High order entropy stable discontinuous Galerkin methods on curvilinear meshes

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Abstract

Things to include: entropy stability for curvilinear meshes, choosing geometric factors to ensure constant state preservation, high order accuracy.

1 Intro/layout

1. Global DG differentiation operators and flux differencing
2. Global DG operators for curvilinear meshes
3. Flux differencing with geometric averaging.

2 Flux differencing for affine meshes

Define the global DG divergence

$$
\begin{aligned}
\left(\nabla_h \cdot \boldsymbol{u}, vw\right)_{\Omega} = &\sum_k \left(\nabla \cdot \Pi_N \boldsymbol{u}, vw\right)_{D^k} \\
&+ \frac{1}{2}\left\langle \boldsymbol{u}^+ - \Pi_N \boldsymbol{u}, vw\boldsymbol{n}\right\rangle_{\partial D^k} + \frac{1}{2}\left\langle \boldsymbol{u} - \Pi_N \boldsymbol{u}, \Pi_N\left(vw\right)\boldsymbol{n}\right\rangle_{\partial D^k}.
\end{aligned}
$$

This is equivalent to defining

$$
\nabla_h \cdot \boldsymbol{u} = \sum_i D_h^i \boldsymbol{u}_i.
$$

Let $\boldsymbol{f}_S^i(\boldsymbol{u}_L, \boldsymbol{u}_R)$ for $i = 1, \ldots, d$ denote the multi-dimensional entropy conservative flux of Tadmor [1], such that

$$
\left(\boldsymbol{v}_L - \boldsymbol{v}_R\right)^T \boldsymbol{f}_S^i(\boldsymbol{u}_L, \boldsymbol{u}_R) = \psi_{i,L} - \psi_{i,R}
$$

where ψ_i denotes the ith component of the entropy potential. We concatenate these components together into a matrix $\boldsymbol{f}_S$ whose ith row is $\boldsymbol{f}_S^i$

$$
\boldsymbol{f}_S = \left[\boldsymbol{f}_S^1, \ldots, \boldsymbol{f}_S^d\right]^T.
$$

Theorem 1. *The semi-discrete DG discretization*

$$
\left(\frac{\partial \boldsymbol{u}}{\partial t} + 2\left.\left(\nabla_h \cdot \boldsymbol{f}_S(\boldsymbol{u}, \boldsymbol{u}')\right)\right|_{\boldsymbol{x}'=\boldsymbol{x}}, \boldsymbol{w}\right)_{\Omega} = 0
$$

is entropy conservative.

Proof. Take $\boldsymbol{w} = \boldsymbol{v}$. The time term reduces to $\frac{\partial S(\boldsymbol{u})}{\partial t}$ by the chain rule. The spatial term is then

$$2\left(\left(\nabla_h \cdot \boldsymbol{f}_S(\boldsymbol{u},\boldsymbol{u}')\right)\big|_{\boldsymbol{x}'=\boldsymbol{x}}, \boldsymbol{v}\right) = \left(\left(\nabla_h \cdot \boldsymbol{f}_S(\boldsymbol{u},\boldsymbol{u}')\boldsymbol{v}'\right)\big|_{\boldsymbol{x}'=\boldsymbol{x}}, 1\right) + \sum_{i=1}^{d}\left(\left(D_h^i \boldsymbol{f}_S^i(\boldsymbol{u},\boldsymbol{u}')\right)\big|_{\boldsymbol{x}'=\boldsymbol{x}}, \boldsymbol{v}\right)$$
$$= \sum_{i=1}^{d}\left\langle \boldsymbol{f}_S^i(\boldsymbol{u},\boldsymbol{u}), \boldsymbol{v}\boldsymbol{n}_i\right\rangle - \left(\left(D_h^i\left((\boldsymbol{v}'-\boldsymbol{v})^T \boldsymbol{f}_S^i(\boldsymbol{u},\boldsymbol{u}')\right)\right)\Big|_{\boldsymbol{x}'=\boldsymbol{x}}, 1\right)$$

The boundary term is simplified by noting that $\boldsymbol{f}_S^i(\boldsymbol{u},\boldsymbol{u}) = \boldsymbol{f}^i(\boldsymbol{u})$. Applying properties of the entropy conservative flux yields

$$(\boldsymbol{v}'-\boldsymbol{v})^T \boldsymbol{f}_S^i(\boldsymbol{u},\boldsymbol{u}') = \psi_i' - \psi_i,$$

which simplifies the final volume term to

$$\left(\left(D_h^i\left(\psi_i' - \psi_i\right)\right)\big|_{\boldsymbol{x}'=\boldsymbol{x}}, 1\right) = \left\langle -\psi_i \boldsymbol{n}_i, 1\right\rangle.$$

□

3 Curved meshes

Assume non-affine geometric factors.

