 + r_path) * MU[1:]
```

```
    return err2
```

Question 2.3 (Part a)

In [27]:

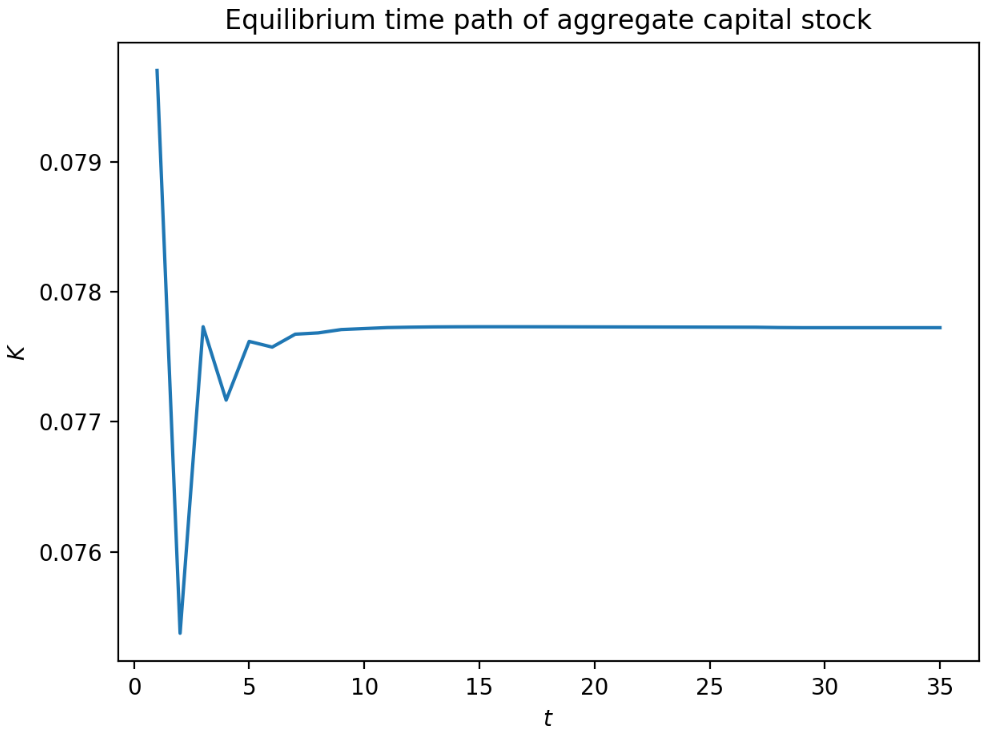
```
dist = 1
TPI_iter = 0
L = nvec.sum()

while dist > TPI_tol and TPI_iter < max_iter:
    TPI_iter += 1
    w_path = get_wage(alpha, A, Kpath, L)
    r_path = get_r(alpha, A, L, Kpath, delta)
    b = np.zeros((2, T + 1))
    b[:, 0] = b1vec
    b32 = opt.root(get_euler_err1, b[1, 0], (nvec[1:], beta, sigma, b[0, 0], w_path[:2], r_path[:2])
)
    b[1, 1] = b32.x
    for t in range(T - 1):
        bvec_guess = np.array([b[0, t], b[1, t + 1]])
        bt = opt.root(get_euler_err2, bvec_guess, (nvec, beta, sigma, w_path[t : t + 3], r_path[t +
1: t + 3]))
        b[0, t + 1] = bt.x[0]
        b[1, t + 2] = bt.x[1]
    Kpath_new = b.sum(axis = 0)
    dist = ((Kpath[:] - Kpath_new[:]) ** 2).sum()
    Kpath = xi * Kpath_new + (1 - xi) * Kpath
    print('iteration:', TPI_iter, ' squared distance: ', dist)
```

