Signals and Systems

ECCE 302

Discrete-Time Signals

Introduction

In general, there exist 2 categories of discrete-domain (e.g. discrete-time) signals:

1. Naturally discrete-domain/time signals, and
2. obtained by *discretization* (*sampling*) of continuous-domain/time signals.

Examples:

i. Number of flights per day (naturally discrete-time)
ii. Digital photographs (naturally discrete-space).

i. Temperature taken once a minute (discrete-time; obtained by sampling)
ii. Digitally measured voltage (discrete-time; obtained by sampling)
iii. Scans of analog photographs (discrete-space; obtained by sampling)

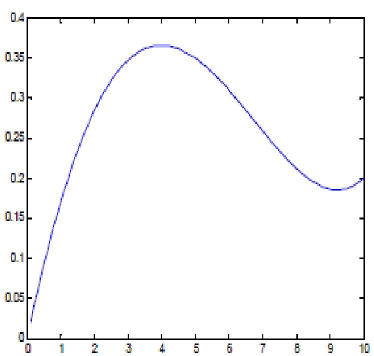
Introduction

- DT (DS) signals $x[k]$ are functions of a variable that takes values from a discrete set $k \in \{\ldots,-2,-1,0,1,2,\ldots\}$ which is the integer time (space) index.

- Values of x are real (if they are integer $\Rightarrow$ the signal is called *digital*)

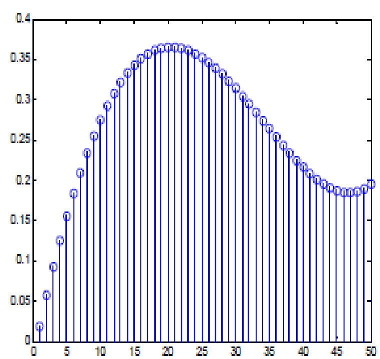
- For discretized signals, the interpretation of k is defined by the *sampling period* T_s (note that $\omega_s=2\pi/T_s$ or $f_s=1/T_s$ is called the *sampling frequency*).

$x(t)$



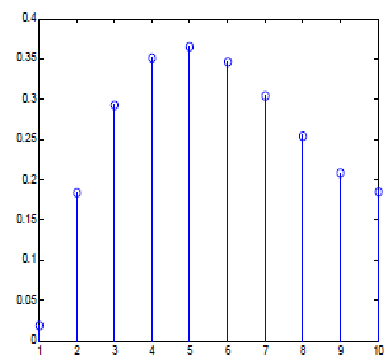
$x[k]$

50 samples



$x[k]$

10 samples

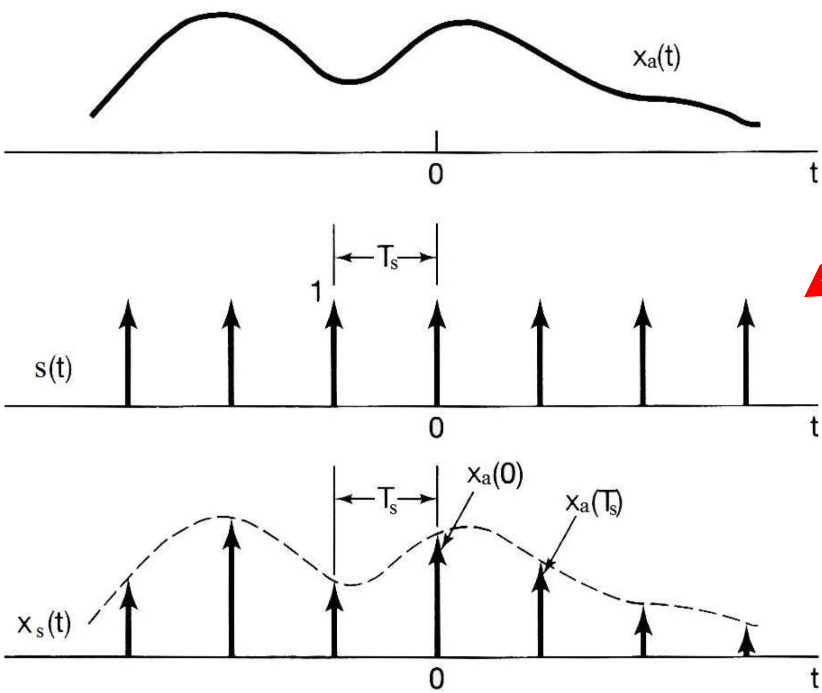


- If $T_s = 1$ (10 samples):
  $x[k] \leftarrow x(1 \cdot k)$

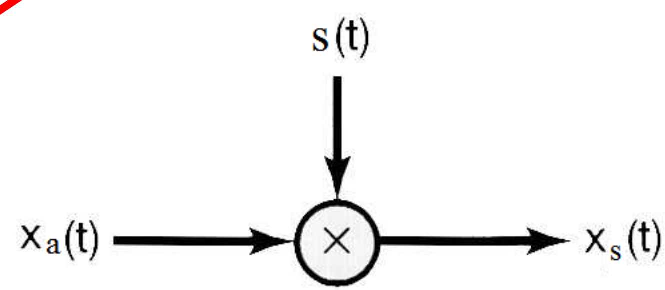
- If $T_s = 0.2$ (50 samples):
  $x[k] \leftarrow x(0.2 \cdot k)$

Discrete-time Signals

Mathematical definition of sampling: The *continuous-time signal* $x_a(t)$ is sampled at regular intervals, $\boldsymbol{T_s}$, which is called the sampling period, by multiplying it by $s(t)$, which is a train of delta functions spaced by $\boldsymbol{T_s}$, as illustrated below.



$$s(t)=\sum_{n=-\infty}^{\infty}\delta(t-nT_s)$$



$$x_s(t)=\sum_{n=-\infty}^{\infty}x_a(nT_s)\delta(t-nT_s)$$

$$x_d[n]$$

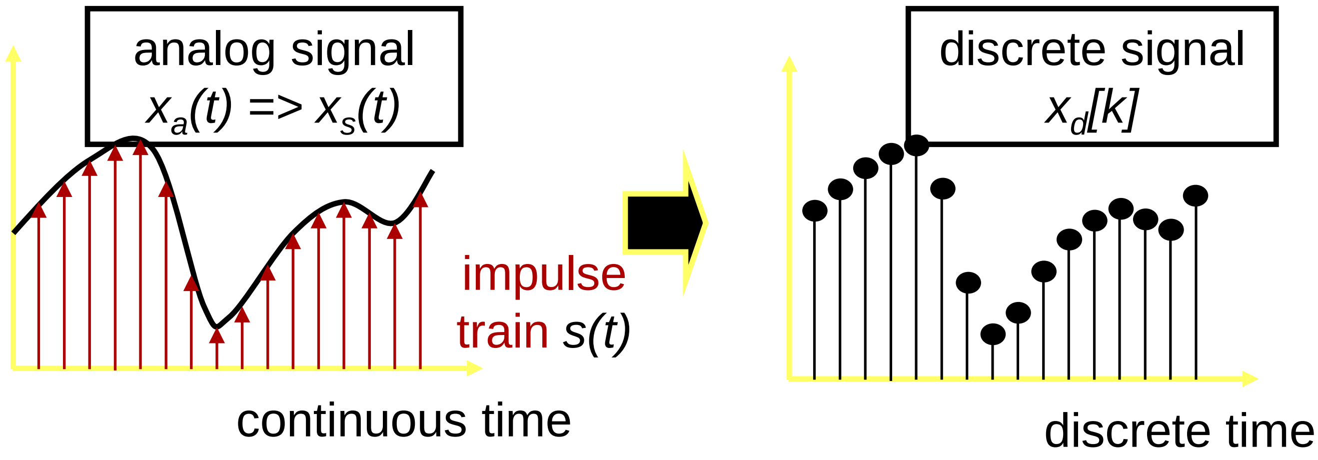
Discrete-time Signals

Introduction

$$x_s(t) = x_a(t) \cdot \sum_{n=-\infty}^{+\infty} \delta\left(t - n \cdot T_s\right) \quad \Rightarrow \quad x_d[n]$$

Thus, in the continuous domain, values of discrete signals are formally considered ***infinite*** (multiplied by δ)!

