Uncharted Waters

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Abstract

Traveling by boat has been an important part of human civilization for millennia. Whether by sail, motor, or both, experienced sailors are able to efficiently get from one location to another subject to any ocean current and wind conditions. In this project, we solve for the optimal sailing control for various scenarios including time-dependent fields for wind and currents. Specifically, we study a simple motor, a high-fidelity motor, and a sailboat problem. We then solve each of these using Pontryagin's Maximum Principle (PMP) or numerical methods, depending on the complexity of the problem.

1 Problem Statement and Motivation

Sailors have navigated the ocean for millennia, successfully piloting boats through dynamic ocean current and weather conditions. Today, the oversea shipping industry continues to be vital for effective supply chains around the world, with an estimated 90% of goods transported by sea [Nag21].

While the idea of shipping may seem simple, there are several complicating factors, including winds and currents. Both of these are always present, and depending on the course of travel can either aid or hinder the boat. In this project we attempt to optimally control motorboats and sailboats through time-variant current and wind fields from a specified port of departure to a specified destination. While our specific problem makes a few simplifying assumptions, such a model can be applicable to modern sea travel and, if implemented, could help to reduce both shipping times and costs.

Due to the importance and practicality of the problem we approach here, it is no surprise that others have attempted similar problems. For example, there are several studies that worked to build a optimal solution for ocean travel given various external conditions. Various techniques were used, including machine learning [YK16], neural networks [FBS94], partical swarm optimization [BZYW18], and other mathematical approaches [SHL17]. Other studies sought to solve related problems, such as underwater travel [ISM05]. The study most related to ours was one which used Pontryagin's Maximum Principle (PMP) and dynamic programming for ship routing, optimizing over travel time while minimizing fuel consumption [Bij02]. Interestingly, aside from the fuel control, their state matched the state of our simple problem [Bij02]. However, their paper focused on forecasting rather than boat control. [Bij02].

2 Mathematical Representation

We developed several mathematical representations of a simple boat model to explore the effects of varying dynamics and boat types. We also experimented with different objective functions to encourage desired optimal behavior.

2.1 Simple Motorboat Problem

We first consider the simplest case. Consider a boat in $\mathbf{R}^2$ propelled by only a motor which contributes a constant velocity of magnitude 1. We seek to control the motorboat's heading angle $u_1 \in [-\pi, \pi]$ to guide it through a time-invariant current field $C(x, y)$ from an initial point (x_0, y_0) to the desired destination (x_f, y_f) as quickly as possible. We begin with this simple case because under favorable conditions on the current field, it has an interesting analytic solution. However, with added intricacies, the problem quickly becomes analytically intractable and we are forced to resort to numerical solutions.

In general we use $\mathbf{x}$ to denote the state vector and $\mathbf{u}$ to denote the control vector and assume our time t to be in $[0, T]$ where T is the maximum allowed time. We can write the simple case as an optimal control problem where we seek to minimize the cost functional

$$J[\mathbf{x}, \mathbf{u}, t_f] = \int_0^{t_f} dt$$

subject to the state equation and boundary conditions

$$\dot{\mathbf{x}} = \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} C_1(x, y) + \cos(u_1) \\ C_2(x, y) + \sin(u_1) \end{bmatrix},$$
$$\mathbf{x}(0) = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}, \mathbf{x}(t_f) = \begin{bmatrix} x_f \\ y_f \end{bmatrix}$$

Where C_1, C_2 represent the first and second components of the current field.

