Ono's theorem on money world and numerical simulation 2018.2.25

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Simple explanation of Ono's theorem. This report is based upon Ono (1994).

A representative individual's utility function at time *t* is assumed

$U_{(t)}= u_{(c(t))} + v_{(m(t))}$ (2.1)

where c is consumption at time t、m is money held at time t, $u_{(c)}$、$v_{(m)}$are utility from consumption and money. Lifelong utility is

$U = \int_0^{\infty}\{u_{(t)} + v_{(m)}\} e^{-\rho t} dt$ (2.2)

Where ρ is subjective discount rate.

The constraint of asset is

$a = m + b$

where a is total asset, b is bond held and it produces nominal interest R.

$\dot{x} = dx/dt$ and income constraint is

$\dot{a} = rb - \pi m - c$

$\Leftrightarrow$ $\dot{a} = ra - c - Rm$ (2.3)

where real interest is r, inflation rate is π.

We introduce following Lagrange using Lagrange multiplier λ

$L = \int_0^T [U_{(t)} + \lambda_{(t)}\{ra_{(t)} - c_{(t)} - Rm - \dot{a}_{(t)}\}] e^{-\rho t} dt$ (2.4)

Hamiltonian $H_{(t)}$ is defined as

$H_{(t)} = U_{(t)} + \lambda_{(t)}\{ra_{(t)} - c_{(t)} - Rm\}$ (2.5)

Then

$L = \int_0^T\{H_{(t)} - \lambda_{(t)}\dot{a}_{(t)}\} e^{-\rho t} dt = \int_0^T H_{(t)} e^{-\rho t}\, dt - \int_0^T \lambda_{(t)}\dot{a}_{(t)} e^{-\rho t}\, dt$ (2.6)

Then partial integral $-\int_0^T\{\lambda_{(t)}\dot{a}_{(t)} e^{-\rho t}\} dt$

$-\int_0^T\{\lambda_{(t)}\dot{a}_{(t)} e^{-\rho t}\} dt = \int_0^T\{\dot{\lambda}_{(t)} a_{(t)} e^{-\rho t} - \rho\lambda_{(t)} a_{(t)} e^{-\rho t}\} dt - [\lambda_{(t)} a_{(t)} e^{-\rho t}]_0^T$ (2.7)

Then

$L = \int_0^T\{H_{(t)}\lambda_{(t)} + \dot{\lambda}_{(t)} a_{(t)} e^{-\rho t} - \rho\lambda_{(t)} a_{(t)} e^{-\rho t}\} dt - \lambda_{(T)} a_{(T)} e^{-\rho T} + \lambda_{(0)} a_{(0)}$ (2.8)

Thus, the first order condition of maximization of $U_{(t)}$ using $f' = \partial f/\partial x$

$\frac{\partial L}{\partial c} = 0 \Leftrightarrow \frac{\partial H}{\partial c} = 0 \Leftrightarrow u'_{(c)} = \lambda$ (2.9)

$\frac{\partial L}{\partial m} = 0 \Leftrightarrow \frac{\partial H}{\partial m} = 0 \Leftrightarrow v'_{(m)} = R\lambda$ (2.10)

$\frac{\partial L}{\partial a} = 0 \Leftrightarrow \dot{\lambda} = -\frac{\partial H}{\partial a} - \frac{-\rho e^{-\rho t}}{e^{-\rho t}}\lambda \Leftrightarrow \dot{\lambda} = -r\lambda + \rho\lambda$

$\Leftrightarrow \dot{\lambda} = (\rho - r)\lambda$ (2.11)

and transversality condition is

$\lim_{t\to\infty} \lambda \quad a \quad e^{-\rho t} = 0$

(2.10) is divided by (2.9)

$R = \frac{v'_{(m)}}{u'_{(c)}}$ (2.12)

Then differentiate both side of (2.7) with respect to t

$\frac{\partial u'_{(c)}}{\partial t} = \dot{\lambda} \Leftrightarrow u''_{(c)}\dot{c} = \dot{\lambda}$ (2.13)

From (2.9), (2.11) and (2.13)

$u''_{(c)}\dot{c} = (\rho - r)u'_{(c)}$

$\Leftrightarrow \frac{u''_{(c)}}{u'_{(c)}}\dot{c} = \rho - r$

$\Leftrightarrow \frac{u''_{(c)}}{u'_{(c)}}\dot{c} = \rho + \pi - R$

$\Leftrightarrow R = -\frac{u''_{(c)}}{u'_{(c)}}\dot{c} + \rho + \pi$ (2.14)

From (2.12) and (2.14), then we get (2.15), these equations are fundamentals of the optimal solutions.

