Mathematics

Algebra

Exponent Laws

$x^a \cdot x^b = x^{a+b}$

$(x^a)^b = x^{ab}$

$(xy)^a = (x^a y^a)$

$x^{-1} = \frac{1}{x}$

$\frac{a^m}{a^n} = a^{m-n}$

Quadratic Formula

➔ Given $ax^2 + bx + c = 0$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Linear Slope Equations

$y = mx + b$

$y - y_1 = m(x - x_1)$

$m = \frac{y_2 - y_1}{x_2 - x_1}$

Factoring

$(a+b)^2 = a^2 + 2ab + b^2$

$a^2 - b^2 = (a+b)(a-b)$

$x^3 + y^3 = (x+y)(x^2 - xy + y^2)$

$x^3 - y^3 = (x-y)(x^2 + xy + y^2)$

Logarithms

$\ln A^x = x \ln A$

$\ln[A \cdot B] = \ln A + \ln B$

$\ln\left[\frac{A}{B}\right] = \ln A - \ln B$

$\ln\left(\frac{1}{x}\right) = -\ln(x)$

$\ln(1) = 0$ $\qquad \ln(e) = 1$

$\ln(e^x) = x$ $\qquad e^{\ln(x)} = x$

Vectors and Matrices

$\vec{u} \cdot \vec{v} = \quad \vec{u} \quad \cdot \quad \vec{v} \quad \cos(\theta)$

$\vec{u} \cdot \vec{v} = u_x v_x + u_y v_y + u_z v_z$

$\hat{u} \cdot \hat{v} = \cos(\theta)$

$\quad \vec{u} \quad = \sqrt{u_x^2 + u_y^2 + u_z^2}$

$\tan^{-1}\left(\frac{r_y}{r_x}\right) = \theta$

➔ Where r_x and r_y are vectors in the x-y plane

$\hat{u} = \frac{\vec{u}}{\vec{u}}$

$\cos\theta = \frac{\vec{u} \cdot \vec{v}}{v \quad u}$

$adj(A) = \begin{vmatrix} d & -b \\ -c & a \end{vmatrix}$

$A = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

$A^{-1} = \frac{1}{ad - bc} \begin{vmatrix} d & -b \\ -c & a \end{vmatrix}$

$\vec{u} \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ u_x & u_y & u_z \\ v_x & v_y & v_z \end{vmatrix}$

$\hat{i} \text{ x } \hat{j} = \hat{k}$ $\qquad \hat{j} \text{ x } \hat{i} = -\hat{k}$

$\hat{j} \text{ x } \hat{k} = \hat{i}$ $\qquad \hat{k} \text{ x } \hat{j} = -\hat{i}$

$\hat{k} \text{ x } \hat{i} = \hat{j}$ $\qquad \hat{i} \text{ x } \hat{k} = -\hat{j}$

Radicals

$\sqrt[n]{a^m} = a^{\frac{m}{n}}$

$\sqrt[n]{ab} = \sqrt[n]{a} \cdot \sqrt[n]{b}$

$\sqrt[m]{\sqrt[n]{a}} = \sqrt[nm]{a}$

$\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$

Geometry

Circles

$A = \pi r^2$

$C = 2\pi r$

$r^2 = (x-a)^2 + (y-b)^2$

$s = r\theta$

$A_{Hoop} = \frac{\pi}{4}(d_o^2 - d_i^2)$

$A_{Hoop} = \pi(r_o^2 - r_i^2)$

➔ (a, b) is the center of the circle.

➔ θ must be in radians.

Cylinders

$A = 2\pi rl + 2\pi r^2$

$V = \pi r^2 l$

Spheres

$A = 4\pi r^2$

$V = \frac{4}{3}\pi r^3$

$r^2 = (x-a)^2 + (y-b)^2 + (z-c)^2$

➔ (a, b, c) is the center of the sphere and (x, y, z) are coordinates on the surface of the sphere.

Right Triangles

$A = \frac{1}{2}bh$

$a^2 + b^2 = c^2$

Equilateral Triangles

$A = \frac{\sqrt{3}}{4}a^2$

$\theta = 60°$

Trigonometry

Right Angle Ratios

$$\sin\theta = \frac{\text{Opposite}}{\text{Hypotenuse}}$$

$$\cos\theta = \frac{\text{Adjacent}}{\text{Hypotenuse}}$$

$$\tan\theta = \frac{\text{Opposite}}{\text{Adjacent}}$$

➔ Using reciprocal identities, the ratios for sec, csc, and cot can be found.