This interpretation is useful in theoretical results relating CT signals and their DT (sampled) variants, but is generally ignored in the engineering practice of discrete signals.

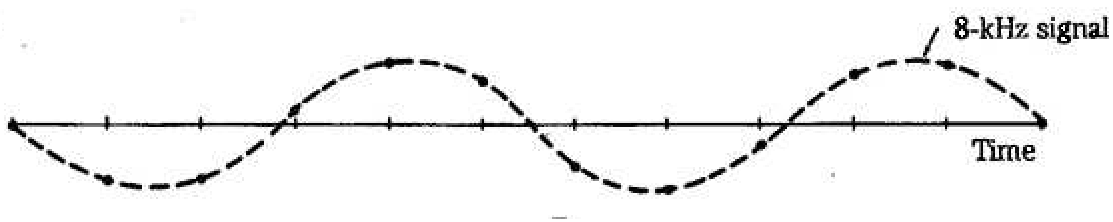


Sampling theorem

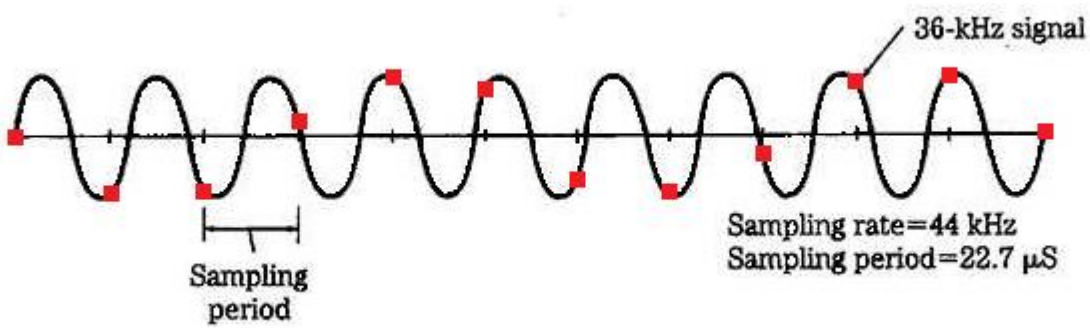
The fundamental question: Is it possible to reconstruct an analog (CT) signal from its sampled sequence?

$$x_d[n] = x_s(nT_s) \quad \overset{possible?}{\Longrightarrow} \quad x_a(t)$$

Sampled signal $x_s(nT)$
$T_s = 22.7\mu sec.$



The "natural" guess.



The actual sampling process.

Discrete-time Signals

Sampling theorem

Representation of sampled signals in the Fourier domain:

$$\mathcal{F}\left(x_s(kT_s)\right)=\mathcal{F}\left(x_a(t)\cdot s(t)\right),\quad \text{where}\quad s(t)=\sum_{k=-\infty}^{+\infty}\delta\left(t-k\cdot T_s\right)$$

General property of the Fourier transform:

$$\mathcal{F}\left(x_a(t)\cdot s(t)\right)=\frac{1}{2\pi}\mathcal{F}\left(x_a(t)\right)*\mathcal{F}\left(s(t)\right)\quad\Longrightarrow\quad \mathcal{F}\left(x_s(kT_s)\right)=\frac{1}{2\pi}X_a(\omega)*\mathcal{S}\left(\omega\right)$$

$$S(\omega)=\mathcal{F}\left(\sum_{k=-\infty}^{+\infty}\delta\left(t-k\cdot T_s\right)\right)\overset{table}{=}\frac{2\pi}{T_s}\sum_{k=-\infty}^{+\infty}\delta\left(\omega-\frac{k\cdot 2\pi}{T_s}\right)=\frac{2\pi}{T_s}\sum_{k=-\infty}^{+\infty}\delta\left(\omega-k\cdot\omega_s\right)$$

Periodic signal of period Ts

$S(\omega)$ is a *train* of ***delta impulses*** separated in the ***frequency domain*** by ***$\omega_s=2\pi/T_s$.***

Discrete-time Signals

Sampling theorem (cont.)

$$X_s(\omega) = \frac{1}{2\pi}\frac{2\pi}{T_s}\sum_{k=-\infty}^{+\infty} X_a(\omega) * \delta\left(\omega - k\omega_s\right)$$

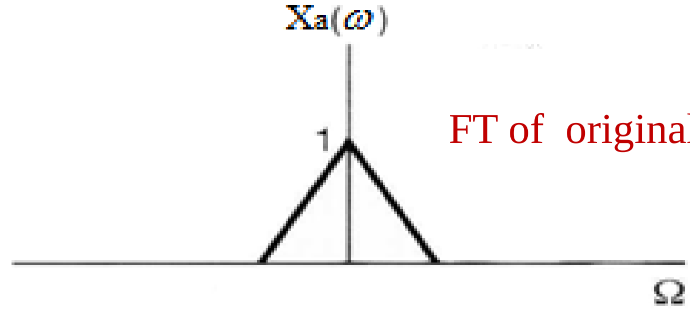
Since a convolution of a function with an impulse $\boldsymbol{\delta(u-u_0)}$ simply shifts the function by $\boldsymbol{u_0}$. Therefore,

$$\boxed{X_s(\omega) = \frac{1}{T_s}\sum_{k=-\infty}^{+\infty} X_a\left(\omega - k\omega_s\right)}$$

$\boldsymbol{X_s(\omega)}$ is a **sum** of an infinite number of *replicas* of $\boldsymbol{X_a(\omega)}$ shifted by integer multiplicities of $\boldsymbol{\omega_s}$ (and additionally multiplied by $\boldsymbol{1/T_s}$).

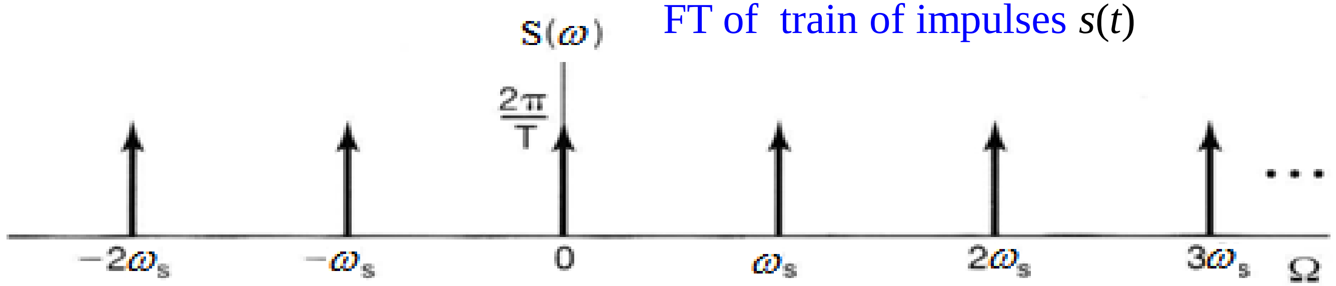
Discrete-time Signals

Sampling theorem (cont.)