Let $\boldsymbol{F}(\boldsymbol{U})$ denote the flux matrix whose rows are

$$\boldsymbol{F}(\boldsymbol{U}) = \begin{pmatrix} \boldsymbol{F}_x \\ \boldsymbol{F}_y \end{pmatrix}.$$

The conservation law of interest is the following

$$\frac{\partial \boldsymbol{U}}{\partial t} + \nabla \cdot \boldsymbol{F}(\boldsymbol{U}) = 0,$$

where the divergence is taken over each column of $\boldsymbol{F}(\boldsymbol{U})$.

Let $\boldsymbol{G}$ denote the Jacobian of the geometric mapping

$$\boldsymbol{G}_{ij} = \frac{\partial \widehat{\boldsymbol{x}}_j}{\partial \boldsymbol{x}_i},$$

and let J denote the determinant of $\boldsymbol{G}$. One can show that

$$\widehat{\nabla} \cdot \left(J\boldsymbol{G}^T\right) = 0, \qquad \widehat{\nabla} \cdot \left(J\boldsymbol{G}^T \boldsymbol{u}\right) = \nabla \cdot \boldsymbol{u}$$

at the continuous level. We also have that the mapped normals obey the property

$$J\boldsymbol{G}\widehat{\boldsymbol{n}}\widehat{J}^f = \boldsymbol{n}J^f.$$

At the continuous level, the physical gradient and divergence satisfy

$$(\nabla u, \boldsymbol{v})_{D^k} = \left(J\boldsymbol{G}\widehat{\nabla} u, \boldsymbol{v}\right)_{\widehat{D}}, \qquad (\nabla \cdot \boldsymbol{u}, v)_{D^k} = \left(\widehat{\nabla} \cdot \left(J\boldsymbol{G}^T u\right), \boldsymbol{v}\right)_{\widehat{D}},$$

as well as a corresponding integration by parts property.

3.1 Flux differencing for curvilinear meshes

We introduce a "reference" global DG derivative, which is defined (for $v \in V_h$ but $u, w \notin V_h$)

$$\left(\widehat{D}_h^i u, vw\right)_\Omega = \sum_k \left(\frac{\partial \Pi_N \boldsymbol{u}}{\partial \widehat{\boldsymbol{x}}_i}, vw\right)_{\widehat{D}} + \frac{1}{2}\left\langle \boldsymbol{u}^+ - \Pi_N \boldsymbol{u}, vw\widehat{\boldsymbol{n}}_i\right\rangle_{\partial\widehat{D}} + \frac{1}{2}\left\langle \boldsymbol{u} - \Pi_N \boldsymbol{u}, \Pi_N\left(vw\right)\widehat{\boldsymbol{n}}_i\right\rangle_{\partial\widehat{D}}.$$

We use this to define a reference DG divergence

$$\widehat{\nabla}_h \cdot \boldsymbol{u} = \sum_{i=1}^d \widehat{D}_h^i \boldsymbol{u}_i.$$

We next introduce two physical DG divergences based on the reference DG divergence. The a "conservative" DG divergence is defined using the reference DG divergence operator

$$\nabla_h^c \cdot \boldsymbol{u} = \widehat{\nabla}_h \cdot \left(J\boldsymbol{G}^T\boldsymbol{u}\right).$$

We also introduce the "non-conservative" divergence

$$\nabla_h^{nc} \cdot \boldsymbol{u} = \left(J\boldsymbol{G}\widehat{\nabla}_h\right) \cdot \boldsymbol{u} = \sum_{i=1}^d \sum_{j=1}^d J\boldsymbol{G}_{ij}\widehat{D}_h^j \boldsymbol{u}_i.$$

Unless $J\boldsymbol{G}$ is element-wise constant,

$$\widehat{\nabla}_h \cdot \left(J\boldsymbol{G}^T\boldsymbol{u}\right) \neq \left(J\boldsymbol{G}\widehat{\nabla}_h\right) \cdot \boldsymbol{u},$$

due to the presence of the projection operator Π_N in the definition of $\widehat{\nabla}_h$.

