Note on local conservation of WADG

WADG is not locally conservative naturally. Let $u \in L^2$. We define the weighted and weight-adjusted projections Π_w, P_w by the problems

$$\left(w \Pi_w u, v\right) = (u, v)\,, \qquad \forall v \in V_h \tag{1}$$

$$\left(T_{1/w}^{-1} P_w u, v\right) = (u, v)\,, \qquad \forall v \in V_h. \tag{2}$$

We first introduce the weighted mass matrix

$$\left(\boldsymbol{M}_w\right)_{ij} = \left(w\phi_j, \phi_i\right).$$

Let $\Pi_w u$ and $P_w u$ be represented by coefficients $\boldsymbol{u}_w, \tilde{\boldsymbol{u}}_w$ in some basis ϕ_j. Discretizing (1) and (2) now yield the matrix equations

$$\boldsymbol{M}_w \boldsymbol{u}_w = \boldsymbol{b}, \qquad \boldsymbol{M}\boldsymbol{M}_{1/w}^{-1}\boldsymbol{M}\tilde{\boldsymbol{u}}_w = \boldsymbol{b}, \qquad \boldsymbol{b}_i = \left(u, \phi_i\right).$$

We refer to the matrix $\boldsymbol{M}\boldsymbol{M}_{1/w}^{-1}\boldsymbol{M}$ as the weight-adjusted mass matrix. Assuming that u is a constant (i.e. $\phi_0 = 1$), both the weighted and weight-adjusted mass matrices can be inverted explicitly to yield explicit formulas for $\Pi_w u, P_w u$

$$\Pi_w u = \frac{\int_{D^k} u}{\int_{D^k} w}, \qquad P_w u = \frac{\int_{D^k} \frac{1}{w}}{|D^k|}\frac{\int_{D^k} u}{|D^k|}.$$

We now decompose $\Pi_w u = u_0 + \tilde{u}(x)$, where $(\tilde{u}w, 1) = 0$. Taking $v = 1$ then shows that the weighted average of $\Pi_w u$ obeys

$$\frac{\int_{D^k} w\Pi_w u}{|D^k|} = u_0 = \frac{\int_{D^k} u}{|D^k|}.$$

To restore local conservation to the weight-adjusted projection, we can modify the weight-adjusted projection by adding and subtracting the weighted mean

$$\tilde{P}_w u = P_w u + \frac{\int_{D^k} u - \int_{D^k} wP_w u}{\int_{D^k} w}.$$

We note that the weighted average of this corrected quantity is the same as the weighted average of the projection $\Pi_w u$

$$\begin{aligned}\left(w\tilde{P}_w u, 1\right) &= \left(w\left(P_w u + \frac{\int_{D^k} u - \int_{D^k} wP_w u}{\int_{D^k} w}\right), 1\right) = \\ &\left(wP_w u + \left(\frac{\int_{D^k} u - \int_{D^k} wP_w u}{\int_{D^k} w}\right)w, 1\right) = (u, 1) = \left(w\Pi_w u, 1\right).\end{aligned} \tag{3}$$

For vector functions $\boldsymbol{u}(\boldsymbol{x})$, the weight can be matrix-valued. Extending the weighted average correction in (3) to the matrix-valued case can be done as follows

$$P_{\boldsymbol{W}}\boldsymbol{u}(\boldsymbol{x}) + \left(\int_{D^k} \boldsymbol{W}\right)^{-1}\left(\int_{D^k} \boldsymbol{u} - \int_{D^k} \boldsymbol{W}P_{\boldsymbol{W}}\boldsymbol{u}\right).$$

The main difference here is that the inverse of the averaged weight matrix $\left(\int_{D^k} \boldsymbol{W}\right)^{-1}$ must be computed. The matrix-weighted average of this corrected quantity is high order accurate and equal to $\int_{D^k} \boldsymbol{W}\boldsymbol{u}$.