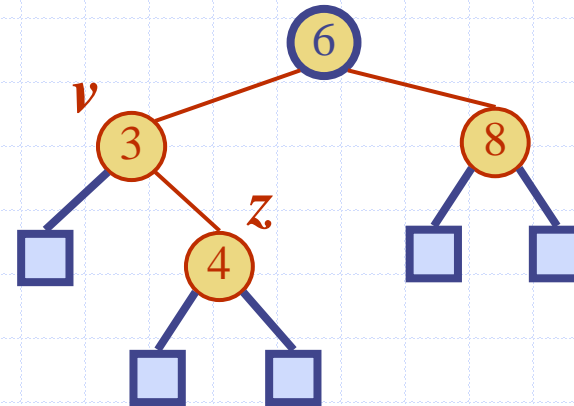# Lecture 9: Red-Black Trees

Perfect Balance and Efficiency

Wholeness Statement

A red-black tree is an implementation of a (2, 4) tree that is optimized for space utilization. The insert and delete operations are also optimized to avoid backtracking; the operations are performed locally yet maintain balance and order in the whole. *Science of Consciousness*: Nature operates in accord with the law of least action while maintaining balance and order in the whole.

Outline and Reading

◆ From (2,4) trees to red-black trees (§3.3.3)

◆ Red-black tree (§ 3.3.3)

- Definition
- Height
- Insertion
  - restructuring
  - recoloring
- Deletion
  - restructuring
  - recoloring
  - adjustment

Balanced Search Trees

- History and development of balanced search trees
  - Started with AVL trees,1962
    - Two Soviet mathematicians (**A**del'son-**V**el'skiĭ and **L**andis)
    - Could have as many as O(log n) rotations
  - 2-3 trees,1970
  - B-trees, 1972
    - Generalization of 2-3 trees to any number keys per node
    - Variations (e.g., B*-tree, B+-tree) became popular file structures
  - Symmetric binary B-trees, 1972
  - Red-Black coloring introduced, 1978
   - Became popular implementation of balanced binary trees in the late 1980's and early 1990's

From (2,4) to Red-Black Trees

- A red-black tree is a representation of a (2,4) tree by means of a binary tree whose nodes are colored red or **black**
- In comparison with its associated (2,4) tree, a red-black tree has
  - same logarithmic time performance
 - simpler implementation with a single node type

5
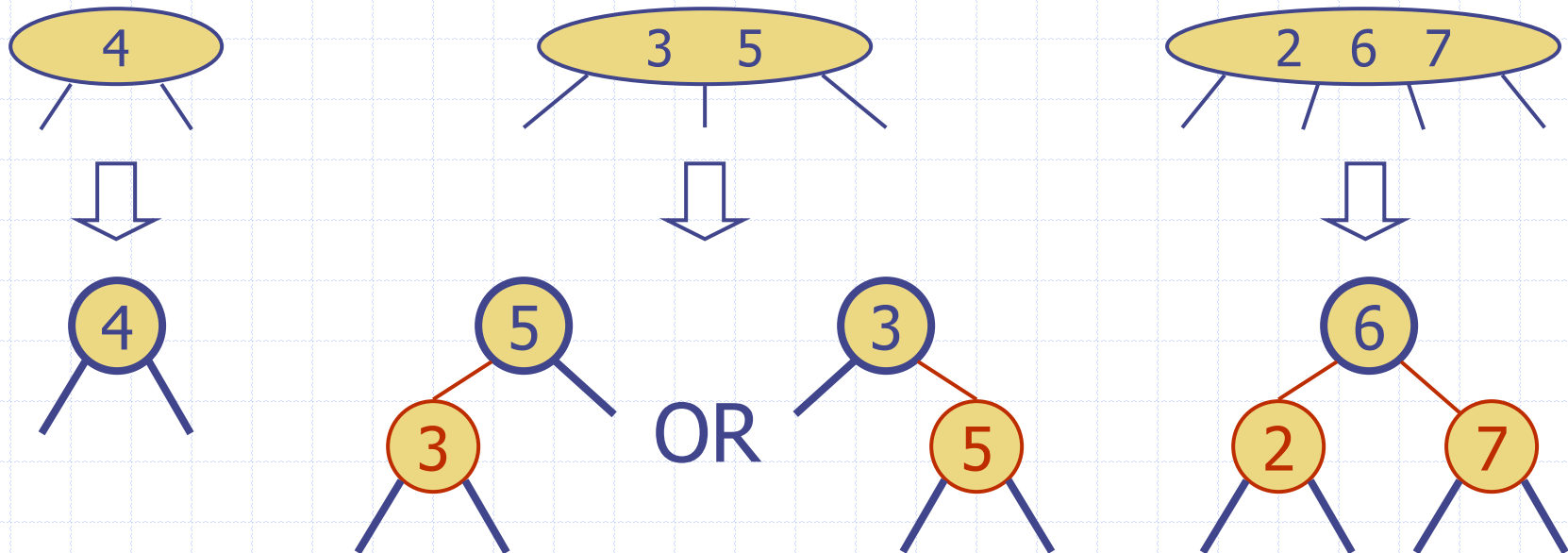
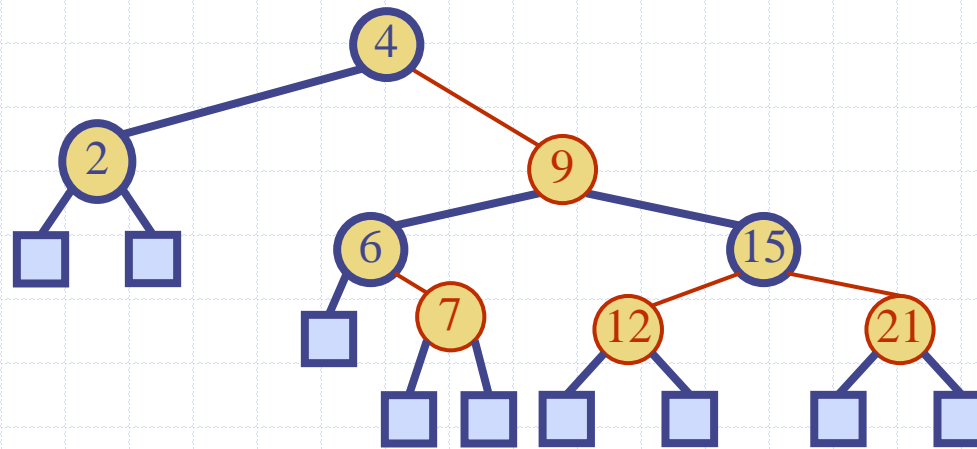# Red-Black Tree

- A red-black tree can also be defined as a binary search tree that satisfies the following properties:
  - Root Property: the root is black
  - External Property: every leaf is black
  - Internal Property: the children of a red node are black
  - Depth Property: all the leaves have the same black depth
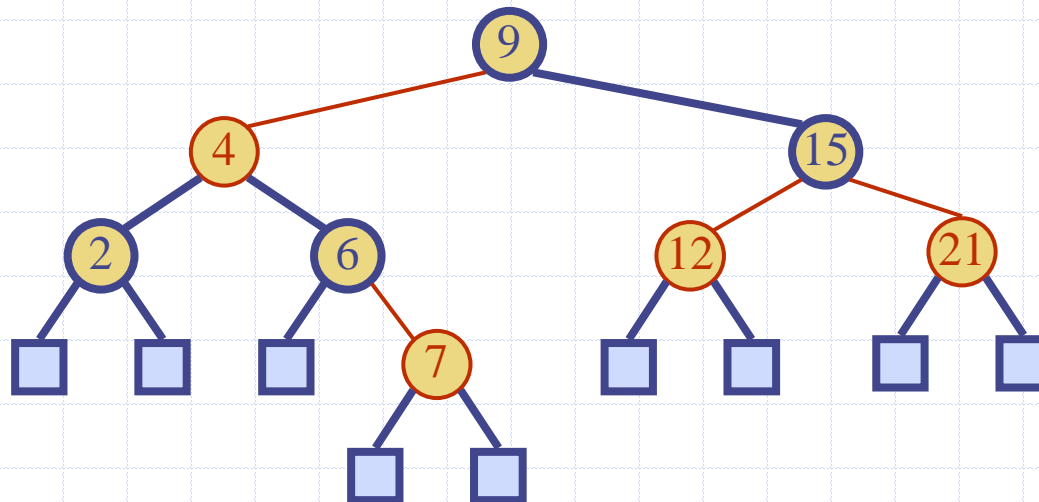
Red-Black Tree to (2,4) Tree
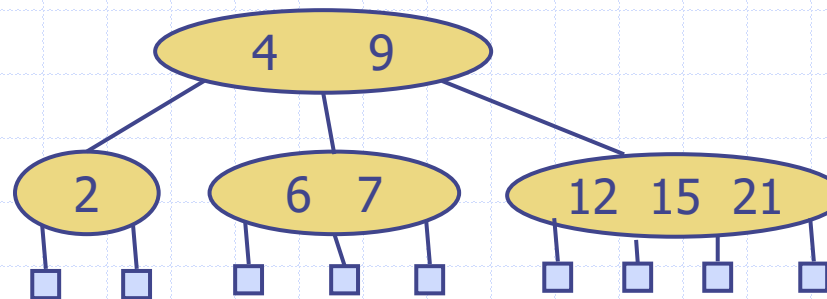
Different Red-Black Tree to Same (2,4) Tree
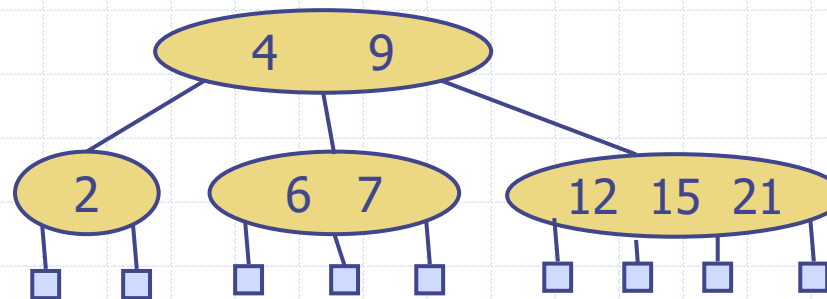
Height of a Red-Black Tree