2.2 Motorboat with Heading and Thrust Controls

Although the simple problem formulation is a good start, we hope to model more interesting scenarios. The next evolution of this representation is to add time-dependence to the current field and a second control for thrust, $u_2 \in [0, u_2^{max}]$, which we penalize in the cost functional. We incorporate the thrust control into the state equations by expanding them to include the velocities of the boat. We also add a fuel consumption penalty to the cost functional with tunable scaling parameter α to discourage fuel consumption. It is important to note that we are interested in modeling the boat dynamics on a large scale such as navigating across an entire ocean. In this case, we assume that changing the heading can be done instantaneously with no cost. The complete formulation of this extended motorboat problem is to minimize the cost functional

$$J[\mathbf{x}, \mathbf{u}, t_f] = \int_0^{t_f} (\alpha u_2^2 + 1) dt$$

subject to the state equation and boundary conditions

$$\dot{\mathbf{x}} = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{v}_x \\ \dot{v}_y \end{bmatrix} = \begin{bmatrix} C_1(x, y, t) + v_x \\ C_2(x, y, t) + v_y \\ u_2 \cos(u_1) \\ u_2 \sin(u_1) \end{bmatrix},$$

$$\mathbf{x}(0) = \begin{bmatrix} x_0 \\ y_0 \\ 0 \\ 0 \end{bmatrix}, \mathbf{x}(t_f) = \begin{bmatrix} x_f \\ y_f \\ 0 \\ 0 \end{bmatrix}.$$

2.3 Sailboat with Heading and Thrust Controls

Our final iteration was to model a sailboat with time-variant wind fields $W(x, y, t)$. Sailboat physics is a difficult problem in itself since the propulsion due to wind forces is a result of wind flow over the sail creating a Bernoulli effect. These forces can cause some unintuitive behaviors such as the ability of some sailboats to travel faster than the wind. Although the fluid dynamics that properly model sailboat physics are a fascinating problem, it is outside of the scope of this project. To compensate with a heuristic, we developed an equation $g(\theta)$ for boat speed as a function of the difference between the boat's heading angle and the angle of the wind. We compute this angle by

$$\theta = u_1 - \arctan\left(\frac{W_2}{W_1}\right)$$

where W_1, W_2 are the components of the wind field W. The heuristic function is then given by

$$g(\theta) = 1 - \sin\left(\theta + \frac{\pi}{2}\right) - \frac{1}{2} e^{-8(\theta - \pi)^2}$$

As seen in Fig 1, this function gives higher speeds at different heading angles. Notably, this function allows our sailboat to travel on zig-zagging "tacks" upwind–a distinct behavior we hoped to capture.

This optimal control problem is very similar to the formulation for the previous motorboat problem, but we also include a term in the velocity equation

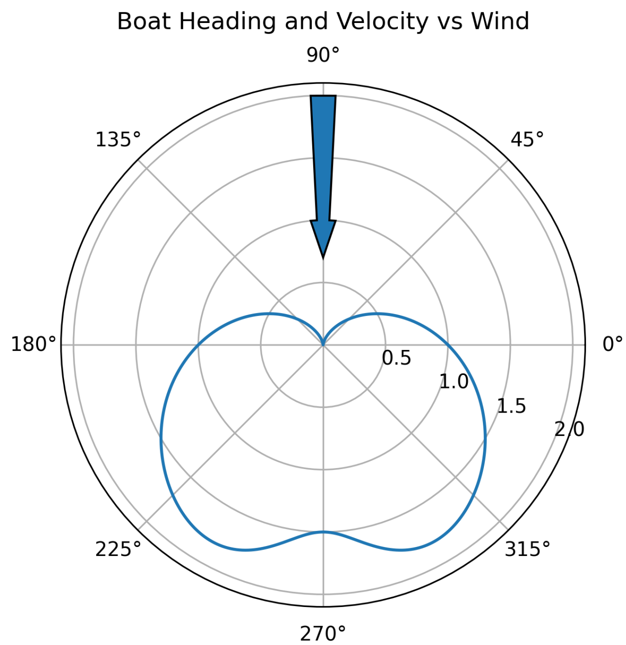

Figure 1: This graph shows boat velocity as a function of wind and heading. The graph has been rotated 90° from the equation given for better visibility. The angle of any point on the curve shows heading, while the radius of the point shows the magnitude of velocity it can obtain as a result of the wind (blue arrow).