```
iteration: 1  squared distance:  0.0004014991152066923
iteration: 2  squared distance:  0.0002826966652592939
iteration: 3  squared distance:  0.00020026875465885345
iteration: 4  squared distance:  0.0001426592082072127
iteration: 5  squared distance:  0.00010212481834275573
iteration: 6  squared distance:  7.343027146808576e-05
iteration: 7  squared distance:  5.30051139372996e-05
iteration: 8  squared distance:  3.8394216775871686e-05
iteration: 9  squared distance:  2.7896278348728884e-05
iteration: 10  squared distance:  2.0323858102749774e-05
iteration: 11  squared distance:  1.4842654861254007e-05
iteration: 12  squared distance:  1.0862919656135854e-05
iteration: 13  squared distance:  7.96547263703129e-06
iteration: 14  squared distance:  5.850890391607051e-06
iteration: 15  squared distance:  4.304341150838523e-06
iteration: 16  squared distance:  3.171075543533071e-06
iteration: 17  squared distance:  2.339231165119519e-06
iteration: 18  squared distance:  1.7276951873671011e-06
iteration: 19  squared distance:  1.2774898003265711e-06
iteration: 20  squared distance:  9.456274233624502e-07
iteration: 21  squared distance:  7.007079291818133e-07
iteration: 22  squared distance:  5.197513827353448e-07
iteration: 23  squared distance:  3.8591146476748656e-07
iteration: 24  squared distance:  2.8681948977131325e-07
iteration: 25  squared distance:  2.133817615712518e-07
iteration: 26  squared distance:  1.5890405994290123e-07
iteration: 27  squared distance:  1.1845278247740205e-07
iteration: 28  squared distance:  8.838801579723708e-08
iteration: 29  squared distance:  6.602158831129704e-08
iteration: 30  squared distance:  4.936630883562093e-08
iteration: 31  squared distance:  3.6951813753668286e-08
iteration: 32  squared distance:  2.768917634344627e-08
iteration: 33  squared distance:  2.0771238815379916e-08
iteration: 34  squared distance:  1.559917728381762e-08
iteration: 35  squared distance:  1.1728338660751221e-08
iteration: 36  squared distance:  8.828256223155284e-09