Reciprocal Identities

$$\sin\theta = \frac{1}{\csc\theta} \qquad \cos\theta = \frac{1}{\sec\theta}$$

$$\tan\theta = \frac{1}{\cot\theta} \qquad \csc\theta = \frac{1}{\sin\theta}$$

$$\sec\theta = \frac{1}{\cos\theta} \qquad \cot\theta = \frac{1}{\tan\theta}$$

Tan/Cotangent Identities

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cot\theta = \frac{\cos\theta}{\sin\theta}$$

Trig Identities

$\sin^2\theta + \cos^2\theta = 1$

$\tan^2\theta + 1 = \sec^2\theta$

$\cot^2\theta + 1 = \csc^2\theta$

$\sin(2x) = 2\sin(x)\cdot\cos(x)$

$\cos(2x) = \cos^2(x) - \sin^2(x)$

$\cos^2(x) = \frac{1}{2}(1 + \cos(2x))$

$\sin^2(x) = \frac{1}{2}(1 - \cos(2x))$

$^*\cos^n\theta = [\cos\theta]^n$

*Valid for all trigonometric functions (sin, cos, tan, cot, sec, csc).

Even/Odd Formulas

$\sin(-\theta) = -\sin\theta$

$\cos(-\theta) = \cos\theta$

$\tan(-\theta) = -\tan\theta$

$\csc(-\theta) = -\csc\theta$

$\sec(-\theta) = \sec\theta$

$\cot(-\theta) = -\cot\theta$

Double Angle Formulas

$\sin 2\theta = 2\sin\theta\cos\theta$

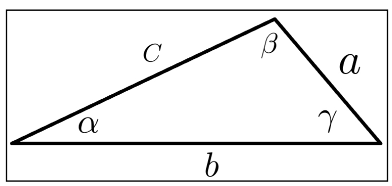
$$\begin{aligned}\cos 2\theta &= \cos^2\theta - \sin^2\theta \\ &= 2\cos^2\theta - 1 \\ &= 1 - 2\sin^2\theta\end{aligned}$$

Degrees to Radians

➔ Where D is an angle is degrees and R is an angle in radians.

$$R = D\cdot\frac{\pi}{180} \qquad D = R\cdot\frac{180}{\pi}$$

Law of Sines



$$\frac{\sin\alpha}{a} = \frac{\sin\beta}{b} = \frac{\sin\gamma}{c}$$

Law of Cosines

$a^2 = b^2 + c^2 - 2bc\cos\alpha$

$b^2 = a^2 + c^2 - 2ac\cos\beta$

$c^2 = a^2 + b^2 - 2ab\cos\gamma$

Small Angle Approx.

$\sin\theta \approx \theta$

$\cos\theta \approx 1 - \frac{\theta^2}{2} \approx 1$

$\tan\theta \approx \theta$

Calculus

Derivative Properties

$\frac{d}{dx}(c) = 0$

$(c\cdot f(x))' = c\cdot f'(x)$

$(f(x) \pm g(x))' = f'(x) \pm g'(x)$

Derivative Power Rule

$\frac{d}{dx}(x^n) = n\cdot x^{n-1}$

Derivative Product Rule

$\frac{d}{dx}\big(f(x)\cdot g(x)\big) = f'(x)\cdot g(x) + f(x)\cdot g'(x)$

Derivative Quotient Rule

$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{f'(x)\cdot g(x) - f(x)\cdot g'(x)}{(g(x))^2}$

