Recent applications of proton-neutron finite-amplitude method

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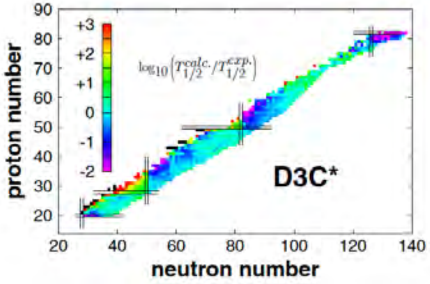
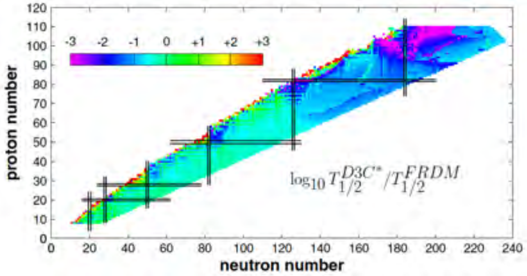
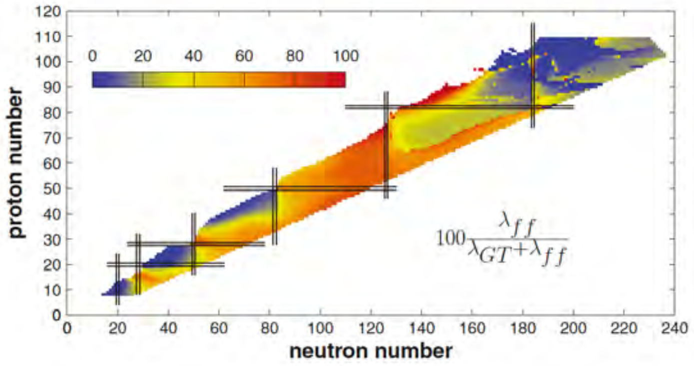
Charge-exchange process: beta decay

- r-process: neutron-star merger, supernovae... beta decay plays important role.
- Theoretical approaches: ab-initio, shell model, DFT

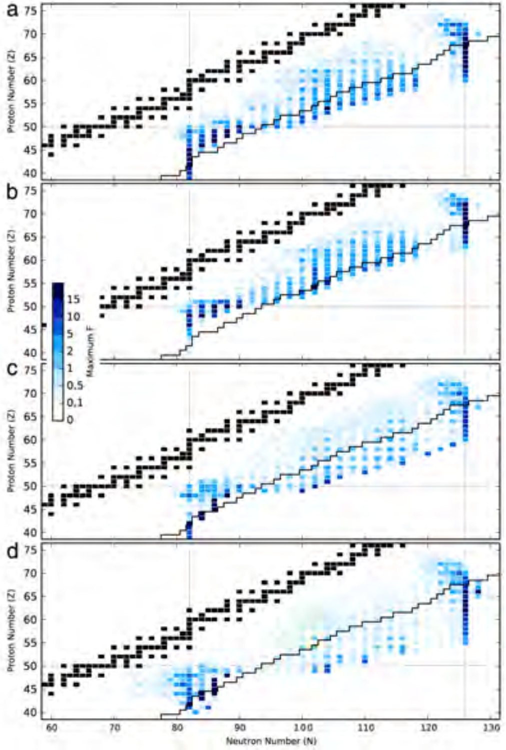
Covariant DFT: Marketin et al., Phys. Rev. C **93**, 025805 (2016)
(Calculation within spherical symmetry, odd: average number to odd)







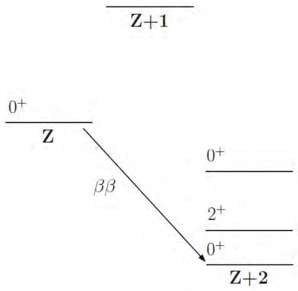
Mumpower et al., Prog. Part. Nucl. Phys. **86**, 86 (2016)



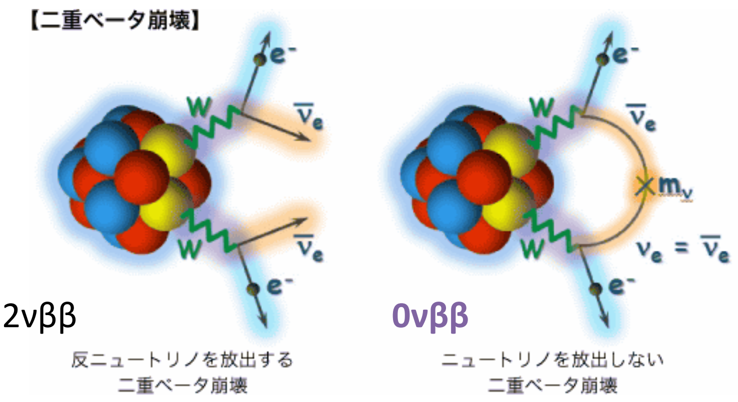
Charge-exchange process: double-beta decay

Neutrinoless double-beta decay (0νββ)

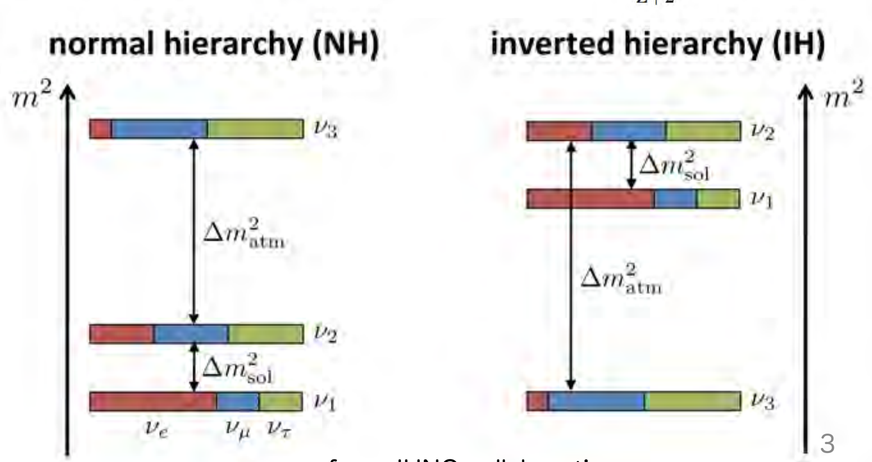
- Neutrino has finite mass
  (confirmed from neutrino oscillation measurement that measured the mass squared difference)
- three mass-eigen states
  - normal hierarchy: two lighter and one heavier
  - inverted hierarchy: one lighter and two heavier
- Neutrino may be Majorana particle (its own antiparticle)



Majorana neutrino can undergo neutrinoless double-beta decay
and the half-life will give the neutrino mass



from CANDLES (Univ. Osaka)



from JUNO collaboration

Charge-exchange process: double-beta decay

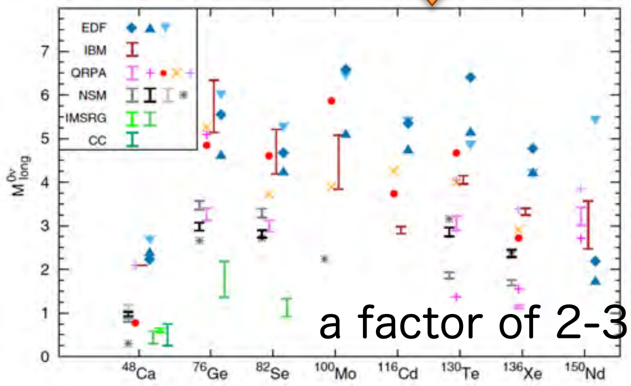
Half-life and nuclear matrix element

Phase space factor

$$(T_{1/2}^{0\nu})^{-1} = G_{0\nu}(Q_{\beta\beta}, Z)|M_{0\nu}|^2\langle m_{\beta\beta}\rangle^2$$

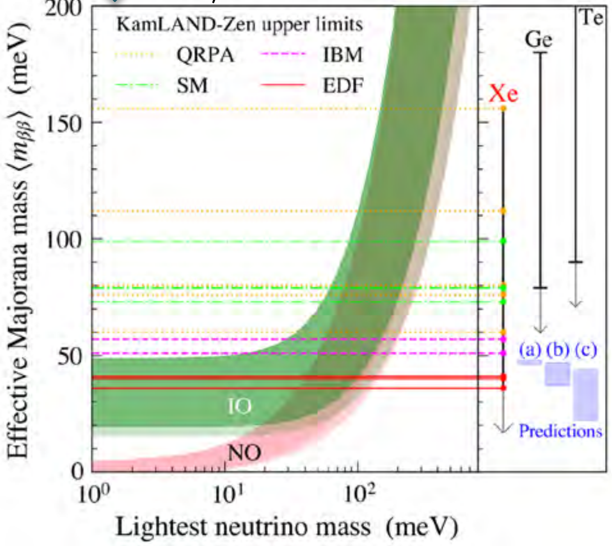
Nuclear matrix element

KamLAND-Zen(^{136}Xe)
$T_{1/2}^{0\nu} > 2.3\times10^{26}$ y



a factor of 2-3 deviation

Agostini et al., Rev. Mod. Phys. **95**, 025002 (2023)



Abe et al., Phys. Rev. Lett. **130**, 051801 (2023)

nuclear structure theories

EDF: generator coordinate method based on energy density functional
IBM: interacting boson model
QRPA: quasiparticle random-phase approximation
NSM: shell model
IMSRG: In-mediums similarity renormalization group
CC: coupled-cluster theory

Other charge-exchange processes

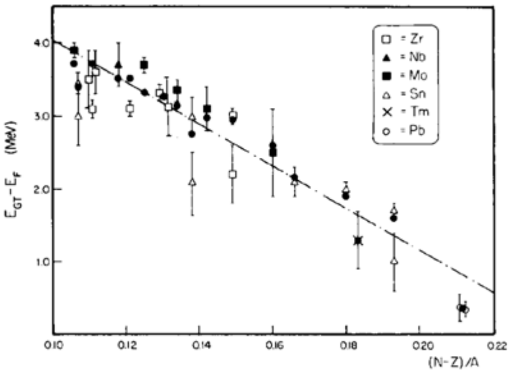
Gamow-Teller resonance and isobaric analogue state

Gamow-Teller resonance: collective states excited by $\sigma\tau^{\pm}$ operator
IAS: states excited with $\tau^{\pm}$ operator (same T, different T_z states)

$$E_{\mathrm{GT}} - E_{\mathrm{IAS}} = \Delta E_{ls} + 2(\tilde{\kappa}_{\sigma\tau} - \tilde{\kappa}_{\tau})\frac{N-Z}{A}$$

$$\hat{H} = \hat{H}_{\mathrm{sp}} + \sum_{i<j}\left[\kappa_\tau \tau_i \cdot \tau_j + \kappa_{\sigma\tau}\tau_i \cdot \tau_j \boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_j + \frac{e^2(1-\tau_{i3})(1-\tau_{j3})}{4|\boldsymbol{r}_i - \boldsymbol{r}_j|}\right] \qquad \kappa_\tau = \frac{\tilde{\kappa}_\tau}{A}, \quad \kappa_{\sigma\tau} = \frac{\tilde{\kappa}_{\sigma\tau}}{A}$$

correlation with spin-isospin part of the effective interaction/EDF

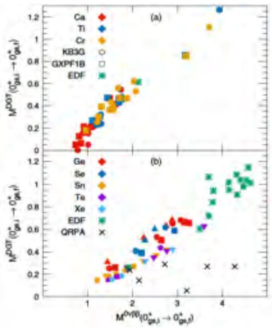


K. Nakayama et al., Phys. Lett. B **114**, 217 (1982)
F. Osterfeld, Rev. Mod. Phys. **64**, 491 (1992)

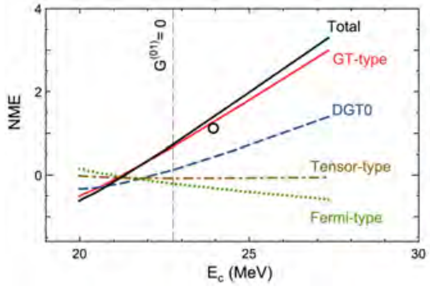
Double Gamow-Teller giant resonance

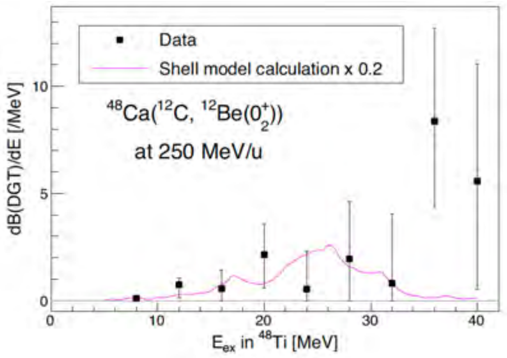
$$B(\mathrm{DGT}^{\pm};\lambda; i \to f) = \frac{1}{2J_i+1}|\langle f|| \left[\sum_j \boldsymbol{\sigma}_j\tau_j^{\pm}\cdot\boldsymbol{\sigma}_j\tau_j^{\pm}\right] ||i\rangle|^2$$

two-phonon states, excited by $(\sigma\tau^-)^2$
correlation with 0νββ decay NME



Shimizu et al., Phys. Rev. Lett. **120**, 142502 (2018)

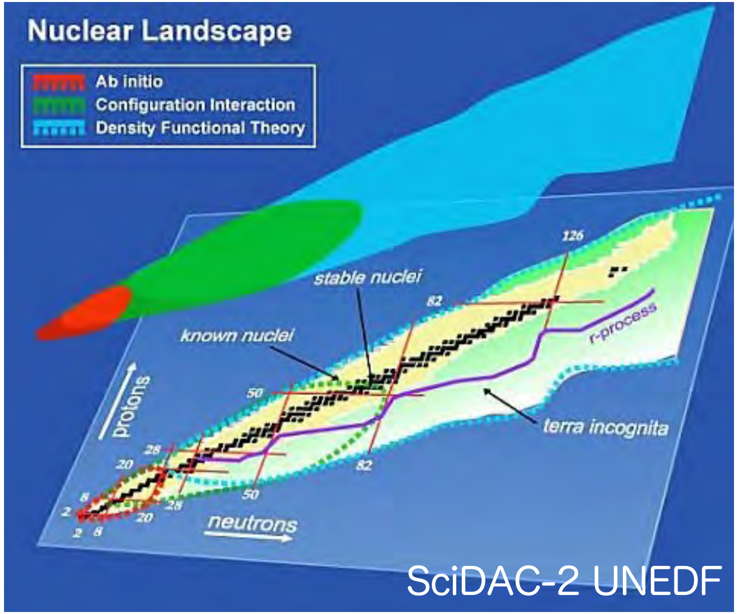




Sakaue et al., Prog. Theor. Exp. Phys. **2024**, 123D03 (2024)

Nuclear density functional theory (DFT)

A microscopic theory that can compute from all unstable nuclei to neutron star



Hartree-Fock-Bogoliubov equation for nuclear ground state

iterative eigenvalue problem (non-linear eq.)

$$\begin{pmatrix} h[\rho,\tilde{\rho}]-\lambda & \Delta[\rho,\tilde{\rho}] \\ -\Delta^*[\rho,\tilde{\rho}] & -h^*[\rho,\tilde{\rho}]+\lambda \end{pmatrix} \begin{pmatrix} U_k \\ V_k \end{pmatrix} = E_k \begin{pmatrix} U_k \\ V_k \end{pmatrix}$$

matrix elements (potential)

quasiparticle wave functions

densities

$$h[\rho,\tilde{\rho}] = \frac{\delta E}{\delta \rho} \quad \Delta[\rho,\tilde{\rho}] = \frac{\partial E}{\partial \tilde{\rho}^*}$$

$$U, V \to \rho, \tilde{\rho}$$

Quasiparticle random-phase approximation (QRPA)

excited states (including beta decay) and dynamics

eigenvalue problem of large dimension (~10^6)

- Input of the nuclear DFT: energy density functional (EDF, E[ρ])
- EDF is determined phenomenologically (not directly derived from the nuclear force)
- EDF: kinetic energy + particle-hole energy + pairing energy + Coulomb

Example of derivation of nuclear EDF

One can derive the energy density functional from the effective interaction

density matrix $\quad \hat{\rho}(\boldsymbol{r}st, \boldsymbol{r}'s't') = \langle \Psi | c^\dagger_{\boldsymbol{r}'s't'} c_{\boldsymbol{r}st} | \Psi \rangle$

nonlocal densities $\quad \rho_k(\boldsymbol{r}, \boldsymbol{r}') = \sum_{stt'} \hat{\rho}(\boldsymbol{r}st, \boldsymbol{r}'st') \tau^k_{t't} \qquad \boldsymbol{s}_k(\boldsymbol{r}, \boldsymbol{r}') = \sum_{ss'tt'} \hat{\rho}(\boldsymbol{r}st, \boldsymbol{r}'s't) \hat{\boldsymbol{\sigma}}_{s's} \hat{\tau}^k_{t't}$

k=0,1,2,3 ($\tau = \delta$ for k=0)

$$\hat{\rho}(\boldsymbol{r}st, \boldsymbol{r}'s't') = \frac{1}{4}\rho_0(\boldsymbol{r}, \boldsymbol{r}')\delta_{ss'}\delta_{tt'} + \frac{1}{4}\vec{\rho}(\boldsymbol{r}, \boldsymbol{r}') \circ \vec{\tau}_{tt'}\delta_{ss'} + \frac{1}{4}\boldsymbol{s}_0(\boldsymbol{r}, \boldsymbol{r}') \cdot \hat{\boldsymbol{\sigma}}_{ss'}\delta_{tt'} + \frac{1}{4}\vec{\boldsymbol{s}}(\boldsymbol{r}, \boldsymbol{r}') \cdot \hat{\boldsymbol{\sigma}}_{ss'} \circ \vec{\tau}_{tt'}$$

Example: delta interaction

$$\hat{v}(\boldsymbol{r}_{12}) = t_0 \hat{\delta}_\sigma \delta(\boldsymbol{r}_{12})$$

$$\sum_{s_1 s_1' s_2 s_2' t_1 t_1' t_2 t_2'} \int d\boldsymbol{r}_1 d\boldsymbol{r}_2 d\boldsymbol{r}_1' d\boldsymbol{r}_2' V(\boldsymbol{r}_1' s_1' t_1', \boldsymbol{r}_2' s_2' t_2', \boldsymbol{r}_1 s_1 t_1, \boldsymbol{r}_2 s_2 t_2) c^\dagger_{\boldsymbol{r}_1' s_1' t_1'} c^\dagger_{\boldsymbol{r}_2' s_2' t_2'} c_{\boldsymbol{r}_1 s_1 t_1} c_{\boldsymbol{r}_2 s_2 t_2}$$

$$V(\boldsymbol{r}_1' s_1' t_1', \boldsymbol{r}_2' s_2' t_2', \boldsymbol{r}_1 s_1 t_1, \boldsymbol{r}_2 s_2 t_2) = \hat{v}(\boldsymbol{r}_{12}, s_1' t_1' s_2' t_2' s_1 t_1 s_2 t_2) \times \left[\hat{\delta}_\sigma \hat{\delta}_\tau \delta(\boldsymbol{r}_1 - \boldsymbol{r}_1')\delta(\boldsymbol{r}_2 - \boldsymbol{r}_2') - \hat{P}_\sigma \hat{P}_\tau \hat{P}_M \delta(\boldsymbol{r}_1 - \boldsymbol{r}_2')\delta(\boldsymbol{r}_2 - \boldsymbol{r}_1')\right]$$

$$E = \frac{t_0}{4} \int d\boldsymbol{r} \sum_{s_1 s_2 s_1' s_2'} \sum_{t_1 t_2 t_1' t_2'} [\hat{\delta}_\sigma \hat{\delta}_\tau - \hat{P}_\sigma \hat{P}_\tau] \langle c^\dagger_{\boldsymbol{r}s_1' t_1'} c^\dagger_{\boldsymbol{r}s_2' t_2'} c_{\boldsymbol{r}s_2 t_2} c_{\boldsymbol{r}s_1 t_1} \rangle$$

$$= \hat{\rho}(\boldsymbol{r}s_1 t_1, \boldsymbol{r}s_1' t_1') \hat{\rho}(\boldsymbol{r}s_2 t_2, \boldsymbol{r}s_2' t_2') - \hat{\rho}(\boldsymbol{r}s_1 t_1, \boldsymbol{r}s_2' t_2') \hat{\rho}(\boldsymbol{r}s_2 t_2, \boldsymbol{r}s_1' t_1')$$

$$= \frac{t_0}{2} \int d\boldsymbol{r} \sum_{ss'tt'} [\hat{\rho}(\boldsymbol{r}st, \boldsymbol{r}st)\hat{\rho}(\boldsymbol{r}s't', \boldsymbol{r}s't') - \hat{\rho}(\boldsymbol{r}st, \boldsymbol{r}s't')\hat{\rho}(\boldsymbol{r}s't', \boldsymbol{r}st)]$$

$$= t_0 \int d\boldsymbol{r} \left[\frac{3}{8}\rho_0^2(\boldsymbol{r}) - \frac{1}{8}\vec{\rho}^{\,2}(\boldsymbol{r}) - \frac{1}{8}\boldsymbol{s}_0^2(\boldsymbol{r}) - \frac{1}{8}\vec{\boldsymbol{s}}^{\,2}(\boldsymbol{r})\right]$$

ρ : local density, s: local spin density

$\rho_k(\boldsymbol{r}) = \rho_k(\boldsymbol{r}, \boldsymbol{r})$ time-even density

$\boldsymbol{s}_k(\boldsymbol{r}) = \boldsymbol{s}_k(\boldsymbol{r}, \boldsymbol{r})$ time-odd density

Time-odd coupling constants

A part of the EDF that involves time-odd density (turned on only in the system without time-reversal symmetry) - excited states, odd-mass nuclei
coupling constants are independent for Skyrme EDF, not well constrained compared to time-even ones
(local gauge invariance connects them with time-even coupling constants)

isoscalar EDF $\quad \chi_0^{\mathrm{odd}}(\boldsymbol{r}) = C_0^s[\rho_0]\boldsymbol{s}_0^2 + C_0^{\Delta s}\boldsymbol{s}_0\cdot\Delta\boldsymbol{s}_0 + C_0^T\boldsymbol{s}_0\cdot\boldsymbol{T}_0 + C_0^j\boldsymbol{j}_0^2 + C_0^{\nabla j}\boldsymbol{s}_0\cdot(\boldsymbol{\nabla}\times\boldsymbol{j}_0) + C_0^{\nabla s}(\boldsymbol{\nabla}\cdot\boldsymbol{s}_0)^2 + C_0^F\boldsymbol{s}_0\cdot\boldsymbol{F}_0$

isovector EDF $\quad \chi_1^{\mathrm{odd}}(\boldsymbol{r}) = \boxed{C_1^s[\rho_0]\vec{\boldsymbol{s}}^2 + C_1^{\Delta s}\vec{\boldsymbol{s}}\cdot\circ\Delta\vec{\boldsymbol{s}}} + \boxed{C_1^T\vec{\boldsymbol{s}}\cdot\circ\vec{\boldsymbol{T}}} + \boxed{C_1^j\vec{\boldsymbol{j}}^2} + \boxed{C_1^{\nabla j}\vec{\boldsymbol{s}}\cdot\circ(\boldsymbol{\nabla}\times\vec{\boldsymbol{j}})} + \boxed{C_1^{\nabla s}(\boldsymbol{\nabla}\cdot\vec{\boldsymbol{s}})^2 + C_1^F\vec{\boldsymbol{s}}\cdot\circ\vec{\boldsymbol{F}}}$

(related to charge-exchange)

$\rho\ \tau$: determined from GDR

spin-current origin terms

spin-orbit origin term

isovector EDF: three isovector components (k=1,2,3, denoted as vector with "→")
k=1, 2: np mixed components (<n+p>, <p+n>), k=3 neutron-proton part (<n+n-p+p>)

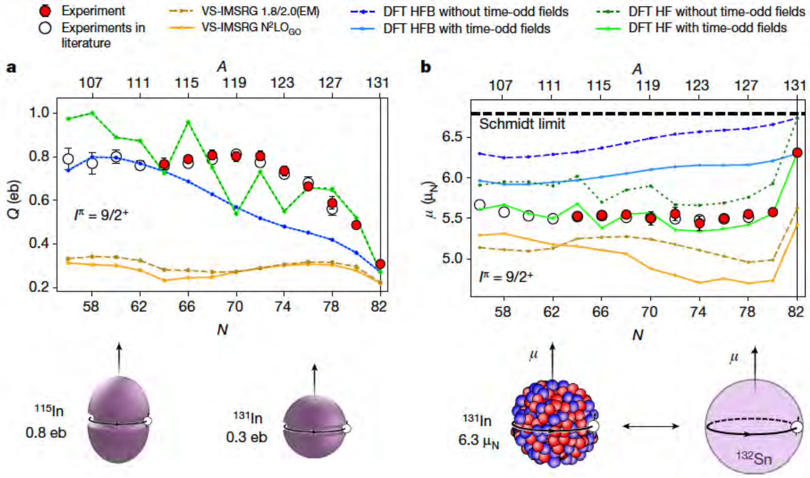
odd-mass ground states(that breaks time-reversal symm.)
excited states of even and odd mass nuclei

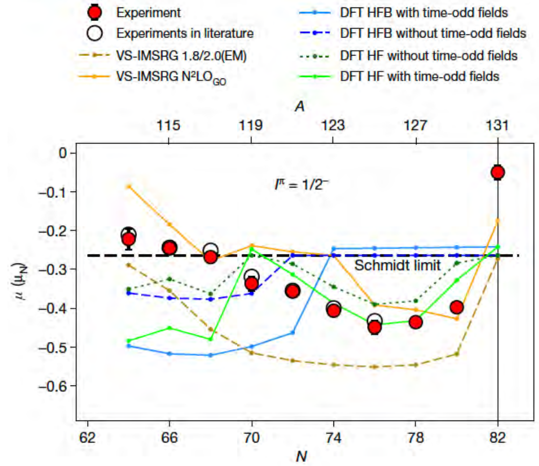
charge-exchange residual interactions (treated within pnQRPA)
neutron-proton mixed ground states (np-pair condensed states)

Time-odd coupling constants

magnetic moment

Vernon et al., Nature **607**, 260 (2022)





Landau-Migdal interaction

$$\tilde{F}(\boldsymbol{k}_1\sigma_1\tau_1\sigma_1'\tau_1';\boldsymbol{k}_2\sigma_2\tau_2\sigma_2'\tau_2') = \frac{\delta^2\mathcal{E}}{\delta\tilde{\rho}(\boldsymbol{k}_1\sigma_1\tau_1\sigma_1'\tau_1')\delta\tilde{\rho}(\boldsymbol{k}_2\sigma_2\tau_2\sigma_2'\tau_2')}$$

(Bender et al. Phys. Rev. C **65**, 054322(2002))

$$= \tilde{f}(\boldsymbol{k}_1,\boldsymbol{k}_2) + \tilde{f}'(\boldsymbol{k}_1,\boldsymbol{k}_2)\vec{\hat{\tau}}_1\circ\vec{\hat{\tau}}_2 + \tilde{g}(\boldsymbol{k}_1,\boldsymbol{k}_2)\hat{\boldsymbol{\sigma}}_1\cdot\hat{\boldsymbol{\sigma}}_2 + \tilde{g}'(\boldsymbol{k}_1,\boldsymbol{k}_2)(\hat{\boldsymbol{\sigma}}_1\cdot\hat{\boldsymbol{\sigma}}_2)(\vec{\hat{\tau}}_1\circ\vec{\hat{\tau}}_2)$$

$$\tilde{g}_0 = \frac{2m^*k_F}{\pi^2\hbar^2}(2C_0^s + 2C_0^T\left(\frac{3\pi^2}{2}\right)^{2/3}\rho_0^{2/3})$$

$$\tilde{g}_0' = \frac{2m^*k_F}{\pi^2\hbar^2}(2C_1^s + 2C_1^T\left(\frac{3\pi^2}{2}\right)^{2/3}\rho_0^{2/3})$$

g_0=0.4, g_0'=1.7 Bonnard et al., Sassarini et al. J. Phys. G **49**, 11LT01(2022)

isovector spin-spin coupling constants (C_1^S, g_0') is important for describing magnetic moment
time-odd coupling constants are poorly constrained

Quasiparticle random-phase approximation (QRPA)

QRPA: Microscopic theory for (the ground and) excited states of nuclei based on the nuclear DFT

QRPA equation (non-Hermitian eigenvalue problem)

$$\begin{pmatrix} A & B \\ B^* & A^* \end{pmatrix} \begin{pmatrix} X^i \\ Y^i \end{pmatrix} = \Omega_i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} X^i \\ Y^i \end{pmatrix}$$

A: Hermitian matrix
B: symmetric matrix
Ω: excitation energy
(X,Y): wave functions

dimension: 10^5-10^6 full diagonalization is computationally demanding

i-th excited state:
(final state in beta decay)

$$|i\rangle = \hat{Q}_i^\dagger |0\rangle \qquad \hat{Q}_i^\dagger = \sum_{\pi\nu} X^i_{\pi\nu} \hat{a}^\dagger_\pi \hat{a}^\dagger_\nu - Y^i_{\pi\nu} \hat{a}_\nu \hat{a}_\pi$$

beta decay: we need transition strength for eigenstates below Q-value

$$\langle i|\hat{F}|0\rangle \approx \langle \mathrm{HFB}| \left[\hat{Q}_i, \hat{F}\right] |\mathrm{HFB}\rangle = \sum_{\pi\nu} (X^i_{\pi\nu} F^{20}_{\pi\nu} + Y^i_{\pi\nu} F^{02}_{\pi\nu})$$

F: decay operators

Solutions of Matrix QRPA:

basis reduction: truncation in the two-quasiparticle space (standard)
Lanczos method (Johnson, et al. Comp. Phys. Commun. **120**, 155 (1999)
iterative Arnoldi method (Toivanen et al. Phys. Rev. C **81**, 034312 (2010))

deformed QRPA calculation for beta-decay is computationally too time-consuming

Finite-amplitude method

Iterative solution of the QRPA

Nakatsukasa et al., Phys. Rev. C **76**, 024318 (2007)

Apply a weak time-dependent external one-body field

$$\hat{F}(t) = \eta \left(\hat{F} e^{-i\omega t} + \hat{F}^\dagger e^{i\omega t} \right)$$

F: one-body operator
η: small number
ω: frequency of the external field (parameter)

$$\hat{F} = \sum_{\mu<\nu} F^{20}_{\mu\nu} \alpha^\dagger_\mu \alpha^\dagger_\nu + F^{02}_{\mu\nu} \alpha_\nu \alpha_\mu$$

time-dependent quasiparticles $\alpha^\dagger_\mu(t) = \sum_k \left[U_{k\mu}(t) c^\dagger_k + V_{k\mu}(t) c_k \right]$ $\alpha_\mu(t) = \{\alpha_\mu + \delta\alpha_\mu(t)\} e^{iE_\mu t}$

infinitesimal displacement from the HFB quasiparticles $\delta\alpha_\mu(t) = \eta \sum_\nu \alpha^\dagger_\nu \left[X_{\nu\mu}(\omega) e^{-i\omega t} + Y^*_{\nu\mu}(\omega) e^{i\omega t} \right]$

time propagation (TDHFB equation)

$$\delta \langle \Phi(t) | \hat{H} - i\hbar \frac{\partial}{\partial t} | \Phi(t) \rangle = 0$$

$$\hat{H}(t) = \hat{H}_0 + \eta \left[\delta\hat{H}(\omega) e^{-i\omega t} + \delta\hat{H}^\dagger(\omega) e^{i\omega t} \right]$$

$$\delta\hat{H}(\omega) = \sum_{\mu<\nu} \left[\delta H^{20}_{\mu\nu}(\omega) \alpha^\dagger_\mu \alpha^\dagger_\nu + \delta H^{02}_{\mu\nu}(\omega) \alpha_\nu \alpha_\mu \right]$$

$$\Rightarrow \quad i \frac{\partial \alpha_\mu(t)}{\partial t} = [\hat{H}(t) + \hat{F}(t), \alpha_\mu(t)]$$

$\Downarrow$

extract first-order terms in η

linear response equation (FAM)

$$\left[\begin{pmatrix} A & B \\ B^* & A^* \end{pmatrix} - \omega \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right] \begin{pmatrix} X(\omega) \\ Y(\omega) \end{pmatrix} = - \begin{pmatrix} F^{20} \\ F^{02} \end{pmatrix}$$

Note: X(ω)、Y(ω) are not the QRPA solutions (eigenvectors)
QRPA is obtained as a small-amplitude limit of the TDHFB (η<<1)

Finite-amplitude method

Nakatsukasa, Inakura, Yabana, Phys. Rev. C **76**, 024318 (2007)
Avogadro and Nakatsukasa, Phys. Rev. C **84**, 014314 (2011)

linear response equations:
linear equation of X and Y

$$\left[\begin{pmatrix} A & B \\ B^* & A^* \end{pmatrix} - \omega \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right] \begin{pmatrix} X(\omega) \\ Y(\omega) \end{pmatrix} = - \begin{pmatrix} F^{20} \\ F^{02} \end{pmatrix}$$

ω and F are parameters

Solution 1: solve it as a simultaneous linear equations of X and Y (two-qp dim)

no need to diagonalize QRPA matrix (A,B)
need the matrix elements of A and B

Solution 2: iteration

displacement of one-body mean-field Hamiltonian

$$(E_\mu + E_\nu - \omega) X_{\mu\nu}(\omega) + \delta H^{20}_{\mu\nu}(\omega) = -F^{20}_{\mu\nu}$$
$$(E_\mu + E_\nu + \omega) Y_{\mu\nu}(\omega) + \delta H^{02}_{\mu\nu}(\omega) = -F^{02}_{\mu\nu}$$

displacement of quasiparticles

$\delta H^{20}(\omega), \delta H^{02}(\omega)$

$X(\omega), Y(\omega)$

$$\delta H^{20}_{\mu\nu}(\omega) = \sum_{\mu'<\nu'} [A_{\mu\nu\mu'\nu'} - (E_\mu + E_\nu)\delta_{\mu\mu'}\delta_{\nu\nu'}] X_{\mu'\nu'}(\omega) + B_{\mu\nu\mu'\nu'} Y_{\mu'\nu'}(\omega)$$
$$\delta H^{02}_{\mu\nu}(\omega) = \sum_{\mu'<\nu'} [A^*_{\mu\nu\mu'\nu'} - (E_\mu + E_\nu)\delta_{\mu\mu'}\delta_{\nu\nu'}] Y_{\mu'\nu'}(\omega) + B^*_{\mu\nu\mu'\nu'} X_{\mu'\nu'}(\omega)$$

need the A and B matrix elements

Finite-amplitude method

Solution 3: FAM (iteration)

Nakatsukasa, Inakura, Yabana, Phys. Rev. C **76**, 024318 (2007)
Avogadro and Nakatsukasa, Phys. Rev. C **84**, 014314 (2011)

$$(E_\mu + E_\nu - \omega) X_{\mu\nu}(\omega) + \delta H^{20}_{\mu\nu}(\omega) = -F^{20}_{\mu\nu}$$
$$(E_\mu + E_\nu + \omega) Y_{\mu\nu}(\omega) + \delta H^{02}_{\mu\nu}(\omega) = -F^{02}_{\mu\nu}$$

displacement of quasiparticles

$X(\omega), Y(\omega)$

displacement of one-body mean-field Hamiltonian

$\delta H^{20}(\omega), \delta H^{02}(\omega)$

$$\delta H^{20}_{\mu\nu}(\omega) = \sum_{\mu'<\nu'} [A_{\mu\nu\mu'\nu'} - (E_\mu + E_\nu)\delta_{\mu\mu'}\delta_{\nu\nu'}] X_{\mu'\nu'}(\omega) + B_{\mu\nu\mu'\nu'} Y_{\mu'\nu'}(\omega)$$
$$\delta H^{02}_{\mu\nu}(\omega) = \sum_{\mu'<\nu'} [A^*_{\mu\nu\mu'\nu'} - (E_\mu + E_\nu)\delta_{\mu\mu'}\delta_{\nu\nu'}] Y_{\mu'\nu'}(\omega) + B^*_{\mu\nu\mu'\nu'} X_{\mu'\nu'}(\omega)$$

A and B matrices are not necessary, but one-body displacement AX+BY and A*Y+B*X (vectors) are

$$A_{\rho\sigma,\mu\nu} = \delta_{\rho\mu}\delta_{\sigma\nu}(E_\mu + E_\nu) + \frac{\partial^2 \mathcal{E}'}{\partial \overline{\kappa}^*_{\rho\sigma} \partial \overline{\kappa}_{\mu\nu}},$$
$$B_{\rho\sigma,\mu\nu} = \frac{\partial^2 \mathcal{E}'}{\partial \overline{\kappa}^*_{\rho\sigma} \partial \overline{\kappa}^*_{\mu\nu}}.$$

$$A, B \approx \frac{\delta^2 E}{\delta \mathcal{R}^2} = \frac{\delta \mathcal{H}}{\delta \mathcal{R}}$$

A,B matrices:
second functional derivative of EDF
(first functional derivative of one-body potential)

displacement of the densities (δR)

$$\delta\rho(\omega) = U X(\omega) V^T + V^* Y^T(\omega) U^\dagger$$
$$\delta\kappa^{(+)}(\omega) = U X(\omega) U^T + V^* Y^T(\omega) V^\dagger$$
$$\delta\kappa^{(-)}(\omega) = V^* X^\dagger(\omega) V^\dagger + U Y^*(\omega) U^T$$

displacement of the one-body potential (δH)

$$\delta h(\omega) = (h[\rho + \eta\delta\rho] - h[\rho])/\eta$$

$$\delta\Delta^{(\pm)}(\omega) = (\Delta[\rho + \eta\delta\rho, \kappa + \eta\delta\kappa^{(\pm)}, \kappa^* + \eta\delta\kappa^{(\mp)*}] - \Delta[\rho, \kappa, \kappa^*])/\eta$$

calculating A and B matrices avoided!

Finite-amplitude method

Nakatsukasa et al., Phys. Rev. C **76**, 024318 (2007)

TDHFB (TDDFT) equation

time-dependent external field ($e^{i\omega t}$, $e^{-i\omega t}$) with frequency ω

$$\boxed{i\frac{\partial \hat{a}_\mu(t)}{\partial t}} = [\boxed{\hat{H}(t)} + \boxed{\hat{F}(t)}, \hat{a}_\mu(t)]$$

displacement of the quasiparticles

$$\hat{a}_\mu(t) = \{\hat{a}_\mu + \delta\hat{a}_\mu(t)\}\, e^{iE_\mu t}$$

$$\delta\hat{a}_\mu(t) = \eta \sum_\nu \hat{a}^\dagger_\nu (X_{\nu\mu}(\omega) e^{-i\omega t} + Y^*_{\nu\mu}(\omega) e^{i\omega t})$$

two-body calc(N^2)

induced field (δH)

$$\hat{H}(t) = \hat{H}_0 + \delta\hat{H}(t)$$

$$\delta\hat{H}(t) = \eta\left\{\delta\hat{H}(\omega) e^{-i\omega t} + \delta\hat{H}^\dagger(\omega) e^{i\omega t}\right\}$$

one-body calc (N)

displacement of densities

$$\rho(t) = \rho_0 + \delta\rho(t)$$

$$\kappa(t) = \kappa_0 + \delta\kappa(t)$$

one-body calc (N)

$h[\rho]$, $\Delta[\rho, \kappa]$

- iterative solution of QRPA
- one-body induced field is calculated through one-body density displacement
- no need to evaluate A and B matrices
- one-body field subroutine for the mean-field calculation can be reused with minor extension

FAM amplitudes and QRPA eigenvectors

NH, Kortelainen, Nazarewicz, Phys. Rev. C **87**, 054309 (2013)

X(ω), Y(ω): FAM amplitudes

$$X_{\mu\nu}(\omega) = -\sum_i \left\{ \frac{X^i_{\mu\nu}\langle i|\hat{F}|0\rangle}{\Omega_i - \omega} + \frac{Y^{i*}_{\mu\nu}\langle 0|\hat{F}|i\rangle}{\Omega_i + \omega} \right\}$$

X^i, Y^i: QRPA eigenvectors
Ω_i: QRPA eigenvalues

$$Y_{\mu\nu}(\omega) = -\sum_i \left\{ \frac{Y^i_{\mu\nu}\langle i|\hat{F}|0\rangle}{\Omega_i - \omega} + \frac{X^{i*}_{\mu\nu}\langle 0|\hat{F}|i\rangle}{\Omega_i + \omega} \right\}$$

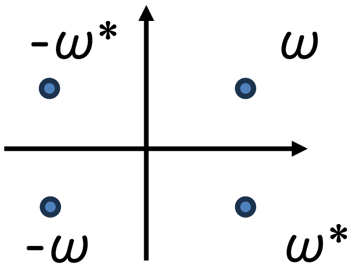
$$S(\hat{F},\omega) = -\sum_i \left[\frac{|\langle i|\hat{F}|0\rangle|^2}{\Omega_i - \omega} + \frac{|\langle 0|\hat{F}|i\rangle|^2}{\Omega_i + \omega} \right]$$

FAM amplitudes are expressed as linear combinations of QRPA eigenvectors

symmetry of QRPA solutions : if ω is an eigenvalue, ω^*, $-\omega$, $-\omega^*$ are also eigenvalues
(Nakada, PTEP **2016**, 063D02(2016))

$$X(\hat{F},-\omega^*) = \left[Y(\hat{F}^\dagger,\omega)\right]^* \qquad X(\hat{F},\omega^*) = \left[X(\hat{F}^\dagger,\omega)\right]^*$$

$$Y(\hat{F},-\omega^*) = \left[X(\hat{F}^\dagger,\omega)\right]^* \qquad Y(\hat{F},\omega^*) = \left[Y(\hat{F}^\dagger,\omega)\right]^*$$



In the case of FAM : if F is a Hermitian operator and XY of QRPA are real
FAM amplitudes at ω contains info on other three points

Note that the external field in charge-exchange process is not Hermitian

Applications of like-particle FAM

Kortelainen, NH, Nazarewicz, Phys. Rev. C **91**, 051302(R) (2015)
Oishi, Kortelainen, NH, Phys. Rev. C **93**, 034329 (2016)

Giant resonance strength distribution

strength function
$$S(\hat{F},\omega) = \sum_{\mu<\nu} F^{20}_{\mu\nu} X_{\mu\nu}(\omega) + F^{02}_{\mu\nu} Y_{\mu\nu}(\omega)$$
$$= -\sum_i \left[\frac{|\langle i|\hat{F}|0\rangle|^2}{\Omega_i - \omega} + \frac{|\langle 0|\hat{F}|i\rangle|^2}{\Omega_i + \omega} \right]$$

Imaginary part of the strength function gives Lorentzian-smeared strength distribution

quadrupole and octupole giant resonances in ^{240}Pu

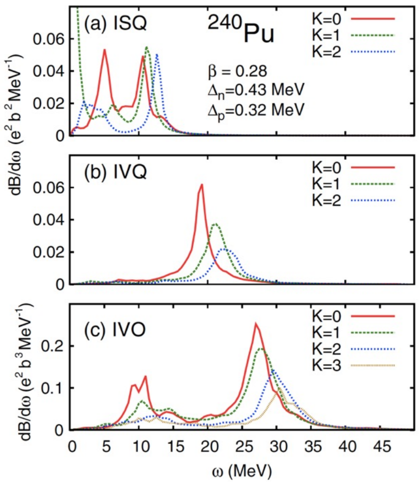
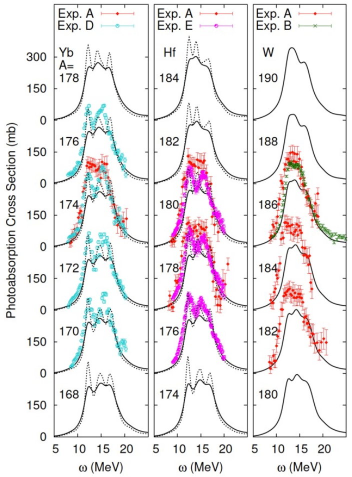


photo absorption cross section in rare-earth region



Low-energy collective excitation

$$X_{\mu\nu}(\omega) = -\sum_i \left\{ \frac{X^i_{\mu\nu}\langle i|\hat{F}|0\rangle}{\Omega_i - \omega} + \frac{Y^{i*}_{\mu\nu}\langle 0|\hat{F}|i\rangle}{\Omega_i + \omega} \right\}$$
$$\frac{1}{2\pi i}\oint_{C_i} X_{\mu\nu}(\omega) d\omega = e^{i\theta}|\langle i|\hat{F}|0\rangle| X^i_{\mu\nu}$$

FAM XY have first-order poles at QRPA energies with QRPA XY as residues

- contour integration around (each) peak
- numerous FAM calculations for searching low-energy peaks
- applications to low-lying states, fission and vibrational inertia

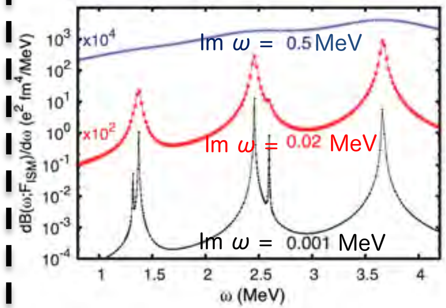


TABLE I. Lowest octupole QRPA modes in ^{154}Sm predicted in our deformed FAM calculations. Shown are the energy ω_1; the IVO transition strength $|\langle 0|f^{IV,+}_{L=3,K}|1\rangle|^2$; and the corresponding $B(E3)$ value. The transition probabilities were computed through the QRPA amplitudes (referred to as FAM-C in Ref. [27]).

K	ω_1 (MeV)	$|\langle 0|f^{IV,+}_{L=3,K}|1\rangle|^2$ ($e^2 fm^6$)	$B(E3)$ (W.u.)
0	0.80	1.57	8.02
1	1.12	26.20	2.75
2	2.40	0.73	2.36
3	2.49	0.47	0.04

proton-neutron FAM

Mustonen et al., Phys. Rev. C **90**, 024308 (2014)

Charge-exchange mode in time-dependent DFT framework

external field $\hat{F}(t) = \eta(\hat{F}e^{-i\omega t} + \hat{F}^\dagger e^{i\omega t})$ $\hat{F} = \sum_{\pi\nu} f_{\pi\nu}\hat{c}_\pi^\dagger \hat{c}_\nu$ (for example, τ-, $\sigma\tau$-)

quasiparticle oscillation in time (neutron-proton mixing)

$$\delta\hat{a}_\nu = \eta\sum_\pi \hat{a}_\pi^\dagger[X_{\pi\nu}(\omega)e^{-i\omega t} + Y^*_{\pi\nu}(\omega)e^{i\omega t}]$$

density oscillation

$$\begin{aligned}
\delta\rho(\omega) =& UX(\omega)V^T + V^*Y^T(\omega)U^\dagger \\
\delta\kappa^{(+)}(\omega) =& UX(\omega)U^T + V^*Y^T(\omega)V^\dagger \\
\delta\kappa^{(-)}(\omega) =& V^*X^\dagger(\omega)V^\dagger + UY^*(\omega)U^T
\end{aligned}$$

$\delta\rho_{\pi\nu}(\omega), \delta\rho_{\nu\pi}(\omega)\cdots$

induced field

$$\delta h(\omega) =(h[\rho+\eta\delta\rho] - h[\rho])/\eta$$

$$\delta\Delta^{(\pm)}(\omega) =(\Delta[\rho+\eta\delta\rho, \kappa+\eta\delta\kappa^{(\pm)}, \kappa^*+\eta\delta\kappa^{(\mp)*}] - \Delta[\rho,\kappa,\kappa^*])/\eta$$

$$h[\rho_0, \vec{\rho}] = C_0^\rho\rho_0 + C_0^s\boldsymbol{s}_0\cdot\hat{\boldsymbol{\sigma}} + C_1^\rho\vec{\rho} + C_1^s\vec{\boldsymbol{s}}\cdot\hat{\boldsymbol{\sigma}} + \cdots$$

$$\delta h(\omega) = C_1^\rho(\delta\rho_1 + \delta\rho_2) + C_1^s(\delta\boldsymbol{s}_1 + \delta\boldsymbol{s}_2)\cdot\hat{\boldsymbol{\sigma}} + \cdots$$

pn-induced field comes from the pn parts (k=1, 2) that are not present in the HFB potentials
induced field does not depend on η

FAM for Gamow-Teller resonance and beta decay

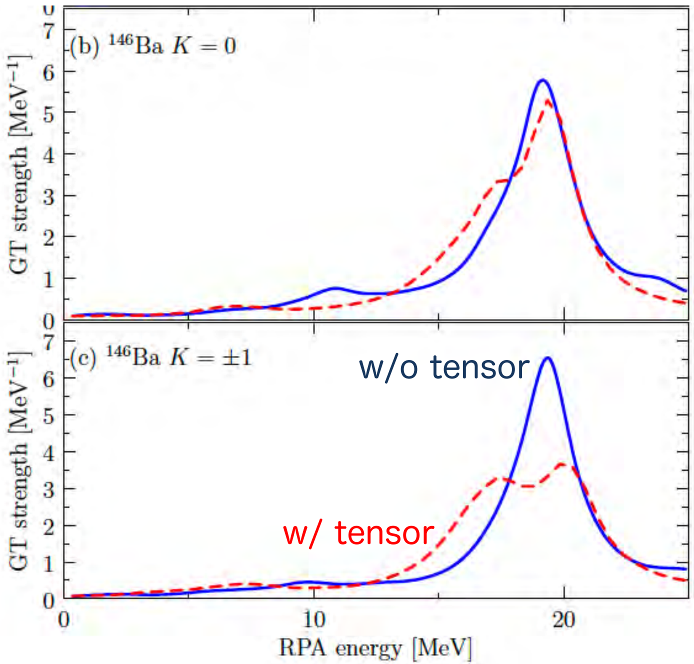
strength function $S(\hat{F},\omega) = \sum_{\mu<\nu} F^{20}_{\mu\nu} X_{\mu\nu}(\omega) + F^{02}_{\mu\nu} Y_{\mu\nu}(\omega)$

Mustonen et al., Phys. Rev. C **90**, 024308 (2014)

(X, Y: FAM amplitudes)

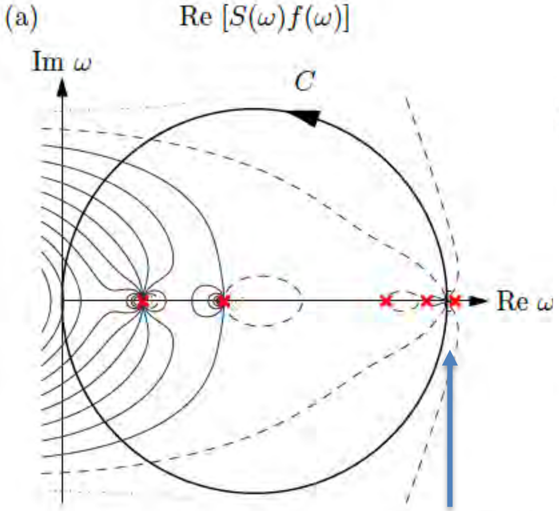
$$\frac{dB}{d\omega}(\hat{F},\omega) = -\frac{1}{\pi}\mathrm{Im}S(\hat{F},\omega) = \frac{\gamma}{\pi}\sum_\nu \left\{ \frac{|\langle\nu|\hat{F}|0\rangle|^2}{(\Omega_i-\omega)^2+\gamma^2} - \frac{|\langle 0|\hat{F}|\nu\rangle|^2}{(\Omega_i+\omega)^2+\gamma^2} \right\}$$

Gamow-Teller resonance



beta decay rate

$$B_n(F) = |\langle n|F|0\rangle|^2 = \mathrm{Res}[S(F),\Omega_n],$$



$$\begin{aligned}\lambda_{1+} &= \frac{\ln 2}{\kappa}\sum_n f(\Omega_n)B_n^{(\mathrm{GT})} \\ &\approx \frac{\ln 2}{\kappa}\sum_n f_{\mathrm{poly}}(\Omega_n)\,\mathrm{Res}[S(\sigma\tau_-),\Omega_n] \\ &= \frac{\ln 2}{\kappa}\sum_n \mathrm{Res}[f_{\mathrm{poly}}S(\sigma\tau_-),\Omega_n] \\ &= \frac{\ln 2}{\kappa}\frac{1}{2\pi i}\oint_C d\omega\, f_{\mathrm{poly}}(\omega)S(\sigma\tau_-;\omega),\end{aligned}$$

$$\omega_{\max} = Q + E_{\mathrm{g.s.}} = \lambda_n - \lambda_p + \Delta M_{\text{n-H}}.$$

Global fit of the pnEDF

Mustonen and Engel, Phys. Rev. C **93**, 014304 (2016)

$$C_1^s \boldsymbol{s}_3^2 + C_1^{\Delta s} \boldsymbol{s}_3 \cdot \Delta \boldsymbol{s}_3 + C_1^T \boldsymbol{s}_3 \cdot \boldsymbol{T}_3 + C_1^j \boldsymbol{j}_3^2 + C_1^{\nabla j} \boldsymbol{s}_3 \cdot (\boldsymbol{\nabla} \times \boldsymbol{j}_3) + C_1^{\nabla s} (\boldsymbol{\nabla} \cdot \boldsymbol{s}_3)^2 + C_1^F \boldsymbol{s}_3 \cdot \boldsymbol{F}_3$$

Set	GT resonances	SD resonances	β-decay half-lives
A	^{208}Pb, ^{112}Sn, ^{76}Ge, ^{130}Te, ^{90}Zr, ^{48}Ca	None	^{48}Ar, ^{60}Cr, ^{72}Ni, ^{82}Zn, ^{92}Kr, ^{102}Sr, ^{114}Ru, ^{126}Cd, ^{134}Sn, ^{148}Ba
B	Same as A	None	^{52}Ti, ^{74}Zn, ^{92}Sr, ^{114}Pd, ^{134}Te, ^{156}Sm, ^{180}Yb, ^{200}Pt, ^{226}Rn, ^{242}U
C	Same as A	None	^{52}Ti, ^{72}Ni, ^{92}Sr, ^{114}Ru, ^{134}Te, ^{156}Nd, ^{180}Yb, ^{204}Pt, ^{226}Rn, ^{242}U
D	Those of A and ^{150}Nd	None	^{58}Ti, ^{78}Zn, ^{98}Kr, ^{126}Cd, ^{152}Ce, ^{166}Gd, ^{204}Pt
E	Same as D	^{90}Zr, ^{208}Pb	^{58}Ti, ^{78}Zn, ^{98}Kr, ^{126}Cd, ^{152}Ce, ^{166}Gd, ^{226}Rn

Fit	Starting point	Target set	Q values	fitted parameters
1A	SkO′	A	Comp.	$V_0 = -173.176$, $C_1^s = 128.279$
1B	SkO′	B	Comp.	$V_0 = -176.614$, $C_1^s = 133.038$
1C	SkO′	C	Comp.	$V_0 = -176.097$, $C_1^s = 126.966$
1D	SkO′	E	Comp.	$V_0 = -209.384$, $C_1^s = 129.297$
1E	SkO′	E	Exp.	$V_0 = -159.397$, $C_1^s = 99.8479$
2	SV-min	D	Comp.	$V_0 = -165.567$, $C_1^s = 132.271$
3A	SkO′	E	Comp.	$V_0 = -195.174$, $C_1^s = 144.833$, $C_1^T = -20.1618$, $C_1^F = -10.3125$
3B	SkO′	E	Exp.	$V_0 = -165.158$, $C_1^s = 120.27$, $C_1^T = -17.7435$, $C_1^F = -17.9902$
4	Fit 3A	E	Comp.	$C_1^j = 54.5$, $C_1^{\nabla j} = -78.7965$, $C_1^{\nabla s} = -87.5$
5	SkO′	E	Comp.	$V_0 = -191.875$, $C_1^s = 146.182$, $C_1^j = -86.4276$

Global fit of the pnEDF

TABLE III. The Jacobian matrix, evaluated at the result of the two-parameter fit 1E. All parameters except for the strength of isoscalar pairing are expressed in natural units. The strength of isoscalar pairing has been scaled by the strength of isovector pairing. The derivatives of the $\log_{10}t$ values are hence dimensionless and those of the resonance energies are in the units of MeV.

$\mathcal{O}$	$d\mathcal{O}/dC_1^s$	$d\mathcal{O}/dV_0$	$d\mathcal{O}/dC_1^F$	$d\mathcal{O}/dC_1^T$	$d\mathcal{O}/dC_1^{\nabla s}$	$d\mathcal{O}/dC_1^{\Delta s}$	$d\mathcal{O}/dC_1^j$	$d\mathcal{O}/dC_1^{\nabla j}$
^{208}Pb $E_{\rm GTR}$	57.261	−0.000	2.434	5.869	0.429	−1.002	0.000	0.143
^{112}Sn $E_{\rm GTR}$	29.498	−1.032	1.432	2.863	0.286	−0.573	0.000	0.000
^{76}Ge $E_{\rm GTR}$	45.115	−7.225	2.004	4.295	0.429	−1.145	0.000	0.000
^{130}Te $E_{\rm GTR}$	53.790	−3.096	2.434	5.297	0.429	−1.002	0.143	0.000
^{90}Zr $E_{\rm GTR}$	29.498	−1.032	1.288	2.720	0.429	−1.002	−0.143	0.143
^{48}Ca $E_{\rm GTR}$	32.968	−0.000	1.432	3.149	0.573	−1.288	0.000	0.000
^{208}Pb $E_{\rm SDR}$	52.055	−0.000	2.291	4.008	0.286	−1.575	−0.143	−0.143
^{90}Zr $E_{\rm SDR}$	29.498	−0.000	1.575	2.004	0.286	−1.432	−0.286	−0.143
^{58}Ti $\log_{10}t$	4.749	−4.318	0.203	0.445	0.045	−0.109	−0.011	−0.002
^{78}Zn $\log_{10}t$	6.889	−2.922	0.256	0.589	0.164	−0.382	0.253	−0.025
^{98}Kr $\log_{10}t$	5.410	−3.252	0.265	0.559	0.050	−0.116	−0.012	−0.003
^{126}Cd $\log_{10}t$	5.583	−4.641	0.252	0.496	0.017	−0.050	0.001	0.007
^{152}Ce $\log_{10}t$	5.409	−2.474	0.293	0.540	0.051	−0.120	0.003	−0.009
^{166}Gd $\log_{10}t$	5.081	−2.924	0.250	0.497	0.035	−0.132	−0.007	−0.010
^{204}Pt $\log_{10}t$	3.755	−3.340	−0.015	0.160	−0.018	−0.316	−0.076	0.026

- beta decay rates mainly correlates with V_0 (isoscalar proton-neutron pairing)
- Gamow-Teller resonance energies correlate with C_1^s spin-spin coupling constants
- weak correlations with other coupling constants

Extension to odd-mass nuclei

Shafer et al., Phys. Rev. C **94**, 055802 (2016)
Ney et al., Phys. Rev. C **102**, 034326 (2020)

odd nuclei: existence of unpaired nucleon:

exact blocking (assume one q.p. level(Λ) is fully occupied)

$$|\Phi_\Lambda\rangle = \hat{\alpha}^\dagger_\Lambda|\Phi\rangle$$

$$\rho_{kk'} = (V^*V^T)_{kk'} + U_{k\Lambda}U^*_{k'\Lambda} - V^*_{k\Lambda}V_{k'\Lambda}$$
$$\kappa_{kk'} = (V^*U^T)_{kk'} + U_{k\Lambda}V^*_{k'\Lambda} - V^*_{k\Lambda}U_{k'\Lambda}$$

equal filling approximation(preserving time-reversal symmetry)

$$\hat{\alpha}^\dagger_\Lambda|\Phi\rangle \qquad \hat{\alpha}^\dagger_{\bar{\Lambda}}|\Phi\rangle$$

$$\rho^{\rm EFA}_{kk'} = (V^*V^T)_{kk'} + \frac{1}{2}(U_{k\Lambda}U^*_{k'\Lambda} + U_{k\bar{\Lambda}}U^*_{k'\bar{\Lambda}} - V^*_{k\Lambda}V_{k'\Lambda} - V^*_{k\bar{\Lambda}}V_{k'\bar{\Lambda}})$$
$$\kappa^{\rm EFA}_{kk'} = (V^*U^T)_{kk'} + \frac{1}{2}(U_{k\Lambda}V^*_{k'\Lambda} + U_{k\bar{\Lambda}}V^*_{k'\bar{\Lambda}} - V^*_{k\Lambda}U_{k'\Lambda} - V^*_{k\bar{\Lambda}}U_{k'\bar{\Lambda}})$$

extension of density operator to an ensemble

$$\hat{\mathcal{D}} = |\Phi\rangle\langle\Phi| + \sum_\mu \hat{\alpha}^\dagger_\mu|\Phi\rangle p_\mu\langle\Phi|\hat{\alpha}_\mu + \frac{1}{2!}\sum_{\mu\nu}\hat{\alpha}^\dagger_\mu\hat{\alpha}^\dagger_\nu|\Phi\rangle p_\mu p_\nu\langle\Phi|\hat{\alpha}_\nu\hat{\alpha}_\mu + \cdots \qquad p_\mu = \begin{cases}1, & \mu\in[\Lambda,\bar{\Lambda}]\\ 0, & \text{otherwise}\end{cases} \qquad \langle\hat{A}\rangle = \frac{\mathrm{Tr}[\hat{\mathcal{D}}\hat{A}]}{\mathrm{T}[\hat{\mathcal{D}}]}$$

$$\langle\hat{O}\rangle_{\rm o-e} = \frac{1}{2}\left(\langle\Phi|\hat{\alpha}_\Lambda\hat{O}\hat{\alpha}^\dagger_\Lambda|\Phi\rangle + \langle\Phi|\hat{\alpha}_{\bar{\Lambda}}\hat{O}\hat{\alpha}^\dagger_{\bar{\Lambda}}|\Phi\rangle\right)$$
$$\langle\hat{O}\rangle_{\rm o-o} = \frac{1}{2}\left(\langle\Phi|\hat{\alpha}_{\Lambda_\nu}\hat{\alpha}_{\Lambda_\pi}\hat{O}\hat{\alpha}^\dagger_{\Lambda_\pi}\hat{\alpha}^\dagger_{\Lambda_\nu}|\Phi\rangle + \langle\Phi|\hat{\alpha}_{\bar{\Lambda}_\nu}\hat{\alpha}_{\bar{\Lambda}_\pi}\hat{O}\hat{\alpha}^\dagger_{\bar{\Lambda}_\pi}\hat{\alpha}^\dagger_{\bar{\Lambda}_\nu}|\Phi\rangle\right)$$

$$\mathcal{R}^{\rm EFA} = \begin{pmatrix}\langle\hat{\alpha}^\dagger\hat{\alpha}\rangle & \langle\hat{\alpha}\hat{\alpha}\rangle\\ \langle\hat{\alpha}^\dagger\hat{\alpha}^\dagger\rangle & \langle\hat{\alpha}\hat{\alpha}^\dagger\rangle\end{pmatrix} = \begin{pmatrix}f & 0\\ 0 & 1-f\end{pmatrix} \qquad f_\mu = \begin{cases}\frac{1}{2}, & \mu\in[\Lambda,\bar{\Lambda}],\\ 0, & \text{otherwise}\end{cases}$$

$$\delta\mathcal{R}(t) = \begin{pmatrix}P_{\pi\nu}(t) & X_{\pi\nu}(t)\\ -X^*_{\pi\nu}(t) & -P^*_{\pi\nu}(t)\end{pmatrix}$$

$$P_{\pi\nu}(t) = P_{\pi\nu}(\omega)e^{-i\omega t} + Q^*_{\pi\nu}(\omega)e^{i\omega t}$$
$$X_{\pi\nu}(t) = X_{\pi\nu}(\omega)e^{-i\omega t} + Y^*_{\pi\nu}(\omega)e^{i\omega t}$$

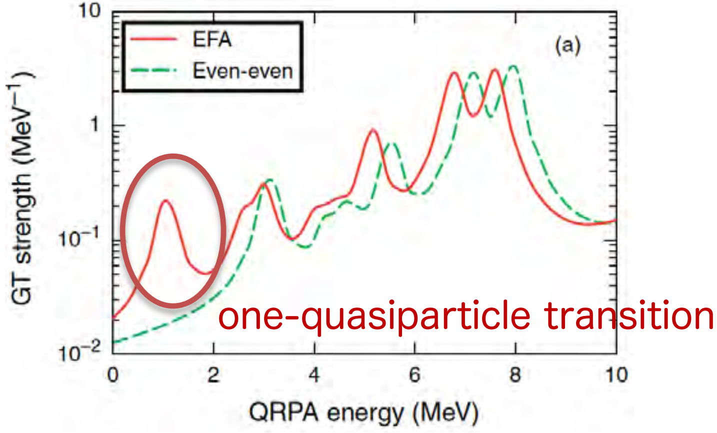
$$X_{\pi\nu}(\omega)[(E_\pi+E_\nu)-\omega] = -(1-f_\nu-f_\pi)[\delta H^{20}_{\pi\nu}(\omega) + F^{20}_{\pi\nu}]$$
$$Y_{\pi\nu}(\omega)[(E_\pi+E_\nu)+\omega] = -(1-f_\nu-f_\pi)[\delta H^{02}_{\pi\nu}(\omega) + F^{02}_{\pi\nu}]$$
$$P_{\pi\nu}(\omega)[(E_\pi-E_\nu)-\omega] = -(f_\nu-f_\pi)[\delta H^{11}_{\pi\nu}(\omega) + F^{11}_{\pi\nu}]$$
$$Q_{\pi\nu}(\omega)[(E_\pi-E_\nu)+\omega] = -(f_\nu-f_\pi)[\delta H^{\overline{11}}_{\pi\nu}(\omega) + F^{\overline{11}}_{\pi\nu}]$$

$$S(F;\omega) = \sum_{\pi\nu}[F^{20*}_{\pi\nu}X_{\pi\nu}(\omega) + F^{20*}_{\pi\nu}Y_{\pi\nu}(\omega) + F^{11*}_{\pi\nu}P_{\pi\nu}(\omega) + F^{\overline{11}*}_{\pi\nu}Q_{\pi\nu}(\omega)]$$

Extension to odd-mass nuclei

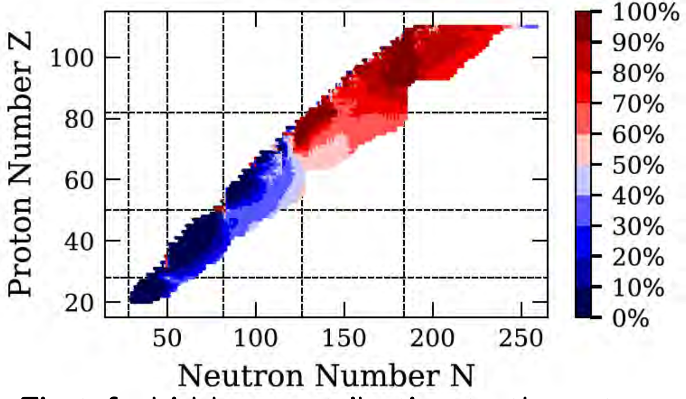
Shafer et al., Phys. Rev. C **94**, 055802 (2016)

^{71}Ga (Z=31, SV-min)

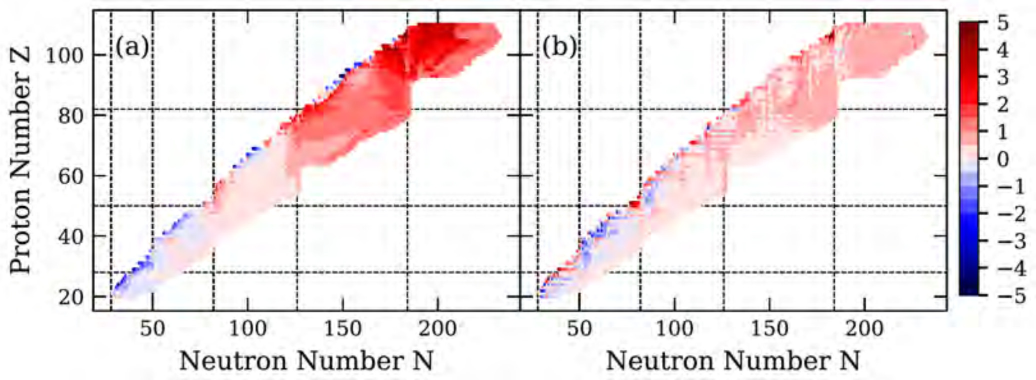


even-even: constrained on averaged number

Ney et al., Phys. Rev. C **102**, 034326 (2020)



First-forbidden contribution to the rates



(a) ratio to Marketin et al., Phys. Rev. C **93**, 025805 (2016) (spherical)
(b) ratio to Moller et al., Phys. Rev. C **67**, 055802 (2003) (FRDM-QRPA)

Quasiparticle-phonon coupling

Liu et al., Phys. Rev. C **109**, 044308 (2024)

QRPA: includes coupling to two-quasiparticle states (Landau damping)
no coupling to more complex configurations (multi-quasiparticles) (spreading width)
in the current HO basis no two-quasiparticle states coupling to continuum (escaping width)
QPVC(quasiparticle vibration coupling): coupling with phonon, a part of spreading width included
GTR: Niu et al., Phys. Rev. C **94**, 064328 (2016), beta decay: Niu et al Phys. Lett. B **780**, 325 (2018)

$$|M\rangle = Q_M^\dagger |0\rangle \qquad Q_M^\dagger = \sum_{\pi\nu} \left(X_{\pi\nu}^M \alpha_\pi^\dagger \alpha_\nu^\dagger - Y_{\pi\nu}^M \alpha_\nu \alpha_\pi + \sum_N \tilde{X}_{\pi\nu N}^M \alpha_\pi^\dagger \alpha_\nu^\dagger Q_N^\dagger - \tilde{Y}_{\pi\nu N}^M Q_N \alpha_\nu \alpha_\pi \right).$$

charge-changing
QRPA level amplitude

beyond QRPA amplitudes
N-th like-particle phonon

In the FAM equations

$$X_{\pi\nu}(\omega) = -\frac{\delta H_{\pi\nu}^{20} + [\tilde{W}(\omega)X(\omega)]_{\pi\nu} + F_{\pi\nu}^{20}}{\varepsilon_\pi + \varepsilon_\nu - \omega}, \qquad [\tilde{W}(\omega)X(\omega)]_{\pi\nu} = \sum_{\pi'\nu'} \tilde{W}_{\pi\nu,\pi'\nu'}(\omega) X_{\pi'\nu'}(\omega),$$

$$Y_{\pi\nu}(\omega) = -\frac{\delta H_{\pi\nu}^{02} + [\tilde{W}^*(-\omega)Y(\omega)]_{\pi\nu} + F_{\pi\nu}^{02}}{\varepsilon_\pi + \varepsilon_\nu + \omega}, \qquad [\tilde{W}^*(-\omega)Y(\omega)]_{\pi\nu} = \sum_{\pi'\nu'} \tilde{W}_{\pi\nu,\pi'\nu'}^*(-\omega) Y_{\pi'\nu'}(\omega).$$

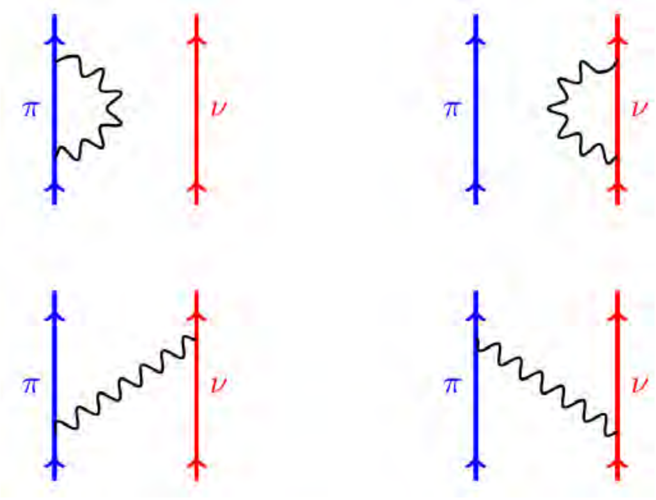
spreading matrix $\tilde{W}(\omega) = W(\omega) - W(0)$ (subtraction procedure. reduces to QRPA at ω=0)

V. I. Tselyaev, Phys. Rev. C **88**, 054301 (2013).

Quasiparticle-phonon coupling

Liu et al., Phys. Rev. C **109**, 044308 (2024)

$$
\begin{aligned}
W_{\pi\nu,\pi'\nu'}(\omega) = \sum_N \Bigg\{ & \sum_{\pi_1} \langle \pi | H | \pi_1, N \rangle \frac{1}{\omega - [\omega_N + (\varepsilon_{\pi_1} + \varepsilon_\nu)]} \langle \pi' | H | \pi_1, N \rangle^* \delta_{\nu'\nu} \\
& + \sum_{\nu_1} \langle \nu | H | \nu_1, N \rangle \frac{1}{\omega - [\omega_N + (\varepsilon_\pi + \varepsilon_{\nu_1})]} \langle \nu' | H | \nu_1, N \rangle^* \delta_{\pi'\pi} + \langle \pi | H | \pi', N \rangle \frac{1}{\omega - [\omega_N + (\varepsilon_{\pi'} + \varepsilon_\nu)]} \langle \nu' | H | \nu, N \rangle^* \\
& + \langle \nu | H | \nu', N \rangle \frac{1}{\omega - [\boxed{\omega_N} + (\varepsilon_\pi + \varepsilon_{\nu'})]} \boxed{\langle \pi' | H | \pi, N \rangle^*} \Bigg\}. \qquad (12)
\end{aligned}
$$



ω_N: phonon energy

<π'|H|π, N>: coupling with proton quasiparticle

Phonon (like-particle QRPA)

Liu et al., Phys. Rev. C **109**, 044308 (2024)

Operator to excite like-particle phonon (isoscalar/isovector multipole operators)

$$G = \sum_{pp'} G_{pp'} a_p^\dagger a_{p'} + \sum_{nn'} G_{nn'} a_n^\dagger a_{n'},$$

$$G_{LK}^{T=0} = \sum_i r_i^L Y_{LK}(\theta_i, \varphi_i), \quad G_{LK}^{T=1} = \sum_i r_i^L Y_{LK}(\theta_i, \varphi_i) \tau_z(i),$$

Coupling with quasiparticles

$$\langle \beta | H | \beta_1, N \rangle = i \lim_{\Delta \to 0} \frac{\Delta}{\langle N | G | 0 \rangle} \delta H_{\beta\beta_1}^{11} (\Omega_N + i\Delta).$$

strength

$$S_G(\omega) = -\sum_N \left(\frac{|\langle N | G | 0 \rangle|^2}{\omega_N - \omega} - \frac{|\langle N | G^\dagger | 0 \rangle|^2}{\omega_N + \omega} \right)$$

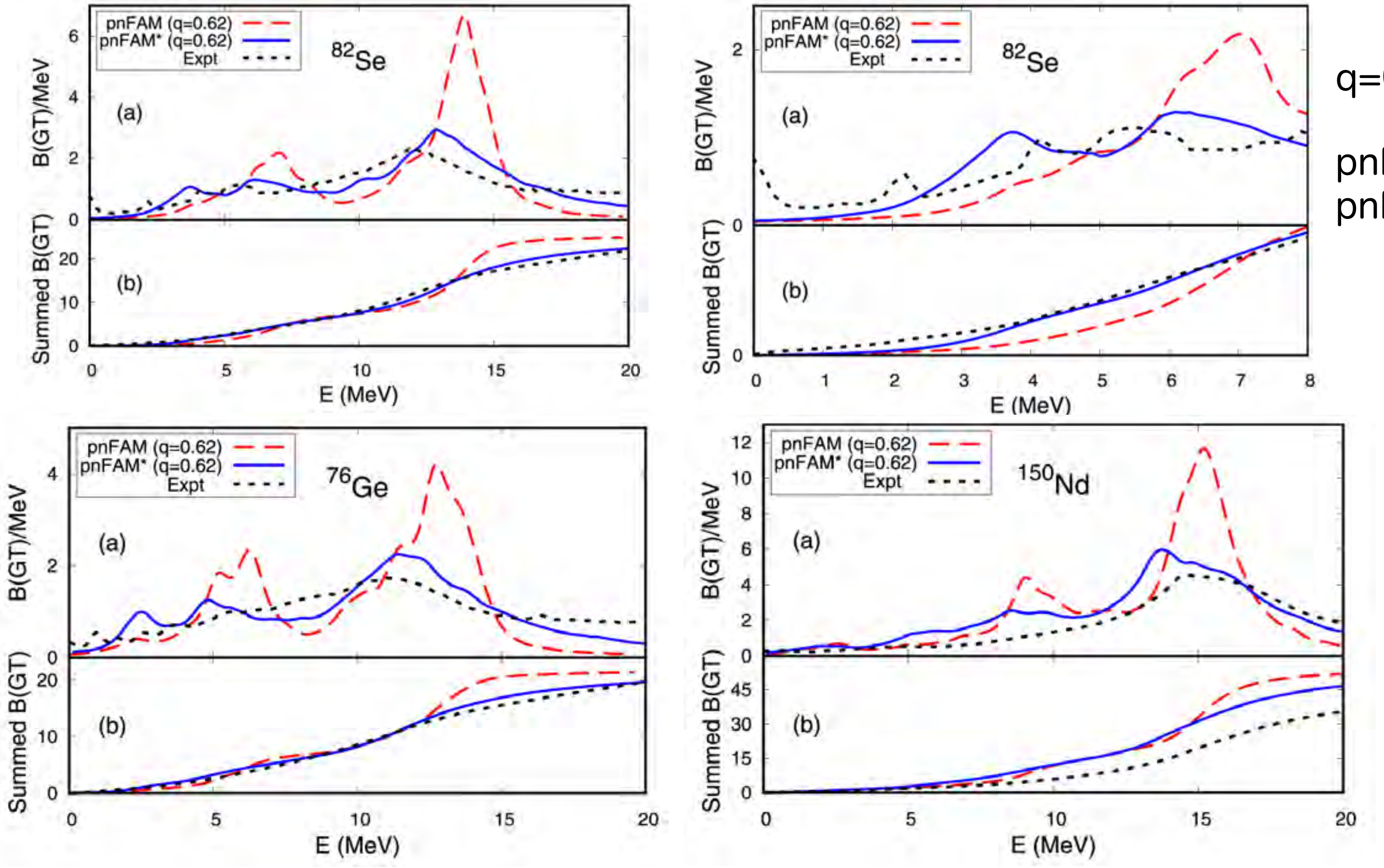
$$\langle N | G | 0 \rangle = \lim_{\Delta \to 0} \sqrt{i \Delta S_G(\omega_N + i\Delta)}.$$

Like-particle phonon search

- FAM with Im ω = 0.5 MeV, 40 points in the range Re ω = 0 - 20 MeV
- for each T=0, 1, L(Lmax=6), K operator
- typical number of phonons: 150 (^{82}Se)

Gamow-Teller resonance

Liu et al., Phys. Rev. C **109**, 044308 (2024)



q=0.62 ~ g_A=1.0

pnFAM: pnQRPA
pnFAM*: pnQRPA+QPVC

Convergence

How many phonons do we need to take into account?

Liu et al., Phys. Rev. C **109**, 044308 (2024)

$$v_N \equiv \frac{\langle V \rangle_N}{\omega_N}.$$

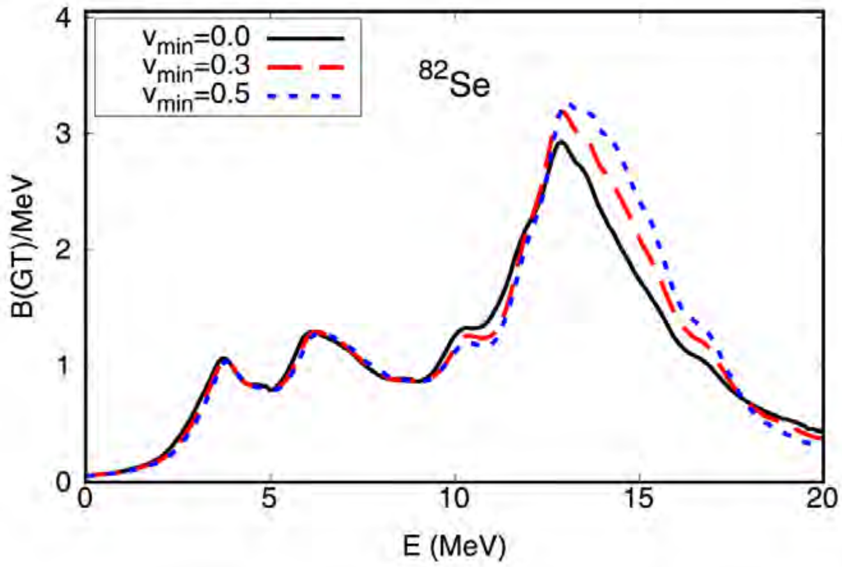
v_n large: collective

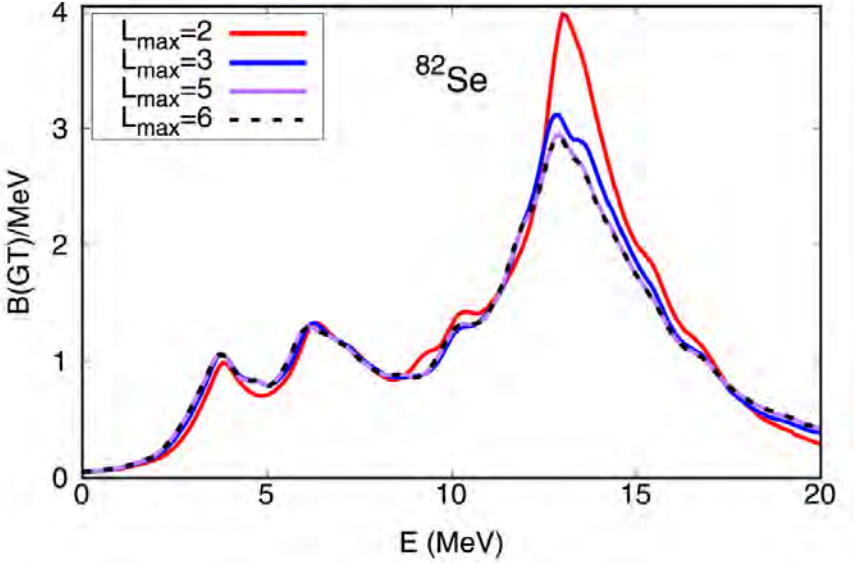
$$v_N = 1 - \frac{1}{\omega_N}\sum_{\alpha\beta}(\varepsilon_\alpha + \varepsilon_\beta)\left(\left|\mathcal{X}^N_{\alpha\beta}\right|^2 + \left|\mathcal{Y}^N_{\alpha\beta}\right|^2\right).$$

FAM amplitudes

$$\mathcal{X}^N_{\alpha\beta} = -i \lim_{\Delta\to 0} \frac{\Delta}{\langle N|G|0\rangle} \boxed{\mathcal{X}_{\alpha\beta}(\omega_N + i\Delta)},$$

$$\mathcal{Y}^N_{\alpha\beta} = -i \lim_{\Delta\to 0} \frac{\Delta}{\langle N|G|0\rangle} \boxed{\mathcal{Y}_{\alpha\beta}(\omega_N + i\Delta)}.$$





- Phonons with small vn contributes to GTR, less contributes to low-lying states
- v_{min}=0.0: 150 phonons, v_{min}=0.3: 62 phonons, v_{min}=0.5: 35 phonons

Beta decay

Liu et al., Phys. Rev. C **109**, 044308 (2024)

- isoscalar pn pairing was globally fitted to beta decay half-life within pnQRPA
  Mustonen and Engel, Phys. Rev. C **93**, 014304 (2016)

TABLE I. Deformation parameter, experimental half-life in seconds, pnFAM half-life with the Skyrme functional SGII, pnFAM* half-life with the same functional, and pnFAM* half-life when phonons with $v_N < 0.3$ are excluded, for 11 deformed isotopes.

Isotope	β	$t_{1/2}^{\mathrm{Exp.}}$(s)	$t_{1/2}^{\mathrm{pnFAM}}$(s)	$t_{1/2}^{\mathrm{pnFAM}^*}$(s)	$t_{1/2}^{\prime\,\mathrm{pnFAM}^*}$(s)
^{78}Zn	0.12	1.47	408	3.77	4.80
^{168}Gd	0.31	3.03	381	37.1	39.7
^{152}Ce	0.29	1.40	93.1	19.0	20.3
^{156}Nd	0.32	5.49	470	53.5	59.2
^{164}Sm	0.33	1.42	142	17.2	18.5
^{154}Ce	0.30	0.30	19.2	7.26	7.89
^{112}Mo	−0.18	0.15	1.92	2.47	2.31
^{94}Kr	−0.22	0.21	1.48	3.23	3.01
^{112}Ru	−0.21	1.75	93	27.0	31.0
^{106}Mo	−0.20	8.73	62.8	38.0	49.6
^{96}Sr	−0.21	1.07	23.8	20.0	25.9

- qp vibration coupling increases allowed beta decay rate (same as isoscalar pairing)
- isoscalar pairing (~strength similar to isovector) reduces
  40% of pnFAM half-life, 20% of pnFAM* half-life (^{112}Ru)
  45% (pnFAM) and 30% (pnFAM*) for ^{95}Sr

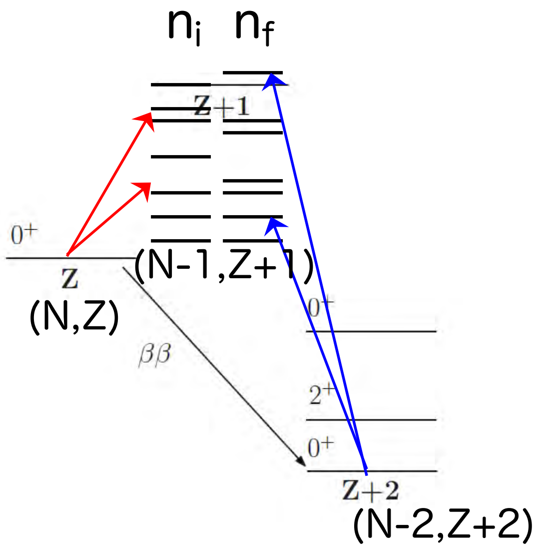
2vββ Double-beta decay

NH, Engel, Phys. Rev. C **105**, 044314 (2022)

two modes : neutrinoless mode (0vββ) and two-neutrino mode (2vββ)

2vββ half-life $$[T^{2\nu}_{1/2}]^{-1} = G_{2\nu}(Q_{\beta\beta}, Z)|M^{2\nu}|^2$$

phase space factor   nuclear matrix element (NME)



NME

$$M^{2\nu}_{\mathrm{GT}} = \sum_n \frac{\langle 0^+_f|\sum_a \boldsymbol{\sigma}_a\tau^-_a|n\rangle \cdot \langle n|\sum_b \boldsymbol{\sigma}_b\tau^-_b|0^+_i\rangle}{E_n - (M_i + M_f)/2}$$

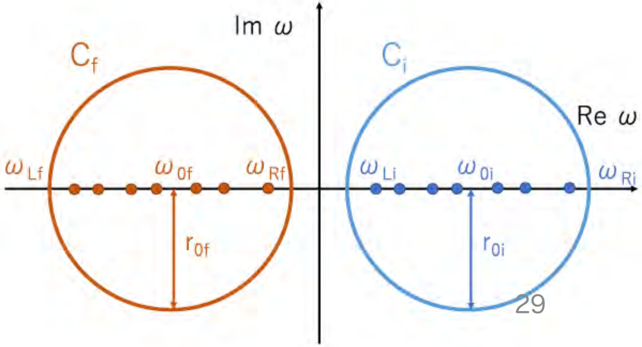
NME can be evaluated by combining two (pn)QRPA

FAM with two contour integrations

Two FAM amplitudes with Gamow-Teller external fields calculated from initial state (ω_i) and final state (ω_f) are combine

$$\mathcal{T}(\alpha;\omega_i,\hat{F}^{T_i}_{K_i};\omega_f,\hat{F}^{T_f}_{K_f}) \equiv \sum_{pn}[\bar{Y}^{(f)}_{pn}(\omega_f,\hat{F}^{T_f}_{K_f})\bar{X}^{(i)}_{pn}(\omega_i,\hat{F}^{T_i}_{K_i}) \; -\alpha\bar{X}^{(f)}_{pn}(\omega_f,\hat{F}^{T_f}_{K_f})\bar{Y}^{(i)}_{pn}(\omega_i,\hat{F}^{T_i}_{K_i})],$$

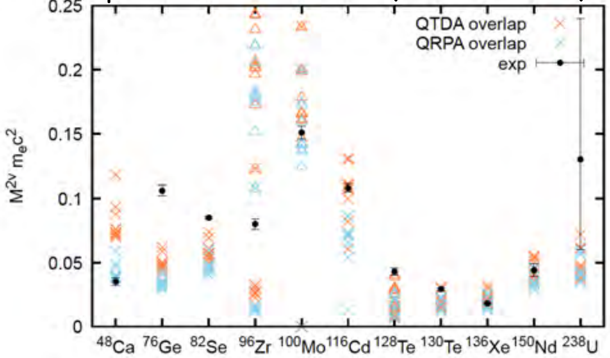
$$M^{2\nu}_{\mathrm{GT}} = -\sum_{K=-1}^{1}(-1)^K\left(\frac{1}{2\pi i}\right)^2 \oint_{C_i} d\omega_i \oint_{C_f} d\omega_f \mathcal{T}(\alpha,\omega_i,\hat{F}^{T_i}_{K_i};\omega_f,\hat{F}^{T_f}_{K_f})\frac{2}{\omega_i-\omega_f}$$



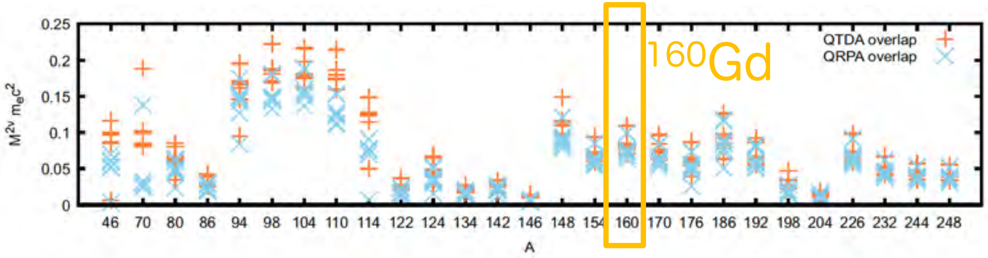
Double-beta decay ($2\nu\beta\beta$)

- First systematic calculation of $2\nu\beta\beta$ NME using DFT based QRPA NH and Engel, Phys. Rev. C **105**, 044314 (2022)
- Calculation with 10 EDF parameters globally fitted to (single) β-decay half-lives
  Mustonen and Engel, Phys. Rev. C **93**, 014304 (2016)

Comparison with data (11 nuclides)

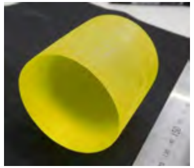


Prediction (27 nuclides)



PIKACHU experiment ($2\nu\beta\beta$ measurement of ^{160}Gd using GAGG crystal)

(Pure Inorganic scintillator experiment in KAmioka for CHallenging Underground sciences)



- Previous measurement in Ukraine: $2\nu\beta\beta$ half-life > 2.1×10^{19} y
  Danevich et al., Nucl. Phys. A **694**, 375 (2001)
- expected $2\nu\beta\beta$ half-life with previous NME： 6×10^{21} y
  Hirsch et al., Phys. Rev. C **66**, 015502 (2002)
- half-life expected from nuclear DFT-QRPA: 8×10^{20} y
- Recent publication: Omori et al., NIM-A **1082**, 171023 (2026)

PIKACHU Collaboration
Takashi Iida (Tsukuba) et al.

Reduced basis method for nuclear DFT response

NH, Zhang, Engel, in preparation

- For QPVC calculations, low-energy like-particle QRPA solutions (with all the excitation energies and amplitudes) in a low-energy range (0-20 MeV) need to be calculated for coupling Hamiltonian and for evaluating v_N
- For 0vββ calculations, (almost all) QRPA solutions will be necessary

conventional method of QRPA: matrix diagonalization (dimension~10^6)

$$\begin{pmatrix} A & B \\ B^* & A^* \end{pmatrix}\begin{pmatrix} X^i \\ Y^i \end{pmatrix} = \Omega_i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} X^i \\ Y^i \end{pmatrix}$$

X_i, Y_i : eigenvectors
Ω_i: eigenvalues

Finite-amplitude method (FAM): iteration of 10^6-dimensional vectors

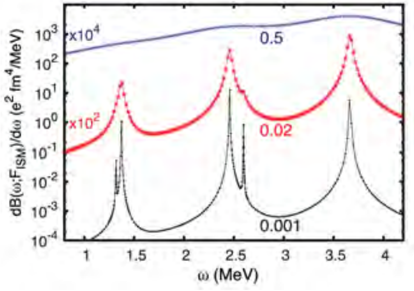
$$\left[\begin{pmatrix} A & B \\ B^* & A^* \end{pmatrix} - \omega \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right]\begin{pmatrix} X(\omega) \\ Y(\omega) \end{pmatrix} = -\begin{pmatrix} F^{20} \\ F^{02} \end{pmatrix}$$

ω: complex energy parameter
F: given external field
AX+BY B^*X+A^*Y: given as a function of X and Y

Low-energy solutions based on FAM: contour integration

$$\frac{1}{2\pi i}\oint_{C_i} X_{\mu\nu}(\omega)d\omega = e^{i\theta}|\langle i|\hat{F}|0\rangle|X^i_{\mu\nu}$$

we need to identify the location of the poles
contour integration for each pole (eigenvalues)



NH et al., Phys. Rev. C **87**, 064309 (2013)

FAM amplitudes and QRPA eigenvectors

NH, Kortelainen, Nazarewicz, Phys. Rev. C **87**, 054309 (2013)

X(ω), Y(ω): FAM amplitudes

$$X_{\mu\nu}(\omega) = -\sum_i \left\{ \frac{X^i_{\mu\nu}\langle i|\hat{F}|0\rangle}{\Omega_i - \omega} + \frac{Y^{i*}_{\mu\nu}\langle 0|\hat{F}|i\rangle}{\Omega_i + \omega} \right\}$$

X^i, Y^i: QRPA eigenvectors
Ω_i: QRPA eigenvalues

$$Y_{\mu\nu}(\omega) = -\sum_i \left\{ \frac{Y^i_{\mu\nu}\langle i|\hat{F}|0\rangle}{\Omega_i - \omega} + \frac{X^{i*}_{\mu\nu}\langle 0|\hat{F}|i\rangle}{\Omega_i + \omega} \right\}$$

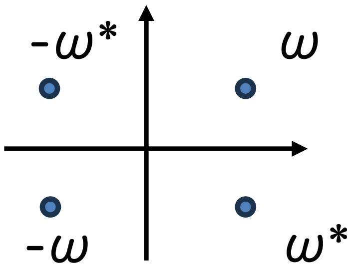
$$S(\hat{F},\omega) = -\sum_i \left[\frac{|\langle i|\hat{F}|0\rangle|^2}{\Omega_i - \omega} + \frac{|\langle 0|\hat{F}|i\rangle|^2}{\Omega_i + \omega} \right]$$

FAM amplitudes are expressed as linear combinations of QRPA eigenvectors

symmetry of QRPA solutions：if ω is an eigenvalue, ω^*, $-\omega$, $-\omega^*$ are also eigenvalues
(Nakada, PTEP **2016**, 063D02(2016))

$$X(\hat{F},-\omega^*) = \left[Y(\hat{F}^\dagger,\omega)\right]^* \qquad X(\hat{F},\omega^*) = \left[X(\hat{F}^\dagger,\omega)\right]^*$$

$$Y(\hat{F},-\omega^*) = \left[X(\hat{F}^\dagger,\omega)\right]^* \qquad Y(\hat{F},\omega^*) = \left[Y(\hat{F}^\dagger,\omega)\right]^*$$



In the case of FAM：if F is a Hermitian operator and XY of QRPA are real
FAM amplitudes at ω contains info on other three points

Note that the external field in charge-exchange process is not Hermitian

Reduced basis method for the FAM

Reduced basis method (RBM)

Review: Drischler et al., Front. Phys. **10**, 1092931 (2023)
Duguet et al., Rev. Mod. Phys. **96**, 031002 (2024)

emulates the solution of large-dimensional problem containing parameters

$$H(\boldsymbol{\theta})|\psi(\boldsymbol{\theta})\rangle = E(\boldsymbol{\theta})|\psi(\boldsymbol{\theta})\rangle \qquad \langle\psi(\boldsymbol{\theta})|\psi(\boldsymbol{\theta})\rangle = 1$$

θ : parameter (ex. interaction strength)

- solution space: constructed with numerical solutions obtained with small sets of θ
- All the solutions at arbitrary values of θ is assumed to be expressed in the solution space
- Weight is determined by variation (n_b dimensional)

$$\{|\psi_i\rangle \equiv |\psi(\boldsymbol{\theta}_i)\rangle\}_{i=1}^{n_b}$$

$$|\tilde{\psi}(\boldsymbol{\theta})\rangle = \sum_{i=1}^{n_b} \beta_i(\boldsymbol{\theta})|\psi_i\rangle \equiv X\vec{\beta}$$

$$X = [|\psi_1\rangle, |\psi_2\rangle, \cdots, |\psi_{n_b}\rangle]$$

RBM for FAM

complex energy ω is regarded as a parameter

$$\begin{pmatrix} X_{\mu\nu}(\omega) \\ Y_{\mu\nu}(\omega) \end{pmatrix} = \sum_{k=1}^{n} a_k(\omega) \begin{pmatrix} X_{\mu\nu}(\omega_k) \\ Y_{\mu\nu}(\omega_k) \end{pmatrix} + b_k(\omega) \begin{pmatrix} Y^*_{\mu\nu}(\omega_k) \\ X^*_{\mu\nu}(\omega_k) \end{pmatrix}$$

FAM solution at $-\omega_k^*$

N: norm kernel (overlap)
H: Hamiltonian kernel

determines a(ω) and b(ω) with Rayleigh-Ritz variational principle

$$\sum_k (\mathcal{H}_{jk} - \omega\mathcal{N}_{jk}) \begin{pmatrix} a_k(\omega) \\ b_k(\omega) \end{pmatrix} = -\begin{pmatrix} S_j^* \\ S_j'^* \end{pmatrix} = -\boldsymbol{S}^*$$

$$S_j = S(\hat{F}, \omega_j)$$
$$S'_j = S(\hat{F}^\dagger, \omega_j)$$

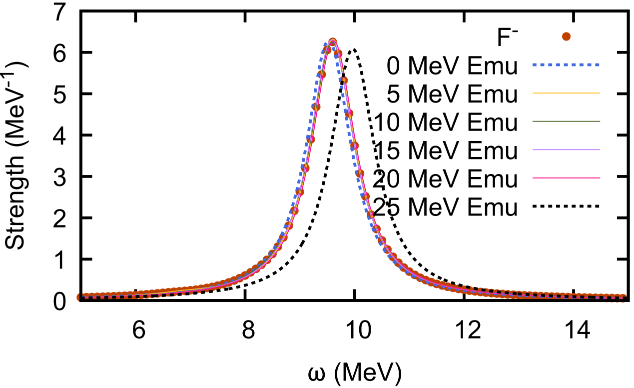
strength at RBM basis energy

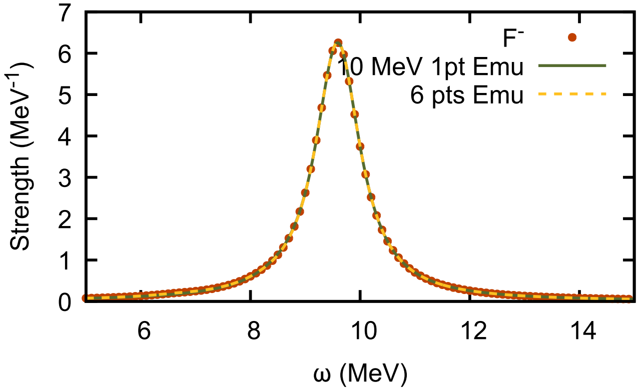
- Essentially the same as Hill-Wheeler equation in GCM (diagonalization of H-ωN)
- Norm kernel is not positive definite, and collective Hamiltonian is not Hermitian

RBM emulator for pnFAM

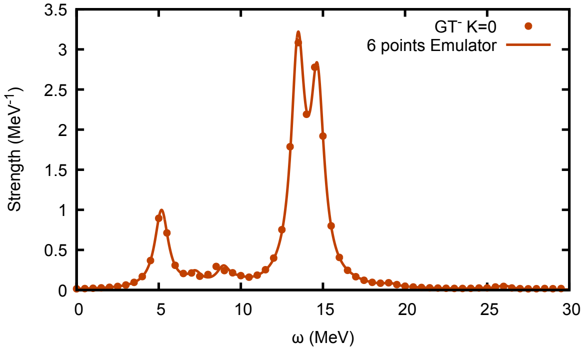
pnFAM : Mustonen et al. Phys. Rev. C **90**, 024308 (2014)
EDF: SkO' + 1B(pn channel) Mustonen and Engel, Phys. Rev. C, **93**, 014304 (2016)
$V_0 = -176.614$ MeV fm^3, $C_1^s = 133.038$ MeV fm^3

^{90}Zr IAS N_{sh}=10 (Im ω = 0.5 MeV)





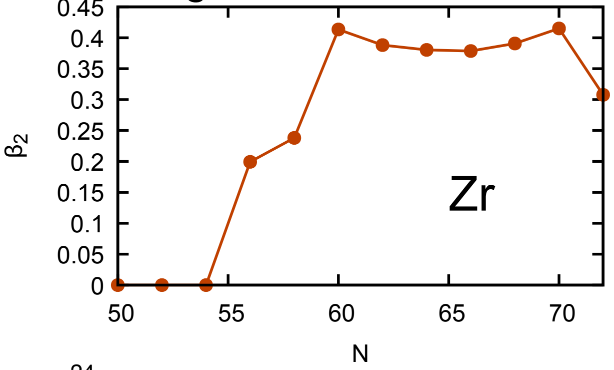
^{90}Zr GTR (K=0)



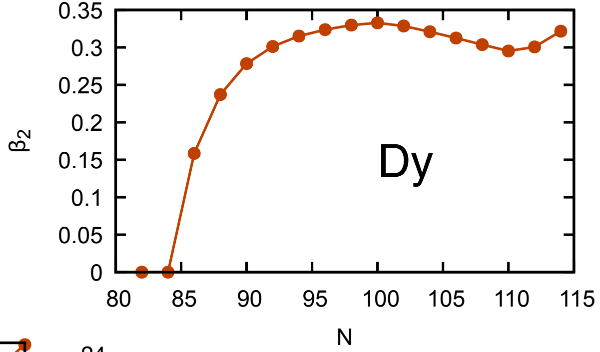
- 6 RBM basis states: (0, 5, 10, 15, 20, 25 + i MeV)
- 1 basis is enough for IAS (if the basis is within 15 MeV from the peak)
- 6 RBM basis can describe the detailed GTR strength distribution

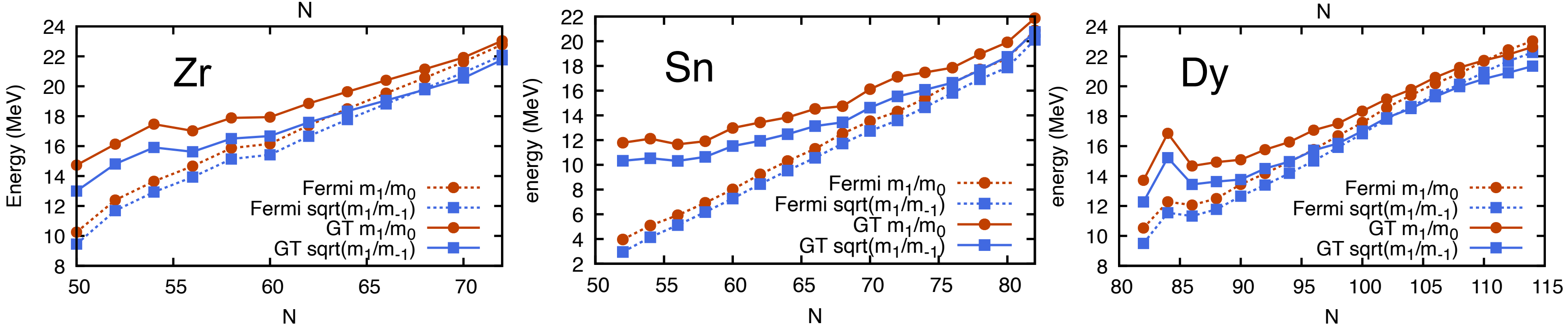
Systematic calculation of IAS and GTR (Zr, Sn, Dy)

- Nsh=20, estimating the peak energy from the sum rule ratio $\frac{m_1}{m_0}$, $\sqrt{\frac{m_1}{m_{-1}}}$
- Energies are measured from the initial state (QRPA energy)



Sn: assumed to be spherical





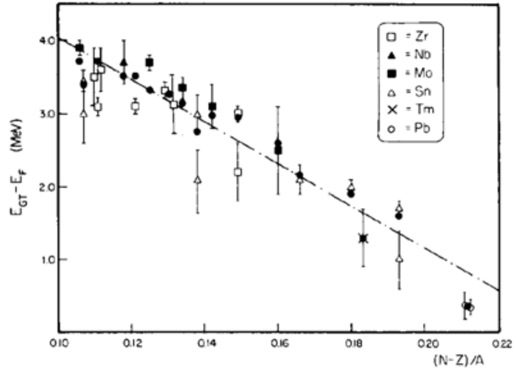
- quadrupole deformation reduces the GTR peak energy
- IAS energy change smoothly with neutron number

Gamow-Teller resonance energy

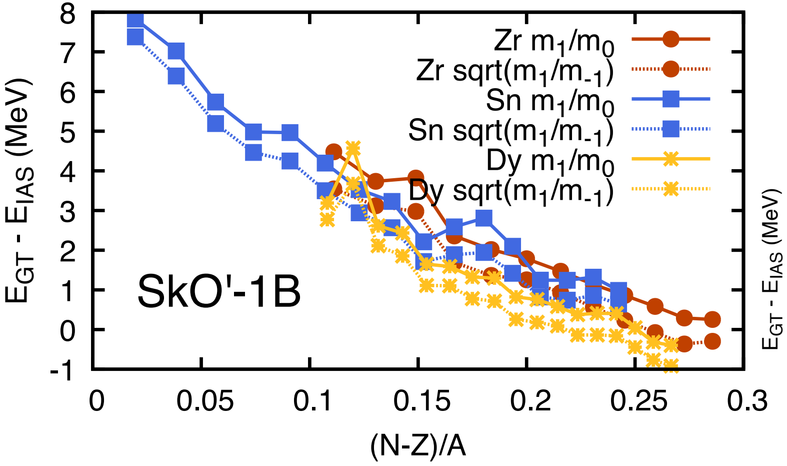
In the separable interaction, (N-Z)/A dependence of the GTR energy is determined by the spin-isospin dependent interaction

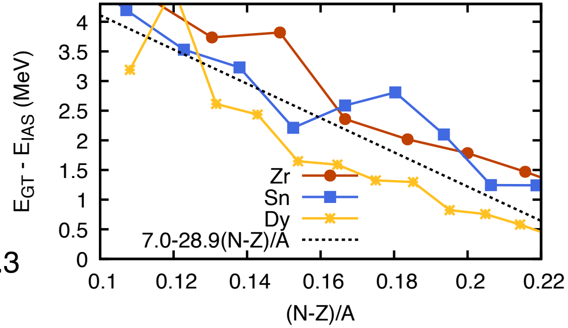
$$E_{\rm GT} - E_{\rm IAS} = \Delta E_{ls} + 2(\tilde{\kappa}_{\sigma\tau} - \tilde{\kappa}_{\tau})\frac{N-Z}{A}$$

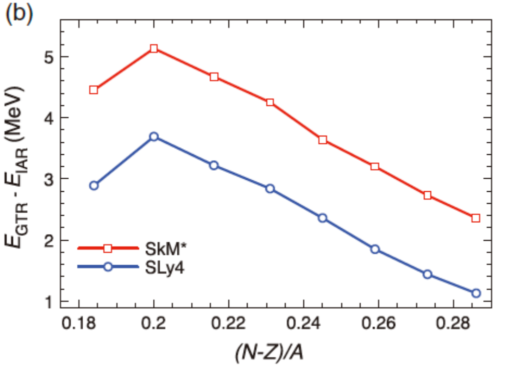
$$\hat{H} = \hat{H}_{\rm sp} + \sum_{i<j}\left[\kappa_\tau \tau_i \cdot \tau_j + \kappa_{\sigma\tau}\tau_i \cdot \tau_j \boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_j + \frac{e^2(1-\tau_{i3})(1-\tau_{j3})}{4|\boldsymbol{r}_i - \boldsymbol{r}_j|}\right] \qquad \kappa_\tau = \frac{\tilde{\kappa}_\tau}{A}, \quad \kappa_{\sigma\tau} = \frac{\tilde{\kappa}_{\sigma\tau}}{A}$$



K. Nakayama et al., Phys. Lett. B **114**, 217 (1982)
F. Osterfeld, Rev. Mod. Phys. **64**, 491 (1992)







K. Yoshida, Prog. Theor. Exp. Phys. **2013**, 113D02 (2013)

Both ΔE_{ls} and κ are consistent with experiments in SkO'-1B

Summary

- Proton-neutron finite-amplitude method for charge-exchange process
  - applications to beta decay, double-beta decay, Gamow-Teller giant resonance
  - Quasiparticle-phonon coupling improves the GT strength distribution, beta decay
  - Reduced basis method to speed up the QRPA phonon calculations

Collaborators

- MSU : Xilin Zhang
- UNC Chapel Hill: Qunqun Liu, Jon Engel
- Jyvaskyla: Markus Kortelainen