◆ Theorem: A red-black tree storing n items has height $O(\log n)$

Proof:

■ The height of a red-black tree is at most twice the height of its associated (2,4) tree, which is $O(\log n)$

◆ The search algorithm for a red-black tree is the same as that for a binary search tree

◆ By the above theorem, searching in a red-black tree takes $O(\log n)$ time

Red-Black Tree Search

Search

- To search for a key k, we trace a downward path starting at the root using our helper *findPosition*

- If we reach a leaf, the key is not found, so we return NO_SUCH_KEY

- Otherwise we return the element associated with the key k

- Example: findElement(4)

Algorithm *findElement*(k)

// The tree T is a field of the receiver *this*

    if *isEmpty*() **then**
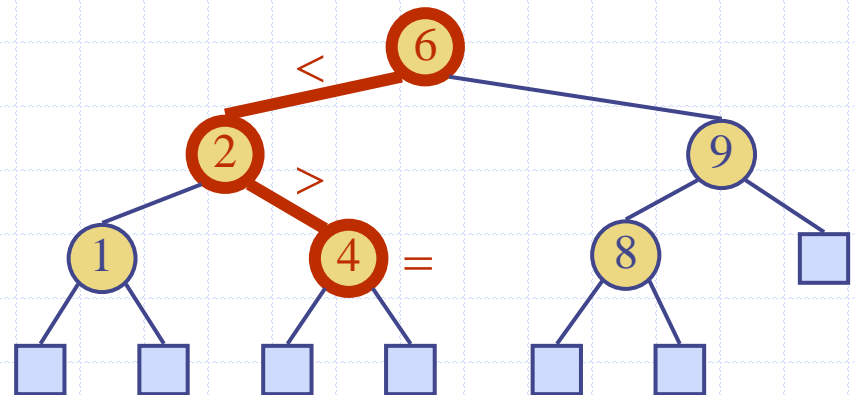
        return *NO_SUCH_KEY*

    $v \leftarrow$ *findPosition*(k, *T.root*()) // Helper method

    if $k \neq key(v)$ **then**

        return *NO_SUCH_KEY*

    else

        return *v.element*()

Helper *findPosition*

- To search for a key k, we trace a downward path starting at the root
- The next node visited is based on the comparison of k with the key of the current node
- If we reach a leaf, the key is not found and we return the parent of the external node
- If we find the key, then we return the node v containing k
- Example: findPosition of key 4 or 5 both return node v

Algorithm *findPosition*(k, v)

Output: the node containing key k or the parent of the node where k would be inserted into tree T

   if $k = key(v)$ **then**

      return v // node containing k

   else if $k < key(v)$ **then**

     if *isExternal* (T.*leftChild*(v)) **then**
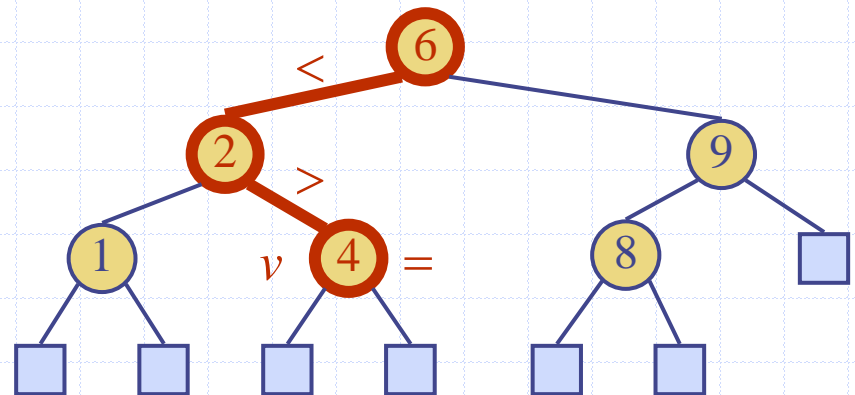
        return v // node where k would be inserted

     else return *findPosition*(k, T.*leftChild*(v))

   else if *isExternal* (T.*rightChild*(v)) **then**

        return v // node where k would be inserted

   **else return** *findPosition*(k, T.*rightChild*(v))

Main Point

1. A red-black tree is an efficient way to implement an ordered dictionary ADT because it achieves logarithmic worst-case running times for both searching and updating (inserting and removing). *Science of Consciousness:* The TM technique is a very simple, effortless way to facilitate contact with the field of total knowledge, where the fulfillment of intellectual study is achieved, i.e., one feels at home with everything and everyone.

Red-Black Tree Insertion

# Recall
Binary Tree Insertion (§3.1.4)

- To perform operation insertItem(k, o), search for key k (k should not be in the tree)
- Let w be the leaf reached by the search
- Insert k at node w and expand w into an internal node

- Example: insert 5
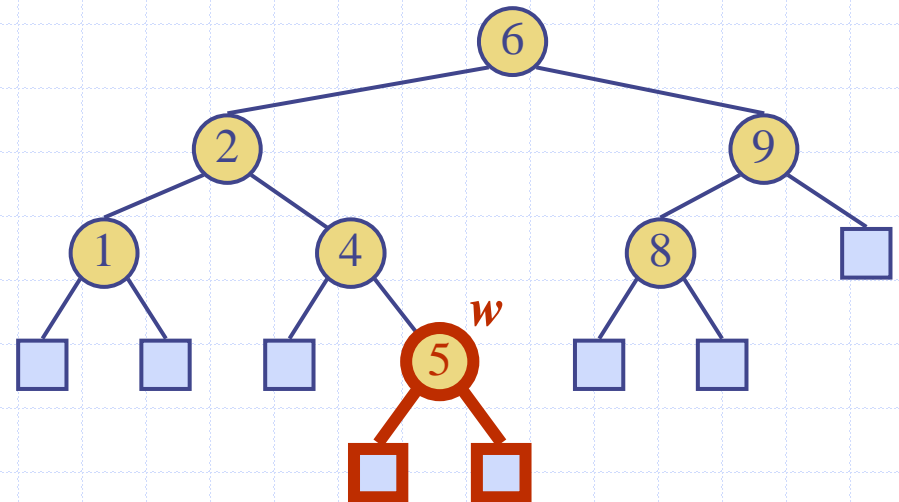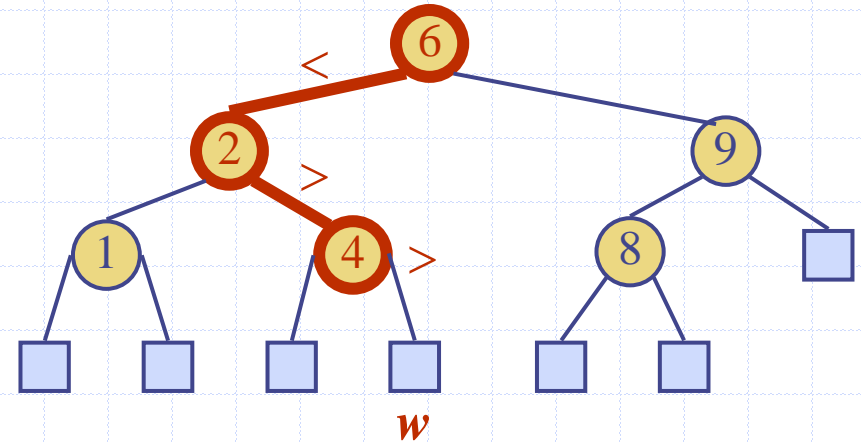
Insertion

◆ To perform operation insertItem(k, o), we execute the insertion algorithm for binary search trees and color the newly inserted node z red unless it is the root. We preserve the root, external, and depth properties.