which is dependent on the wind field. We use the same cost functional and boundary conditions, but the state update equation becomes

$$\dot{\mathbf{x}} = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{v}_x \\ \dot{v}_y \end{bmatrix} = \begin{bmatrix} C_1(x,y,t) + W_s(x,y,t,u_1)\cos(u_1) + v_x \\ C_2(x,y,t) + W_s(x,y,t,u_2)\sin(u_1) + v_y \\ u_2\cos(u_1) \\ u_2\sin(u_1) \end{bmatrix}$$

where the windspeed is given by

$$W_s(x,y,t,u_1) = g\left(u_1 - \arctan\left(\frac{W_2(x,y,t)}{W_1(x,y,t)}\right)\right).$$

3 Solution

3.1 Analytic Solution to Simple Motorboat Problem

Consider the simple motorboat problem with

$$C(x, y) = \begin{bmatrix} y \\ 0 \end{bmatrix}, \mathbf{x}(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \mathbf{x}(t_f) = \begin{bmatrix} 10 \\ 0 \end{bmatrix}.$$

We seek to solve this problem analytically for the optimal control u_1 using Pontryagin's Maximum Principle (PMP). The Hamiltonian for this system is given by

$$H = p_1(y + \cos(u_1)) + p_2 \sin(u_1) - 1.$$

PMP guarantees that the optimal control u_1^* satisfies $\frac{\partial H}{\partial u_1} = 0$. We compute

$$\frac{\partial H}{\partial u_1} = -p_1 \sin(u_1) + p_2 \cos(u_1).$$

Setting this to 0 and solving for u_1 yields

$$p_1 \sin(u_1) = p_2 \cos(u_1),$$

so we have

$$u_1^* = \arctan\left(\frac{p_2}{p_1}\right).$$

To solve for the costates p_1, p_2 we use Hamilton's canonical equations

$$\begin{aligned} \dot{p_1} &= -\frac{\partial H}{\partial x} = 0 \\ \dot{p_2} &= -\frac{\partial H}{\partial y} = -p_1. \end{aligned}$$

Solving these differential equations gives

$$\begin{aligned} p_1 &= c_1 \\ p_2 &= -c_1 t + c_2. \end{aligned}$$

Substituting these into our equation for our optimal control we have

$$u^* = \arctan\left(\frac{-c_1 t + c_2}{c_1}\right) = \arctan\left(\frac{c_2}{c_1} - t\right).$$

Substituting this control into our state equation for y yields

$$\dot{y} = \sin\left(\arctan\left(\frac{c_2}{c_1} - t\right)\right).$$

Solving the ODE for y and applying initial condition $y(0) = 0$ we obtain

$$y(t) = \sqrt{\left(\frac{c_2}{c_1}\right)^2 + 1} - \sqrt{\left(t - \frac{c_2}{c_1}\right)^2 + 1}.$$

Applying the endpoint condition $y(t_f) = 0$ and solving for t_f gives the relation $t_f = \frac{2c_2}{c_1}$.

Next we substitute our equation for the optimal control into our state equation for x

$$\dot{x} = \cos\left(\arctan\left(\frac{c_2}{c_1} - t\right)\right) + \sqrt{\left(\frac{c_2}{c_1}\right)^2 + 1} - \sqrt{\left(t - \frac{c_2}{c_1}\right)^2 + 1}.$$

Solving the ODE for x and applying initial condition $x(0) = 0$ gives

$$\begin{aligned} x(t) = -\frac{1}{2}\text{arcsinh}\left(\frac{c_2}{c_1} - t\right) + \frac{1}{2}\text{arcsinh}\left(\frac{c_2}{c_1}\right) + t\sqrt{\frac{c_2^2}{c_1^2} + 1} - \frac{c_2}{2c_1}\sqrt{\frac{c_2^2}{c_1^2} + 1} \\ + \frac{c_2}{2c_1}\sqrt{\left(t - \frac{c_2}{c_1}\right)^2 + 1} - \frac{1}{2}t\sqrt{\left(t - \frac{c_2}{c_1}\right)^2 + 1}. \end{aligned}$$

Applying the relation for t_f gives

$$x(t_f) = \frac{1}{2}\text{arcsinh}\left(\frac{t_f}{2}\right) - \frac{1}{2}\text{arcsinh}\left(-\frac{t_f}{2}\right) + \frac{1}{2}t_f\sqrt{\frac{t_f^2}{4} + 1}.$$

Finally, using the endpoint condition on x, $x(t_f) = 10$ and solving for t_f we get $t_f = 5.5736$. The optimal control is then given by

$$u^* = \arctan(2.7868 - t)$$

We checked this solution with our numerical approximation for the solution and they agreed, as seen in Figure 2.