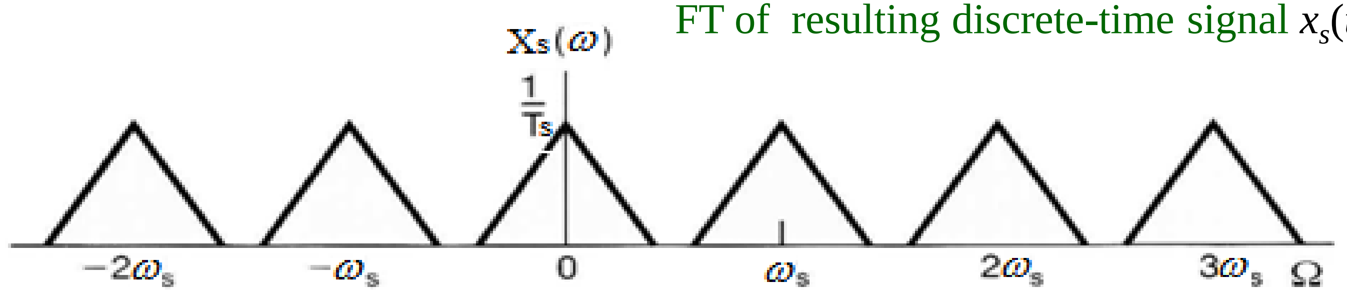


FT of original continuous-time signal $x_a(t)$

FT of train of impulses $s(t)$



FT of resulting discrete-time signal $x_s(t)$



Discrete-time Signals

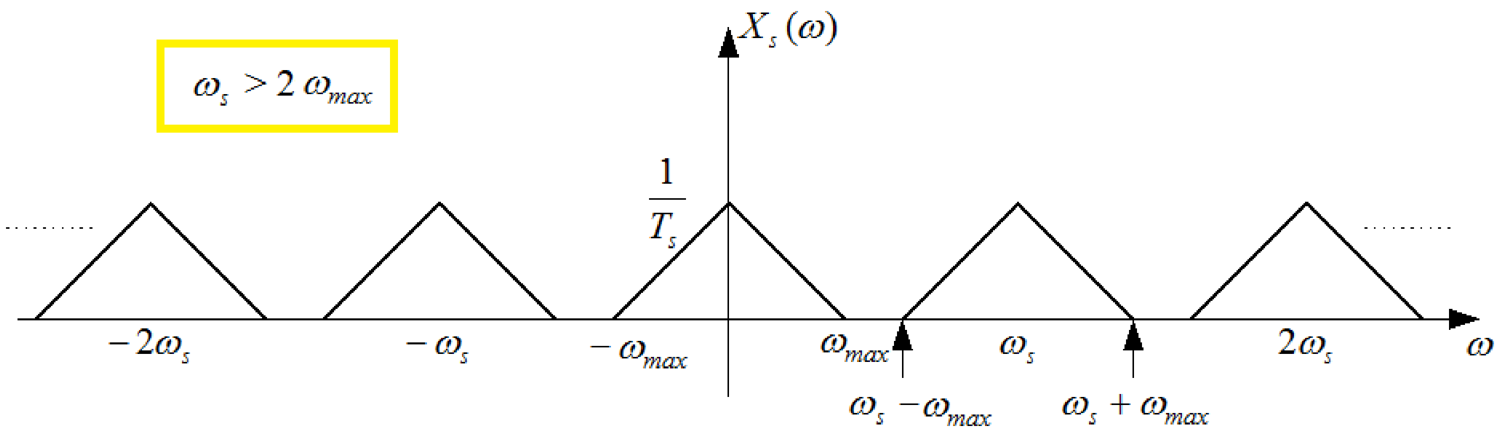
Sampling theorem (cont.)

Assume the signal $x_a(t)$ has a limited bandwidth, i.e.

$$X_a(\omega) \rightarrow \begin{cases} \neq 0 & |\omega| \leq \omega_{\max} \\ = 0 & |\omega| > \omega_{\max} \end{cases}$$

Two cases are possible:

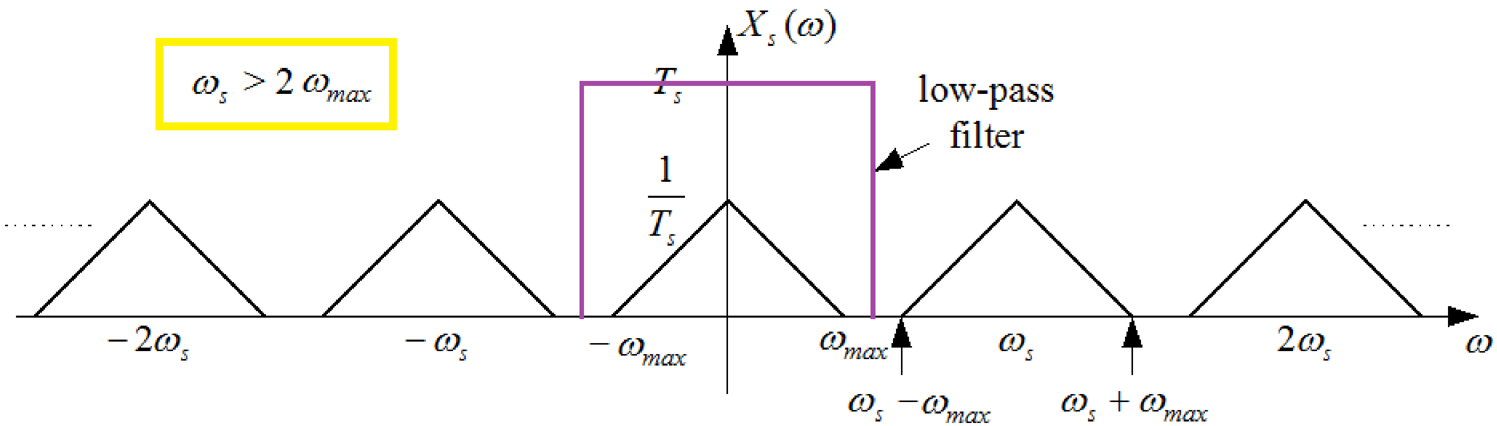
Case 1: $\omega_s > 2\omega_{max}$



Discrete-time Signals

Sampling theorem (cont.)

- ✓ There is no overlap between shifted replicas of $\boldsymbol{X_a(\omega)}$.
- ✓ Therefore, the original spectrum $\boldsymbol{X_a(\omega)}$ can be ***recovered*** from the spectrum of sampled signals $\boldsymbol{X_s(\omega)}$.
- ✓ This can be achieved using an ***ideal low-pass filter*** (with T_s gain) and a cut-off frequency $\omega_{max} < \omega_c < \omega_s - \omega_{max}$ (**there are practical problems!**)
- ✓ This filter is called a ***reconstruction filter*** or ***anti-aliasing filter***.



Discrete-time Signals

Sampling theorem (cont.)

Theorem (Nyquist, Kotelnikov, Whittaker, Shannon)

A signal $x(t)$ with the maximum frequency ω_{max} can be fully reconstructed from its discretized version, if it is sampled with T_s period satisfying $2\pi/T_s = \omega_s > 2\omega_{max}$.