  ▪ If the parent v of z is black, we preserve the internal property and we are done…

  ▪ …else (v is red) we have a double red (i.e., a violation of the internal property), which requires a reorganization of the tree

◆ For example, the insertion of 4 causes a double red:
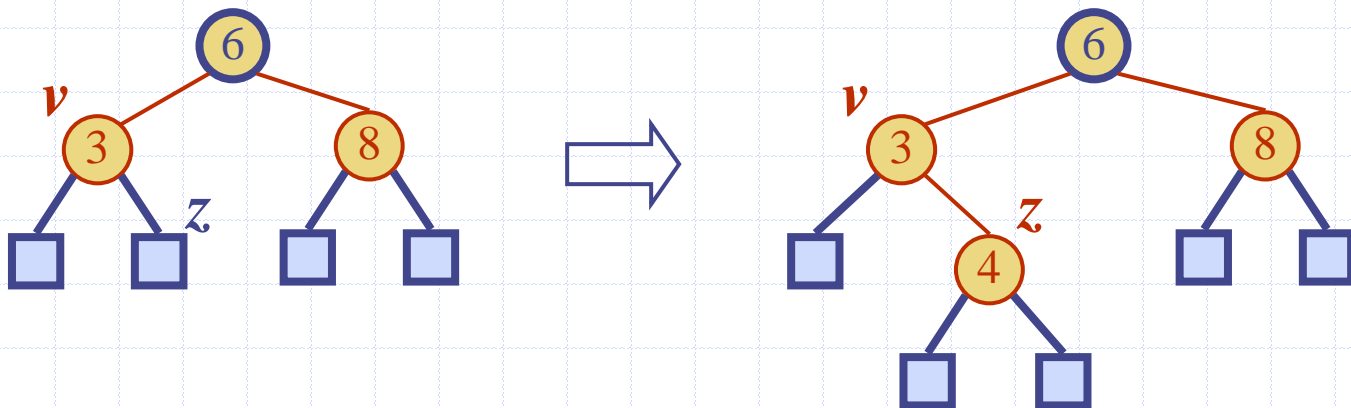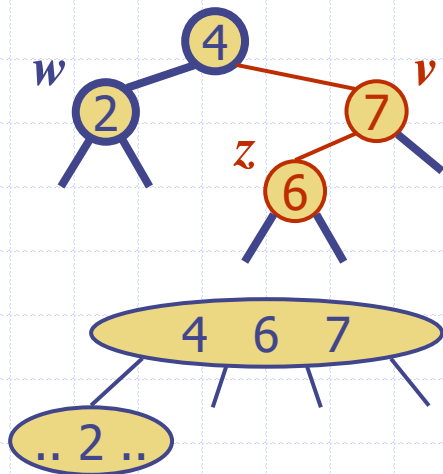
Remedying a Double Red

- Consider a double red with child z and parent v, and let w be the sibling of v

Case 1: z's uncle w is black

- The double red is an incorrect replacement of a 4-node
- Restructuring: we change the 4-node replacement

Case 2: z's uncle w is red

- The double red corresponds to an overflow
- Recoloring: we perform the equivalent of a split
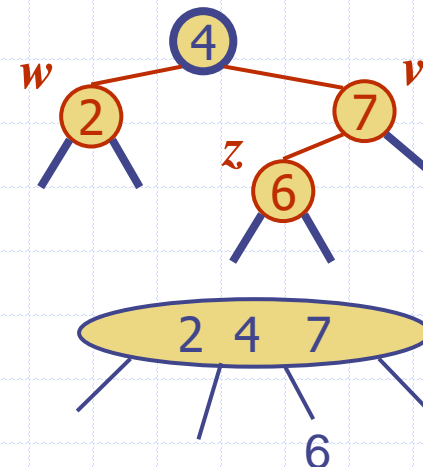
Case 1: Restructuring

- A restructuring remedies a child-parent double red when the parent red node has a black sibling
- It is equivalent to restoring the correct replacement of a 4-node
- The internal property is restored and the other properties are preserved
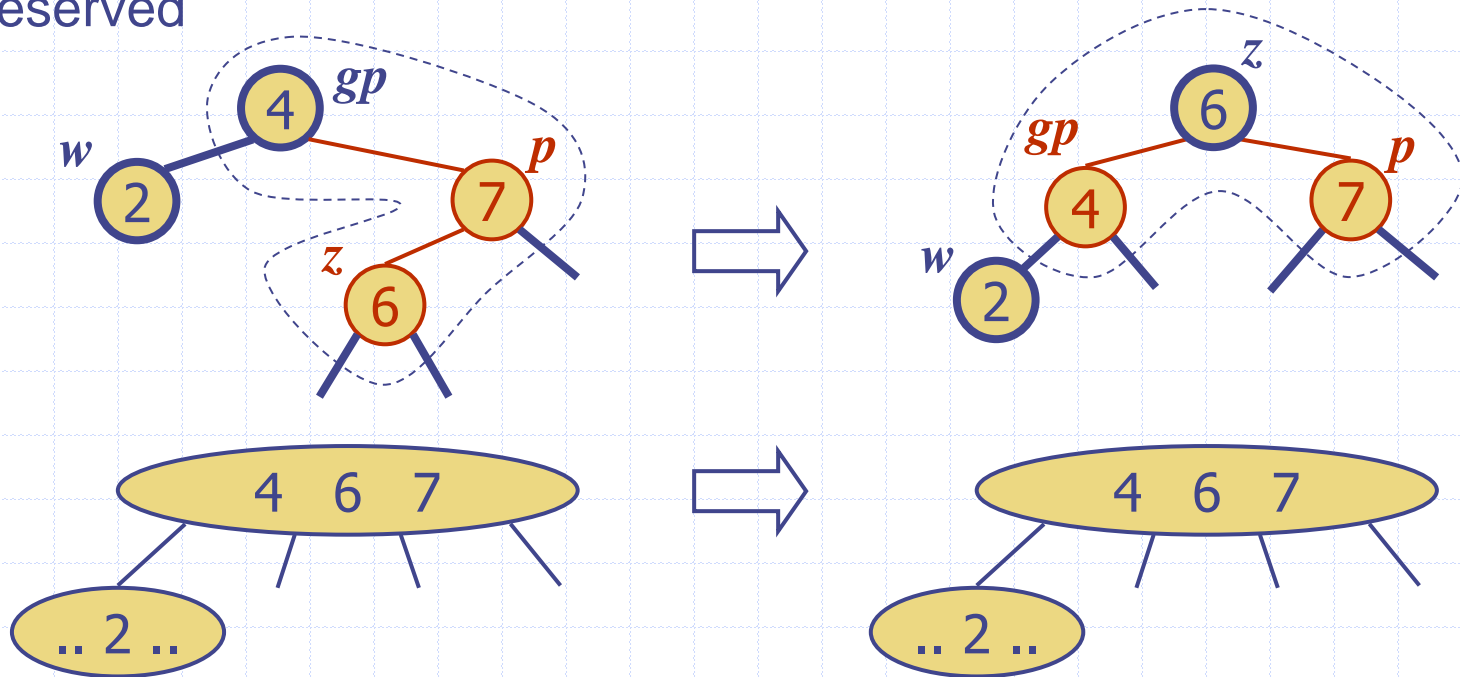
Restructuring (cont.)

◆ There are four restructuring configurations depending on whether the double red nodes are left or right children
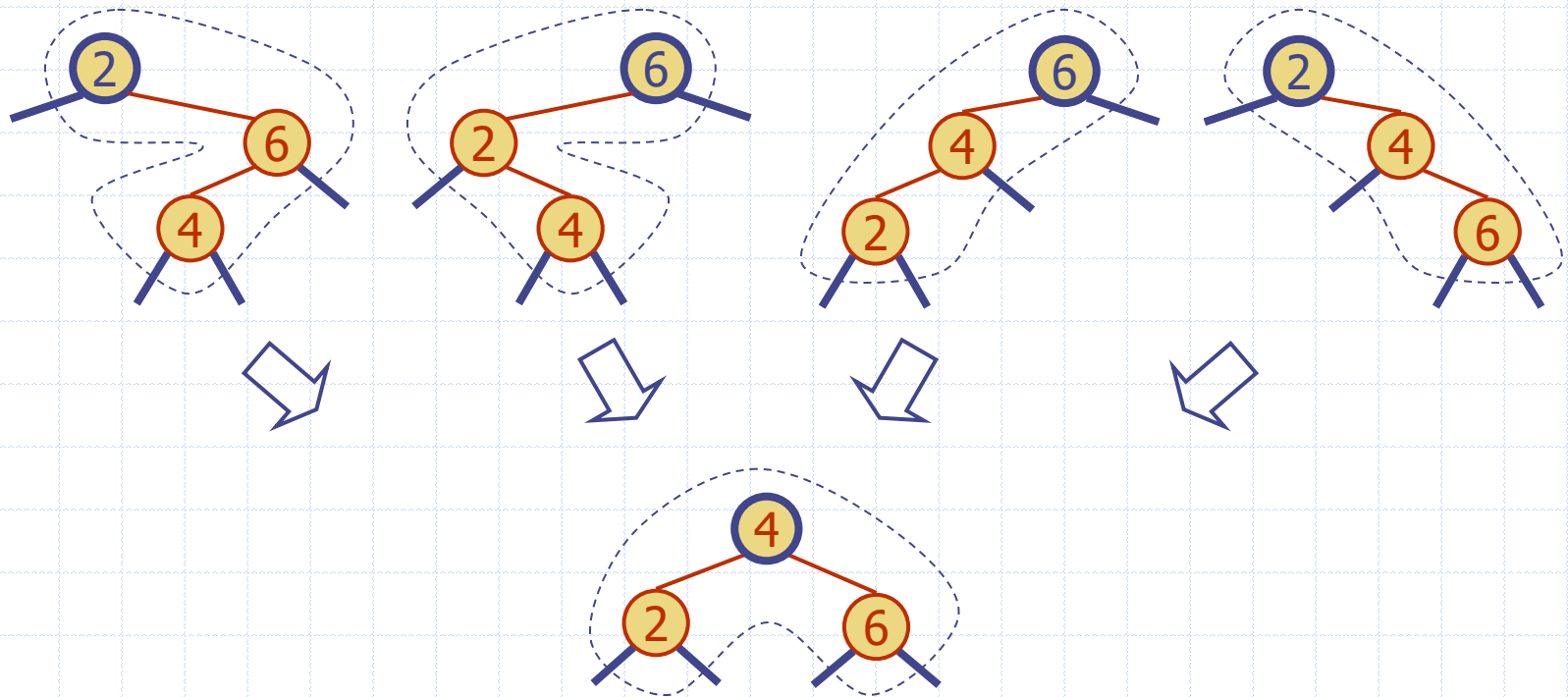
Case 2: Recoloring

