Lecture 7: Dictionaries (Maps)

Sequential Unfoldment of Knowledge

Wholeness Statement

The Dictionary ADT stores a searchable collection of *key-value* items that represents either an unordered or an ordered collection. Hashing solves the problem of item-lookup by providing a table whose size is not unreasonably large, yet it can store a large range of keys such that the element associated with each key can be accessed quickly ($O(1)$). SCI provides systematic techniques for accessing and experiencing total knowledge of the Universe to enhance individual life.

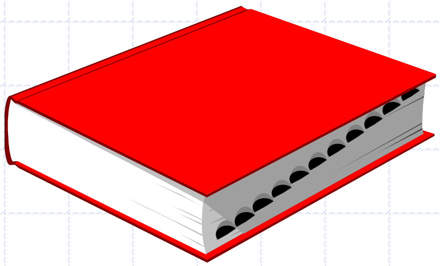
The Dictionary ADT

Two Types of Dictionaries

1. Unordered (§2.5.1)
2. Ordered (§3.1)

- Both use a key to identify a specific element/value
- Stores items, i.e., key-value pairs
- For the sake of generality, multiple items can have the same key

Unordered Dictionary ADT (§2.5.1)



- The dictionary ADT models a searchable collection of key-element items
- The main operations of a dictionary are searching, inserting, and deleting items
- Multiple items with the same key are allowed
- Applications:
  - address book
  - credit card authorization
  - mapping host names (e.g., cs16.net) to internet addresses (e.g., 128.148.34.101)

- Dictionary ADT methods:
  - findValue(k): if the dictionary has an item with key k, returns its element, else, returns the special element NO_SUCH_KEY
  - insertItem(k, o): inserts item (k, o) into the dictionary
  - removeItem(k): if the dictionary has an item with key k, removes it from the dictionary and returns its element, else returns the special element NO_SUCH_KEY
  - size(), isEmpty()
  - keys() , values(), items()

Log Files (§2.5.1)

- A log file (or audit trail) is a dictionary implemented by means of an unsorted sequence
  - Items are stored in the dictionary in a sequence in arbitrary order
  - Based on doubly-linked lists or a circular array
- Performance:
  - insertItem takes $O(1)$ time since we can insert the new item at the beginning or at the end of the sequence
  - findValue and removeItem take $O(n)$ time since in the worst case (the item is not found) we traverse the entire sequence to look for an item with the given key

Log File

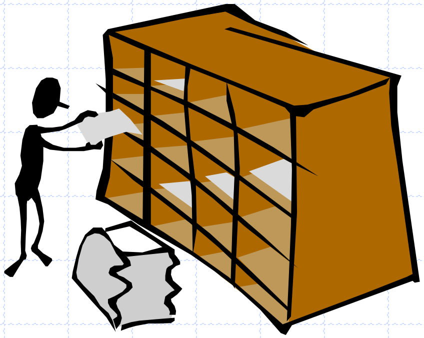
- Effective only for dictionaries of small size or
- For dictionaries on which insertions are the most common operations, while searches and removals are rarely performed

  (e.g., historical record of logins to a workstation)

- What do we do if we need to do frequent searches and removals in a large dictionary?

Hash Tables

Hash Tables and Hash Functions (§2.5.2)