Alternative notation:

- sampling frequency (hertz): $f_s = \omega_s / 2\pi$
- sampling interval: $T_s = 1/ f_s = 2\pi / \omega_s$
- Nyquist frequency (hertz): $f_N = 2 f_{max}$

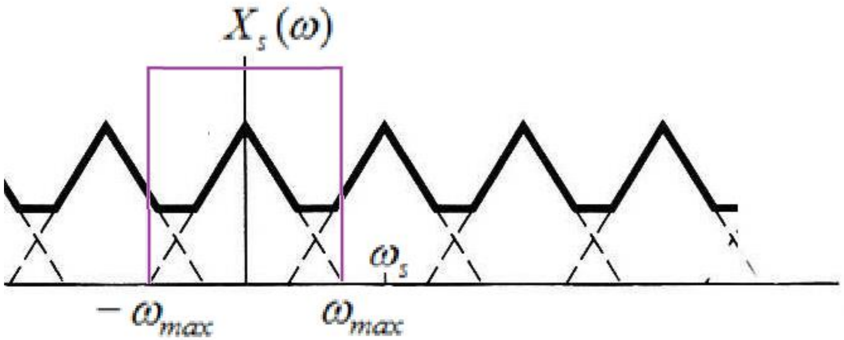
e.g. CD audio: $f_{max} \approx 20$ kHz $\rightarrow f_s = 44.1$ kHz $\approx f_N$

Discrete-time Signals

Sampling theorem (cont.)

Case 2: $\omega_s < 2\omega_{max}$

The sampling frequency is too low.



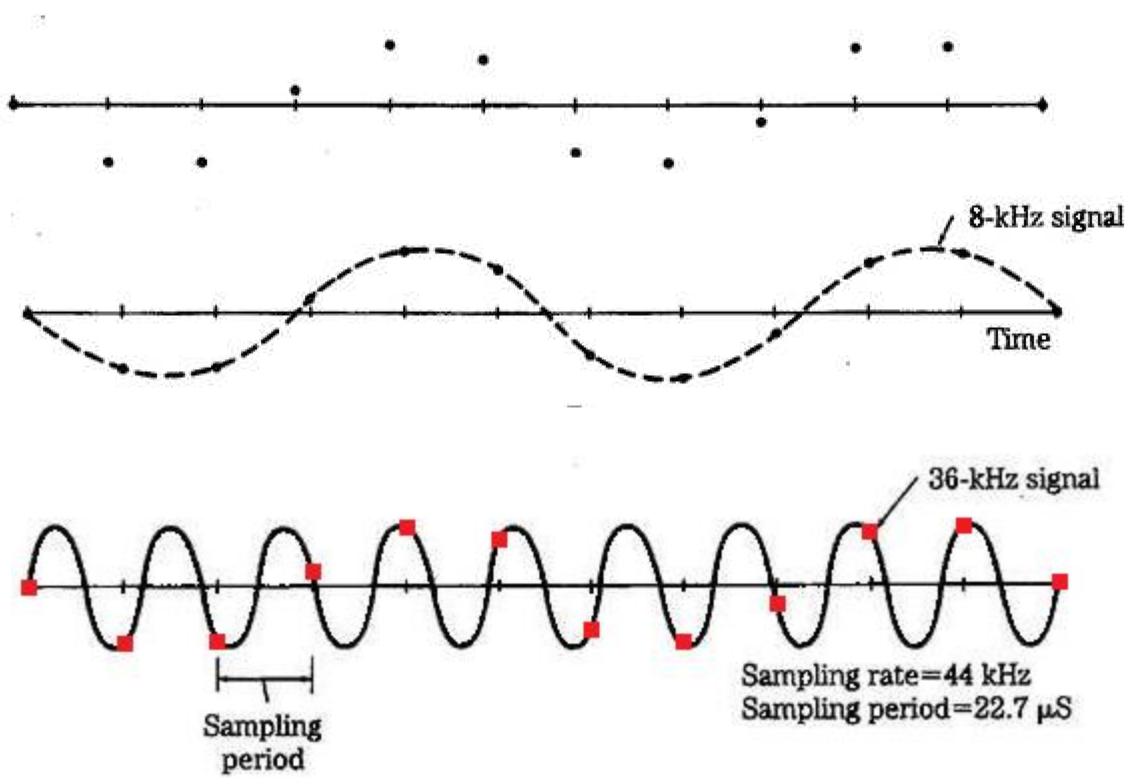
- ✓ The shifted replicas of $\boldsymbol{X_a(\omega)}$ overlap.
- ✓ Therefore, the original spectrum $\boldsymbol{X_a(\omega)}$ **CANNOT** be reconstructed from the spectrum of sampled signal $\boldsymbol{X_s(\omega)}$.
- ✓ The phenomenon of such overlapping replicas is called ***aliasing***.

Discrete-time Signals

Sampling theorem (cont.)

Aliasing in the time-domain (exemplary effect).

The ***signal*** and its *alias* may fit to (i.e. pass through) the ***same set of samples***.



The signal is sampled at a sampling period T_s =22.7μs (i.e., a sampling frequency f_s = 44kHz.)

The 'natural' guess is **8kHz** (the curve **cos16000π** fits the samples).

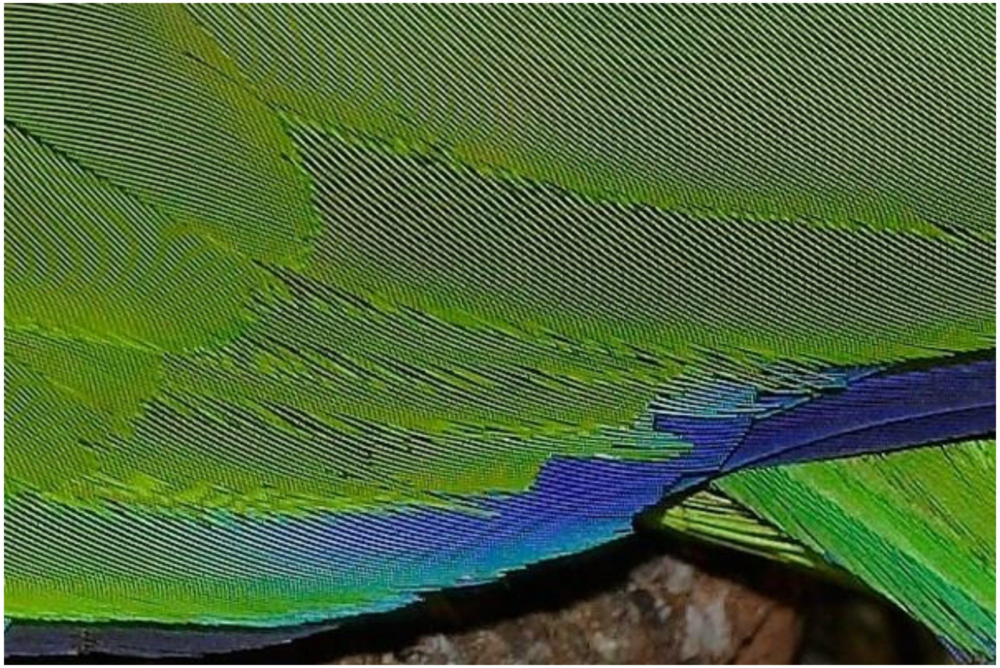
The actual signal frequency is **36kHz (72000π)** (observe that here there is aliasing since f_s is < the Nyquist frequency = $2f_{max}$ = **72kHz (144000π)**).

Discrete-time Signals

Sampling theorem (cont.)

