Entropy stable notes

Jesse Chan, Lucas Wilcox

1 Curved meshes

Assume non-affine geometric factors.

Let $\boldsymbol{F}(\boldsymbol{U})$ denote the flux matrix whose rows are

$$\boldsymbol{F}(\boldsymbol{U}) = \begin{pmatrix} \boldsymbol{F}_x \\ \boldsymbol{F}_y \end{pmatrix}.$$

The conservation law of interest is the following

$$\frac{\partial \boldsymbol{U}}{\partial t} + \nabla \cdot \boldsymbol{F}(\boldsymbol{U}) = 0,$$

where the divergence is taken over each column of $\boldsymbol{F}(\boldsymbol{U})$.

Let $\boldsymbol{G}$ denote the Jacobian of the geometric mapping

$$\boldsymbol{G}_{ij} = \frac{\partial \widehat{\boldsymbol{x}}_j}{\partial \boldsymbol{x}_i},$$

and let J denote the determinant of $\boldsymbol{G}$. One can show that

$$\widehat{\nabla} \cdot \left(J\boldsymbol{G}^T\right) = 0$$

at the continuous level. We also have that the mapped normals obey the property

$$J\boldsymbol{G}\widehat{\boldsymbol{n}}\widehat{J}^f = \boldsymbol{n}J^f.$$

At the continuous level, the physical gradient and divergence satisfy

$$(\nabla u, \boldsymbol{v})_{D^k} = \left(J\boldsymbol{G}\widehat{\nabla} u, \boldsymbol{v}\right)_{\widehat{D}}, \qquad (\nabla \cdot \boldsymbol{u}, v)_{D^k} = \left(\widehat{\nabla} \cdot \left(J\boldsymbol{G}^T u\right), \boldsymbol{v}\right)_{\widehat{D}},$$

as well as a corresponding integration by parts property.

2 Flux differencing

Replace the flux derivative with

$$(\nabla \cdot \boldsymbol{F}(\boldsymbol{U}), \boldsymbol{V})_{D^k} \approx \left(\nabla \cdot \boldsymbol{F}_S(\boldsymbol{U}(\boldsymbol{x}), \boldsymbol{U}(\boldsymbol{x}'))|_{\boldsymbol{x}'=\boldsymbol{x}}, \boldsymbol{V}\right)_{D^k}.$$

It is possible to remove the effect of geometric aliasing by evaluating the above term via

$$\left(\widehat{\nabla} \cdot \boldsymbol{F}_S^k(\boldsymbol{U}(\boldsymbol{x}), \boldsymbol{U}(\boldsymbol{x}'))\Big|_{\boldsymbol{x}'=\boldsymbol{x}}, \boldsymbol{V}\right)_{\widehat{D}}, \qquad \boldsymbol{F}_S^k = \{\{J\boldsymbol{G}\}\}\, \boldsymbol{F}_S\left(\boldsymbol{U}(\boldsymbol{x}), \boldsymbol{U}(\boldsymbol{x}')\right)$$

Another alternative is to treat curvilinear grids using a skew-symmetric form, though this may not be strictly locally conservative.

2.1 Weight-adjusted DG

We define the curvilinear L^2 projection Π_N^J such that

$$\left(\Pi_N^J u, v\right)_{D^k} = \left(J\Pi_N^J u, v\right)_{\widehat{D}} = (uJ, v)_{\widehat{D}} = (u, v)_{D^k}.$$

Let the weight-adjusted projection operator P_N be defined as $P_N u = \Pi_N\left(\frac{1}{J}\Pi_N(uJ)\right)$. Note that P_N is self-adjoint with respect to the J-weighted L^2 inner product

$$(JP_N u, v) = \left(\Pi_N\left(\frac{1}{J}\Pi_N(uJ)\right), vJ\right) = \left(uJ, \Pi_N\left(\frac{1}{J}\Pi_N(vJ)\right)\right) = (uJ, P_N v).$$

Let $\Pi_N^J u$ be the L^2 projection of u with respect to the weighted (curvilinear) L^2 inner product. We observe in numerical experiments that for a fixed geometric mapping, $\left\|\Pi_N^J u - P_N u\right\|_{L^2(\Omega)} = O(h^{N+2})$. Because the difference between the L^2 and WADG projection is superconvergent, the results are indistinguishable for a fixed geometric mapping.

We can prove this bound using results from [1]. We first define the operator T_w^{-1} as follows

$$\left(wT_w^{-1}u, v\right)_{\widehat{D}} = (u, v)_{\widehat{D}}, \qquad \forall v \in V_h.$$

This forms the basis of the weight-adjusted approximation of weighted L^2 inner products. The first theorem we need shows that $T_{1/J}^{-1}$ can be used to approximate weighted curvilinear L^2 inner products with order $(2N+2)$ accuracy.

Theorem 1. *Let $u \in W^{N+1,2}\left(D^k\right)$ and $v \in V_h\left(D^k\right)$. Then,*

$$\left|(u, vJ)_{\widehat{D}} - \left(T_{1/J}^{-1}u, v\right)_{\widehat{D}}\right| \leq$$
$$Ch^{2N+2}\|J\|_{L^\infty(D^k)}\left\|\frac{1}{J}\right\|^2_{L^\infty(D^k)}\|J\|^2_{W^{N+1,\infty}(D^k)}\|u\|_{W^{N+1,2}(D^k)}\|v\|_{W^{N+1,2}(D^k)}.$$

Proof. The proof involves straightforward adaptations of Theorem 4, Theorem 5, and Theorem 6 in [1] to the reference element $\widehat{D}$. □

The next result we need is a generalized inverse inequality.