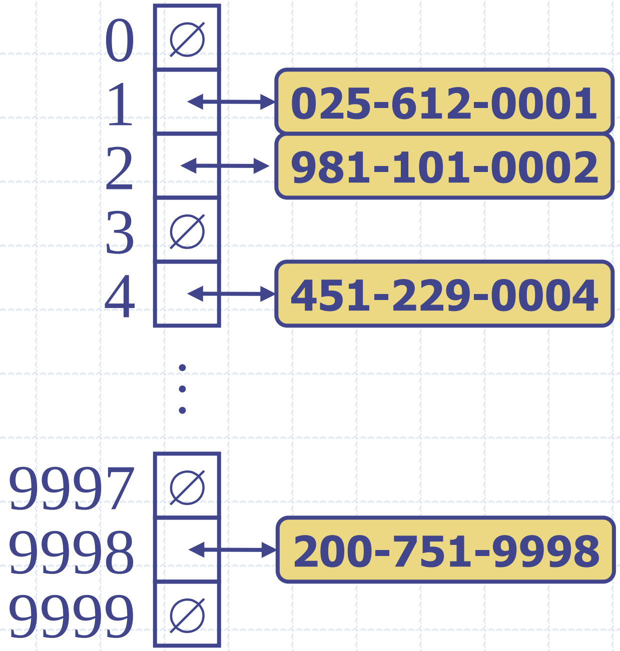
- A hash table for a given key type consists of
  - Hash function h
  - Array (called table) of size N
- A hash function h maps keys of a given type to integers in a fixed interval $[0, N - 1]$
- Example:
  $$h(k) = k \bmod N$$
  is a hash function for integer keys
- The integer $h(k)$ is called the hash value of key k

Goals of Hash Functions

1. Store item (***k***, ***o***) at index ***i*** = ***h***(***k***) in the table
2. Avoid collisions as much as possible
   - Collisions occur when two keys hash to the same index i
   - The average performance of hashing depends on how well the hash function distributes the set of keys (i.e., avoids collisions)

Example

- Design a hash table for a dictionary storing items (SSN, Name), where SSN (social security number) is a nine-digit positive integer
- Our hash table uses an array of size $N = 10{,}000$ and the hash function
  $h(x)$ = last four digits of x



Hash Functions (§ 2.5.3)

- A hash function is usually specified as the composition of two functions:

  Hash code map:

  h_1: keys $\rightarrow$ integers

  Compression map:

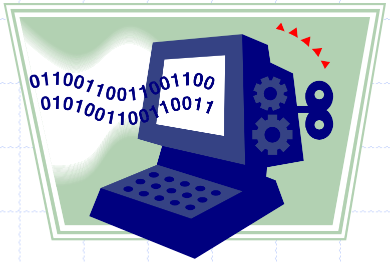
  h_2: integers $\rightarrow [0, N-1]$

- The hash code map is applied first, and the compression map is applied next on the result, i.e.,

  $$h(x) = h_2(h_1(x))$$

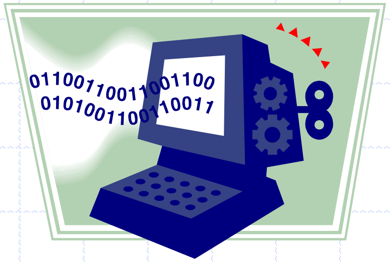
- The goal of the hash function is to "disperse" the keys in an apparently random way

Hash Code Maps (§2.5.3)



- **Memory address:**
  - We reinterpret the memory address of the key object as an integer
    - (default hash code of all Java objects)
  - Good in general, except for numeric and string keys
- **Integer cast:**
  - We reinterpret the bits of the key as an integer
  - Suitable for keys of length less than or equal to the number of bits of the integer type (e.g., byte, short, int, and float in Java)
- **Component sum:**
  - We partition the bits of the key into components of fixed length (e.g., 16 or 32 bits)
  - Then we sum the components (ignoring overflows)
  - Suitable for numeric keys of fixed length greater than or equal to the number of bits of the integer type (e.g., long and double in Java)

Hash Code Maps (cont.)



- Polynomial accumulation:
  - We partition the bits of the key into a sequence of components of fixed length (e.g., 8, 16 or 32 bits)
    $$a_0\, a_1\, \dots\, a_{n-1}$$
  - We evaluate the polynomial
    $$p(z) = a_0 + a_1 z + a_2 z^2 + \dots$$
    $$\dots + a_{n-1} z^{n-1}$$
    at a fixed value z, ignoring overflows
  - Especially suitable for strings (e.g., the choice $z = 33$ gives at most 6 collisions on a set of 50,000 English words)
- Polynomial $p(z)$ can be evaluated in $O(n)$ time using Horner's rule:
  - The following polynomials are successively computed, each from the previous one in $O(1)$ time
    $$p_0(z) = a_{n-1}$$
    $$p_i(z) = a_{n-i-1} + z p_{i-1}(z)$$
    $$(i = 1, 2, \dots, n-1)$$
- We have $p(z) = p_{n-1}(z)$