Derivative Chain Rule

$\frac{d}{dx}\Big(f\big(g(x)\big)\Big) = f'\big(g(x)\big)\cdot g'(x)$

Standard Derivatives

$\frac{d}{dx}(x) = 1$

$\frac{d}{dx}\big(\sin(x)\big) = \cos(x)$

$\frac{d}{dx}\big(\cos(x)\big) = -\sin(x)$

$\frac{d}{dx}\big(\tan(x)\big) = \sec^2(x)$

$\frac{d}{dx}\big(\sec(x)\big) = \sec(x)\cdot\tan(x)$

$\frac{d}{dx}\big(\csc(x)\big) = -\csc(x)\cdot\cot(x)$

$\frac{d}{dx}\big(\cot(x)\big) = -\csc^2(x)$

$\frac{d}{dx}\big(\sin^{-1}(x)\big) = \frac{1}{\sqrt{1-x^2}}$

$\frac{d}{dx}\big(\cos^{-1}(x)\big) = -\frac{1}{\sqrt{1-x^2}}$

$$\frac{d}{dx}\left(\tan^{-1}(x)\right) = \frac{1}{1+x^2}$$

$$\frac{d}{dx}\left(n^x\right) = n^x \cdot \ln(n)$$

$$\frac{d}{dx}\left(e^{nx}\right) = n \cdot e^{nx}$$

$$\frac{d}{dx}\left(\ln(x)\right) = \frac{1}{x}\ , x > 0$$

$$\frac{d}{dx}\left(\ln\ x\ \right) = \frac{1}{x}\ , x \neq 0$$

Chain Rule Variations

$$\frac{d}{dx}\left(\left[f(x)\right]^n\right) = n \cdot \left[f(x)\right]^{n-1} \cdot f'(x)$$

$$\frac{d}{dx}\left(e^{f(x)}\right) = f'(x) \cdot e^{f(x)}$$

$$\frac{d}{dx}\left(\ln\left[f(x)\right]\right) = \frac{f'(x)}{f(x)}$$

$$\frac{d}{dx}\left(\sin[f(x)]\right) = f'(x) \cdot \cos\left[f(x)\right]$$

$$\frac{d}{dx}\left(\cos\left[f(x)\right]\right) = -f'(x) \cdot \sin\left[f(x)\right]$$

$$\frac{d}{dx}\left(\tan\left[f(x)\right]\right) = f'(x) \cdot \sec^2\left[f(x)\right]$$

$$\frac{d}{dx}\left(\sec\left[f(x)\right]\right) = f'(x) \cdot \sec\left[f(x)\right] \cdot \tan\left[f(x)\right]$$

$$\frac{d}{dx}\left(\tan^{-1}\left[f(x)\right]\right) = \frac{f'(x)}{1+\left[f(x)\right]^2}$$

Integral Properties

$$\int f(x) \pm g(x)\, dx = \int f(x)\, dx \pm \int g(x)\, dx$$

$$\int_a^b f(x)\, dx = \int_a^c f(x)\, dx + \int_c^b f(x)\, dx$$

$$\int_a^a dx = 0$$

$$\int_a^b f(x)\, dx = -\int_b^a f(x)\, dx$$

$$\int_a^b C \cdot f(x)\, dx = C \cdot \int_a^b f(x)\, dx$$

$$\int_a^b C \cdot\ dx = C \cdot (b-a)$$

Integral Power Rule

$$\int x^n\, dx = \frac{x^{n+1}}{n+1} + C$$

Standard Integrals

$$\int k\ dx = k \cdot x + C$$

$$\int e^n x\ dx = \frac{1}{n} e^x + C$$

$$\int \cos(x)\, dx = \sin(x) + C$$

$$\int \sin(x)\, dx = -\cos(x) + C$$

$$\int \sec^2(x)\, dx = \tan(x) + C$$

$$\int \csc^2(x)\, dx = -\cot(x) + C$$

$$\int \frac{1}{ax+b}\, dx = \frac{1}{a} \ln\ ax+b\ + C$$

$$\int \sec(x) \cdot \tan(x)\, dx = sec(x) + C$$

$$\int \csc(x) \cdot \cot(x)\, dx = -\csc(x) + C$$

$$\int \sec(x)\, dx = \ln\ \sec(x) + \tan(x)\ + C$$

$$\int \csc(x)\, dx = -\ln\ \csc(x) + \cot(x)\ + C$$

$$\int x^{-1}\, dx = \int \frac{1}{x}\, dx = \ln\ x\ + C$$

$$\int \ln(x)\, dx = x \cdot \ln(x) - x + C$$

$$\int \tan(x)\, dx = \ln\ \sec(x)\ + C$$

$$\int \tan(x)\, dx = -\ln\ \cos(x)\ + C$$

$$\int \frac{1}{a^2+x^2}\, dx = \frac{1}{a} \arctan\left(\frac{x}{a}\right) + C$$

Integration Techniques

Topics include U-Sub, Integration by parts, trigonometric integrals, trig. sub, and PFD.

U-Substitution

Take an "X" term to be u, and then take du of that u term. Solve the integral in terms of u, and then re-substitute into the equation.