- A recoloring remedies a child-parent double red when the parent red node has a red sibling
- The parent v and its sibling w become black and the grandparent u becomes red, unless it is the root
- It is equivalent to performing a split on a 5-node
- The double red violation may propagate to the grandparent u
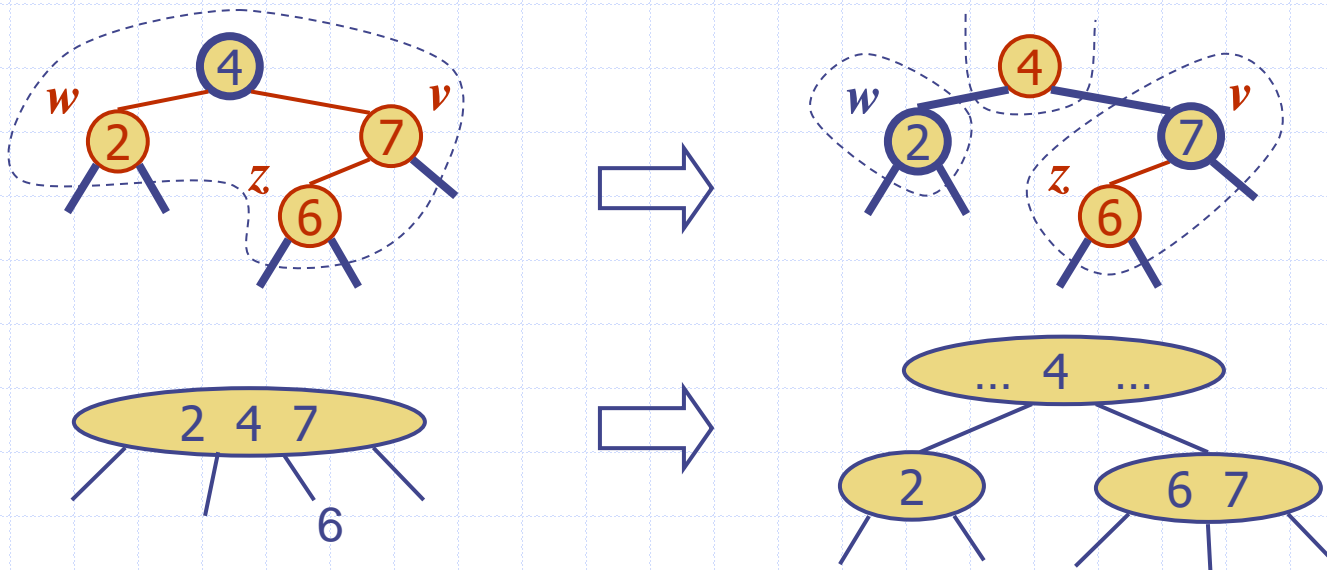
Example:

- Insert the following into an initially empty red-black tree in this order:
  (22, 5, 16, 45, 2, 10, 18, 30, 50, 12, 13, 33)

Analysis of Insertion

Algorithm *insertItem*(k, o)

1. Search for key k to locate the insertion node z

2. Add the new item (k, o) at node z and color z red

3. **while** *doubleRed*(z)

    if *isBlack*(*sibling*(*parent*(z)))

        $z \leftarrow$ *restructure*(z)

        return

    else { *sibling*(*parent*(z)) is red }

        $z \leftarrow$ *splitRecolor*(z)

◆ Recall that a red-black tree has $O(\log n)$ height

◆ Step 1 takes $O($? $)$ time

◆ Step 2 takes $O($? $)$ time

◆ Step 3 takes $O($? $)$ time

◆ Thus, an insertion in a red-black tree takes $O($? $)$ time

Analysis of Insertion

Algorithm *insertItem*(k, o)

1. Search for key k to locate the insertion node z using *findPosition*
2. Add the new item (k, o) at node z and color z red
3. **while** *doubleRed*(z)

   if *isBlack*(*sibling*(*parent*(z)))

   $gpz \leftarrow parent(parent(z))$

   restructure(z)

   setColor(*gpz, RED*)

   setColor(*parent(gpz), BLK*)

   return

   else { *sibling*(*parent*(z)) is red }

   $z \leftarrow splitRecolor(z)$

- Recall that a red-black tree has $O(\log n)$ height
- Step 1 takes $O(\log n)$ time because we visit $O(\log n)$ nodes
- Step 2 takes $O(1)$ time
- Step 3 takes $O(\log n)$ time because we perform
  - $O(\log n)$ recolorings, each taking $O(1)$ time, and
  - at most one restructuring taking $O(1)$ time
- Thus, an insertion in a red-black tree takes $O(\log n)$ time

Analysis of Insertion

Algorithm *isDoubleRed*(*z*)
    if *isRoot*(*z*) **then**
        setColor(*z, BLK*)
        return False
    else
        return isRed(parent(z))

Algorithm *splitRecolor*(*z*)
    pz ← parent(z)
    setColor(*pz, BLK*)
    setColor(*sibling*(*pz*), *BLK*)
    gpz ← parent(*pz*)
    setColor(*gpz, RED*)
    return *gpz*

Algorithm *restructure*(*z*)
    pz ← parent(z)
    if *isLeft*(*z*) **then**
        if *isLeft*(*pz*) **then**
            rotateRight(*pz*)
        else
            rotateRight(*z*)
            rotateLeft(*z*)
    else { z is a right child }
        if *isLeft*(*pz*) **then**
            rotateLeft(*z*)
            rotateRight(*z*)
        else
            rotateLeft(*pz*)

Red-Black Tree Deletion

Recall
Binary Tree Deletion (Case 1)

◆ To perform operation removeItem(k), first search for key k

◆ Assume key k is in the tree, and let u be the node storing k

◆ Two cases:
   1. Node u has a leaf child w
   2. Node u has no leaf child

◆ If node u has a leaf child w, we remove u and w from the tree with operation remove(u)

