Entropy stable notes

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1 Curved meshes

Assume non-affine geometric factors.

Let $\boldsymbol{F}(\boldsymbol{U})$ denote the flux matrix whose rows are

$$\boldsymbol{F}(\boldsymbol{U}) = \begin{pmatrix} \boldsymbol{F}_x \\ \boldsymbol{F}_y \end{pmatrix}.$$

The conservation law of interest is the following

$$\frac{\partial \boldsymbol{U}}{\partial t} + \nabla \cdot \boldsymbol{F}(\boldsymbol{U}) = 0,$$

where the divergence is taken over each column of $\boldsymbol{F}(\boldsymbol{U})$.

Let $\boldsymbol{G}$ denote the Jacobian of the geometric mapping

$$\boldsymbol{G}_{ij} = \frac{\partial \widehat{\boldsymbol{x}}_j}{\partial \boldsymbol{x}_i},$$

and let J denote the determinant of $\boldsymbol{G}$. One can show that

$$\widehat{\nabla} \cdot \left(J\boldsymbol{G}^T\right) = 0$$

at the continuous level. We also have that the mapped normals obey the property

$$J\boldsymbol{G}\widehat{\boldsymbol{n}}\widehat{J}^f = \boldsymbol{n}J^f.$$

At the continuous level, the physical gradient and divergence satisfy

$$(\nabla u, \boldsymbol{v})_{D^k} = \left(J\boldsymbol{G}\widehat{\nabla} u, \boldsymbol{v}\right)_{\widehat{D}}, \qquad (\nabla \cdot \boldsymbol{u}, v)_{D^k} = \left(\widehat{\nabla} \cdot \left(J\boldsymbol{G}^T u\right), \boldsymbol{v}\right)_{\widehat{D}},$$

as well as a corresponding integration by parts property.

2 Flux differencing

Replace the flux derivative with

$$(\nabla \cdot \boldsymbol{F}(\boldsymbol{U}), \boldsymbol{V})_{D^k} \approx \left(\nabla \cdot \boldsymbol{F}_S(\boldsymbol{U}(\boldsymbol{x}), \boldsymbol{U}(\boldsymbol{x}'))|_{\boldsymbol{x}'=\boldsymbol{x}}, \boldsymbol{V}\right)_{D^k}.$$

It is possible to remove the effect of geometric aliasing by evaluating the above term via

$$\left(\widehat{\nabla} \cdot \boldsymbol{F}_S^k(\boldsymbol{U}(\boldsymbol{x}), \boldsymbol{U}(\boldsymbol{x}'))\Big|_{\boldsymbol{x}'=\boldsymbol{x}}, \boldsymbol{V}\right)_{\widehat{D}}, \qquad \boldsymbol{F}_S^k = \{\{J\boldsymbol{G}\}\}\, \boldsymbol{F}_S\left(\boldsymbol{U}(\boldsymbol{x}), \boldsymbol{U}(\boldsymbol{x}')\right)$$

Another alternative is to treat curvilinear grids using a skew-symmetric form, though this may not be strictly locally conservative.

2.1 Weight-adjusted DG

Let u_p be the L^2 projection of u with respect to the J-weighted L^2 norm. We observe in numerical experiments that for a fixed geometric mapping, $\left\|u_p - \Pi_N\left(\frac{1}{J}\Pi_N(uJ)\right)\right\|_{L^2(\Omega)} = O(h^{N+2})$. Because the difference between the L^2 and WADG projection is superconvergent, the results are indistinguishable for a fixed geometric mapping.

Define the weight-adjusted projection $P_N u = \Pi_N\left(\frac{1}{J}\Pi_N(uJ)\right)$. We note that it's self-adjoint with respect to the J-weighted L^2 inner product

$$(JP_N u, v) = \left(\Pi_N\left(\frac{1}{J}\Pi_N(uJ)\right), vJ\right) = \left(uJ, \Pi_N\left(\frac{1}{J}\Pi_N(vJ)\right)\right) = (uJ, P_N v).$$

Let u_p be the L^2 projection over the curved element D^k

$$(u_p J, v) = (uJ, v), \qquad \forall v \in V_h.$$

One idea: let v then be the difference between the L^2 and weight-adjusted projection. We can estimate it as

$$\|v\|^2_{L^2(D^k)} = (Jv, v) = (Ju_p, v) - (JP_N u, v) = (Ju, v - P_N v) \leq \|u\|_{L^2(D^k)} \|v - P_N v\|_{L^2(D^k)}.$$

If $\|v - P_N v\|_{L^2(D^k)} \leq Ch^{N+2}\|v\|_{L^2(D^k)}$, then this yields a superconvergent error estimate.

3 Limiting

Ignoring high order terms in the Taylor expansion, we have

$$u(\Pi_N v) - u = \frac{\partial u}{\partial v}(\Pi_N v - v).$$

We want to limit u with a compression towards the mean $u = \bar{u} + \theta(u - \bar{u})$ such that $|u(\Pi_N v)| \leq C|u|$.

One option: ensure $|u(\Pi_N v) - u(\bar{v})| \approx |u - u(\bar{v})|$ based on Taylor series.

$$u(\Pi_N v) - u(\bar{v})$$

Limit $\tilde{u} = \bar{u} + \theta(u - \bar{u})$ such that

$$\min \Pi_N v \geq \min v, \qquad \max \Pi_N v \leq \max v.$$

Other options

- Adaptively choose time-step size?
- Expensive option: bisection or Newton algorithm for finding theta.