If needed, find new limits of integration using the substitution. Example:

$$\int_1^2 5x^2 \cos(x^3)\, dx$$

so

$$u = x^3 \therefore du = 3x^2\, dx$$

or

$$\frac{1}{3}\, du = x^2\, dx$$

resulting in

$$= 5 \int_*^{**} \frac{1}{3} \cos(u)\, du$$

Notice the substitution chosen allows for all x terms to be turned into u terms.

The integral can now easily be solved through standard methods. Once solved, replace u with the substitution above and replace the limits of integration as well. Solve as normal.

It is possible to complete u-sub without suppressing the limits of integration, you will just need to plug the given limits into the u term to find the new limits of integration.

For example, the lower would become $(1^3) = 1$ and the upper would become $(2^3) = 8$. Note that either method works and produces the same solution.

Integration by Parts

The standard formula for integration by parts is as follows:

$$\int u\,dv = uv - \int v\,du$$

Find u and dv in the original equation, then solve for du and v. Plug into the formula above and solve.
The u term can be found according to ILATE: inverse trigonometric, logarithmic, algebraic, trigonometric and exponential.

Example:

$$\int xe^{-x}\,dx$$

so

$$u = x \qquad dv = e^{-x}$$
$$du = dx \qquad v = -e^{-x}$$

using the equation above:

$$= -xe^{-x} + \int e^{-x}\,dx$$

resulting in

$$= -xe^{-x} - e^{-x} + C$$

Trigonometric Integrals

When solving an integral with trigonometric functions (usually involving powers and multiple trig functions multiplied together), a u-sub may not be able to be applied.

Instead, the integral will need to be separated into multiples of the trig function, apply a trig identity, and then complete the u-sub.

Example:

$$\int \sin^6 x \cos^3 x\,dx$$

separating $\cos^3 x$ into $\cos^2 x \cdot \cos x$ and applying an identity:

$$= \int \sin^6 x(1-\sin^2 x)\cos x\,dx$$

take $u = \sin x \therefore du = \cos x\,dx$ and perform the remaining u-sub:

$$= \int u^6(1-u^2)\,du$$

ending with:

$$= \frac{1}{7}\sin^7 x - \frac{1}{9}\sin^9 x + C$$

Note that while $\sin^2 x + \cos^2 x = 1$ is a common substitution, it is also common for other identities such as $\tan^2 x + 1 = \sec^2 x$ to be used as well.

Trigonometric Substitution

In certain cases, an integral may contain one of the following roots. In such situation, the following substitutions and formulas will be used to solve the integral.

Case I:

$$\sqrt{a^2 - b^2x^2} \Rightarrow x = \frac{a}{b}\sin\theta$$

uses $\cos^2(\theta) = 1 - \sin^2(\theta)$

Case II:

$$\sqrt{b^2x^2 - a^2} \Rightarrow x = \frac{a}{b}\sec(\theta)$$

uses $\tan^2(\theta) = \sec^2(\theta) - 1$

Case III:

$$\sqrt{a^2 + b^2x^2} \Rightarrow x = \frac{a}{b}\tan(\theta)$$

uses $\sec^2(\theta) = \tan^2(\theta) + 1$

Example:

$$\int \frac{1}{(1-x^2)^{3/2}}\,dx$$

Because this is a case I problem, use the substitution

$$x = \sin\theta \therefore dx = \cos\theta$$

Apply the substitution(s) back into the original equation:

$$\int \frac{1}{(1-\sin^2(\theta))^{3/2}} \cdot \cos\theta\,d\theta$$

From here, the integral can be simplified and solved readily:

$$= \int \frac{1}{(\cos^2\theta)^{3/2}} \cdot \cos\theta\,d\theta$$

$$= \int \frac{1}{\cos^3\theta} \cdot \cos\theta\,d\theta$$

$$= \int \frac{1}{\cos^2\theta}\,d\theta$$

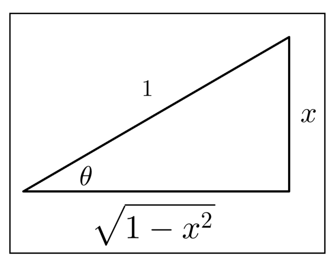
$$= \int \sec^2\theta\,d\theta$$

$$= \tan\theta + C$$

Although temping to assume so, the problem is not solved. Because a substitution was applied near the beginning, the final answer must be in terms of x, not θ.

$$\sin\theta = \frac{x}{1} = \frac{\text{opposite}}{\text{hypotenuse}}$$

By creating a right triangle with this definition, the adjacent side a can be solved:



Recall that $a^2 + b^2 = c^2$ and as such $(x)^2 + (a)^2 = (1)^2$, resulting in

$$a = \sqrt{1-x^2}$$

The final result can finally be expressed in terms of x as

$$= \frac{x}{\sqrt{1-x^2}} + C$$

Partial Fractions

Occasionally an integral will involve a fraction which may be difficult to be solved by standard substitution methods.
Using PFD, the integral can be broken up into simpler fractions which can be easier solved.