◆ Example: remove 4
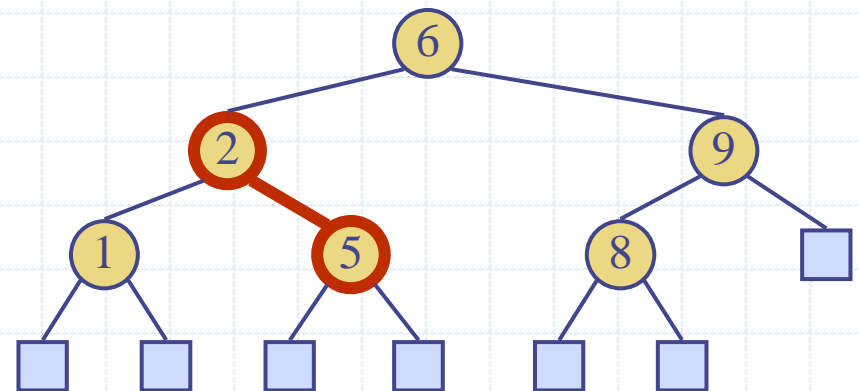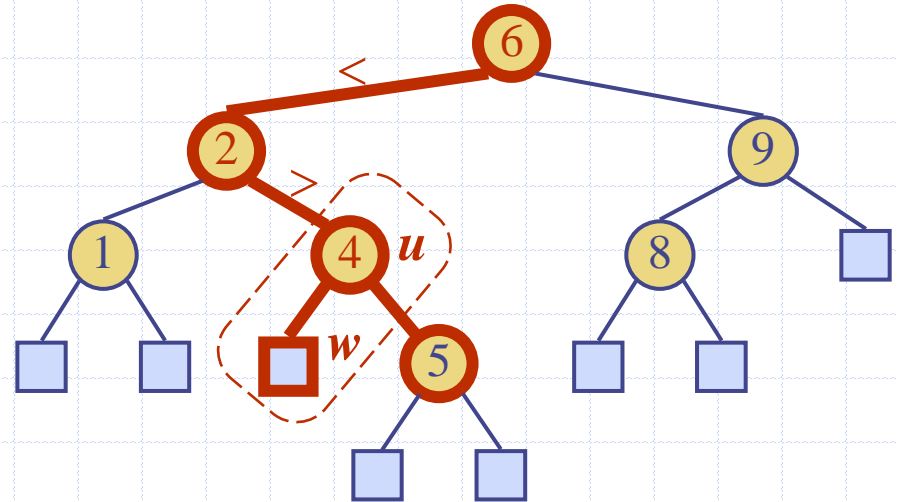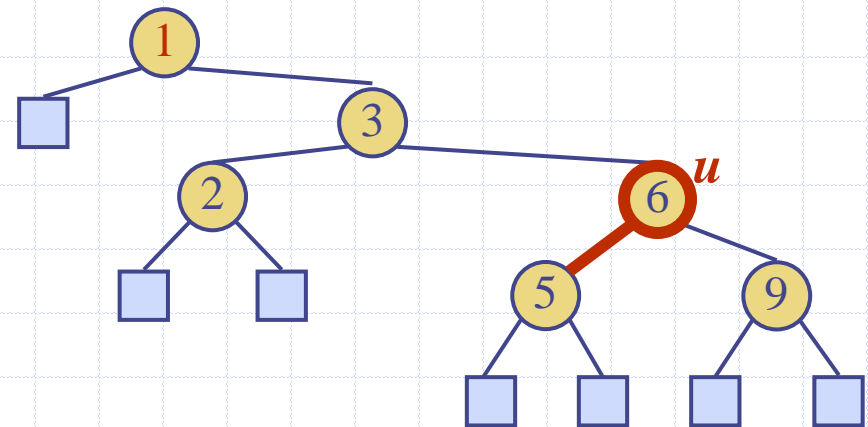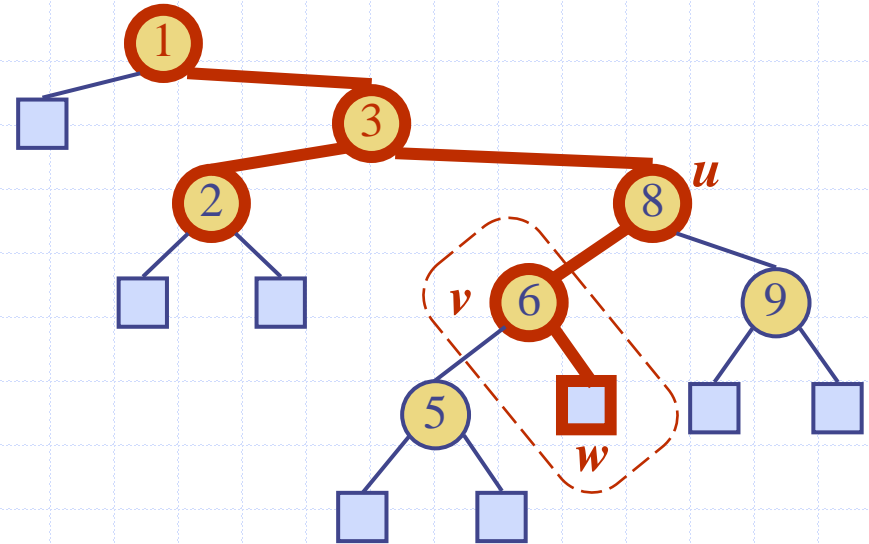
Recall
Binary Tree Deletion (Case 2)

◆ We consider the case where the key k to be removed is stored at a node u whose children are both internal

  - we find the internal node v that precedes u in an in-order traversal (v in the example)
  - we copy $key(v)$ into node u
  - we remove node v and its external child w by means of operation remove(v)

◆ Example: remove 8

Deletion

◆ To perform operation removeItem(k), first execute the deletion algorithm for binary search trees

- If node to be removed, u, does not have an external child, find next internal node by inorder traversal, called v, and move key at v to u, then remove v.

- Thus, every removal occurs at an internal node with an external child.
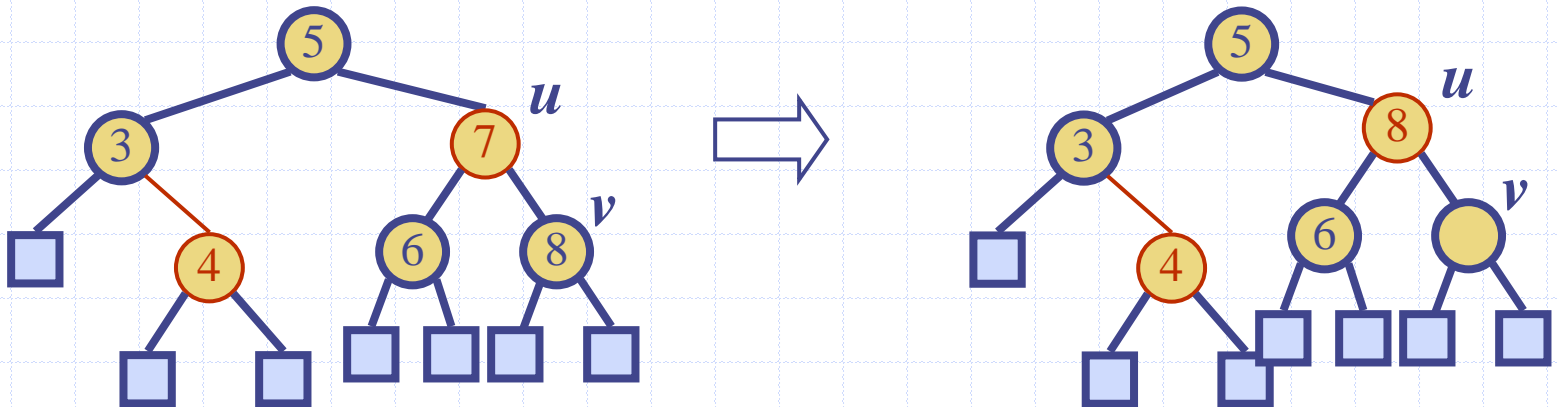
Deletion

◆ Let *v* be the internal node removed, *w* the external node removed, and *r* the sibling of *w*

- ■ If either *v* or *r* was red, we color *r* black and we are done
- ■ Else (*v* and *r* were both black), so we color *r* **double black** (a fictitious color), which is a violation of the internal property requiring a reorganization of the tree (denotes underflow)

◆ For example, the deletion of 8 causes a double black:
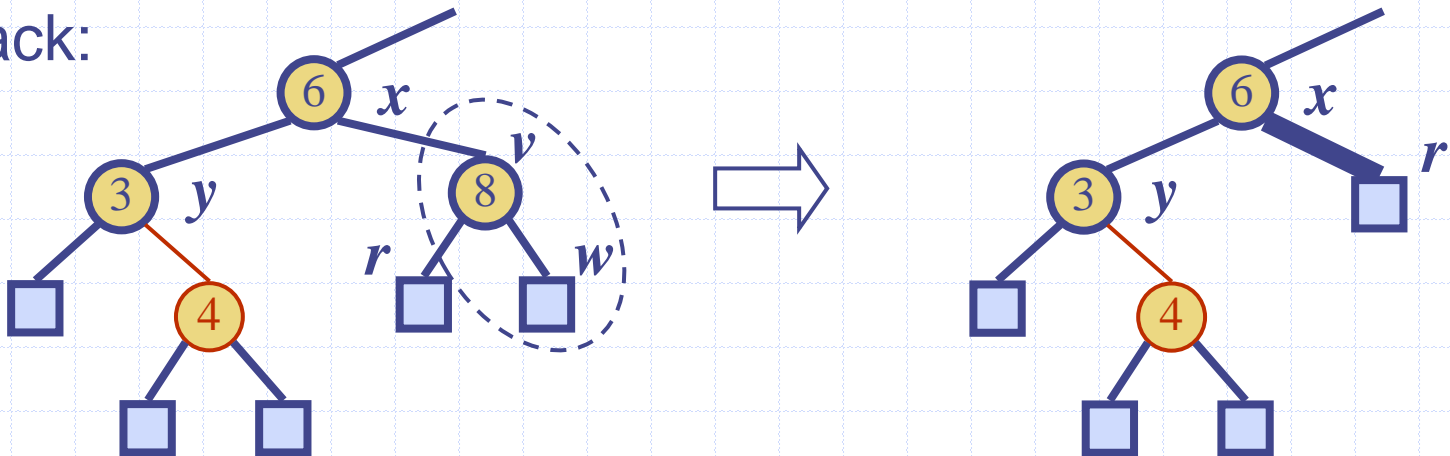
Remedying a Double Black

- The algorithm for remedying a double black node r with sibling y considers three cases