Aliasing in the time-domain (exemplary high-frequency effects)

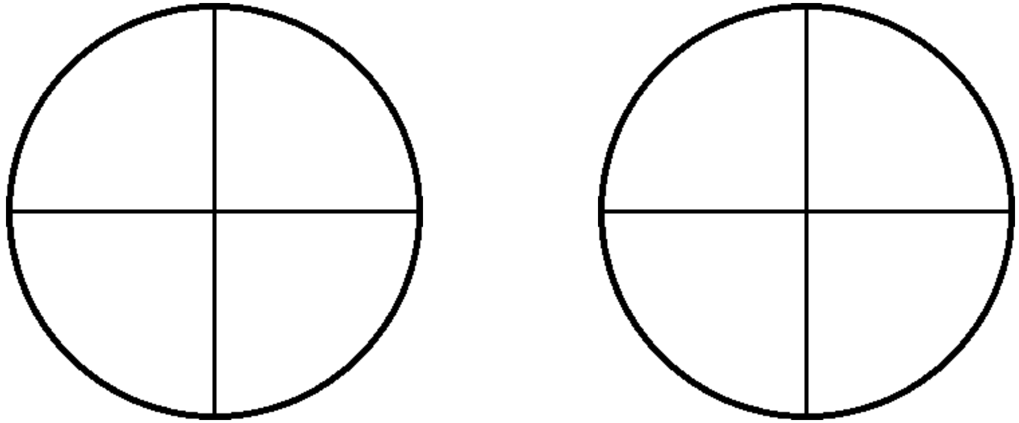
Moire patterns (high-frequency noise on digital images)



Discrete-time Signals

Sampling theorem (cont.)

Aliasing in the time-domain (exemplary low-frequency effects)



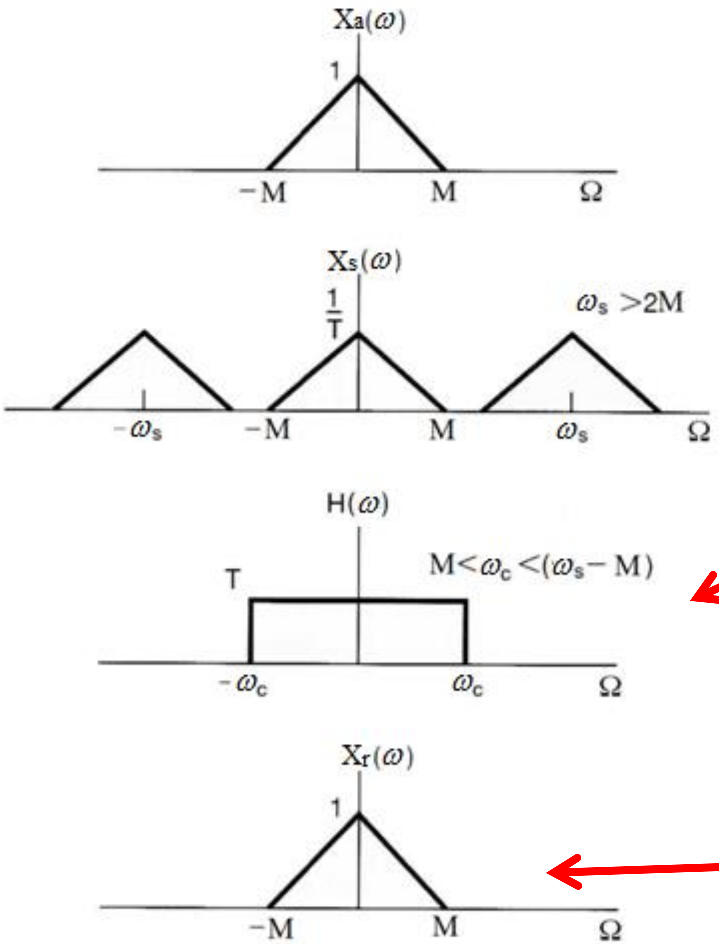
www.youtube.com/watch?v=jHS9JGkEOmA

When the vehicle slows down, wheels rotate ‘normally’.

When the vehicle accelerates, wheels ‘rotate’ in the opposite direction.

The sampling frequency (25/30/50 frames/sec) is too low.

Signal Reconstruction: Spectral Perspective



The ideal reconstruction system is an ***ideal low-pass filter*** with ***amplifier***.

$$H(\omega) = \begin{cases} T, & |\omega| \le \omega_c = \omega_s / 2 \\ 0, & |\omega| > \omega_c = \omega_s / 2 \end{cases}$$

The Fourier transform of the reconstructed signal is.

$$X_r(\omega) = X_s(\omega) \cdot H(\omega)$$

Discrete-time Signals

Quantization of DT signals

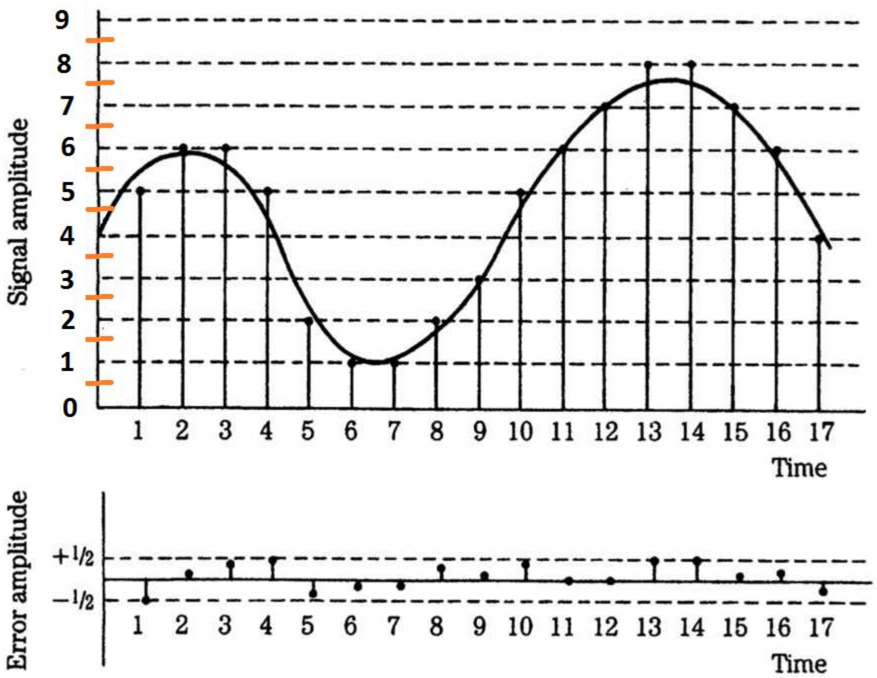
Discrete *versus* Digital Signals

- A ***discrete-time signal*** obtained by *sampling* a continuous-time continuous-amplitude signal has a ***discrete*** time but ***continuous*** amplitude.
- A ***digital signal*** has a ***discrete time*** and ***discrete amplitude***.
- A ***digital signal*** can be obtained by ***quantization*** of the ***amplitude*** of a sampled (discrete) signal.