Compression Maps (§2.5.4)

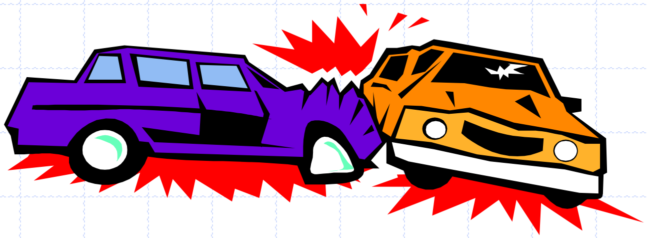
- Division:
  - $h_2(y) = y \bmod N$
  - The size N of the hash table is usually chosen to be a prime
    - The reason has to do with number theory and is beyond the scope of this course
- Multiply, Add and Divide (MAD):
  - $h_2(y) = (ay + b) \bmod N$
  - a and b are nonnegative integers such that
    $a \bmod N \neq 0$
  - Otherwise, every integer would map to the same value b

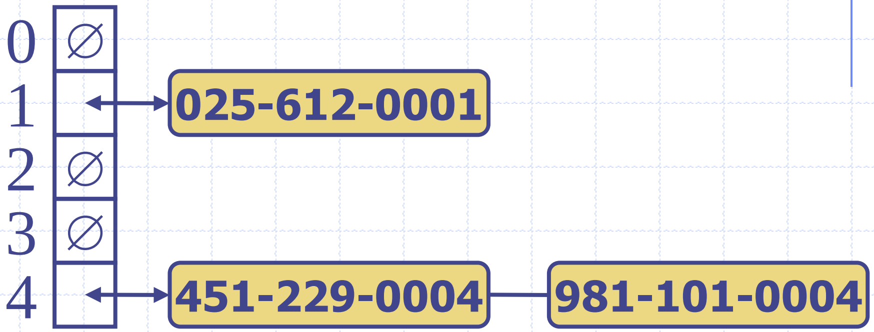
Main Point

1. The hash function solves the problem of fast table-lookup, i.e., it allows the element associated with each key to be accessed quickly (in $O(1)$ time). A hash function is composed of a hash code function and a compression function that transforms (in constant time) each key into a specific location in the table.

   Science of Consciousness: Through a process of self-referral, the unified field transforms itself into all the values of creation without making mistakes.

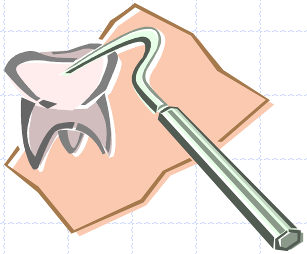
Collision Handling (§ 2.5.5)





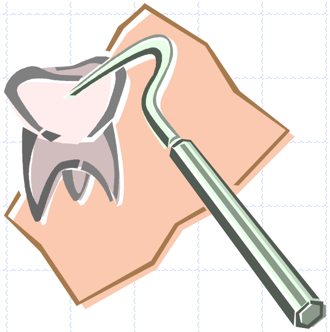
- Collisions occur when different elements are mapped to the same cell
- **Chaining**: let each cell in the table point to a linked list of elements that map there
- Chaining is simple, but requires additional memory outside the table

Linear Probing (§2.5.5)



- Open addressing: the colliding item is placed in a different cell of the table
- **Linear probing** handles collisions by placing the colliding item in the next (circularly) available table cell
- Each table cell inspected is referred to as a "probe"
- Colliding items lump together, causing future collisions to cause a longer sequence of probes

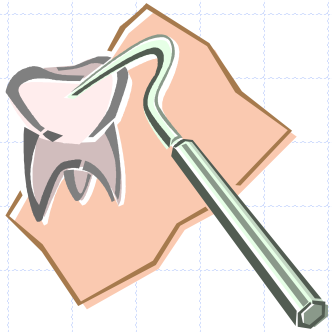
Linear Probing (§2.5.5)