  Case 1: sibling y is black and has a red child z

  Case 2: sibling y is black and its children are both black

  Case 3: sibling y is red

Case 1a – sibling y is black and has a red child z

Trinode Restructuring
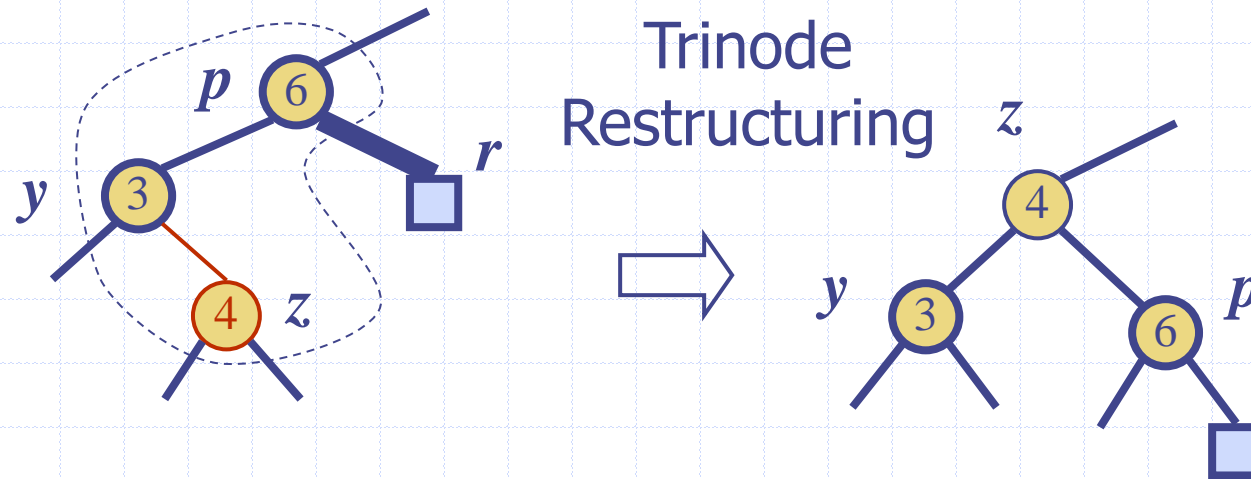
Color y and p black, give z the former color of p, and set r to black

Case 1b – sibling y is black and has a red child z

Trinode Restructuring
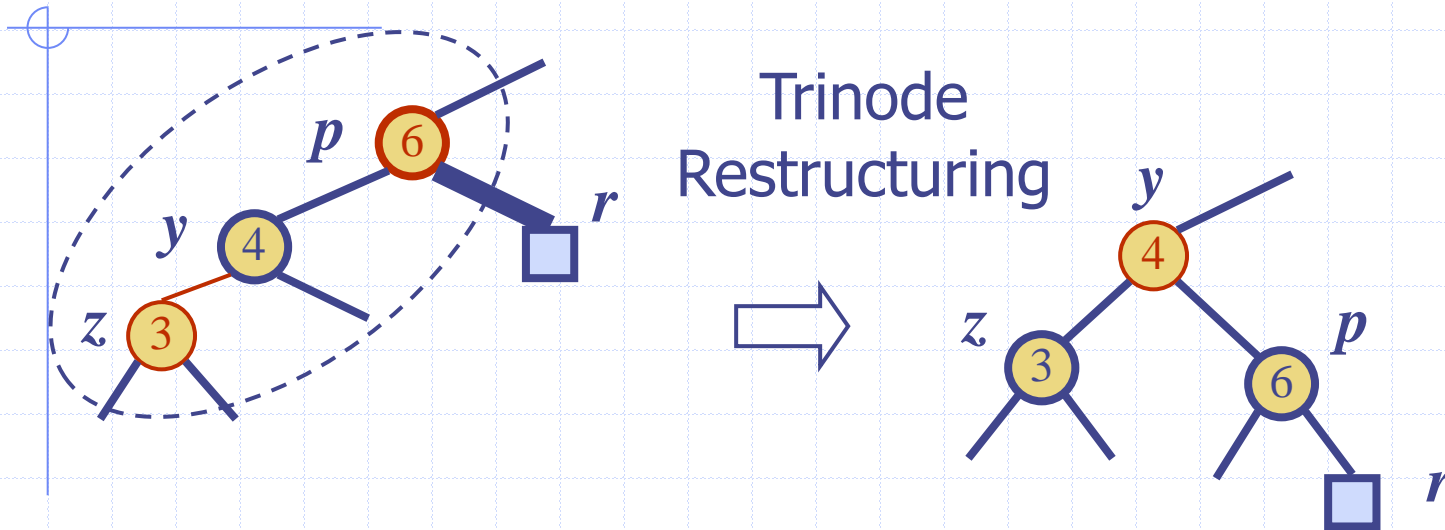
Color z and p black, give y the former color of p, and set r to black

Case 2 – sibling y is black and its children are both black

Recolor

4

8 *x*

6 *y*

r

⇨

4

8 *x*

6 *y*

r
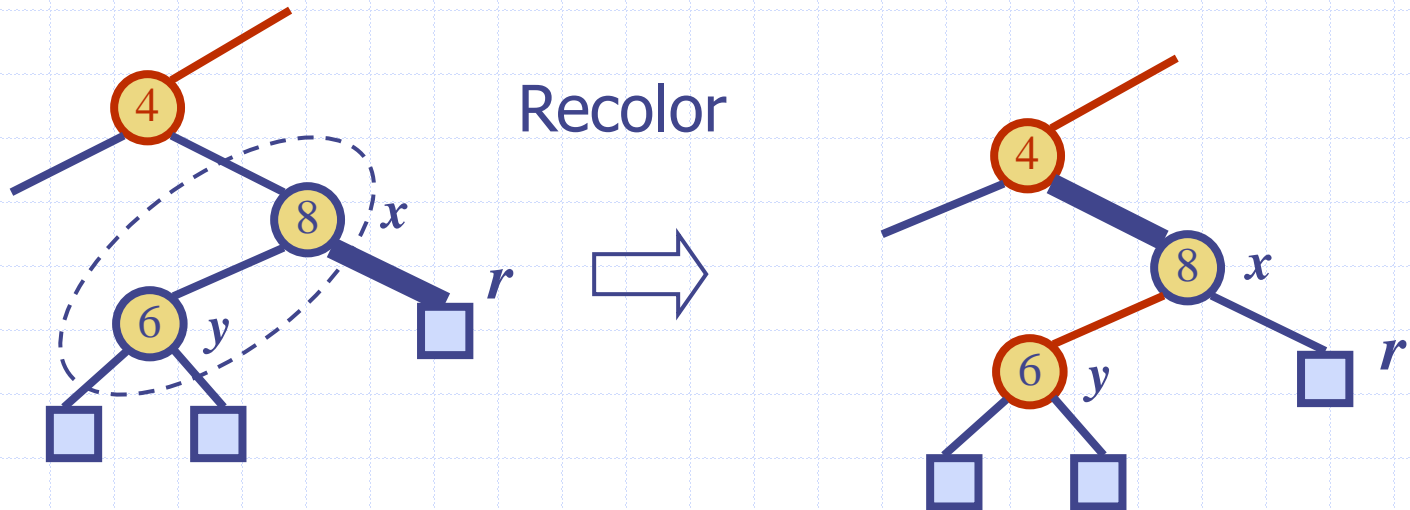
Color r black and y red;
if x is red, color x black else
color x double black

Case 3 – sibling y is red

Adjustment

Execute the trinode restructure operation that makes y the parent of x; Color y black and x red;

Then apply Case 1 or 2, with new y.

Remedying a Double Black

- ◈ The algorithm for remedying a double black node r with sibling y considers three cases

  Case 1: sibling y is black and has a red child
  - Perform a restructuring, equivalent to a transfer, and we are done

  Case 2: sibling y is black and its children are both black
  - Perform a recoloring, equivalent to a fusion, which may propagate the double black violation up to parent

  Case 3: sibling y is red
  - We perform an adjustment, equivalent to choosing a different representation of a 3-node, after which either Case 1 or Case 2 applies

- ◈ Deletion in a red-black tree takes $O(\log n)$ time

Restructuring (as Single Rotations)