Discrete-time Signals

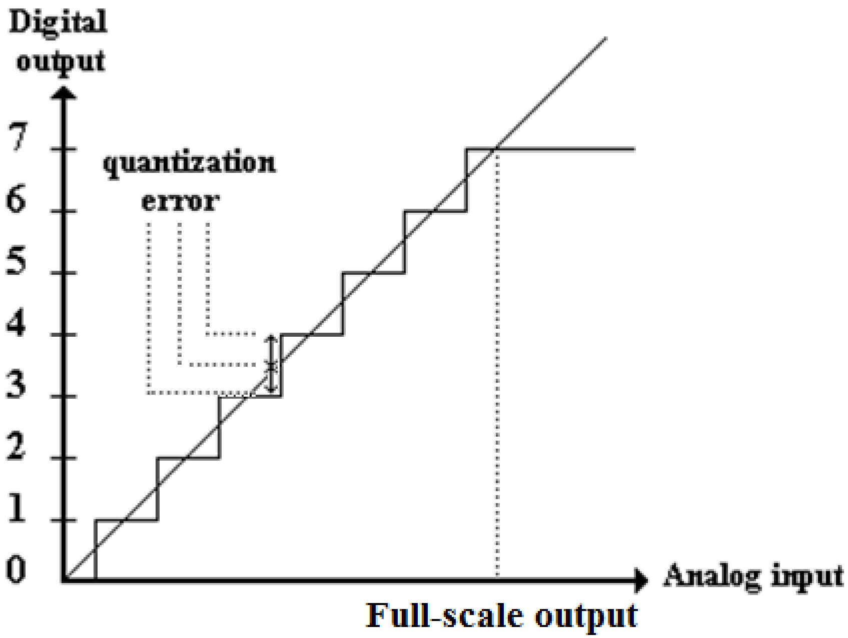
Quantization of DT signals

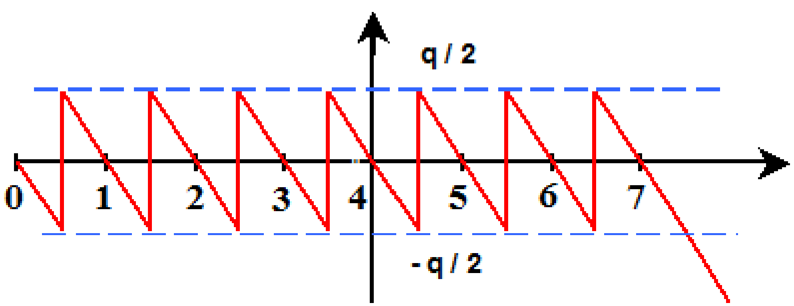
Quantization is the process of converting the signals values into a finite number $\boldsymbol{L}$ of (digital) values. The number of possible values is usually defined by the number of bits $\boldsymbol{b}$ used in the quantization, i.e. $\boldsymbol{L = 2^b}$.



Discrete-time Signals

Quantization of DT signals





Quantization error

- A finite set of ***L*** *quantisation levels* is defined.
- If the interval (step-size) ***Q*** between the quantization levels is fixed, we have ***uniform quantisation***.
- The amplitude of each sample is ***rounded*** to the ***nearest level*** in the defined set.
- Quantisation is ***irreversible***. The *loss of information* during quantization is called the ***quantisation error***.
- The ***maximum quantisation error*** is ***Q*/2**.
- The quantisation error can be reduced by ***reducing the size of the quantisation interval*** ***Q***, and consequently by ***increasing the number of quantisation levels*** ***L*.**

Discrete-time Signals

Quantization of DT signals - coding

- If we have ***L*** quantisation levels (and the probability of each level is assumed the same) each sample has to be represented (encoded) by at least $\boldsymbol{B = \log_2 L}$ bits.

- Increasing the number of quantisation levels ***L*** reduces the quantisation error but also increases the code length ***B*** (which leads to an increase in storage and processing requirements).

- The **reverse** process of encoding is called ***decoding*****.**

- The coding (encoding-decoding) process is ***reversible*****.**

Discrete-time Signals

Quantization of DT signals - coding

Evaluation of the quantization/coding error

➢ Assume a signal ranging from **0** to ***R***.

➢ With ***B*** bits available, the signal is quantizated into $\boldsymbol{L = 2^B}$ levels, where the stepsize (quantization interval) is $\boldsymbol{Q = R/2^B}$.

➢ We define the level of **quantization noise** $\boldsymbol{N_Q}$ as the "quantization error x 2" (because the error can be positive or negative). Thus, $\boldsymbol{N_Q}$ is just equal to the quantization interval, i.e. $\boldsymbol{N_Q = Q = R/2^B}$.

➢ The signal-to-noise ratio (***SNR***) can be defined as the ratio between the **signal range** ***R*** and the **quantization noise** $\boldsymbol{N_Q = Q}$. In the decibel scale:

$$SNR = 20\log_{10}\left(\frac{R}{Q}\right) = 20\log_{10}\left(2^B\right) = 20B\log_{10}2 \approx B\cdot 6.021[dB]$$

This is so-called ***6dB per bit*** rule.

Discrete-time Signals

Classification and basic properties of DT signals

Even and Odd Signals:

- A signal is called an *even* signal if:

$$x[k]=x[-k]$$

- An *even* signal is *symmetric*.
- A signal is called an *odd* signal if:

$$x[k]=-x[-k]$$

- An *odd* signal is *antisymmetric*.

Discrete-time Signals

Classification and basic properties of DT signals

Even and **Odd** Signals:

- Any *signal* **x[*k*]** can be expressed as a *sum* of *odd* and *even* signals. Thus:

$$x[k] = x_e[k] + x_o[k]$$

where

$$x_e[k] = \frac{x[k] + x[-k]}{2}$$

$$x_o[k] = \frac{x[k] - x[-k]}{2}$$

Discrete-time Signals

Classification and basic properties of DT signals

Even and **Odd** Signals:

- The above definitions of symmetry and antisymmetry apply only to ***infinite sequences*** or to ***finite sequences defined over a symmetric interval –M ≤ 0 ≤ +M***.
- For finite sequences defined over the interval $0 \le k \le N\text{-}1$, the definition of symmetry is slightly different.
- In this case, a sequence $x[k]$ can be expressed as a *sum* of so-called *periodic even part*, $x_{pe}[k]$, and *periodic odd part*, $x_{po}[k]$.

$$x[k] = x_{pe}[k] + x_{po}[k]$$

$$x_{pe}[k] = \frac{x[k] + x[N-1-k]}{2} \qquad x_{po}[k] = \frac{x[k] - x[N-1-k]}{2}$$

Discrete-time Signals

Classification and basic properties of DT signals

Energy and Power DT Signals:

- The ***total energy*** of a discrete-time signal is given by

$$E_x = \sum_{k=-\infty}^{+\infty} \left| x[k] \right|^2$$

- The ***average power*** of an ***aperiodic signal*** is

$$P_x = \lim_{K \to \infty} \frac{1}{2K+1} \sum_{k=-K}^{+K} \left| x[k] \right|^2$$

Discrete-time Signals

Classification and basic properties of DT signals

- The ***average power*** of a ***periodic signal*** with a fundamental period N is given by

$$P_x = \frac{1}{N}\sum_{k=0}^{N-1}\left|x[k]\right|^2$$

- A signal is referred to as an ***energy signal*** if it has a ***finite total energy*** and ***zero average power***, i.e. $0<E_x<\infty$ and $P_x=0$.
- A signal is referred to as a ***power signal*** if it has a ***finite average power*** and ***infinite total energy***, i.e. $0<P_x<\infty$ and $E_x=\infty$.
- A signal can be neither power nor energy signal, when $P_x=\infty$ and $E_x=\infty$.

Discrete-time Signals

Classification and basic properties of DT signals

<u>Causal, Anticausal and Non-Causal Signals:</u>

- **Causal** signal is zero for all negative times.

$$x[k] = 0 \qquad k<0$$

- **Anticausal** signal is zero for all positive times.

$$x[k] = 0 \qquad k>0$$

- **Non-Causal** signal has nonzero values in both positive and negative discrete times.

Discrete-time Signals

Classification and basic properties of DT signals

- **Amplitude Scaling**: (also called scalar multiplication)

$$y[k] = c\,x[k]$$

- **Addition**:

$$y[k] = x_1[k] + x_2[k]$$

- **Time Reversal**: (also called reflection, flipping or folding)

$$y[k] = x[-k]$$

- **Time Shift**:

$$y[k] = x[k + K]$$

where K is an <u>integer</u>. If $K > 0$ then $x[k]$ will be *advanced* or *shifted to the left* by K time units. If, however, $K < 0$ then $x[k]$ will be *delayed* or *shifted to the right*.