- Exercise:
  - $h(x) = x \bmod 13$
  - Insert keys 18, 41, 22, 44, 59, 32, 31, 73, in this order

0	1	2	3	4	5	6	7	8	9	10	11	12

Linear Probing (§2.5.5)

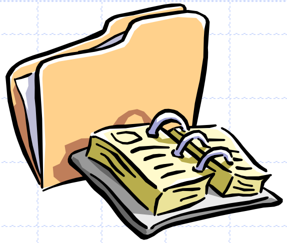


- Example:
  - $h(x) = x \bmod 13$
  - Insert keys 18, 41, 22, 44, 59, 32, 31, 73, in this order

0	1	2	3	4	5	6	7	8	9	10	11	12

		41			18	44	59	32	22	31	73	
0	1	2	3	4	5	6	7	8	9	10	11	12

Search with Linear Probing



- Consider a hash table A that uses linear probing
- findValue(k)
  - We start at cell $h(k)$
  - We probe consecutive locations until one of the following occurs
    - An item with key k is found, or
    - An empty cell is found, or
    - N cells have been unsuccessfully probed

```
Algorithm findValue(k)
    item ← findItem(k)
    return item.element()

Algorithm findItem(k)
i ← h(k)
p ← 0
while p < N do
    x ← (i + p) mod N
    item ← A[x]
    if item = ∅ then
        return NO_SUCH_KEY
    else if item.key () = k then
        return item
    else
        p ← p + 1

return NO_SUCH_KEY
```

Updates with Linear Probing

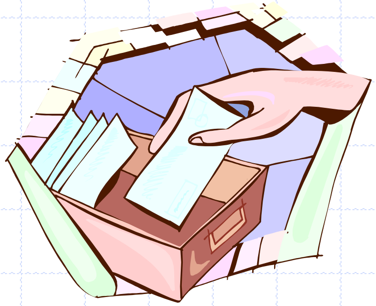
- To handle insertions and deletions, we introduce a special object, called ***AVAILABLE***, which replaces deleted elements
- removeItem(***k***)
  - We search for an item with key ***k*** (***findItem***(***k***))
  - If such an item (***k, o***) is found, we replace it with the special item ***AVAILABLE*** and we return element ***o***
  - Else, we return ***NO_SUCH_KEY***
- insert Item(***k, o***)
  - We throw an exception if the table is full
  - We start at cell ***h***(***k***)
  - We probe consecutive cells until one of the following occurs
    - A cell ***i*** is found that is either empty or stores ***AVAILABLE***, or
    - ***N*** cells have been unsuccessfully probed
  - We store item (***k, o***) in cell ***i***

Quadratic Probing

- Start with the hash value $i=h(k)$,
- Then search $A[(i + j^2) \bmod N]$

  for $j = 0, 1, 2, \ldots$ until an empty slot is found
- Disadvantages
  - Complicates removal even more
  - Secondary clustering

Double Hashing



- Double hashing uses a secondary hash function $d(k)$ and handles collisions by placing an item in the first available cell of the series

  $$(i + j*d(k)) \bmod N$$

  for $j = 0, 1, \ldots, N-1$
- The secondary hash function $d(k)$ cannot have zero values
- The table size N must be a prime to allow probing of all the cells
- Common choice of compression map for the secondary hash function:

  $$d(k) = q - (k \bmod q)$$

  where
  - $q < N$
  - q is a prime
- The possible values for $d(k)$ are

  $$1, 2, \ldots, q$$

Example of Double Hashing

- Consider a hash table storing integer keys that handles collision with double hashing
  - $N = 13$
  - $h(k) = k \bmod 13$
  - $d(k) = 7 - (k \bmod 7)$
- Insert keys 18, 41, 22, 44, 59, 32, 31, 73, in this order