Instead of choosing one definition over the other, it was shown in [2] that stability is achieved when using the average of these two definitions of the physical gradient. The following lemma describes how to implement this within the flux differencing framework.

Lemma 1. *Let $\boldsymbol{u}^k(\boldsymbol{x}, \boldsymbol{x}')$ be defined as*

$$\boldsymbol{u}^k(\boldsymbol{x}, \boldsymbol{x}') = \{\{J\boldsymbol{G}^T\}\}\,\boldsymbol{u} = \left(\frac{J(\boldsymbol{x})\boldsymbol{G}^T(\boldsymbol{x}) + J(\boldsymbol{x}')\boldsymbol{G}^T(\boldsymbol{x}')}{2}\right)\boldsymbol{u}.$$

Applying the reference global DG divergence to $\boldsymbol{u}^k$ is equivalent to averaging both definitions of the global DG divergence

$$\left.\left(\widehat{\nabla}_h \cdot \boldsymbol{u}^k(\boldsymbol{x}, \boldsymbol{x}')\right)\right|_{\boldsymbol{x}'=\boldsymbol{x}} = \frac{1}{2}\left(\widehat{\nabla}_h \cdot \left(J\boldsymbol{G}^T\boldsymbol{u}\right) + \left(J\boldsymbol{G}\widehat{\nabla}_h\right)\cdot\boldsymbol{u}\right) = \frac{1}{2}\left(\nabla_h^c \cdot \boldsymbol{u} + \nabla_h^{nc} \cdot \boldsymbol{u}\right).$$

Proof. TBD. □

We now have the following curvilinear generalization of Theorem 1

Theorem 2. *The semi-discrete DG discretization*

$$\left(\frac{\partial \boldsymbol{u}}{\partial t} + 2\left.\left(\widehat{\nabla}_h \cdot \boldsymbol{f}_S^k(\boldsymbol{u}, \boldsymbol{u}')\right)\right|_{\boldsymbol{x}'=\boldsymbol{x}}, \boldsymbol{w}\right) = 0, \qquad \boldsymbol{f}_S^k(\boldsymbol{u}_L, \boldsymbol{u}_R) = \{\{J\boldsymbol{G}^T\}\}\,\boldsymbol{f}_S(\boldsymbol{u}_L, \boldsymbol{u}_R).$$

is entropy conservative on curvilinear meshes.

Proof. Take $\boldsymbol{w} = \boldsymbol{v}$. The time term is the same as before; however, we now have two spatial terms corresponding to the conservative and non-conservative DG divergences. Essentially, integrating the conservative divergence terms by parts results in volume contributions with the non-conservative divergence, and vice-versa. Contributions are swapped between each type of divergence, so that the last steps of the affine entropy conservation proof can be repeated for two volume terms (involving the conservative and non-conservative divergence, respectively).

□

4 Free stream preservation

We seek conditions for which free-stream preservation

$$\frac{\partial \boldsymbol{u}}{\partial t} = \nabla_h \cdot \boldsymbol{u} = 0$$

is satisfied if $\boldsymbol{u} = \mathbf{1}$. Free stream preservation is not always maintained at the discrete level in 3D due to the fact that geometric factors are higher degree polynomials than the corresponding discrete space [3, 4]. For curvilinear meshes, $\widehat{\nabla} \cdot \boldsymbol{J}\boldsymbol{G}^T \neq 0$ due to polynomial aliasing of geometric factors.

This can be remedied by using Kopriva's interpolation of the curl-conservative form of the geometric factors, which ensures that $\widehat{\nabla} \cdot \left(\boldsymbol{J}\boldsymbol{G}^T\right) = 0$ locally. Then, noting that $\boldsymbol{J}\boldsymbol{G}^T\boldsymbol{u} \in \left(P^N\left(D^k\right)\right)^d$ for $\boldsymbol{u}$ constant, we have that

$$\widehat{\nabla} \cdot \left(\boldsymbol{J}\boldsymbol{G}^T\boldsymbol{u}\right) = \left(\widehat{\nabla} \cdot \left(\boldsymbol{J}\boldsymbol{G}^T\right)\right) \cdot \boldsymbol{u} = 0, \qquad \boldsymbol{J}\boldsymbol{G}^T\boldsymbol{u} = \Pi_N\left(\boldsymbol{J}\boldsymbol{G}^T\boldsymbol{u}\right).$$