Discrete-time Signals

Classification and basic properties of DT signals

- Time Scaling
  - downsampling

$$y[k] = x[Mk], \text{ where } \boldsymbol{M} \text{ is a positive integer}$$

This is corresponding to time compression.

  - upsampling

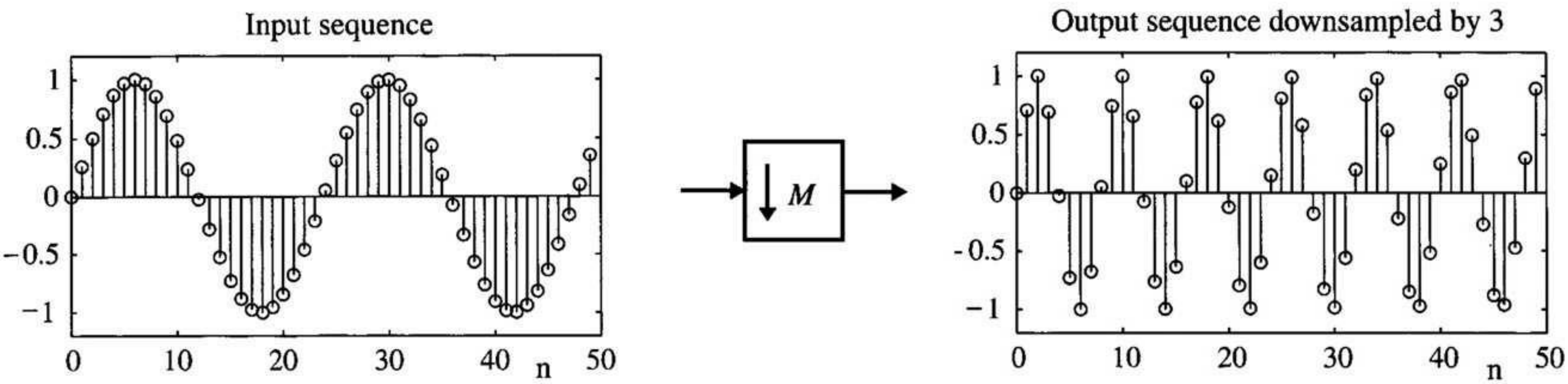
$$y[k] = x[k/L], \text{ where } \boldsymbol{L} \text{ is a positive integer}$$

This is corresponding to time expansion.

Discrete-time Signals

Classification and basic properties of DT signals

Downsampling



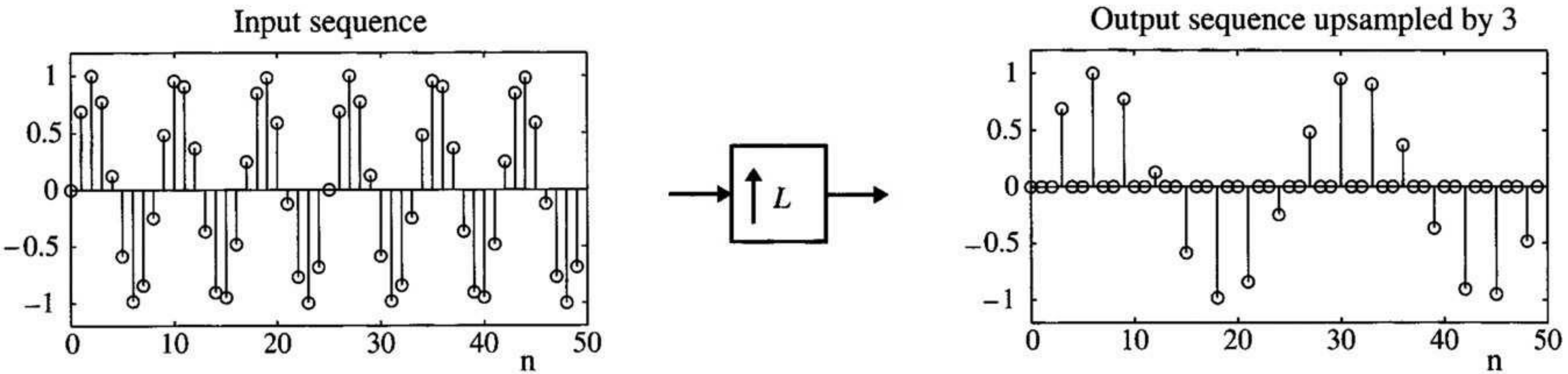
Downsampling is equivalent to sampling the original analog signal with $\boldsymbol{M \times T_s}$ sampling period instead of $\boldsymbol{T_s}$ period.

Compared to the original sequence, a downsampled sequence contains less information (data loss).

Discrete-time Signals

Classification and basic properties of DT signals

Upsampling



Upsampling is NOT equivalent to sampling the original analog signal with T_s/L sampling period instead of T_s period.

Actually, upsampling creates a sequence with many "gaps" (zero samples). In order to fill these gaps, we can use interpolation.

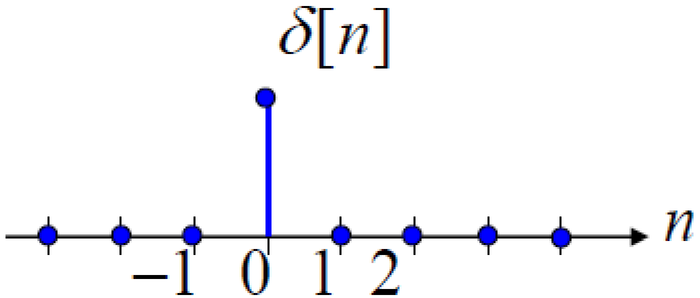
Compared to the original sequence, an upsampled sequence contains the same amount of information (no loss of data).

Discrete-time Signals

Elementary DT signals

Discrete unit impulse signal (Kronecker delta function)

$$\delta[n] = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases}$$



Sampling property of Kronecker delta function

$$\sum_{n=-\infty}^{+\infty} x[n]\delta[n] = x[0]$$

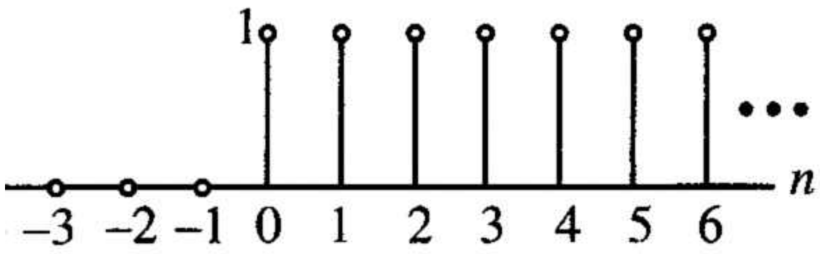
$$\sum_{n=-\infty}^{+\infty} x[n]\delta[n-n_0] = x[n_0]$$