3.2 Solving Complicated Variations of the Problem

As evidenced in the previous section, even the simplest cases of our problems are very involved to solve analytically. As we extend to time-dependent fields and higher-dimensional states, analytic solutions are not even guaranteed to exist. To solve more complex problems we relied on powerful numerical solvers. We found the Julia interior point optimization package IPOPT especially useful in solving high-dimensional, constrained systems. By discretizing our control problems and optimizing over each timestep for our states and controls subject to the given constraints, IPOPT quickly computed good numerical approximations for our state dynamics and controls.

4 Results

4.1 Analytic Solution for Time-Invariant Current

As derived in section 3.1, the analytic solution and the numerical solution to the simple motorboat problem with time-invariant current are equivalent, displayed in Figure 2. This confirms that IPOPT is performing as we expect.

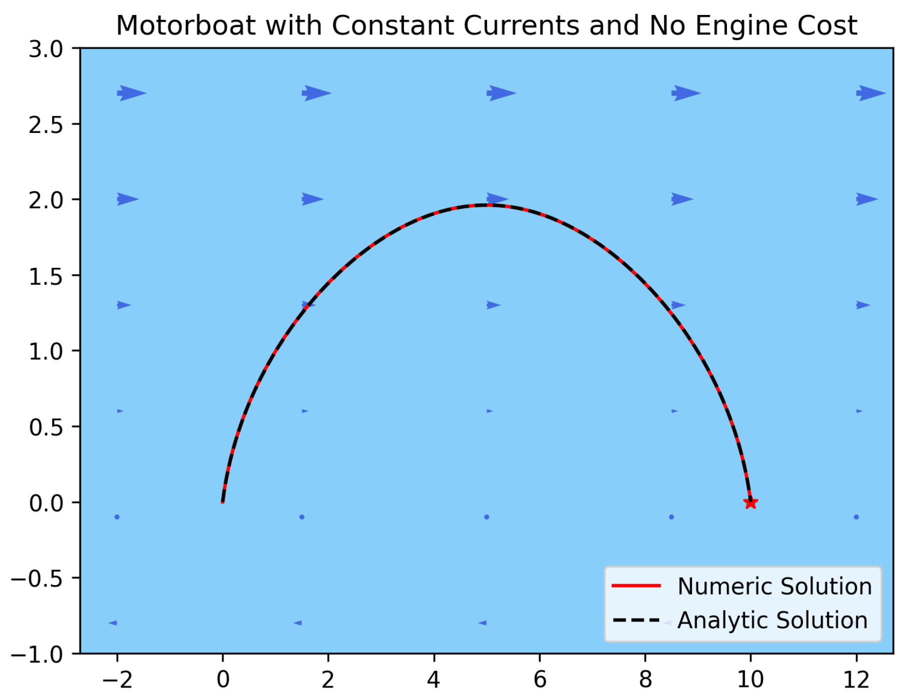

Figure 2: This chart demonstrates the equivalence between our analytic and numeric solutions for the simple motorboat case. The blue arrows represent the current while the star on the right is our target destination.

4.2 Simple Motorboat with Time-Varying Current

We first modeled a simple motor with heading control but no throttle control subject to forces from changing currents. Figure 3 shows the plotted trajectory of an optimal boat given the time-varying current function:

$$C_x = 0.1y \cdot \sin(0.5t)$$
$$C_y = 0.1x \cdot \cos(0.5t)$$

where C_x and C_y are the x and y components of the current, respectively.

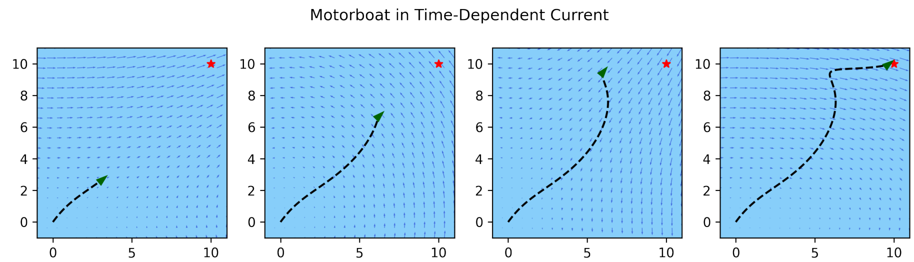

Figure 3: Each plot in this image is a frame from an animation. The black line shows the trajectory taken by the boat, the green arrow shows the boat's position and heading, the red star is the target destination, and the small blue arrows are the current at the given time.

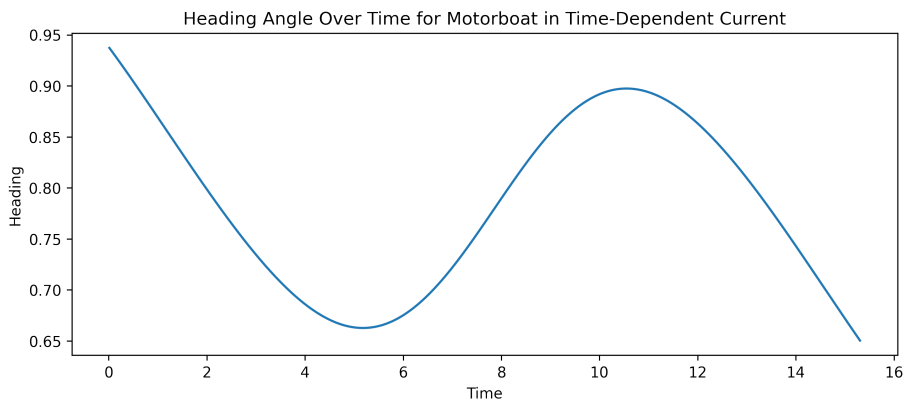

Figure 4: Angle in radians of the boat heading in Figure 3 at each time step.

Note that the boat rapidly approached the target destination while responding to the changing current in realistic ways. The optimal control shown in Figure 4 is smooth, avoiding any rapid, discontinuous, or unrealistic changes.

4.3 Variable Throttle Motorboats

We then proceeded to model a slightly more complicated version of the previous problem. Now our motorboat has variable amount of throttle that can be applied, but we penalize using the throttle in our cost function to conserve gas and limit sudden acceleration. Additionally, we require the boat to have zero velocity at our target destination. The optimal paths for boats with two differently weighted costs for throttle usage are shown in Figure 5 and Figure 6. In both of these scenarios, we use the same current defined in the simpler case.

In both cases, we see that the the boat powered toward the destination before turning around and letting the current carry it the rest of the way while

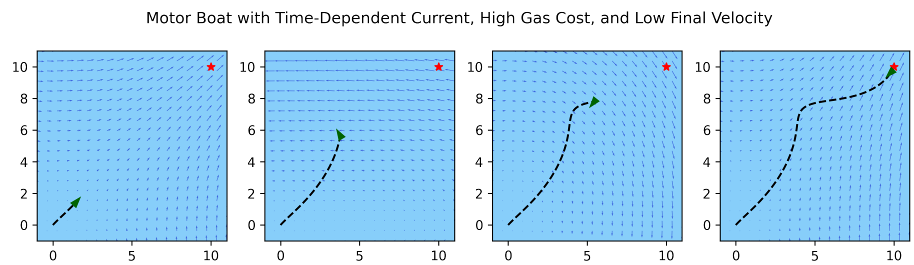

Figure 5: Optimal path of motorboat with heavier throttle penalty.

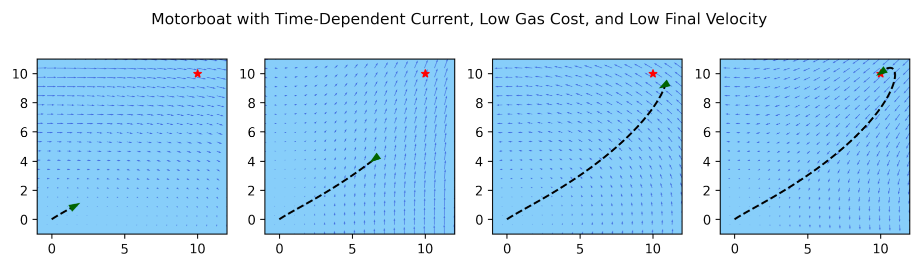

Figure 6: Optimal path of motorboat with lower throttle penalty.

accelerating against the current in order to arrive with minimal velocity. A comparison of the amount of throttle used is shown in Figure 7.

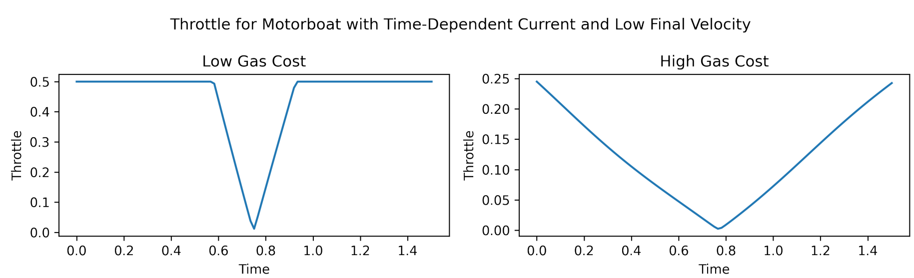

Figure 7: Amount of throttle used by the motorboat in the higher and lower cost scenarios.

4.4 Sailboat

Lastly, we model a sailboat that experiences wind but is not affected by current, since we are curious to see if our model can perform tacking maneuvers. The path of the sailboat from its initial position to its target is shown in Figure 8.

Note that the sailboat is able to successfully tack against a headwind.

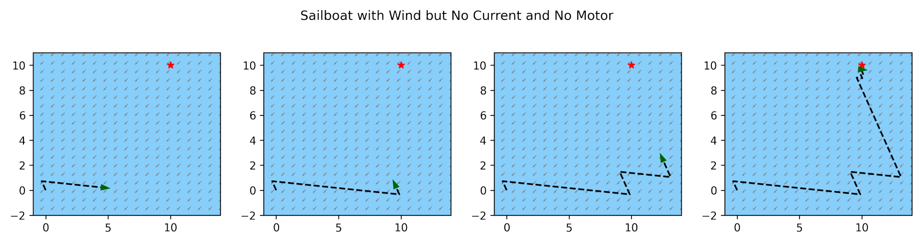

Figure 8: Sailboat tacking against constant headwind represented by the gray arrows.

5 Conclusion

Optimal control for motorboats and sailboats in time-varying conditions is difficult, but our results show that our formulations and analysis successfully capture the dynamics of the problem. Even the simplest versions of these problems demonstrated significant complexity to approach analytically. With the help of numerical solvers, we were able to get impressive, physical solutions. Our models express realistic behavior such as utilizing current flows for speed management and tacking against the wind.

Opportunities for further research involve additional constraints. First, for many large ships (such as container ships), the friction experienced by water against the bow is considerable, rendering sudden changes in direction very difficult. Modeling this phenomenon might require penalizing changes in the angle and investigating whether the qualitative properties of the solutions change. Second, even in the ocean, obstacle avoidance plays a major role, such as avoiding storms, maintaining proximity to other ships/shipping lanes, respecting sovereign waters, etc. Finally, we would be very interested in getting current and wind data fields for the the world's oceans to determine the optimal sailing path between any two points on the globe.

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6 FERPA Release

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