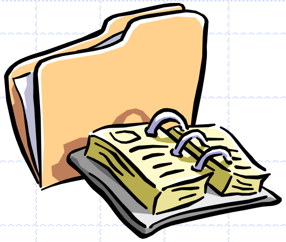
k	$h(k)$	$d(k)$	Probes		
18	5	3	5		
41	2	1	2		
22	9	6	9		
44	5	5	5	10	
59	7	4	7		
32	6	3	6		
31	5	4	5	9	0
73	8	4	8		

0	1	2	3	4	5	6	7	8	9	10	11	12

⇩

31		41			18	32	59	73	22	44		
0	1	2	3	4	5	6	7	8	9	10	11	12

Linear Probing



Algorithm *findItem*(k)

$i \leftarrow h(k)$

$p \leftarrow 0$

while $p < N$ **do**

    $x \leftarrow (i + p) \bmod N$

    $item \leftarrow A[x]$

    if $item = \varnothing$ **then**

        return *NO_SUCH_KEY*

    else if $item.key() = k$ **then**

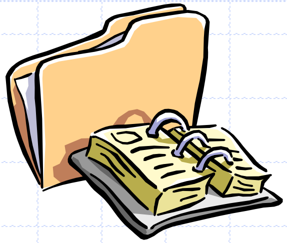
        return *item*

    else

        $p \leftarrow p + 1$

return *NO_SUCH_KEY*

Probing Algorithms



Quadratic Probing

Algorithm *findItem*(k)
$i \leftarrow h(k)$
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while $p < N$ **do**
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    $item \leftarrow A[x]$
    if $item = \varnothing$ **then**
        return *NO_SUCH_KEY*
    else if $item.key() = k$ **then**
        return *item*
    else
        $p \leftarrow p + 1$

return *NO_SUCH_KEY*

Double Hashing

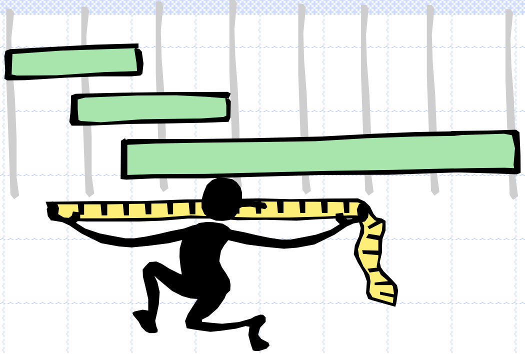
Algorithm *findItem*(k)
$i \leftarrow h(k)$
$p \leftarrow 0$
while $p < N$ **do**
    $x \leftarrow (i + p*d(k)) \bmod N$
    $item \leftarrow A[x]$
    if $item = \varnothing$ **then**
        return *NO_SUCH_KEY*
    else if $item.key() = k$ **then**
        return *item*
    else
        $p \leftarrow p + 1$

return *NO_SUCH_KEY*

Load Factors and Rehashing

- Load factor is n/N where n is the number items in the table and N is the table size
- When the load factor goes above .75, the table is resized and the items are rehashed

Performance of Hashing

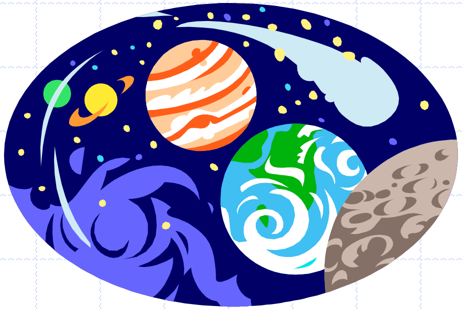


- In the worst case, searches, insertions and removals on a hash table take $O(n)$ time
- The worst case occurs when all the keys inserted into the dictionary collide
- The load factor $\alpha = n/N$ affects the performance of a hash table
- Assuming that the hash values are like random numbers, it can be shown that the expected number of probes for an insertion with open addressing is
  $$1 / (1 - \alpha)$$
- The expected running time of all the dictionary ADT operations in a hash table is $O(1)$
- In practice, hashing is very fast provided the load factor is not close to 100%
- Applications of hash tables:
  - small databases
  - compilers
  - browser caches