As a result, because $\boldsymbol{u}^+ = \boldsymbol{u}$ for $\boldsymbol{u}$ globally constant,

$$\begin{aligned}(\nabla_h \cdot \boldsymbol{u}, vw)_\Omega = \sum_k \left(\widehat{\nabla} \cdot \left(\boldsymbol{J}\boldsymbol{G}^T\boldsymbol{u}\right), vw\right)_{\widehat{D}} + \\ \frac{1}{2}\left\langle \boldsymbol{J}\boldsymbol{G}^T - \Pi_N\left(\boldsymbol{J}\boldsymbol{G}^T\right), vw\widehat{\boldsymbol{n}}\right\rangle_{\partial\widehat{D}} + \frac{1}{2}\left\langle \boldsymbol{J}\boldsymbol{G}^T - \Pi_N\left(\boldsymbol{J}\boldsymbol{G}^T\right), \Pi_N(vw)\,\widehat{\boldsymbol{n}}\right\rangle_{\partial\widehat{D}} = 0\end{aligned}$$

for any $w \in H^1(\Omega_h)$ and $v \in V_h$.

4.1 Weight-adjusted DG

We define the curvilinear L^2 projection Π_N^k such that

$$\left(\Pi_N^k u, v\right)_{D^k} = \left(J\Pi_N^k u, v\right)_{\widehat{D}} = (uJ, v)_{\widehat{D}} = (u, v)_{D^k}.$$

Let the weight-adjusted projection operator P_N be defined as $P_N u = \Pi_N\left(\frac{1}{J}\Pi_N(uJ)\right)$. Note that P_N is self-adjoint with respect to the J-weighted L^2 inner product

$$(JP_N u, v) = \left(\Pi_N\left(\frac{1}{J}\Pi_N(uJ)\right), vJ\right) = \left(uJ, \Pi_N\left(\frac{1}{J}\Pi_N(vJ)\right)\right) = (uJ, P_N v).$$

Let $\Pi_N^k u$ be the L^2 projection of u with respect to the weighted (curvilinear) L^2 inner product. We observe in numerical experiments that for a fixed geometric mapping, $\left\|\Pi_N^k u - P_N u\right\|_{L^2(\Omega)} = O(h^{N+2})$. Because the difference between the L^2 and WADG projection is superconvergent, the results are indistinguishable for a fixed geometric mapping.

We can prove this bound using results from [5]. We first define the operator T_w^{-1} as follows

$$\left(wT_w^{-1}u, v\right)_{\widehat{D}} = (u, v)_{\widehat{D}}, \qquad \forall v \in V_h.$$

This forms the basis of the weight-adjusted approximation of weighted L^2 inner products. The first theorem we need shows that $T_{1/J}^{-1}$ can be used to approximate weighted curvilinear L^2 inner products with order $(2N+2)$ accuracy.

Theorem 3. *Let $u \in W^{N+1,2}\left(D^k\right)$ and $v \in V_h\left(D^k\right)$. Then,*

$$\begin{aligned}&\left|(u, vJ)_{\widehat{D}} - \left(T_{1/J}^{-1}u, v\right)_{\widehat{D}}\right| \leq \\ &Ch^{2N+2}\|J\|_{L^\infty(D^k)}\left\|\frac{1}{J}\right\|_{L^\infty(D^k)}^2 \|J\|_{W^{N+1,\infty}(D^k)}^2 \|u\|_{W^{N+1,2}(D^k)}\|v\|_{W^{N+1,2}(D^k)}.\end{aligned}$$

Proof. The proof involves straightforward adaptations of Theorem 4, Theorem 5, and Theorem 6 in [5] to the reference element $\widehat{D}$. $\square$

The next result we need is a generalized inverse inequality.

Lemma 2. *Let $v \in P^N\left(D^k\right)$, and let $h = \operatorname{diam}\left(D^k\right)$. Then,*

$$\|v\|_{W^{N+1,2}(D^k)} \leq C_N h^{-N} \|v\|_{L^2(D^k)}.$$

where C_N is independent of h.

