Note on local conservation of WADG

WADG is not locally conservative naturally. Let $u \in L^2$. We define the weighted and weight-adjusted projections Π_w, P_w by the problems

$$\left(w \Pi_w u, v\right) = (u, v), \qquad \forall v \in V_h \tag{1}$$

$$\left(T_{1/w}^{-1} P_w u, v\right) = (u, v), \qquad \forall v \in V_h. \tag{2}$$

We first introduce the weighted mass matrix

$$\left(\boldsymbol{M}_w\right)_{ij} = \left(w \phi_j, \phi_i\right).$$

Let $\Pi_w u$ and $P_w u$ be represented by coefficients $\boldsymbol{u}_w, \tilde{\boldsymbol{u}}_w$ in some basis ϕ_j. Discretizing (1) and (2) now yield the matrix equations

$$\boldsymbol{M}_w \boldsymbol{u}_w = \boldsymbol{b}, \qquad \boldsymbol{M} \boldsymbol{M}_{1/w}^{-1} \boldsymbol{M} \tilde{\boldsymbol{u}}_w = \boldsymbol{b}, \qquad \boldsymbol{b}_i = \left(u, \phi_i\right).$$

We refer to the matrix $\boldsymbol{M} \boldsymbol{M}_{1/w}^{-1} \boldsymbol{M}$ as the weight-adjusted mass matrix. Assuming that u is a constant (i.e. $\phi_0 = 1$), both the weighted and weight-adjusted mass matrices can be inverted explicitly to yield explicit formulas for $\Pi_w u, P_w u$

$$\Pi_w u = \frac{(u, 1)}{(w, 1)}, \qquad P_w u = \frac{\left(\frac{1}{w}, 1\right)}{(1, 1)} \frac{(u, 1)}{(1, 1)}.$$

We now decompose $\Pi_w u = u_0 + \tilde{u}(x)$, where $(\tilde{u} w, 1) = 0$. Taking $v = 1$ then shows that the weighted average of $\Pi_w u$ obeys

$$\frac{\left(w \Pi_w u, 1\right)}{(1, 1)} = u_0 = \frac{(u, 1)}{(1, 1)}.$$

To restore local conservation to the weight-adjusted projection, we can modify the weight-adjusted projection by adding and subtracting the weighted mean

$$P_w u + \frac{(u, 1) - \left(w P_w u, 1\right)}{(w, 1)}.$$

We note that the weighted average of this corrected quantity is the same as the weighted average of the projection $\Pi_w u$

$$\left(w \left(P_w u + \frac{(u, 1) - \left(w P_w u, 1\right)}{(w, 1)}\right), 1\right) = \left(w P_w u + \left(\frac{(u, 1) - \left(w P_w u, 1\right)}{(w, 1)}\right) w, 1\right) = (u, 1) = \left(w \Pi_w u, 1\right). \tag{3}$$

For vector functions $\boldsymbol{u}(\boldsymbol{x})$, the weight can be matrix-valued. Extending the weighted average correction in (3) to the matrix-valued case can be done as follows

$$P_{\boldsymbol{W}} \boldsymbol{u}(\boldsymbol{x}) + \left(\int_{D^k} \boldsymbol{W}\right)^{-1} \left((\boldsymbol{u}, 1) - \left(\boldsymbol{W} P_{\boldsymbol{W}} \boldsymbol{u}, 1\right)\right).$$

The main difference here is that the inverse of the averaged weight matrix $\left(\int_{D^k} \boldsymbol{W}\right)^{-1}$ must be computed. The matrix-weighted average of this corrected quantity is high order accurate and equal to $(\boldsymbol{W} \boldsymbol{u}, 1)$.