$R = \frac{v'_{(m)}}{u'_{(c)}} = -\frac{u''_{(c)}}{u'_{(c)}}\dot{c} + \rho + \pi$ (2.15)

Since $\lim_{c\to\infty} u'_{(c)} = 0$, and $\lim_{c\to\infty} u''_{(c)} < 0$, we assume

$u_{(c)} = \ln c$ (2.16) as a simplest case

Then from (2.15),

$\dot{c} = v'_{(m)}c^2 - (\rho + \pi)c$ (2.17)

We assume that inflation rate π as $\alpha\left(\frac{c}{y} - 1\right)$ where y is maximum capacity of production, and α is the speed of adjustment and positive constant.

$$\dot{m} = -\pi m = -\alpha\left(\frac{c}{y} - 1\right)m \quad (2.18)$$

$$\dot{c}/c = (v'_{(m)} - \alpha/y)c \quad - (\rho - \alpha) \quad (2.19)$$

(2.17) and (2.18), Ono explains the four possible paths of equilibrium in Figure 1.

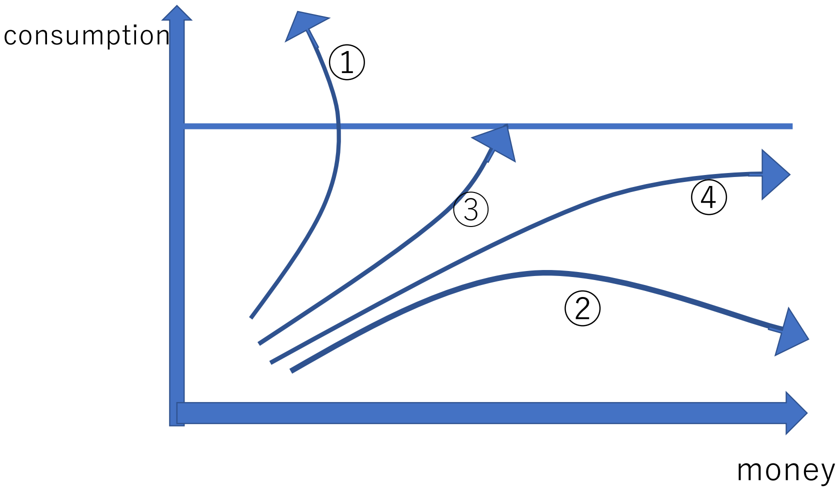


Figure.1 Ono's $c - m$ field and paths

As to the figure, there might exist four paths, ① excessive demand causes inflation, ② lack of effective demand causes deflation, ③ full equilibrium toward neo-classical equilibrium and, ④permanent stagnation of Keynesian. As to③and④,when (2.19) is satisfied then③,when (2.20) is satisfied then④

$\rho > \beta y \; and \; \alpha < \rho$ (2.19)

$\rho < \beta y \; and \; \alpha < \rho$ (2.20)

where

$\lim_{m\to\infty} v'_{(m)} = \beta$ (2.21)

as to ③, when $\dot{c} = 0$ and $\dot{m} = 0$ we can get

$c = \frac{\rho+\pi}{v'_{(m)}}$ (2.22)

$\pi = 0 \Leftrightarrow c = \mathrm{y}$ (2.23)

Then, when $m \to \infty$

$\bar{c} = \mathrm{y} = \frac{\rho}{\beta}$ (2.24)

this is ③'s equilibrium point. When $\frac{\rho}{\beta} < y, m \to \infty$ then, $\mathrm{c} < \mathrm{y}$ ④occurs.