Universal Hashing

- If allowed to pick the keys to be hashed, then a malicious adversary can choose n keys that all hash to the same slot
  - Any fixed hash function is vulnerable to this sort of worst-case behavior
- The only effective way to improve the situation
  - Choose the hash function randomly in a way that is independent of the keys to be stored
- This approach is called universal hashing
  - The hash function is chosen randomly at beginning of execution
- Yields good performance no matter what keys are chosen by an adversary

Universal Hashing (§ 2.5.6)



- A family of hash functions is **universal**

  if, for any $0 \le j,k \le M-1$,
  $Pr(h(j)=h(k)) \le 1/N$.

- Keys are in the range [0, M-1]
- A hash function maps to the range [0, N-1]

- Theorem: The set of all functions, h, as defined here, is universal.

- Choose p as a prime between M and 2M.
- Randomly select $0<a<p$ and $0 \le b<p$, and define

  $h(k)=(ak+b \bmod p) \bmod N$

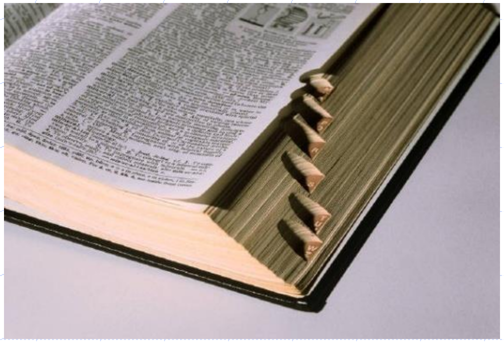
Main Point

2. A hash table is an example of a highly efficient implementation of an unordered Dictionary ADT (its operations have complexity $O(1)$). However, efficiency is only possible if the issues related to implementation are handled, e.g., resizing, handling collisions.

   Science of Consciousness: Access to Pure Consciousness is simple, effortless, easy, and spontaneous through the introduction of the proper techniques.

Ordered Dictionaries

Ordered Dictionaries



- Keys are assumed to come from a total order, i.e., the keys can be sorted.
- Constraints of iterator operations:
  - keys()

    Returns an iterator of the keys in sorted order
  - values()

    Returns the element of the items in key-sorted order
  - items()

    Returns the (k, e) items in sorted order by key (k) of the item

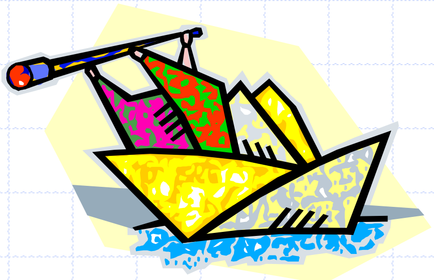
Lookup Tables (§3.1.1)

Lookup Table (§3.1.1)

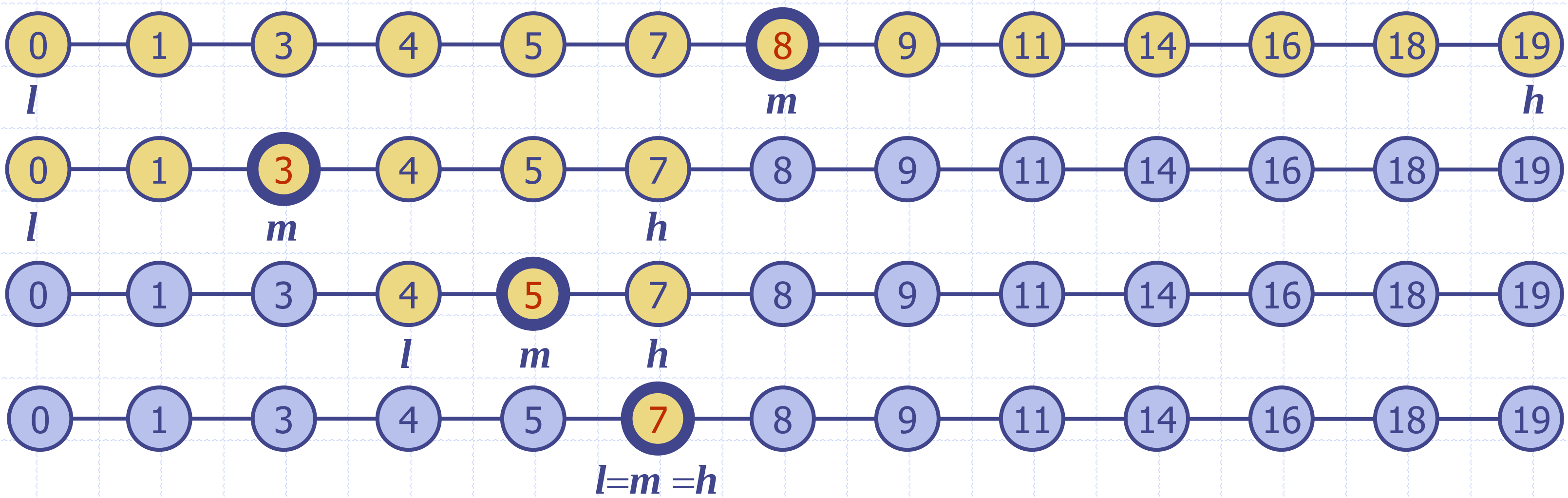
A dictionary implemented by means of a sorted sequence
- store the items of the dictionary in an array-based sequence, sorted by key
- use an external comparator for the keys

When would a table like this be useful?
- only useful if primarily used for lookup and rarely updated with added items or removals
- when the input to the table is processed or comes in in sorted order

Binary Search (§3.1.1)



- Binary search performs operation findValue(k) on a dictionary implemented by means of an array-based sequence, sorted by key
  - similar to the high-low game
  - at each step, the number of candidate items is halved
  - terminates after O(log n) steps
- Example: findValue(7)



Binary Search Algorithm (recursive)

Algorithm BinarySearch(*S, k, low, high*):

Input: *An ordered vector S storing n items, accessed by keys()*

Output: *An element of S with key k and rank between low & high.*

```
if low > high then
    return NO_SUCH_KEY
else
    mid ← (low + high)/2
    if k = key(mid) then
        return elem(mid)
    else if k < key(mid) then
        return BinarySearch(S, k, low, mid-1)
    else
        return BinarySearch(S, k, mid + 1, high)
```

Running Time of Binary Search

- Running time proportional to number of recursive calls performed.
- Recurrence equation:

$$T(n) = \begin{cases} b & \text{if } n = 0 \\ T(n/2) + b & \text{else,} \end{cases}$$

Exercise: solve this recurrence equation (use the master method).

Running Time of Binary Search

- Recurrence equation:

$$T(n) \le \begin{cases} b & \text{if } n < 2 \\ T(n/2) + b & \text{else,} \end{cases}$$

Guess: T(n) is $O(\log n)$

Assume $T(n/2) \le c \log n/2$

$T(n) \le c \log n/2 + b$

$= c(\log n - \log 2) + b$

$= c(\log n - 1) + b$

$= c \log n - c + b$

$\le c \log n \quad$ for $b \le c$

Binary Search Algorithm (iterative) for use in a Lookup Table

```
Algorithm BinarySearch(S, k):
  Input: An ordered Sequence S storing n items, ordered by keys()
  Output: The rank in S where key k is stored; if not in table, then the rank
     where an item containing k should be inserted.
  low ← 0
  high ← S.size() - 1
  while low <  high do
       mid ← (low + high)/2
        item ← S.elemAtRank(mid)
        if  item.key() < k then    // only one key comparison per iteration
              low ← mid + 1
        else  //  item.key() ≥ k (mid is not eliminated yet)
              high ← mid

  return low  // the rank where an item with key k is located or should be
              // inserted because every item at rank < low, the item.key() < k
```

findValue using a Binary Search to find an item in a Lookup Table

Algorithm findValue(*k*):

Input : *An ordered (by keys()) Sequence S storing n items is a private internal field inside the Lookup Table*

Output: *the element associated with k in the sequence of items in S if it is in the table.*

rank ← **binarySearch*(S, k)***

if S.size() ≤ rank **then return NO_SUCH_KEY** // handles empty S

item ← S.elemAtRank(rank)

if k = item.key() **then**

  return item.value()

else // key k is not in the Dictionary

  return NO_SUCH_KEY

insertItem using a Binary Search to find where to insert new item

Algorithm insertItem(*k, e*):

Input: *An ordered (by keys()) Sequence S storing n items is a private internal field inside the Lookup Table*

Output: *the element associated with k in the sequence of items in S if it is in the table.*

```
rank ← binarySearch(S, k)
if rank = S.size() then // also handles the case when S is empty
        S.insertLast( (k, e) )
        return e
item ← S.elemAtRank(rank)
if k = item.key() then      // key k is in the Dictionary, so replace item
        old ← item.value()  // element/value of the old item to be returned
        S.replaceElement( (k, e) )  // replace old item with the new one
        return old
else  //  key k is not in the Dictionary, so insert the new item
        S.insertAtRank(rank, (k, e) )
        return e
```

removeItem using Binary Search to find where to remove

Algorithm removeItem(*k*):

Input: *An ordered (by keys()) Sequence S storing n items as a private internal field inside the Lookup Table*

Output: *the item containing k in the sequence of items in S is removed from the table if it exists.*

```
rank ← binarySearch(S, k)
if S.size() ≤ rank then return NO_SUCH_KEY // handles empty S
item ← S.elemAtRank(rank)
if k = item.key() then   // key k is in the Dictionary
    old ← item.value()  // element of the old item is to be returned
    S.removeAtRank(rank)  // remove old item
    return old
else  //  key k is not in the Dictionary
    return NO_SUCH_KEY
```

Lookup Table (§3.1.1)

- A dictionary implemented by means of a sorted sequence
  - store the items of the dictionary in an array-based sequence, sorted by key
  - use an external comparator for the keys
- Performance:
  - findValue(k)
  - insertItem(k, e)
  - removeItem(k)

Lookup Table (§3.1.1)

- A dictionary implemented by means of a sorted sequence
  - store the items of the dictionary in an array-based sequence, sorted by key
  - use an external comparator for the keys
- Performance:
  - findValue takes $O(\log n)$ time, using binary search
  - insertItem takes $O(n)$ time since in the worst case we have to shift n items to make room for the new item
  - removeItem take $O(n)$ time since in the worst case we have to shift $n/2$ items to compact the items after the removal

Lookup Table (§3.1.1)

- Effective only
  - for dictionaries of small size or
  - for dictionaries on which
    - searches are the most common operation, and
    - insertions and removals are rarely performed
    - (e.g., credit card authorizations)

- What do we do if this is not the case?

Main Point

3. A lookup table is an example of an ordered Dictionary ADT allowing elements to be efficiently accessed in order by key. When implemented as an ordered sequence, searching for a key is relatively efficient, O(log n), but insertion and deletion are not, O(n).
   Science of Consciousness: The unified field of natural law always operates with maximum efficiency.

Connecting the Parts of Knowledge with the Wholeness of Knowledge

1. A hash table is a very efficient way of implementing an unordered Dictionary ADT; the running time of search, insertion, and deletion is expected O(1) time.

2. To achieve efficient behavior of the hash table operations takes a careful choice of table size, load factor, hash function, and handling of collisions.

3. **Transcendental Consciousness** is the silent field of perfect efficiency and frictionless flow for coordinating all activity in the universe.
4. **Impulses within Transcendental Consciousness**: The dynamic natural laws within this unbounded field create and maintain the order and balance in creation, all spontaneously without effort.
5. **Wholeness moving within itself:** In Unity Consciousness, the diversity of creation is experienced as waves of intelligence, perfectly efficient fluctuations of one’s own self-referral consciousness.