Discrete-time Signals

Elementary DT signals

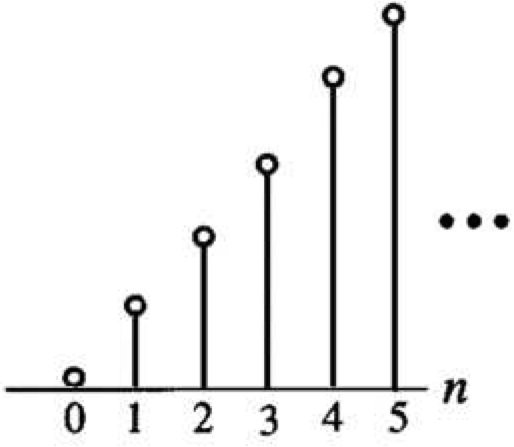
1. Discrete unit-step signal

$$u[k] = \begin{cases} 0 & k < 0 \\ 1 & k \geq 0 \end{cases}$$



2. Discrete unit-ramp signal

$$r[k] = \begin{cases} 0 & k < 0 \\ k & k \geq 0 \end{cases}$$

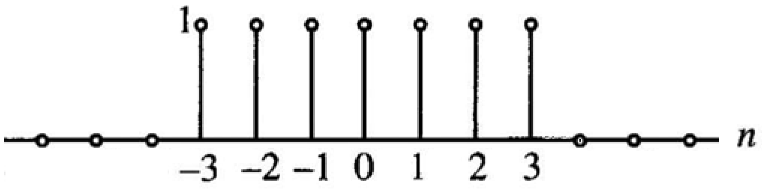


Discrete-time Signals

Elementary DT signals

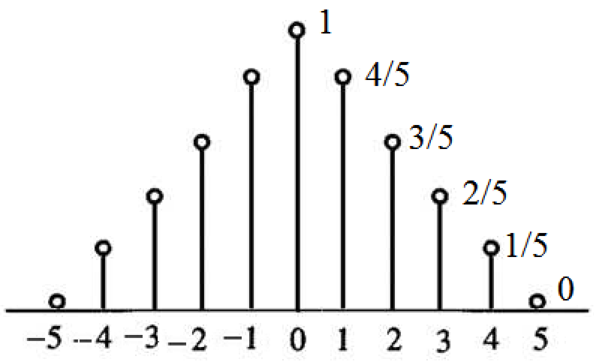
3. Discrete rectangular signal

$$\Pi_{2K}[k]=\begin{cases}1, & -K\le k\le K\\ 0, & \text{otherwise}\end{cases}$$



4. Discrete triangular signal

$$\Delta_{2K}[k]=\begin{cases}0\ , & k<-K\\ 1+\dfrac{k}{K}, & -K\le k<0\\ 1-\dfrac{k}{K}, & 0\le k\le K\\ 0\ , & k>K\end{cases}$$



Discrete-time Signals

Discrete periodic signals

A discrete signal ***x* [*k*]** is periodic with the fundamental period ***N*** if

x* [*k*] = *x* [*k*+*N*]** (and ***x* [*k*] = *x* [*k*+*mN*]** for any integer ***m)

Given a periodic discrete signal ***x* [*k*]** with the fundamental period ***N***, we define the ***fundamental frequency*** **Ω** as

$$\Omega = \frac{2\pi}{N} \quad \text{or (in } \textit{hertz} \text{ domain)} \quad F = \frac{1}{N}$$

IMPORTANT NOTE: The fundamental frequencies **Ω** of discrete signals are always from **(0; π>** range (or **(0;0.5>** range in the *hertz* domain). In other words, in the discrete signal domain only such frequencies can exist.

Discrete-time Signals

Note: If we create a DT sinusoid by sampling a CT sinusoid, *their periods may not be the same*. In fact, the *DT sequence may not be periodic*.

Let a DT sinusoid

$$g[n] = A\cos(2\pi K n + \theta)$$

be related to a CT sinusoid

$$g(t) = A\cos(2\pi f_0 t + \theta)$$

through $g[n] = g(nT_s)$ where T_s is the *sampling period*, i.e.,

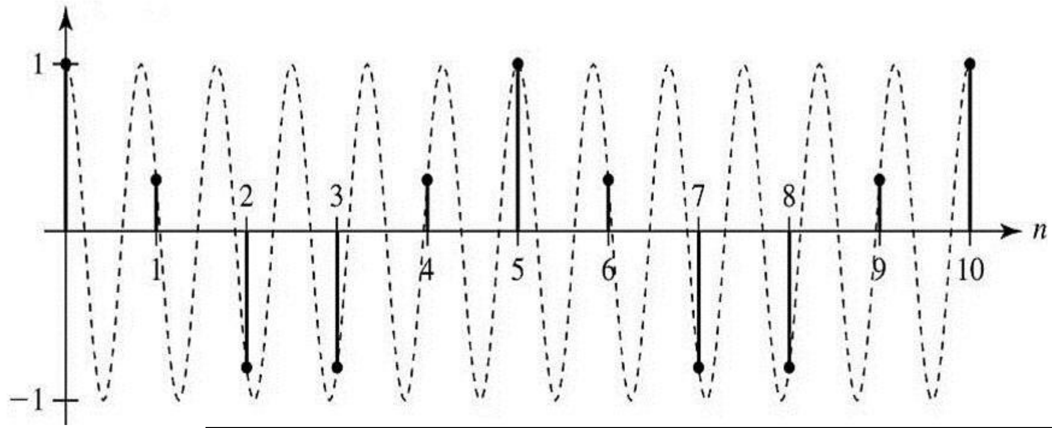
$$g[n] = g(nT_s) = A\cos(2\pi f_0 n T_s + \theta).$$

This implies that: $K = f_0 T_s = f_0/f_s = T_s/T_o$

Discrete-time Signals

Sampling analog periodic signals

On the other hand, *for a sequence to repeat, N sampling intervals T_s must fit **exactly** into M periods T_0 of the analog signal being sampled*.



Here, we have $N = 5$ samples in $M = 6$ analog periods.

That is,

$$\boxed{NT_s = MT_0 \Longrightarrow \frac{N}{M} = \frac{T_0}{T_s} = \frac{f_s}{f_0} = \frac{\omega_s}{\omega_0} = \frac{1}{K}.}$$

CONCLUSION: A *cosine* signal remains periodic after sampling *if and only if* the sampling frequency f_s (ω_s) is a rational multiplicity (i.e., *a ratio of 2 integers*) of the original frequency f_0 (ω_0).

Discrete-time Signals

Sampling analog periodic signals

Thus, some *cosine* signals (and, similarly, other signals periodic in the analog domain) may not remain periodic after sampling.

Such signals are usually referred to as ***quasi-periodic*** signals.

From the common-sense (engineering) perspective they “behave” periodically, but they do not satisfy the mathematical definition of periodicity.

Sampling Analog Periodic Signals

Example: A DT signal is defined as

$$g[n] = \cos(2n).$$

Is this sequence periodic?

Solution: We have

$$\begin{aligned} g[n] &= \cos(2n) = \cos\left(2\pi\frac{2}{2\pi}n\right) \\ &= \cos\left(2\pi\frac{1}{\pi}n\right) \Longrightarrow K = \frac{1}{\pi} \neq \frac{M}{N}. \end{aligned}$$

As K is irrational (*not a ratio of 2 integers*), this DT signal is *not periodic*.

Sampling Analog Periodic Signals

Example: A DT signal is defined as

$$x[n] = \cos\left(\frac{n4\pi}{5}\right).$$

Is this sequence periodic?

Solution: We have

$$x[n] = \cos\left(\frac{n4\pi}{5}\right) = \cos\left(2\pi\frac{2}{5}n\right) \Longrightarrow K = \frac{2}{5} = \frac{M}{N}.$$

Therefore,

- $x[n]$ is a periodic signal, and
- $N = 5$ and $M = 2$ ➔ The sequence repeats every 5 samples, and that these 5 samples are collected over 2 complete cycles of the analog signal being sampled.

Sampling Analog Periodic Signals

Examples: other non-periodic digital signals.

$$\cos(k), \quad \sin\left(\frac{k}{10}\right), \quad \cos\left(\frac{\pi k}{4}\right) + \cos(k)$$

Supplementary information can be found in the textbook:

- **Sections 9.1 (selected fragments)**
- **Sections 9.2-9.4.**