As to ① since $\mathrm{c} > \mathrm{y}$ then $-\alpha\left(\frac{c}{y} - 1\right)m$ becomes negative within the finite period of time, thus it is infeasible. As to ②, from (2.9) and (2.16),
$u'_{(c)} = \lambda = 1/c$, and (2.18), (2.19), and since

$\dot{\lambda m} = \dot{\lambda} m + \lambda \dot{m}$ $\frac{\dot{\lambda m}}{\lambda m} = \frac{\dot{\lambda} m + \lambda \dot{m}}{\lambda m} = \frac{\dot{m}}{m} + \frac{\dot{\lambda}}{\lambda} = \frac{\dot{m}}{m} + \frac{u''_{(c)}\dot{c}}{\lambda} = \frac{\dot{m}}{m} - \frac{\dot{c}}{c^2} * c$

Then,

$\rho - cv'_{(m)} = \frac{\dot{m}}{m} - \frac{\dot{c}}{c} = \frac{\dot{\lambda m}}{\lambda m}$

Since c approaches zero, λm expands at a rate of ρ, and $a > m$, this imply that
$\lim_{t\to\infty} \lambda \ \ a \ \ e^{-\rho t} \geq \lim_{t\to\infty} \lambda \ \ m \ \ e^{-\rho t} > 0$, thus, the transversality condition
$\lim_{t\to\infty} \lambda \ \ a \ \ e^{-\rho t} = 0$ cannot be satisfied. Thus, this path cannot exist.
Only equilibrium we have are ③and④, since both cases $\rho - cv'_{(m)} < \rho$, the transversality condition is satisfied.
Then, Ono shows the classification of this dynamic process by parameters $\beta y, \rho, \alpha$ in the following figure.

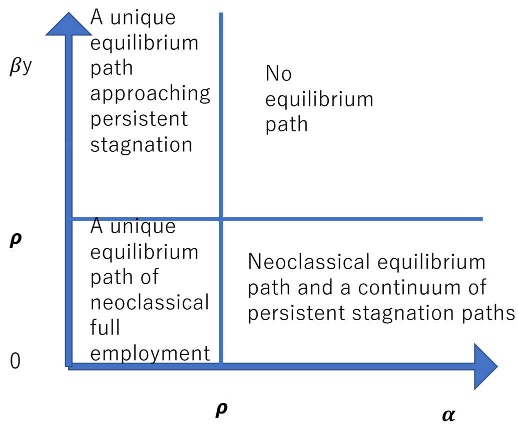


Figure 2 Types of dynamic equilibrium

The important point here is that there exists the threshold which distinguishes neo-classical equilibrium and persistent stagnation equilibrium. When $\rho < \beta y$, this means that very high production capacity y may exist and marginal utility of money might be not so extremely small, i.e. not zero, and also our subjective discount rate of utility in terms of time is small, we will have the situation that stagnation continues and only money assets increases.

Example of Ono's c-m path

When we assume $v'_{(m)} = \beta$ then from (2.19) $\dot{c} = (\beta - \alpha/y)c^2 - (\rho - \alpha)c$

This is a Bernoulli's deferential equation, we can solve it with respect to t.

Then we can solve (2.18), then we can solve R with respect to t and λ in (2.12) with respect to t.

$$\frac{\dot{c}}{c^2} = (\beta - \frac{\alpha}{y}) - \frac{(\rho - \alpha)}{c}$$

suppose $z = \frac{1}{c}$

then,

$$\dot{z} = (\rho - \alpha)z - \left(\beta - \frac{\alpha}{y}\right)$$

$$z = \frac{\left(\beta - \frac{\alpha}{y}\right)}{\rho - \alpha} + C_0 e^{(\rho-\alpha)t} \quad (C_0 \text{ is constant})$$

Therefore,

$$c(t) = \frac{\rho - \alpha}{\left(\beta - \frac{\alpha}{y}\right) + C_0(\rho-\alpha)e^{(\rho-\alpha)t}} \qquad (2.25) \qquad \left(\lambda(t) = \frac{1}{c} = \frac{\left(\beta - \frac{\alpha}{y}\right)}{\rho - \alpha} + C_0 e^{(\rho-\alpha)t}\right)$$