Lemma 1. *Let $v \in P^N\left(D^k\right)$, and let $h = \operatorname{diam}\left(D^k\right)$. Then,*

$$\|v\|_{W^{N+1,2}(D^k)} \leq C_N h^{-N}\|v\|_{L^2(D^k)}.$$

where C_N is independent of h.

Proof. The result is the consequence of a scaling argument and a Rayleigh quotient bound involving the largest eigenvalue of the generalized eigenvalue problem

$$\boldsymbol{K}_N\boldsymbol{u} = \lambda\boldsymbol{M}\boldsymbol{u},$$

where $\boldsymbol{M}$ is the L^2 mass matrix over D^k and $\boldsymbol{K}_N$ is the Gram matrix corresponding to the Sobolev inner product for $W^{N+1,2}\left(D^k\right)$. We note that the constant C_N depends on the largest eigenvalue, which in turn depends on the order N and dimension d. □

We can now prove that $P_N u$ is superconvergent to the curvilinear L^2 projection $\Pi_N^J u$

Theorem 2. *Let $u \in W^{N+1,2}\left(D^k\right)$. The difference between the L^2 projection $\Pi_N u$ and the weight-adjusted projection $P_N u$ is*

$$\left\|\Pi_N u - P_N u\right\|_{L^2(D^k)} \leq C \left\|J\right\|_{L^\infty(D^k)} \left\|\frac{1}{J}\right\|^2_{L^\infty(D^k)} \left\|J\right\|^2_{W^{N+1,\infty}(D^k)} h^{N+2} \left\|u\right\|_{W^{N+1,2}(D^k)}.$$

where C is a mesh-independent constant.

Proof. Let $\Pi_N^J u$ be the L^2 projection over the curved element D^k, such that

$$\left(J\Pi_N^J u, v\right) = \left(uJ, v\right), \qquad \forall v \in V_h.$$

By definition, $P_N u$ satisfies a similar property

$$\left(T^{-1}_{1/J} P_N u, v\right) = \left(uJ, v\right), \qquad \forall v \in V_h.$$

Then, we have that

$$\left\|\Pi_N^J u - P_N u\right\|^2_{L^2(D^k)} = \left(\Pi_N^J u - P_N u, vJ\right)_{\widehat{D}}, \qquad v = \Pi_N^J u - P_N u.$$

Because $v \in P^N\left(D^k\right)$, we can also evaluate the squared error as

$$\begin{aligned}\left\|\Pi_N^J u - P_N u\right\|^2_{L^2(D^k)} &= \left|\left(\Pi_N^J, vJ\right)_{\widehat{D}} - \left(P_N u, vJ\right)_{\widehat{D}}\right| \\ &= \left|\left(u, vJ\right)_{\widehat{D}} - \left(P_N u, vJ\right)_{\widehat{D}}\right| = \left|\left(T^{-1}_{1/J} P_N u, v\right)_{\widehat{D}} - \left(J P_N u, v\right)_{\widehat{D}}\right|.\end{aligned}$$

Applying Theorem 1 and Lemma 1 with $M = N$ then yields that

$$\begin{aligned}\left\|\Pi_N u - P_N u\right\|^2_{L^2(D^k)} &\leq C h^{2N+2} C_J \left\|u\right\|_{W^{N+1,2}(D^k)} \left\|v\right\|_{W^{N+1,2}(\widehat{D})} \\ &\leq C h^{N+2} C_J \left\|u\right\|_{W^{N+1,2}(D^k)} \left\|v\right\|_{L^2(D^k)},\end{aligned}$$

where C_J is a mesh-independent constant depending on J. Dividing through by

$$\left\|v\right\|_{L^2(D^k)} = \left\|\Pi_N u - P_N u\right\|_{L^2(D^k)}$$

gives the desired result. □

3 Limiting

Ignoring high order terms in the Taylor expansion, we have

$$u\left(\Pi_N v\right) - u = \frac{\partial u}{\partial v}\left(\Pi_N v - v\right).$$

We want to limit u with a compression towards the mean $u = \bar{u} + \theta(u - \bar{u})$ such that $\left|u\left(\Pi_N v\right)\right| \leq C\left|u\right|$.

One option: ensure $\left|u\left(\Pi_N v\right) - u(\bar{v})\right| \approx \left|u - u(\bar{v})\right|$ based on Taylor series.

$$u\left(\Pi_N v\right) - u(\bar{v})$$

Limit $\tilde{u} = \bar{u} + \theta(u - \bar{u})$ such that

$$\min \Pi_N v \geq \min v, \qquad \max \Pi_N v \leq \max v.$$

Other options

- Adaptively choose time-step size?
- Expensive option: bisection or Newton algorithm for finding theta.

References

[1] Jesse Chan, Russell J Hewett, and T Warburton. Weight-adjusted discontinuous Galerkin methods: wave propagation in heterogeneous media. *arXiv preprint arXiv:1608.01944*, 2016.

[2] Jesse Chan, Russell J Hewett, and T Warburton. Weight-adjusted discontinuous Galerkin methods: curvilinear meshes. *arXiv preprint arXiv:1608.03836*, 2016.