Entropy stable notes

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1 Curved meshes

Non-affine geometric factors

Let $\boldsymbol{F}(\boldsymbol{U})$ denote the flux matrix whose rows are

$$\boldsymbol{F}(\boldsymbol{U}) = \begin{pmatrix} \boldsymbol{F}_x \\ \boldsymbol{F}_y \end{pmatrix}.$$

The conservation law we're interested in is the following

$$\frac{\partial \boldsymbol{U}}{\partial t} + \nabla \cdot \boldsymbol{F}(\boldsymbol{U}) = 0,$$

where the divergence is taken over each column of $\boldsymbol{F}(\boldsymbol{U})$.

Let $\boldsymbol{G}$ denote the Jacobian of the geometric mapping

$$\boldsymbol{G}_{ij} = \frac{\partial \widehat{\boldsymbol{x}}_j}{\partial \boldsymbol{x}_i},$$

and let J denote the determinant of $\boldsymbol{G}$. One can show (Kopriva?) that

$$\widehat{\nabla} \cdot \left(J\boldsymbol{G}^T \right) = 0.$$

We also have that the mapped normals obey the property

$$J\boldsymbol{G}\widehat{\boldsymbol{n}}\widehat{J}^f = \boldsymbol{n}J^f.$$

At the continuous level, the physical gradient and divergence satisfy

$$(\nabla u, \boldsymbol{v})_{D^k} = \left(J\boldsymbol{G}\widehat{\nabla} u, \boldsymbol{v} \right)_{\widehat{D}}, \qquad (\nabla \cdot \boldsymbol{u}, v)_{D^k} = \left(\widehat{\nabla} \cdot \left(J\boldsymbol{G}^T u \right), \boldsymbol{v} \right)_{\widehat{D}},$$

as well as a corresponding integration by parts property.

2 Flux differencing

Replace the flux derivative with

$$\left(\nabla \cdot \boldsymbol{F}_S(\boldsymbol{U}(\boldsymbol{x}), \boldsymbol{U}(\boldsymbol{x}'))|_{\boldsymbol{x}'=\boldsymbol{x}} \right)_{D^k}.$$

It is possible to remove the effect of geometric aliasing by evaluating the above term via

$$\left(\widehat{\nabla} \cdot \boldsymbol{F}_S^k(\boldsymbol{U}(\boldsymbol{x}), \boldsymbol{U}(\boldsymbol{x}'))\Big|_{\boldsymbol{x}'=\boldsymbol{x}} \right)_{\widehat{D}}, \qquad \boldsymbol{F}_S^k = \{\{J\boldsymbol{G}\}\}\, \boldsymbol{F}_S\left(\boldsymbol{U}(\boldsymbol{x}), \boldsymbol{U}(\boldsymbol{x}') \right)$$

We can evaluate the new geometrically de-aliased flux using the discrete gradient of Chen and Shu

$$(\nabla_h \cdot \boldsymbol{u}, v)_\Omega = \sum_k \left(\widehat{\nabla} \cdot J\boldsymbol{G}^T \Pi_N \boldsymbol{u}, v \right)_{\widehat{D}} + \frac{1}{2} \left\langle J\boldsymbol{G}^T\boldsymbol{u}^+ - \Pi_N J\boldsymbol{G}^T\boldsymbol{u}, v\widehat{\boldsymbol{n}} \right\rangle_{\partial\widehat{D}} + \frac{1}{2} \left\langle J\boldsymbol{G}^T\boldsymbol{u} - \Pi_N J\boldsymbol{G}^T\boldsymbol{u}, \Pi_N\left(v\widehat{\boldsymbol{n}} \right) \right\rangle_{\partial\widehat{D}}$$

Another alternative is to treat curvilinear grids using a direct skew-symmetric form.

3 Limiting

Ignoring high order terms in the Taylor expansion, we have

$$u\left(\Pi_N v\right) - u = \frac{\partial u}{\partial v}\left(\Pi_N v - v\right).$$

We want to limit u with a compression towards the mean $u = \bar{u} + \theta(u - \bar{u})$ such that $\left|u\left(\Pi_N v\right)\right| \leq C\left|u\right|$.

3.1 Option 1

One option: ensure $\left|u\left(\Pi_N v\right) - u(\bar{v})\right| \approx \left|u - u(\bar{v})\right|$ based on Taylor series.

$$u\left(\Pi_N v\right) - u(\bar{v})$$

Limit $\tilde{u} = \bar{u} + \theta(u - \bar{u})$ such that

$$\min \Pi_N v \geq \min v, \qquad \max \Pi_N v \leq \max v.$$

Another option: estimate or filter $\Pi_N v$

3.2 Other options

- Adaptively choose time-step size?
- Expensive option: bisection-type algorithm for finding theta.