iteration: 37  squared distance:  6.653106577910857e-09
iteration: 38  squared distance:  5.019863081914466e-09
iteration: 39  squared distance:  3.79212741149397e-09
iteration: 40  squared distance:  2.8681537781363875e-09
iteration: 41  squared distance:  2.171971427745118e-09
iteration: 42  squared distance:  1.6467985190721587e-09
iteration: 43  squared distance:  1.2501519398283977e-09
iteration: 44  squared distance:  9.502133229330115e-10
```

Question 2.3 (Part b)

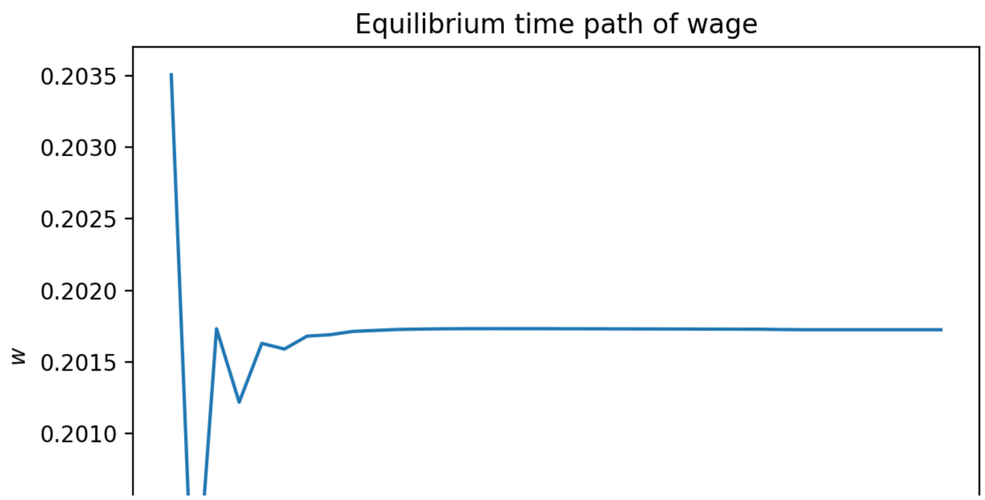
In [28]:

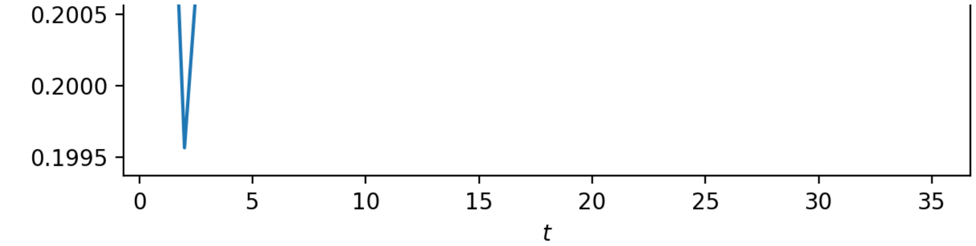
```
%matplotlib notebook
xval = np.arange(1, T + 6)
yval = np.append(Kpath[:T-2], K_ss * np.ones(7))
plt.plot(xval, yval)
plt.title('Equilibrium time path of aggregate capital stock', fontsize=12)
plt.xlabel(r'$t$')
plt.ylabel(r'$K$')
plt.tight_layout()
```



In [29]:

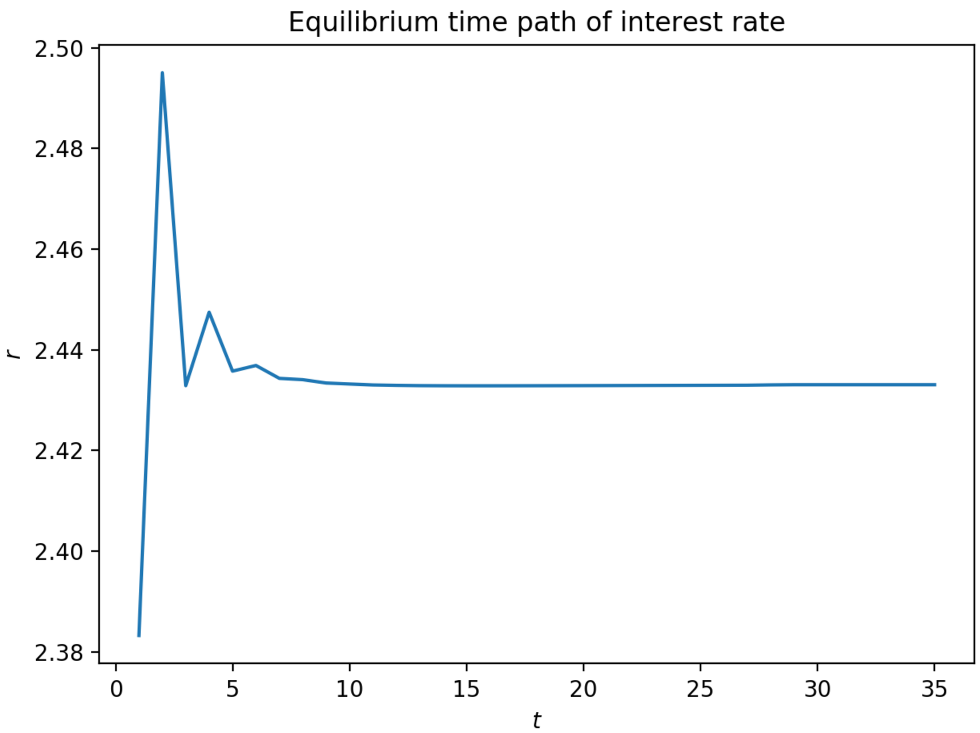
```
%matplotlib notebook
xval = np.arange(1, T + 6)
yval = np.append(w_path[:T-2], w_ss * np.ones(7))
plt.plot(xval, yval)
plt.title('Equilibrium time path of wage', fontsize=12)
plt.xlabel(r'$t$')
plt.ylabel(r'$w$')
plt.tight_layout()
```





In [30]:

```
%matplotlib notebook
xval = np.arange(1, T + 6)
yval = np.append(r_path[:T-2], r_ss * np.ones(7))
plt.plot(xval, yval)
plt.title('Equilibrium time path of interest rate', fontsize=12)
plt.xlabel(r'$t$')
plt.ylabel(r'$r$')
plt.tight_layout()
```



In [31]:

```
k_first = [k for k in Kpath if abs(k - K_ss) < 0.00001][0]
t1 = np.where(Kpath == k_first)[0][0]
print('first instance at time : ', t1+1)

k_last = [k for k in Kpath[:-3] if abs(k - K_ss) > 0.00001][-1]
t2 = np.where(Kpath == k_last)[0][0]
print('never out of bounds after time : ', t2+1)
```

```
first instance at time :  3
never out of bounds after time :  9
```

Question 2.3 (Part c)

The economy was within 0.00001 of the steady-state aggregate capital stock with 3 time periods.
And, after the 9th period aggregate capital stock is within 0.00001 margin of the steady-state capital stock value.

In [ ]: