Towards consistent nuclear interactions from chiral Lagrangians II: Symmetry preserving regularization[1]

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[1] Phys. Rev. C 110, 044004

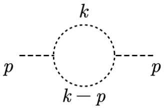
Outline

A brief introduction to regularization

The method of expressing a divergent integral as the limit of finite terms is called **regularization**.

Consider the scattering amplitude of scalar one loop:

$$
\begin{aligned}
i\mathcal{M}_{\text{loop}} &= \frac{(ig)^2}{2}\int \frac{\mathrm{d}^4 k}{(2\pi)^4}\frac{i}{(k-p)^2-m^2}\frac{i}{k^2-m^2} \\
&\simeq \mathcal{O}\left(\int \frac{\mathrm{d}^4 k}{k^4}\right) \qquad \text{log-divergent!}
\end{aligned}
$$



Ways to deal with divergent integral:

(1) UV cutoff: $\int \mathrm{d}^4 q\,(\cdots) \to \int^{\Lambda} \mathrm{d}^4 q\,(\cdots)$

(2) Pauli-Villars regularization: Introduce a ghost particle with mass $\Lambda \gg m$:

$$\frac{1}{k^2-m^2} \to \frac{1}{k^2-m^2} - \frac{1}{k^2-\Lambda^2}$$

(3) Dimensional regularization(DR): Change dimension of integral from 4 to $4-\epsilon$.

Problem: Common regularizations can NOT be employed in applications of nonperturbative systems, especially in applications of χEFT to nuclear systems[2].

Purpose: Discuss symmetry preserving regularization of chiral Lagrangian, which satisfies:

(1) Preserves the chiral and gauge symmetries.

(2) Gaussian-type regulator.

(3) Results in finite pion-exchange contributions to the nuclear forces.

Content: Two regularizations: Higher derivative regularization and Gradient flow regularization.

[2] arXiv:1911.11875, 1908.01538

Construction of π effective chiral Lagrangian

Stuffs in hand:

- Meson fields: $U \xrightarrow{g} RUL^\dagger, \quad u = \sqrt{U}$
- External sources: $r_\mu = v_\mu + a_\mu, l_\mu = v_\mu - a_\mu, s, p$
- Axial vector flow: $u_\mu = iu^\dagger \nabla_\mu U u^\dagger, \nabla_\mu U = \partial_\mu U - ir_\mu U + iUl_\mu$
- Scalar & pseudo-scalar sources: $\chi_\pm = u^\dagger \chi u^\dagger \pm u\chi^\dagger u, \quad \chi = 2B\,(s + ip)$

The LO π Lagrangian takes the form:

$$\mathcal{L}_\pi^{\mathrm{E}(2)} = \frac{F^2}{4}\mathrm{Tr}\left[(\nabla_\mu U)^\dagger (\nabla_\mu U) - U^\dagger \chi - \chi^\dagger U\right] \tag{1}$$

Consider LO π Lagrangian with chiral limit and no external sources :

$$\mathcal{L}_{\pi}^{\mathrm{E}(2)} = \frac{F^2}{4}\mathrm{Tr}\left[\partial_\mu U^\dagger \partial_\mu U\right] \;\to\; \mathcal{L}_{\pi,\Lambda}^{\mathrm{E}(2)} = \frac{F^2}{4}\mathrm{Tr}\left[\partial_\mu U^\dagger e^{-\partial^2/\Lambda^2}\partial_\mu U\right] \tag{2}$$

σ-guage parametrization of matric U in terms of π:

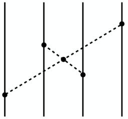
$$U = \sqrt{1-\frac{\pi^2}{F^2}} + i\frac{\boldsymbol{\tau}\cdot\boldsymbol{\pi}}{F} \tag{3}$$

Form of the regularized Lagrangian:

$$\mathcal{L}_{\pi,\Lambda}^{\mathrm{E}(2)} = -\frac{1}{2}\boldsymbol{\pi}\cdot\partial^2 e^{-\partial^2/\Lambda^2}\boldsymbol{\pi} + \frac{1}{2F^2}\boldsymbol{\pi}\cdot\partial_\mu\boldsymbol{\pi} e^{-\partial^2/\Lambda^2}\boldsymbol{\pi}\cdot\partial_\mu\boldsymbol{\pi} + \mathcal{O}\left(\boldsymbol{\pi}^6\right) \tag{4}$$

with LO πN Lagrangian, we obtain the following result for the right figure[a]:

$$V_\Lambda^{4N} = \frac{g^4}{128F^6}\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2\boldsymbol{\tau}_3\cdot\boldsymbol{\tau}_4\frac{\vec{\sigma}_1\cdot\vec{q}_1\vec{\sigma}_2\cdot\vec{q}_2\vec{\sigma}_3\cdot\vec{q}_3\vec{\sigma}_4\cdot\vec{q}_4}{\vec{q}_1^{\,2}\vec{q}_2^{\,2}\vec{q}_3^{\,2}\vec{q}_4^{\,2}}\vec{q}_{12}^{\,2} \times e^{-\frac{\vec{q}_{13}^{\,2}}{\Lambda^2}} e^{-\frac{\vec{q}_{14}^{\,2}}{\Lambda^2}} + 23\text{perm.} \tag{5}$$



[a] Detailed calculation: Eur.Phys.J.A34:197-214,2007

$$V_{\Lambda}^{4N}=\frac{g^4}{128F^6}\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2\boldsymbol{\tau}_3\cdot\boldsymbol{\tau}_4\frac{\vec{\sigma}_1\cdot\vec{q}_1\vec{\sigma}_2\cdot\vec{q}_2\vec{\sigma}_3\cdot\vec{q}_3\vec{\sigma}_4\cdot\vec{q}_4}{\vec{q}_1^2\vec{q}_2^2\vec{q}_3^2\vec{q}_4^2}\vec{q}_{12}^2 e^{-\frac{\vec{q}_{13}^2}{\Lambda^2}}e^{-\frac{\vec{q}_{14}^2}{\Lambda^2}}+23\text{perm}$$

where $\vec{q}_{ij}=\vec{q}_i-\vec{q}_j$. The result is not sufficiently regularized. The 4NF should be regularized in at least 3 independent combinations $(\sum\alpha_i\vec{q}_i)^2$ to ensure a converget behavior.

This problem can be mitigated using an ansatz for the additional cutoff-dependent terms in the Lagrangian by requiring them to be proportional to the EOM[3].

$$\mathcal{L}_{\pi,\Lambda}^{\mathrm{E}(2)}=\mathcal{L}_{\pi}^{\mathrm{E}(2)}-\frac{F^2}{4}\mathrm{Tr}\left[\mathrm{EOM}\frac{1-\exp\left(\frac{-\mathrm{ad}_{D\mu}\mathrm{ad}_{D\mu}+\frac{1}{2}\chi_+}{\Lambda^2}\right)}{-\mathrm{ad}_{D\mu}\mathrm{ad}_{D\mu}+\frac{1}{2}\chi_+}\mathrm{EOM}\right]\tag{6}$$

where $\mathrm{ad}_A B=[A,B]$, EOM is derived from LO π Lagrangian[4]:

$$\mathrm{EOM}=0,\quad \mathrm{EOM}:=[D_\mu,u_\mu]+\frac{i}{2}\chi_- -\frac{i}{4}\mathrm{Tr}\left(\chi_-\right)\tag{7}$$

[3] Nucl. Phys. B 31, 301-315 (1971)

[4] Detailed calculation: Appendix B

Higher derivative regularization

The scalar source is set to a constant value equal to the light-quark mass m_q, switch off other external sources, do the same thing as before[5],

$$
\begin{aligned}
\mathcal{L}^{\mathrm{E}(2)}_{\pi,\Lambda} = & -F^2M^2 + \frac{1}{2}\boldsymbol{\pi}\cdot\left(-\partial^2+M^2\right)e^{\frac{-\partial^2+M^2}{\Lambda^2}}\boldsymbol{\pi} + \frac{\boldsymbol{\pi}^2}{8F^2}\left[6\partial_\mu\boldsymbol{\pi}\cdot\partial_\mu\boldsymbol{\pi} + 3\boldsymbol{\pi}\cdot\partial^2\boldsymbol{\pi} + \boldsymbol{\pi}\cdot\left(-\partial^2+M^2\right)\boldsymbol{\pi}\right] \\
& - \frac{\alpha}{F^2}\boldsymbol{\pi}^2\boldsymbol{\pi}\cdot\left(-\partial^2+M^2\right)e^{\frac{-\partial^2+M^2}{\Lambda^2}}\boldsymbol{\pi} - \frac{1}{F^2}\left(\partial_\mu\boldsymbol{\pi}\cdot\partial_\mu\boldsymbol{\pi} + \frac{1}{2}\boldsymbol{\pi}\cdot\partial^2\boldsymbol{\pi}\right)\boldsymbol{\pi}\cdot e^{\frac{-\partial^2+M^2}{\Lambda^2}}\boldsymbol{\pi} \\
& + \frac{1}{4F^2\Lambda^2}\int_0^1 \mathrm{d}s\left\{M^2\boldsymbol{\pi}^2\left[\left(1-e^{(1-s)\frac{-\partial^2+M^2}{\Lambda^2}}\right)\boldsymbol{\pi}\right]\cdot\left[\left(-\partial^2+M^2\right)e^{s\frac{-\partial^2+M^2}{\Lambda^2}}\boldsymbol{\pi}\right]\right. \\
& \left.+\left(\boldsymbol{\pi}\times\partial_\mu\boldsymbol{\pi}\right)\cdot\left[\left(e^{(1-s)\frac{-\partial^2+M^2}{\Lambda^2}}\left(-\partial^2+M^2\right)\boldsymbol{\pi}\right)\times\overleftrightarrow{\partial}_\mu\left(1-e^{(1-s)\frac{-\partial^2+M^2}{\Lambda^2}}\right)\boldsymbol{\pi}\right]\right\} + \mathcal{O}\left(\boldsymbol{\pi}^6\right) \qquad (8) \\
V^{4N}_\Lambda = & -\frac{g^4}{128F^6}\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2\boldsymbol{\tau}_3\cdot\boldsymbol{\tau}_4\frac{\vec{\sigma}_1\cdot\vec{q}_1\vec{\sigma}_2\cdot\vec{q}_2\vec{\sigma}_3\cdot\vec{q}_3\vec{\sigma}_4\cdot\vec{q}_4}{(\vec{q}_2^{\,2}+M^2)(\vec{q}_3^{\,2}+M^2)(\vec{q}_4^{\,2}+M^2)}\left[2\vec{\sigma}_1\cdot\vec{q}_{12}f^{234}_\Lambda + \vec{\sigma}_1\cdot\vec{q}_1\left(2f^{123}_\Lambda - f^{134}_\Lambda - f^{234}_\Lambda\right)\right] \\
& + \frac{g^4}{128F^6}\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2\boldsymbol{\tau}_3\cdot\boldsymbol{\tau}_4\frac{\vec{\sigma}_1\cdot\vec{q}_1\vec{\sigma}_2\cdot\vec{q}_2\vec{\sigma}_3\cdot\vec{q}_3\vec{\sigma}_4\cdot\vec{q}_4}{(\vec{q}_1^{\,2}+M^2)(\vec{q}_2^{\,2}+M^2)(\vec{q}_3^{\,2}+M^2)(\vec{q}_4^{\,2}+M^2)} \\
& \times\left[-M^2f_\Lambda + \left(2M^2+\vec{q}_{12}^{\,2}\right)f^{123}_\Lambda + 2M^2\left(M^2+\vec{q}_1^{\,2}\right)\frac{f^{134}_\Lambda - f^{234}_\Lambda}{\vec{q}_1^{\,2}-\vec{q}_2^{\,2}} + 2\left(M^2+\vec{q}_2^{\,2}\right)\left(\vec{q}_{13}^{\,2}-\vec{q}_{12}^{\,2}\right)\frac{f^{124}_\Lambda - f^{134}_\Lambda}{\vec{q}_2^{\,2}-\vec{q}_3^{\,2}}\right] \\
& + 23\mathrm{perm.} \qquad (9)
\end{aligned}
$$

the regulator functions:

$$
f_\Lambda = e^{-\frac{\vec{q}_1^{\,2}+M^2}{\Lambda^2}}e^{-\frac{\vec{q}_2^{\,2}+M^2}{\Lambda^2}}e^{-\frac{\vec{q}_3^{\,2}+M^2}{\Lambda^2}}e^{-\frac{\vec{q}_4^{\,2}+M^2}{\Lambda^2}}, \qquad f^{ijk}_\Lambda = e^{-\frac{\vec{q}_i^{\,2}+M^2}{\Lambda^2}}e^{-\frac{\vec{q}_j^{\,2}+M^2}{\Lambda^2}}e^{-\frac{\vec{q}_k^{\,2}+M^2}{\Lambda^2}} \qquad (10)
$$

There is nonsingular for any values of the momentum transfers.

[5] Detailed calculation: Appendix C

As $\Lambda \to \infty$, $f_\Lambda \sim 1 + \mathcal{O}\left(\Lambda^{-2}\right)$, $f_\Lambda^{ijk} \sim 1 + \mathcal{O}\left(\Lambda^{-2}\right)$, 4NF potential:

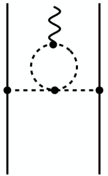
$$V_\infty^{4N} = -\frac{g^4}{64F^6}\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2\boldsymbol{\tau}_3\cdot\boldsymbol{\tau}_4\frac{\vec{\sigma}_1\cdot\vec{q}_1\vec{\sigma}_2\cdot\vec{q}_2\vec{\sigma}_3\cdot\vec{q}_3\vec{\sigma}_4\cdot\vec{q}_4}{\left(\vec{q}_2^{\,2}+M^2\right)\left(\vec{q}_3^{\,2}+M^2\right)\left(\vec{q}_4^{\,2}+M^2\right)}\vec{\sigma}_1\cdot\vec{q}_{12} + \frac{g^4}{128F^6}\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2\boldsymbol{\tau}_3\cdot\boldsymbol{\tau}_4\frac{\vec{\sigma}_1\cdot\vec{q}_1\vec{\sigma}_2\cdot\vec{q}_2\vec{\sigma}_3\cdot\vec{q}_3\vec{\sigma}_4\cdot\vec{q}_4}{\left(\vec{q}_1^{\,2}+M^2\right)\left(\vec{q}_2^{\,2}+M^2\right)\left(\vec{q}_3^{\,2}+M^2\right)\left(\vec{q}_4^{\,2}+M^2\right)}\left(M^2+\vec{q}_{12}^{\,2}\right) + 23\text{perm.} \tag{11}$$

The same as the result of Phys. J. A 34, 197-214 (2007). But...

Let l_1, l_2denote the 4-momenta of π, $l_2 = l_1 + k$ with k denoting the photon momentum. The overall effect of the regulator for the loop contribution is

$$\exp\left(\frac{l_1^2+M^2}{\Lambda^2}\right)\exp\left(-\frac{l_2^2+M^2}{\Lambda^2}\right) = \exp\left(-\frac{2k\cdot l_1 + k^2}{\Lambda^2}\right), \tag{12}$$

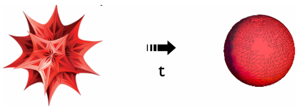
can NOT be regularized except for photon momentum $k = 0$.

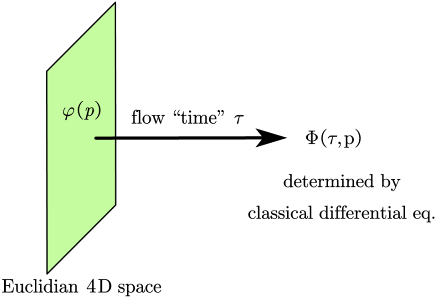


The ansatz of Higher derivative regularization appears to be too restricted for our purposes.

Gradient flow as a regulator

- Behaves like heat equation for smooth manifolds.
- Smooths out bump.





$$\begin{cases} \partial_\tau \Phi(\tau,x) = \Box\Phi(\tau,x) \\ \Phi(0,x) = \varphi(x) \end{cases}$$

$$\Downarrow$$

$$\Phi(\tau,p) = e^{-\tau p^2}\varphi(p)$$

- $\Phi(\tau,p)$ is a **Gaussian smearing** of $\varphi(p)$.
- $1/\tau$ has dimension m^2 and serves as cutoff.

Extension of building blocks:

- Pion field: $U(x) \to W(x,\tau),\ W(x,\tau) = w^2(x,\tau)$
- Scalar & pseudo-scalar sources: $\chi_{\pm}(\tau) = w^\dagger \chi w \pm w \chi^\dagger w$
- $\text{EOM}(\tau) = [D_\mu, w_\mu] + \frac{i}{2}\chi_-(\tau) - \frac{i}{4}\text{Tr}\chi_-(\tau)$

Chirally covariant version of the gradient flow equation[6]:

$$\partial_\tau W = -iw\text{EOM}(\tau)\, w \tag{13}$$

with boundary condition $W(x,0) = U(x)$.

How to solve and analyze?

[6] D. Kaplan, Gradient flow for chiral effective theories, talk at the Workshop on HHIQCD.

The most general parametrization of W can be written as

$$W = 1 + i\boldsymbol{\tau}\cdot\boldsymbol{\phi}\left(1-\alpha\boldsymbol{\phi}^2\right) - \frac{\boldsymbol{\phi}^2}{2}\left[1+\left(\frac{1}{4}-2\alpha\right)\boldsymbol{\phi}^2\right] + \mathcal{O}\left(\boldsymbol{\phi}^5\right) \tag{14}$$

Introduce a metric

$$g_{ab} = -\frac{1}{2}\mathrm{Tr}\left(W^\dagger\frac{\partial W}{\partial\phi_a}W^\dagger\frac{\partial W}{\partial\phi_b}\right) \tag{15}$$

Rewrite the gradient flow equation in a form:

$$\partial_\tau\phi_a = \frac{i}{2}\left[g^{-1}\right]_{ab}\mathrm{Tr}\left[\frac{\partial W}{\partial\phi_b}w^\dagger\mathrm{EOM}\left(\tau\right)w^\dagger\right] \tag{16}$$

Expand ϕ in the form of a power series in $1/F$:

$$\phi_b = \sum_{n=0}^{\infty}\frac{\phi_b^{(n)}}{F^n} \tag{17}$$

We obtain a series of recursive differential equations.

Solution up to order 3:

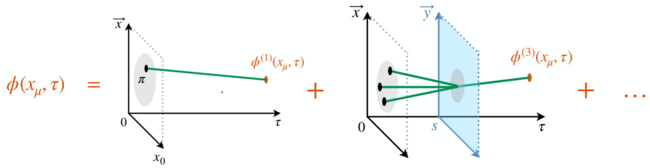
$$\phi_b^{(1)}(x,\tau) = \int d^4y G(x-y,\tau)\pi_b(y) \tag{18}$$

$$\phi_b^{(2)}(x,\tau) = 0 \tag{19}$$

$$\begin{aligned}\phi_b^{(3)}(x,\tau) = \int_0^\tau ds \int d^4y G(x-y,\tau-s)\Big[&(1-2\alpha)\partial_\mu\boldsymbol{\phi}^{(1)}(y,s)\cdot\partial_\mu\boldsymbol{\phi}^{(1)}(y,s)\phi_b^{(1)}(y,s) \\ &-4\alpha\partial_\mu\boldsymbol{\phi}^{(1)}(y,s)\cdot\boldsymbol{\phi}^{(1)}(y,s)\partial_\mu\phi_b^{(1)}(y,s)+\frac{M^2}{2}(1-4\alpha)\boldsymbol{\phi}^{(1)}(y,s)\cdot\boldsymbol{\phi}^{(1)}(y,s)\phi_b^{(1)}(y,s)\Big]\end{aligned} \tag{20}$$

where G is Green's function:

$$G(x,\tau) = \theta(\tau)\int \frac{d^4q}{(2\pi)^4} e^{-\tau(q^2+M^2)} e^{-iq\cdot x} = \frac{\theta(\tau)}{16\pi^2\tau^2} e^{-\frac{x^2+4M^2\tau^2}{4\tau}} \tag{21}$$



The field $\phi(x,\tau)$ is expressed in terms of an increasing number of smeared of pion fields that live on the boundary $\tau = 0$.

The corresponding momentum-space expression is given by:

$$\tilde{\phi}_b^{(1)}(q,\tau) = e^{-\tau\left(q^2+M^2\right)}\tilde{\pi}_b(q) \tag{22}$$

$$\begin{aligned}\tilde{\phi}_b^{(3)}(q,\tau) = & \int \frac{\mathrm{d}^4 q_1}{(2\pi)^4}\frac{\mathrm{d}^4 q_2}{(2\pi)^4}\frac{\mathrm{d}^4 q_3}{(2\pi)^4}(2\pi)^4\,\delta^4(q-q_1-q_2-q_3)\int_0^\tau \mathrm{d}s e^{-(\tau-s)\left(q^2+M^2\right)}e^{-s\sum_{j=1}^{3}\left(q_j^2+M^2\right)} \\ & \times\left[4\alpha q_1\cdot q_2-(1-2\alpha)\,q_1\cdot q_2+\frac{M^2}{2}(1-4\alpha)\right]\tilde{\boldsymbol{\pi}}(q_1)\,\tilde{\boldsymbol{\pi}}(q_2)\,\tilde{\pi}_b(q_3)\end{aligned} \tag{23}$$

In the limit $\tau \to 0$, all multi-pion contributions to ϕ get suppressed and the field ϕ turns to the pion field $\boldsymbol{\pi}$.

Next, we will construct regularized Lagrangian $\mathcal{L}^{\mathrm{E}}_{\pi N,\tau}$ using the gradient flow method.

Building blocks at $\tau \neq 0$

- Replace pion matrix U by $W(\tau)$
- Nucleon field N, transforms according to $N \to K(\tau)N$, where $K(\tau) = \sqrt{LW^\dagger R^\dagger} Rw$.

Using the solution in $\phi_b^{(1)}(x,\tau)$ and $\phi_b^{(3)}(x,\tau)$, along with the Green's function, the regularized LO πN Lagrangian takes the form:(?)

$$\mathcal{L}_{\pi N,\tau}^{\mathrm{E}(1)} = N^\dagger \left(D_0^w + g w_\mu S_\mu\right) N \tag{24}$$

with

$$D_\mu^w = \partial_\mu + \Gamma_\mu^w, \; \Gamma_\mu^w = \frac{1}{2}\left[w^\dagger, \partial_\mu w\right] - \frac{i}{2} w^\dagger r_\mu w - \frac{i}{2} w l_\mu w^\dagger, \; S_\mu = \left(0, \frac{\vec{\sigma}}{2}\right) \tag{25}$$

Gradient flow regularization

$$
\begin{aligned}
\mathcal{L}_{\pi N,\tau}^{\mathrm{E}(1)} =& N^{\dagger}(x)\Big[-\frac{g}{2F}\vec{\sigma}\cdot\vec{\nabla}_x\mathcal{G}[\delta\boldsymbol{\pi}](x,\tau)\cdot\boldsymbol{\tau}+\frac{i}{4F^2}\left(\boldsymbol{\tau}\times\mathcal{G}[\delta\boldsymbol{\pi}](x,\tau)\right)\cdot\partial_x^0\mathcal{G}[\delta\boldsymbol{\pi}](x,\tau)\\
&+\frac{g}{4F^3}\vec{\sigma}\cdot\vec{\nabla}_x\Big(M^2(4\alpha-1)\mathcal{G}[\theta\mathcal{G}[\delta\boldsymbol{\pi}]\cdot\mathcal{G}[\delta\boldsymbol{\pi}]\mathcal{G}[\delta\boldsymbol{\pi}]](x,\tau)+2(2\alpha-1)\mathcal{G}[\theta\mathcal{G}[\delta\partial_\mu\boldsymbol{\pi}]\cdot\mathcal{G}[\delta\partial_\mu\boldsymbol{\pi}]\mathcal{G}[\delta\boldsymbol{\pi}]](x,\tau)\\
&+8\alpha\mathcal{G}[\theta\mathcal{G}[\delta\boldsymbol{\pi}]\cdot\mathcal{G}[\delta\partial_\mu\boldsymbol{\pi}]\cdot\mathcal{G}[\delta\boldsymbol{\pi}]\mathcal{G}[\delta\partial_\mu\boldsymbol{\pi}]](x,\tau)+2\alpha\mathcal{G}[\delta\boldsymbol{\pi}](x,\tau)\cdot\mathcal{G}[\delta\boldsymbol{\pi}](x,\tau)\mathcal{G}[\delta\boldsymbol{\pi}](x,\tau)\Big)\cdot\boldsymbol{\tau}\\
&-\frac{g}{8F^3}\boldsymbol{\tau}\cdot\mathcal{G}[\delta\boldsymbol{\pi}](x,\tau)\,\vec{\sigma}\cdot\vec{\nabla}_x\left(\mathcal{G}[\delta\boldsymbol{\pi}](x,\tau)\cdot\mathcal{G}[\delta\boldsymbol{\pi}](x,\tau)\right)\Big]N(x)+\mathcal{O}\left(\pi^4\right)
\end{aligned}
\tag{26}
$$

here, we have introduced the short-hand notation:

$$
\mathcal{G}[f](x,\tau)=\int_{-\infty}^{\tau}\mathrm{d}s e^{-(\tau-s)\left(-\partial_x^2+M^2\right)}f(x,s) \tag{27}
$$

When taking $\tau\to 0$, the regularized above turns into unregularized Lagrangian:

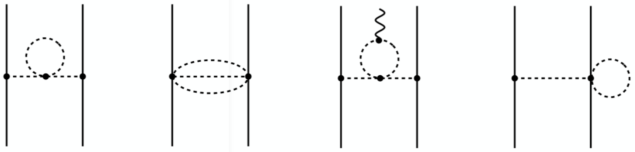
$$
\begin{aligned}
\mathcal{L}_{\pi N}^{\mathrm{E}(1)} =& N^{\dagger}\Big[\partial_0-\frac{g}{2F}\vec{\sigma}\cdot\vec{\nabla}\boldsymbol{\pi}\cdot\boldsymbol{\tau}+\frac{i}{4F^2}(\boldsymbol{\tau}\times\boldsymbol{\pi})\cdot\partial_0\boldsymbol{\pi}+\frac{g}{2F^3}\left(2\alpha-\frac{1}{2}\right)\boldsymbol{\tau}\cdot\boldsymbol{\pi}\boldsymbol{\pi}\vec{\sigma}\cdot\vec{\nabla}\boldsymbol{\pi}\\
&+\frac{g}{2F^3}\alpha\boldsymbol{\pi}^2\vec{\sigma}\cdot\vec{\nabla}\boldsymbol{\pi}\cdot\boldsymbol{\tau}\Big]N+\cdots
\end{aligned}
\tag{28}
$$

$$
\begin{aligned}
V_{\Lambda}^{4N} =& \frac{g^4}{64F^6}\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2\boldsymbol{\tau}_3\cdot\boldsymbol{\tau}_4 \frac{\vec{\sigma}_1\cdot\vec{q}_1\vec{\sigma}_2\cdot\vec{q}_2\vec{\sigma}_3\cdot\vec{q}_3\vec{\sigma}_4\cdot\vec{q}_4}{\left(\vec{q}_2^{\,2}+M^2\right)\left(\vec{q}_3^{\,2}+M^2\right)\left(\vec{q}_4^{\,2}+M^2\right)}\left[\vec{\sigma}_1\cdot\vec{q}_1\left(2g_\Lambda-4f_\Lambda^{123}+2f_\Lambda^{134}-f_\Lambda^{234}\right)-\vec{\sigma}_1\cdot\vec{q}_2 f_\Lambda^{234}\right.\\
&+2\vec{\sigma}_1\cdot\vec{q}_1\left(5M^2+\vec{q}_1^{\,2}+\vec{q}_2^{\,2}+\vec{q}_3^{\,2}+\vec{q}_4^{\,2}+\vec{q}_{34}^{\,2}\right)\frac{g_\Lambda-f_\Lambda^{134}}{2M^2+\vec{q}_1^{\,2}+\vec{q}_3^{\,2}+\vec{q}_4^{\,2}-\vec{q}_2^{\,2}}\\
&\left.-4\vec{\sigma}_1\cdot\vec{q}_1\left(3M^2+\vec{q}_1^{\,2}+\vec{q}_2^{\,2}+\vec{q}_3^{\,2}+\vec{q}_4^{\,2}-\vec{q}_{34}^{\,2}\right)\frac{g_\Lambda-f_\Lambda^{124}}{2M^2+\vec{q}_1^{\,2}+\vec{q}_2^{\,2}+\vec{q}_4^{\,2}-\vec{q}_3^{\,2}}\right]\\
&+\frac{g^4}{128F^6}\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2\boldsymbol{\tau}_3\cdot\boldsymbol{\tau}_4\frac{\vec{\sigma}_1\cdot\vec{q}_1\vec{\sigma}_2\cdot\vec{q}_2\vec{\sigma}_3\cdot\vec{q}_3\vec{\sigma}_4\cdot\vec{q}_4}{\left(\vec{q}_1^{\,2}+M^2\right)\left(\vec{q}_2^{\,2}+M^2\right)\left(\vec{q}_3^{\,2}+M^2\right)\left(\vec{q}_4^{\,2}+M^2\right)}\left(M^2+\vec{q}_{12}^{\,2}\right)\left(4f_\Lambda^{123}-3g_\Lambda\right)+23\text{perm.} \qquad (29)
\end{aligned}
$$

where

$$
g_\Lambda = e^{-\frac{\vec{q}_1^{\,2}+M^2}{2\Lambda^2}}e^{-\frac{\vec{q}_2^{\,2}+M^2}{2\Lambda^2}}e^{-\frac{\vec{q}_3^{\,2}+M^2}{2\Lambda^2}}e^{-\frac{\vec{q}_4^{\,2}+M^2}{2\Lambda^2}} \qquad (30)
$$

Taking $\Lambda\to\infty$, Eq.(29) reduces to Eq.(11). Gradient flow method does not eliminate all UV divergent in calculation of loop, but we can use an additional regularization, e.g. DR.



Summary and conclusions

- Modify the LO Lagrangian for pion by replacing Tr $\left[\partial_\mu U^\dagger \partial_\mu U\right]$ $\rightarrow$ Tr $\left[\partial_\mu U^\dagger e^{-\partial^2/\Lambda^2} \partial_\mu U\right]$. But this method does not sufficiently regularized.
- Consider the additional cutoff-dependent terms in the Lagrangian are taken to be proportional to the EOM. It faces limitations when applied to processes involving external sources.
- Construct Lagrangian with the artificial flow time τ, which ensures that no problematic expontially increasing factors emerge. The gradient flow method is found to comply with all requirements we impose on the regularizaton scheme.