Example:

$$\int \frac{3x+2}{x^2+x}\,dx$$

This integral is difficult by itself, due to the fact that an easy u-sub is not available.

To help with this, it can be broken down into simpler integrals. Begin by observing the fraction only and factoring the denominator:

$$\frac{3x+2}{x(x+1)}$$

This fraction can now re-written, with the factors of the denominator for each fraction:

$$\frac{3x+2}{x(x+1)} = \frac{A}{x} + \frac{B}{x+1}$$

Because the numerators are not known, variables A and B are put in place. Note the original factored fraction goes on the left.

From here the denominator of the left (in this case $x(x+1)$) is multiplied through the equation:

$$^*\ 3x+2 = A(x+1) + B(x)$$

Make note that parts of the denominators of terms A and B canceled, resulting in a much simpler expression than what was started with.

Multiplying terms:

$$3x+2 = Ax + A + Bx$$

Group terms based on their order (or "power"):

$$3x+2 = (A+B)x + A$$

From here, the coefficient matching game is played. Match the coefficients from the left (with respect to exponents/powers) to the coefficients of the right.

$$3 = A + B$$
$$2 = A$$

Notice it is just the raw coefficients and A/B terms in the new set of equations. From here, it is seem that $A = 2$ and $B = 1$.

This conclusion could also be reached by revisiting equation *. Because the equation is true for any value of x, the equation can be solved by picking "0" as x and solving from there.

$$3(0) + 2 = A(0+1) + B(0)$$
$$2 = A$$

The same could be done for finding B (notice that A cancels this time):

$$3(-1) + 2 = A(-1+1) + B(-1)$$
$$-1 = -B \therefore B = 1$$

Once the numerators are realized, they can be plugged back into the first decomposition:

$$\frac{3x+2}{x(x+1)} = \frac{2}{x} + \frac{1}{x+1}$$

Because of this, the starting integral can now be replaced as well:

$$\int \frac{3x+2}{x(x+1)}\,dx = \int \frac{2}{x} + \frac{1}{x+1}\,dx$$

This is now a much easier integral, and can be readily solved using standard methods:

$$\int \frac{2}{x}\,dx + \int \frac{1}{x+1}\,dx$$

The Laplace Transform

$$\mathscr{L}\{f(t)\} = \int_0^\infty e^{-st} \cdot f(t)\,dt$$

Laplace Transforms

$$\mathscr{L}\{1\} = \frac{1}{s}$$
$$\mathscr{L}\{t^n\} = \frac{n!}{s^{n+1}}, n = 1,2,3,...$$
$$\mathscr{L}\{e^{at}\} = \frac{1}{s-a}$$
$$\mathscr{L}\{\sin kt\} = \frac{k}{s^2+k^2}$$
$$\mathscr{L}\{\cos kt\} = \frac{s}{s^2+k^2}$$
$$\mathscr{L}\{\sinh kt\} = \frac{k}{s^2-k^2}$$
$$\mathscr{L}\{\cosh kt\} = \frac{s}{s^2-k^2}$$

Inverse Laplace Transforms

$$\mathscr{L}^{-1}\left\{\frac{1}{s}\right\} = 1$$
$$\mathscr{L}^{-1}\left\{\frac{n!}{s^{n+1}}\right\} = t^n, n = 1,2,3,...$$
$$\mathscr{L}^{-1}\left\{\frac{1}{s-a}\right\} = e^{at}$$
$$\mathscr{L}^{-1}\left\{\frac{k}{s^2+k^2}\right\} = \sin kt$$
$$\mathscr{L}^{-1}\left\{\frac{s}{s^2+k^2}\right\} = \cos kt$$
$$\mathscr{L}^{-1}\left\{\frac{k}{s^2-k^2}\right\} = \sinh kt$$
$$\mathscr{L}^{-1}\left\{\frac{s}{s^2-k^2}\right\} = \cosh kt$$