Proof. The result is the consequence of a scaling argument and a Rayleigh quotient bound involving the largest eigenvalue of the generalized eigenvalue problem

$$\boldsymbol{K}_N \boldsymbol{u} = \lambda \boldsymbol{M} \boldsymbol{u},$$

where $\boldsymbol{M}$ is the L^2 mass matrix over D^k and $\boldsymbol{K}_N$ is the Gram matrix corresponding to the Sobolev inner product for $W^{N+1,2}\left(D^k\right)$. We note that the constant C_N depends on the largest eigenvalue, which in turn depends on the order N and dimension d. $\square$

We can now prove that $P_N u$ is superconvergent to the curvilinear L^2 projection $\Pi_N^k u$

Theorem 4. *Let $u \in W^{N+1,2}\left(D^k\right)$. The difference between the L^2 projection $\Pi_N u$ and the weight-adjusted projection $P_N u$ is*

$$\left\|\Pi_N^k u - P_N u\right\|_{L^2(D^k)} \leq C \|J\|_{L^\infty(D^k)} \left\|\frac{1}{J}\right\|^2_{L^\infty(D^k)} \|J\|^2_{W^{N+1,\infty}(D^k)} h^{N+2} \|u\|_{W^{N+1,2}(D^k)}.$$

where C is a mesh-independent constant.

Proof. Let $\Pi_N^k u$ be the L^2 projection over the curved element D^k, such that

$$\left(J \Pi_N^k u, v\right) = (uJ, v), \qquad \forall v \in V_h.$$

By definition, $P_N u$ satisfies a similar property

$$\left(T^{-1}_{1/J} P_N u, v\right) = (uJ, v), \qquad \forall v \in V_h.$$

Then, we have that

$$\left\|\Pi_N^k u - P_N u\right\|^2_{L^2(D^k)} = \left(\Pi_N^k u - P_N u, vJ\right)_{\widehat{D}}, \qquad v = \Pi_N^k u - P_N u.$$

Because $v \in P^N\left(D^k\right)$, we can also evaluate the squared error as

$$\begin{aligned}\left\|\Pi_N^k u - P_N u\right\|^2_{L^2(D^k)} &= \left|\left(\Pi_N^k, vJ\right)_{\widehat{D}} - \left(P_N u, vJ\right)_{\widehat{D}}\right| \\ &= \left|(u, vJ)_{\widehat{D}} - (P_N u, vJ)_{\widehat{D}}\right| = \left|\left(T^{-1}_{1/J} P_N u, v\right)_{\widehat{D}} - (J P_N u, v)_{\widehat{D}}\right|.\end{aligned}$$

Applying Theorem 3 and Lemma 2 then yields that

$$\begin{aligned}\left\|\Pi_N^k u - P_N u\right\|^2_{L^2(D^k)} &\leq C h^{2N+2} C_J \|u\|_{W^{N+1,2}(D^k)} \|v\|_{W^{N+1,2}(\widehat{D})} \\ &\leq C h^{N+2} C_J \|u\|_{W^{N+1,2}(D^k)} \|v\|_{L^2(D^k)},\end{aligned}$$

where C_J is a mesh-independent constant depending on J. Dividing through by

$$\|v\|_{L^2(D^k)} = \left\|\Pi_N^k u - P_N u\right\|_{L^2(D^k)}$$

gives the desired result. $\square$

5 Limiting

Ignoring high order terms in the Taylor expansion, we have

$$u\left(\Pi_N v\right) - u = \frac{\partial u}{\partial v}\left(\Pi_N v - v\right).$$

We want to limit u with a compression towards the mean $u = \bar{u} + \theta(u - \bar{u})$ such that $\left|u\left(\Pi_N v\right)\right| \leq C\left|u\right|$.

One option: ensure $\left|u\left(\Pi_N v\right) - u(\bar{v})\right| \approx \left|u - u(\bar{v})\right|$ based on Taylor series.

$$u\left(\Pi_N v\right) - u(\bar{v})$$

Limit $\tilde{u} = \bar{u} + \theta(u - \bar{u})$ such that

$$\min \Pi_N v \geq \min v, \qquad \max \Pi_N v \leq \max v.$$

Other options

- Adaptively choose time-step size?
- Expensive option: bisection or Newton algorithm for finding theta.

References

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