◆ Single Left Rotation around **y**:

single rotation
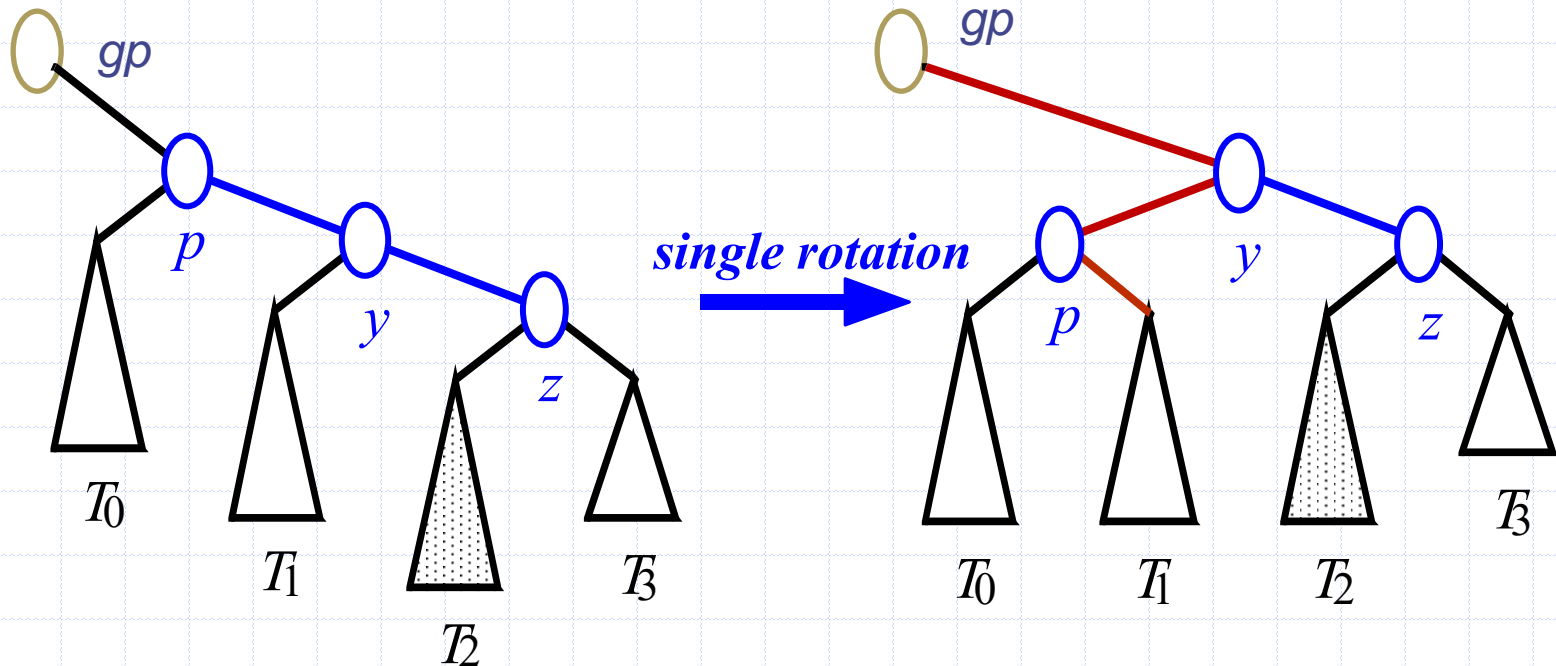
Left Rotation

Algorithm *rotateLeft*(T, *y*)
    Input Binary Tree T and node *y* in T
    Output a left rotation around node *y* is performed

    if *T.isRoot(y)* **then** **throw** InvalidLeftRotation
    p ← T.parent(y)
    gp ← T.parent(p)

    T.setRightChild(p, T.leftChild(y))
    if *T.isInternal(T.rightChild(y))* **then** {external node is a null reference}
        T.setParent(T.rightChild(y), p)

    T.setLeftChild(y, p)
    T.setParent(p, y)

    if *T.isRoot(p)*
        then *T.setRoot(y)*
        else
            if *T.rightChild(gp) = p*
                then *T.setRightChild(gp, y)*
                else *T.setLeftChild(gp, y)*
            T.setParent(y, gp)

Exercise:
Do Single Right Rotation

single rotation

gp

p

y

z

T_0

T_1

T_2

T_3

gp

z

y

p

T_0

T_1

T_2

T_3
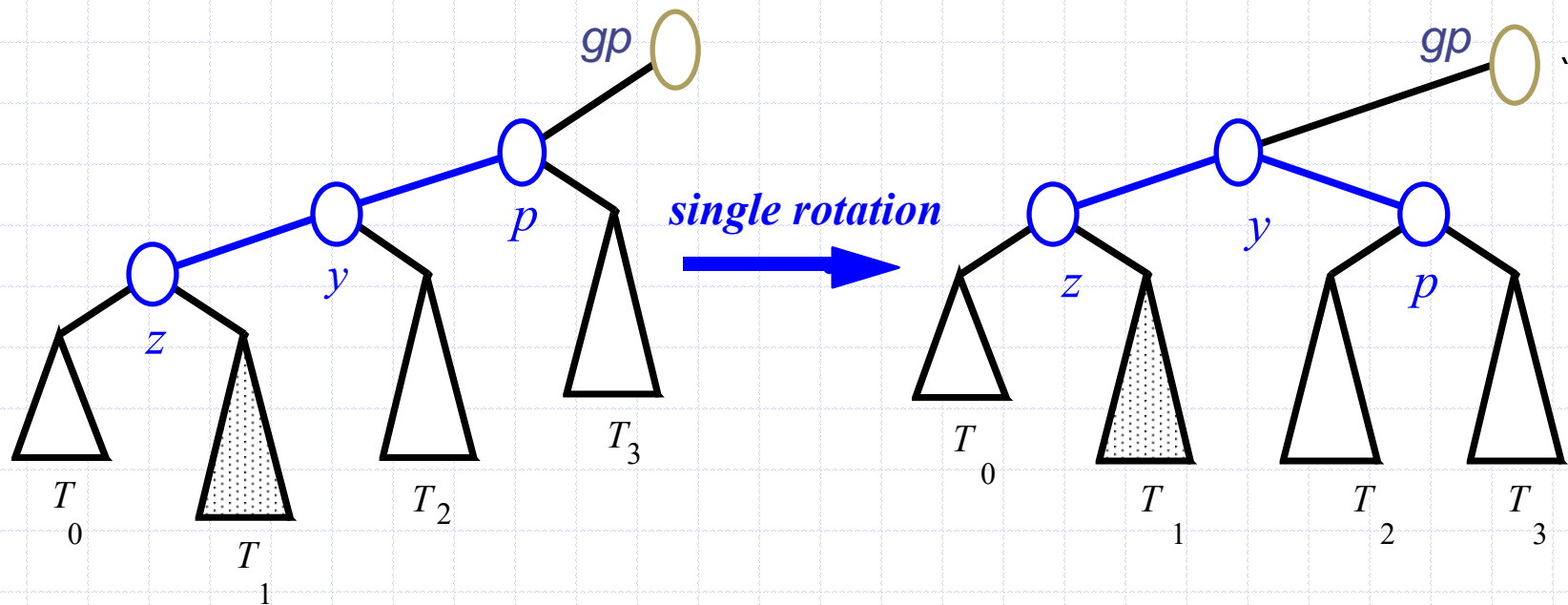
Right Rotation

Algorithm *rotateRight*(T*, y*)
 Input Binary Tree T and node *y* in T
 Output a left rotation around node *y* is performed

  if *T.isRoot(y)* **then** **throw** InvalidLeftRotation
  p ← T.parent(y)
  gp ← T.parent(p)

  T.setLeftChild(p, T.rightChild(y))
  if *T.isInternal*(*T.leftChild(y)*) **then** {external node is a null reference}
   T.setParent(*T.leftChild(y), p*)

  T.setRightChild(y, p)
  T.setParent(p, y)

  if *T.isRoot(p)*
   then *T.setRoot(y)*
   else
    if *T.rightChild(gp) = p*
     then *T.setRightChild(gp, y)*
     else *T.setLeftChild(gp, y)*
   T.setParent(y, gp)

Restructuring (as Double Rotations)