Also, from (2.20) and (2.19), we can derive m(t) as follows:

$$\dot{m} = -\pi m = -\alpha\left(\frac{c}{y} - 1\right)m$$

$$\frac{dm}{m} = -\alpha\left(\frac{c}{y} - 1\right)dt$$

$$\ln(m) = -\alpha\left(\frac{1}{y}\int \frac{\rho - \alpha}{\left(\beta - \frac{\alpha}{y}\right) + C_0(\rho - \alpha)e^{(\rho-\alpha)t}}\, dt - t + C_1\right)$$

$$m = \exp\left(-\alpha\left(\frac{1}{y}\int \frac{\rho - \alpha}{\left(\beta - \frac{\alpha}{y}\right) + C_0(\rho - \alpha)e^{(\rho-\alpha)t}}\, dt - t + C_1\right)\right)$$

$$= C_2 \exp\left(-\alpha\left(\frac{1}{y}\int \frac{\rho - \alpha}{\left(\beta - \frac{\alpha}{y}\right) + C_0(\rho - \alpha)e^{(\rho-\alpha)t}}\, dt - t\right)\right)$$

We can calculate the integrated part as follows:

$$(highlighted\ part) \times \frac{e^{(\alpha-\rho)t}}{e^{(\alpha-\rho)t}} = \int \frac{(\rho-\alpha)e^{(\alpha-\rho)t}}{\left(\beta - \frac{\alpha}{y}\right)e^{(\alpha-\rho)t} + C_0(\rho-\alpha)}\, dt$$

We then substitute $x = \left(\beta - \frac{\alpha}{y}\right) e^{(\alpha-\rho)t} + C_0(\rho - \alpha)$

$$\left(\quad dx = \left(\left(\beta - \frac{\alpha}{y}\right)(\rho - \alpha)e^{(\alpha-\rho)t}\right) dt \quad \right)$$

Then,

$$\int \frac{(\rho-\alpha)e^{(\alpha-\rho)t}}{\left(\beta-\frac{\alpha}{y}\right)e^{(\alpha-\rho)t}+C_0(\rho-\alpha)}\, dt \;=\; \int \frac{\frac{1}{\left(\beta-\frac{\alpha}{y}\right)}}{x}\, dx = \frac{y}{\beta y-\alpha}\ln(x)$$

$$= \frac{y}{\beta y-\alpha}\ln\left(\left(\beta - \frac{\alpha}{y}\right) e^{(\alpha-\rho)t} + C_0(\rho - \alpha)\right)$$

Therefore,

$$m(t) = C_2 \exp\left(-\alpha \left(\frac{\ln\left(\left(\beta - \frac{\alpha}{y}\right) e^{(\alpha-\rho)t} + C_0(\rho - \alpha)\right)}{\beta y - \alpha} \right) - t \right)$$

$$= C_2 e^{-\alpha t}\left(\left(\beta - \frac{\alpha}{y}\right) e^{(\alpha-\rho)t} + C_0(\rho - \alpha)\right)^{\frac{\alpha}{\alpha-\beta y}} \quad (C_0 \text{ and } C_2 \text{ are constants})$$

$$m(t) = C_2 e^{-\alpha t}\left(\left(\beta - \frac{\alpha}{y}\right) e^{(\alpha-\rho)t} + C_0(\rho - \alpha)\right)^{\frac{\alpha}{\alpha-\beta y}} \quad (2.27)$$

We have done simulation of c-m paths with specific values within table 1.

Simulation Result:

Initial value:

c=350 (trillion yen)

m=50 (trillion yen)

table 1　Simulation Value

	Price adjustment speed	coefficient value in v(m)	Subjective discount rate	Production
Parameters	α	β	ρ	y
Value	0.03	(N) 0.00007 (K) 0.000083	0.04	500—600
Meaning	$\alpha < \rho$ $value\ near\ \rho$	(N) $\rho > \beta y$ (K) $\rho < \beta y$	Official discount rate(JPN)	GDP of Japan (trillion yen)