◆ double rotations:
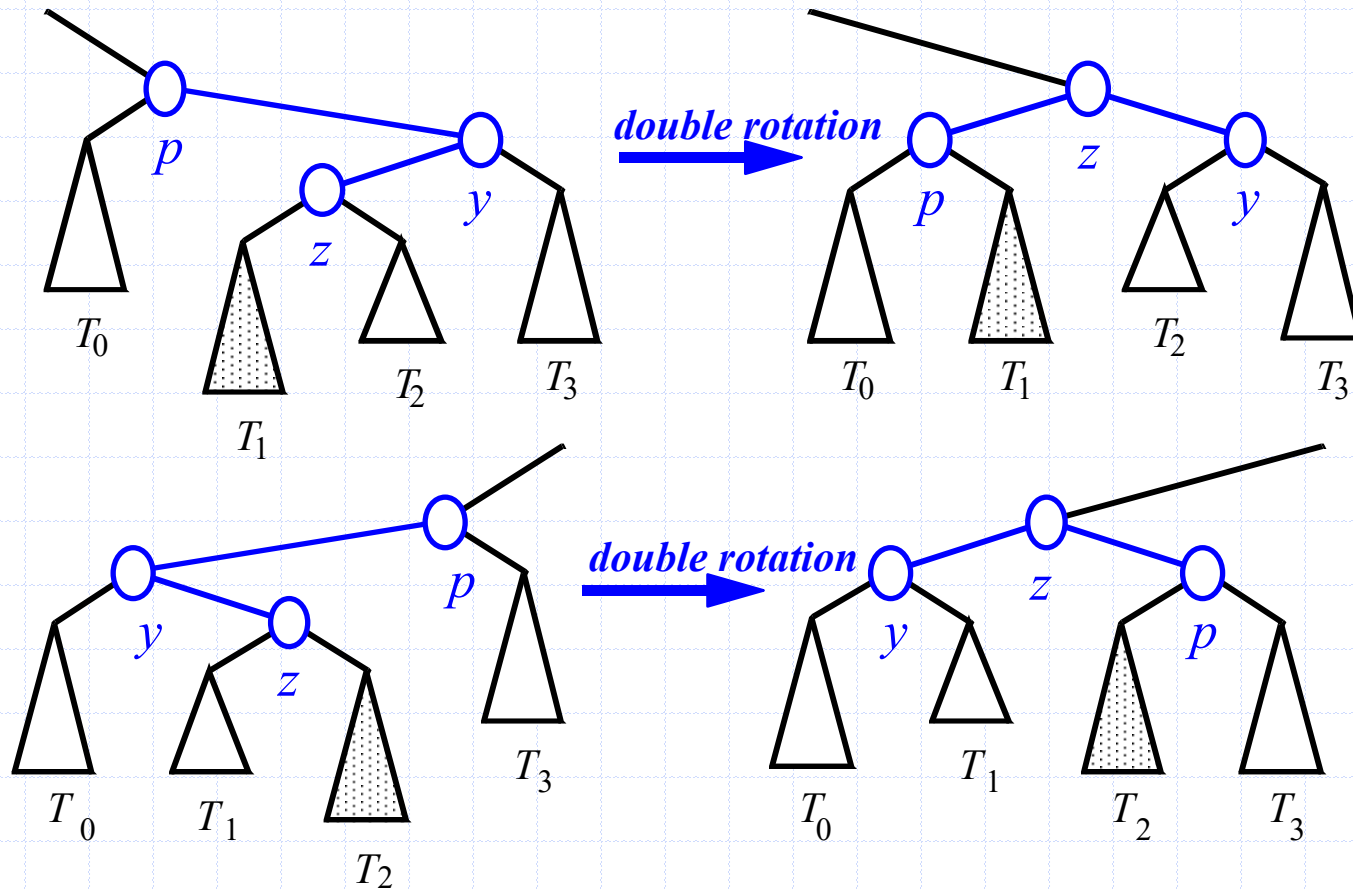
Right Left

Left Right

Deletion

◆ Let *v* be the internal node removed, *w* the external node removed, and *r* the sibling of *w*

   ■ If either *v* or *r* was red, we color *r* black and we are done

   ■ Else (*v* and *r* were both black), so we color *r* **double black** (a fictitious color), which is a violation of the internal property requiring a reorganization of the tree (denotes underflow)

◆ For example, the deletion of 8 causes a double black:
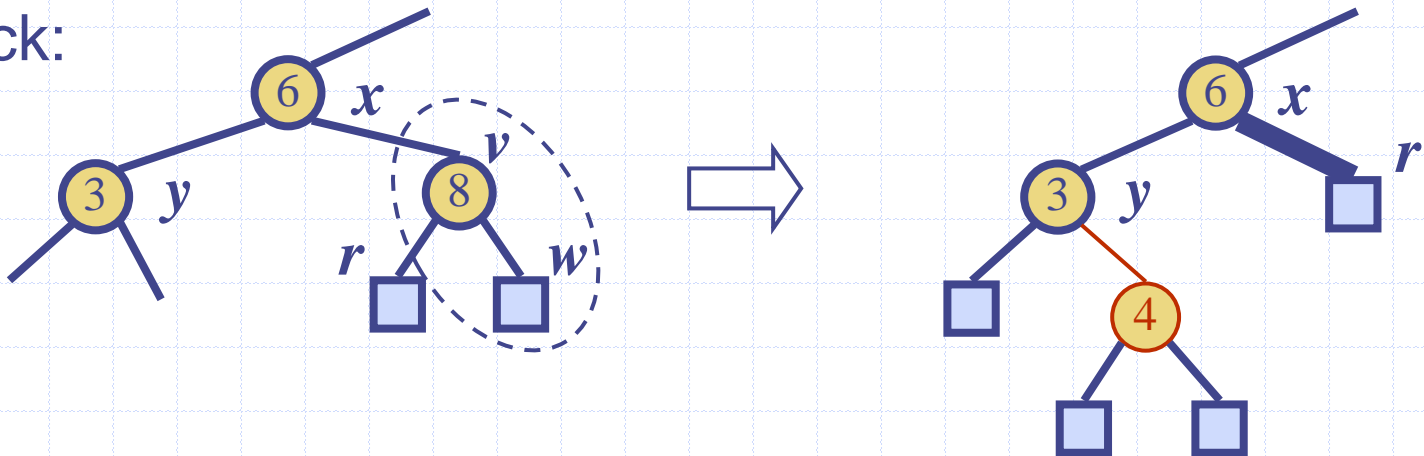
Analysis of Deletion

Algorithm *removeItem*(k)

1. $v \leftarrow$ *findNode2Remove*(k) Search for key k, then find deletion node v with external child w. Let y=*sibling*(v) and r=*sibling*(w).
2. Remove node v (removes v and w and returns r=sibling(w)). If v and r are both black then color r double black else r was red so color r black and we're done
3. **while** *isDoubleBlack*(r)
   $y \leftarrow$ *sibling*(r)
   if *isRed*(y) **then**
        $y \leftarrow$ *adjustment*(y)
   if *hasRedChild*(y) **then** //**transfer**
        $r \leftarrow$ *restructure*(r)
        return
   else {*sibling*(r) has no red child}
        $r \leftarrow$ *fusionRecolor*(r)

- Recall that a red-black tree has $O(\log n)$ height
- Step 1 takes $O(\log n)$ time because we visit $O(\log n)$ nodes
- Step 2 takes $O(1)$ time
- Step 3 takes $O(\log n)$ time because we perform
  - $O(\log n)$ recolorings, each taking $O(1)$ time, and
  - at most two restructurings taking $O(1)$ time each
- Thus, a deletion in a red-black tree takes $O(\log n)$ time

Analysis of Deletion (details)

Algorithm *findNode2Remove*(*k*)
  Input returns node *r* containing key k, node r has at least one external node
     v ← *findPosition*(*k*, *root*())
     r ← *v*
     if *isInternal*(*leftChild*(*v*)) ∧ *isInternal*(*rightChild*(*v*)) **then** // one child must be external
       r ← *findPosition*(*k*, *leftChild*(v))   // finds node containing predecessor of k
       swapElements(*v, r*)   // swaps items so r is node containing key being deleted
     return *r*   // r is the node to be deleted and contains key k unless k is not in tree

Algorithm *fusionRecolor*(*y, p, r*)
  Input *r* and *y* are siblings, *p* is their parent
     setColor(*y, RED*)
     if *isRed*(*p*) **then**
       setColor(*p, BLK*)
     else
       setColor(*p, DOUBLE_BLACK*)
     if *isInternal*(*r*) **then**
       setColor(*r, BLK*)
     return *p*

Analysis of Deletion (details)

Algorithm *removeDoubleBlack*(*y, r*)
  Input *r* is the double black node and *y* is *sibling*(*r*)
  if *isDoubleBlack*(*r*) **then**
    if *isRed*(*y*) **then** {Case 3: when *y,* the sibling of r, is red}
      y ← *adjustment*(*y*)
    p ← *parent*(*y*)
    z ← *redChildOf*(*y*)
    if *isBlack*(*z*) **then** {Case 1: when *y* has no red child}
      r ← *fusionRecolor*(*y, p, r*)
      if *isRoot*(*r*) **then**
        setColor(*r, BLK*)
      else
        removeDoubleBlack(*sibling*(*r*), *r*) // recursive call
    else {Case 2: *y* has a red child *z*, so we do a transfer}
      restructure(*z*)
      setColor(*parent*(*p*), *color*(*p*))
      setColor(*p, BLK*)
      setColor(*z, BLK*)
      if *isInternal*(*r*) **then**
        setColor(*r, BLK*) // make sure r is not external/null

Main Point

2.   Restoring balance after insertion or deletion in a red-black tree only requires a constant number of trinode restructurings (0, 1, or 2) and at most O(log n) recolorings. The red-black tree is slightly more complicated than a (2,4) tree because of restructuring, but has a major advantage in space requirements and simplifies splitting and fusion of nodes.

Science of Consciousness: The TM technique is a simple, effortless technique that restructures the physiology to a more balanced state.

Red-Black Tree Reorganization

Insertion	remedy double red	
Red-black tree action	**(2,4) tree action**	**result**
restructuring	correcting of 4-node representation	double red removed
recoloring	split	double red removed or propagated up

Deletion	remedy double black	
Red-black tree action	**(2,4) tree action**	**result**
restructuring	transfer	double black removed
recoloring	fusion	double black removed or propagated up
adjustment	change of 3-node representation	restructuring or recoloring follows

Connecting the Parts of Knowledge with the Wholeness of Knowledge

1. A (2, 4) tree offers a simple and effective way of maintaining balance in a dynamic tree structure.

2. A red-black tree offers a refinement of the (2, 4) tree by eliminating data slots and optimizing operations.

3. **<u>Transcendental Consciousness</u>** is the unbounded field of pure order and balance and is the basis of order and balance in creation.

4. **<u>Impulses within Transcendental Consciousness</u>**: The dynamic natural laws within this unbounded field create and maintain the order and balance in creation.

5. **<u>Wholeness moving within itself</u>**: In Unity Consciousness, the diversity of creation is experienced as waves of intelligence, perfectly orderly fluctuations of one's own self-referral consciousness.