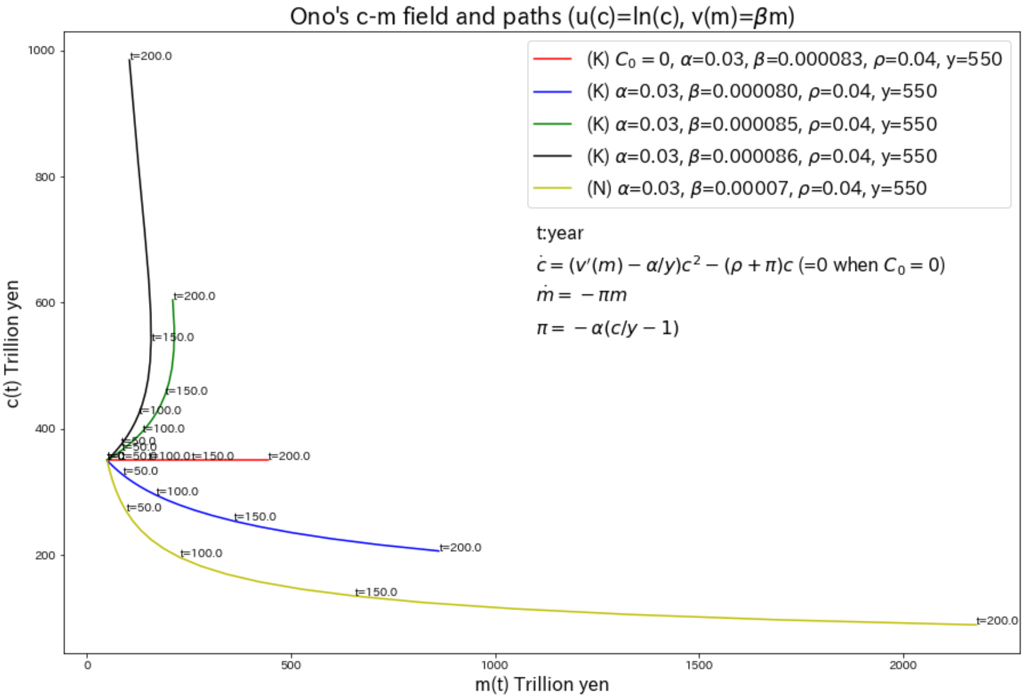


Figure 3 Ono's c-m field and paths

We can see from the Figure 3 that paths other than C_0=0 are either paths of inflation or deflation. (From(2.25), $c(t) = \frac{\rho-\alpha}{\left(\beta-\frac{\alpha}{y}\right)+C_0(\rho-\alpha)e^{(\rho-\alpha)t}}$. If $C_0 \neq 0$, $c(t)$ converges to 0 which does not satisifiy transversality condition. Therefore (2.25) satisfies the transversality condition only when $C_0 = 0$.)

For further research, we need to do more simulation with different assumption of u(c) and v(m).

Ono,Y 1994, Money, Interest, and Stagnation, Oxford University Press

報告書提出後の記録（3/10/2018）

<u>上の例では新古典派均衡が存在しない理由</u>

境界関数は、

$$\dot{c} = 0 : c = \left\{\frac{\rho - \alpha}{v'(m)y - \alpha}\right\} y \tag{1}$$

$$\dot{m} = 0 : c = y \tag{2}$$

である。

教科書 p79 3.1 の新古典派的完全均衡経路の条件は、

$$\alpha < \rho \, , \beta y < \rho \tag{3}$$

この時(1),(2)は交点が存在する⇔ $\frac{\rho-\alpha}{v'(m)y-\alpha} = 1$なる m が存在する。

（すなわち、$\dot{c} = 0$ かつ$\dot{m} = 0$の点が存在しこの点が均衡点である）

ここで、

$v'(m) = \beta$ とすると$\frac{\rho-\alpha}{v'(m)y-\alpha} = \frac{(\rho-\alpha)}{\beta y-\alpha} > 1$ (定数) となり(1),(2)の交点が存在しない。従って、$\dot{c} = 0$ かつ$\dot{m} = 0$なる点は存在しない。

<u>ここで、$\beta y = \rho$と条件を緩めることで均衡直線は実現できるが、教科書には触れられていない</u>。

<u>v(m) 別の関数の仮定</u>

ここで、

$$v'(m) = \beta + \frac{\beta_2}{m} \quad \left(v(m) = \beta m + \beta_2 \ln(m) + C \, , C_m は定数\right) \tag{4}$$

とすると、均衡解が存在する。

m が十分小さい時、

$$\frac{\rho-\alpha}{(v'(m)y-\alpha}=\frac{\rho-\alpha}{\beta+\frac{\beta_2}{m}-\alpha}<1$$

であり、

$$m\to\infty \text{ の時、} \frac{\rho-\alpha}{v'(m)y-\alpha}=\frac{(\rho-\alpha)}{\beta y-\alpha}>1$$

である。従って、$\frac{\rho-\alpha}{v'(m)y-\alpha}=1$なる m が存在する。（交点が存在する）

次の微分方程式を backward で解く。（runge-kutta 法）

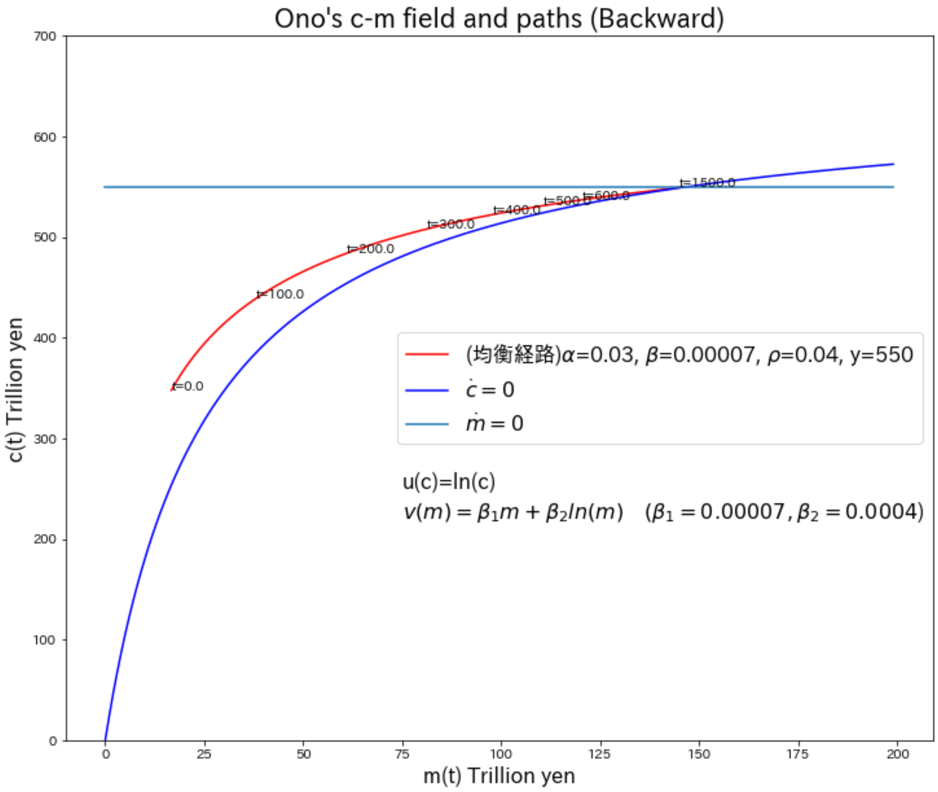
$\dot{c}=(\beta+\beta_2/m-\alpha/y)c^2-(\rho-\alpha)c$

$\dot{m}=-\pi m$ $=-\alpha\left(\frac{c}{y}-1\right)m$

赤色線均衡経路である。ここで、c(T), m(T)は境界線の交点、すなわち均衡経路にあたり、backward シミュレーションは c(T),m(T)からスタートする。

下の図だとある 1500 年程度かかることが見られるが、これは境界線の交点を初期時刻の値 c(0),m(0)に近づけることで少なくすることができる。(例えば 50 年など)

新古典派的均衡経路



実際にはこの関数で不況均衡経路を作ることも可能である。

ソースコード(python)：

```
//ライブラリの導入

%matplotlib inline  import matplotlib

import matplotlib.pyplot as plt

import numpy as np

from matplotlib.font_manager import FontProperties

from pylab import rcParams


rcParams['figure.figsize'] = 12,10

tlist=[] eps=0.00001 alpha=0.03 beta=0.00007 rho=0.04 y=550

//cbound は dc/dt=0 の境界線

def cbound(m):

    return y*(rho-alpha)/((beta+0.0004/m)*y-alpha) //主要部


def func(alpha,beta,rho,y,c0,m0,mlist,clist,check):

//dm/dt=f, dc/dt=g である

    def f(t,m,c):

        return -alpha*(c/y-1)*m

    def g(t,m,c):

        return (beta+0.0004/m-alpha/y)*c*c-(rho-alpha)*c //連立微分方程式 Runge-Kutta 法

    def RK(f,g,finaltime,tscale): //dt は計算における時刻の刻み幅

        dt = 0.01 //tt は計算における初期時刻（backward なのでここから引いて行く。実用上は十分大
        きな値をとっていれば大丈夫）

        tt=3000

        m_old=m0

        c_old=c0

        for time in range(int(finaltime/dt)+1):

        //finaltime/dt は時刻 finaltime までに dt 刻みで何回計算させるかという意味

        //tscale 毎に各値を記録

            if(tt%tscale==0):
```

```
if (len(tlist)<=finaltime/tscale): tlist.append(round(tt,3))

    mlist.append(m_old) clist.append(c_old)

    //4 次のルンゲクッタの方法。m と c に対し並列に行うことで差分を随時更新している。 k11=f(tt,m_old,c_old)*dt k12=g(tt,m_old,c_old)*dt

    k21=f(tt-dt/2,m_old-k11/2,c_old-k12/2)*dt k22=g(tt-dt/2,m_old-k11/2,c_old-k12/2)*dt

    k31=f(tt-dt/2,m_old-k21/2,c_old-k22/2)*dt k32=g(tt-dt/2,m_old-k21/2,c_old-k22/2)*dt

    k41=f(tt-dt,m_old-k31,c_old-k32)*dt

    k42=g(tt-dt,m_old-k31,c_old-k32)*dt

    km=(k11+2*k21+2*k31+k41)/6

    kc=(k12+2*k22+2*k32+k42)/6

    収束条件（更新する値 km,kc が eps より小さい場合収束したと考え、計算をやめる）

    if((km < eps) and (kc <eps)):

          break

            値の更新

    else:

        m_new=m_old-km

        c_new=c_old-kc

        m_old=m_new

        c_old=c_new

        tt=round(tt-dt,3) //round で少数第 3 桁まで記録


   RK(f,g,1500,1)

NUM=1

mlist = [[] for j in range(NUM)]

clist = [[] for j in range(NUM)]

//計算の実行 (func(alpha,beta,rho,y,c0,m0,mlist,clist,check)

func(0.03,0.00007,0.04,550,550,145,mlist[0],clist[0],1)

fig=plt.figure()

ax = fig.add_subplot(111)

ax.ticklabel_format(useOffset=False)
```

```
ax.plot(mlist[0],clist[0],'r',label=r"均衡経路
()␣$¥alpha$=0.03,␣$¥beta$=0.00007,␣$¥rho$=0.04,␣y=550")

x=np.arange(200)

ax.plot(x,list(map(cbound,x)), label=r"$¥dot␣c=0$")

ax.plot(x,[y for j in range(len(x))],label=r"$¥dot␣m=0$")

plt.title(r"Ono's c-m field and paths (Backward)",fontsize="20")

plt.xlabel("m(t)␣Trillion␣yen", fontsize="16")

plt.ylabel("c(t)␣Trillion␣yen",fontsize="16")



tstamp=100 //tstamp*tscale 毎に時刻をふる

tlast=tlist[len(tlist)-1]

for i,txt in enumerate(tlist[::tstamp]):

  if (i>8 or i==0): //(backward で解くため i=0 が定常点での値になる)

    for t in range(NUM):

      ax.annotate("t={tl}".format(tl=txt-tlast),(mlist[t][tstamp*i],clist[t][tstamp*i]))

print(mlist[0][1000],clist[0][1000])

plt.text(75,250,r"u(c)=ln(c)",fontsize="16")

plt.text(75,220,r"$v(m)=¥beta_{1}␣m+¥beta_{2}ln(m)␣¥quad␣(¥beta_{1}=0.00007,␣¥beta_
{2}=0.0004$)",fontsize="16")

plt.legend(prop={'size': 16},loc='right')

plt.ylim(0,700)

plt.show()

 図の保存

fig.savefig('onos-cm-field(backward).png') # save the figure to file
```