Lecture 10c: Reasoning About Correctness

Spontaneous Right Action

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So far in the course

- Important basic data structures
  - Arrays, Lists, Sequences, Trees, Priority Queues, Heaps, Dictionaries, Hash Tables, and Binary Search Trees
  - Search Trees
- Important algorithms
  - Sorting (insertion, heap, PQ, merge, Quick, bucket, radix)
  - Searching (Dictionary: binary search, hash table, BST)
  - Selection (Quick, deterministic) not covered this time
- Design strategies
  - Exhaustive Search, Divide-and-Conquer, Prune-and-Search, and randomization
- Solution to recurrences
- Amortized analysis (average behavior over a large number of times running the algorithm)

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Reasoning About Loops

- Make sure the loop has a goal and it is progressing toward that goal each time through the loop
  - Make sure the loop invariant holds every time at the start and end of the loop body
- Make sure the loop terminates
  - Make sure the loop is making progress toward the terminating condition
- Check boundary conditions
  - E.g., check size 0, 1, n

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What is the loop invariant?

- An assertion that is necessarily true immediately before and immediately after each iteration of a loop
- Could be false part way through the loop, but must be re-established before the end of the loop body
- **The invariant at termination of the loop should imply the goal of the loop has been achieved!!!!**

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Binary Search Algorithm (What's wrong)

Algorithm BinarySearch(*S, k*):
Input: *key k and Sequence S storing n items, sorted by item.key()*
Output: *the value associated with key k or* **NO_SUCH_KEY**

```
low ← 0
high ← S.size() - 1
while low <  high do
    mid ← (low + high)/2
    if k = key(S.elemAtRank(mid))  then
        return  value(S.elemAtRank(mid))
    if k < key(S.elemAtRank(mid))  then
        high ← mid - 1
    else
        low ← mid + 1
return NO_SUCH_KEY
```

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Error in binary search

- Does not handle the case when low equals high (boundary condition)
  - When the segment is size 1, the key may not be found because we do not enter the loop

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Binary Search Algorithm (Corrected)

```
Algorithm BinarySearch(S, k):
 Input: key k and Sequence S storing n items, sorted by item.key()
 Output: the value associated with key k or NO_SUCH_KEY
 low ← 0
 high ← S.size() - 1
 while low ≤ high do
     mid ← (low + high)/2
     if k = key(S.elemAtRank(mid))  then
        return  value(S.elemAtRank(mid))
     if k < key(S.elemAtRank(mid))  then
        high ← mid - 1
     else
        low ← mid + 1
 return NO_SUCH_KEY
```

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Introducing Errors through Copy-Paste

- We wish to have only one key comparison during each iteration of the loop
- So we copy from above version then modify as described
  - Move the check for equality after the loop
  - Now we do not exit the loop early however we do half the key comparisons during each iteration

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Binary Search Algorithm (what's wrong? Look at red)

```
Algorithm BinarySearch(S, k):
  Input: An ordered Sequence S storing n items, accessed by keys()
  Output: An element of S with key k.
  low ← 0
  high ← S.size() - 1
  while low ≤ high do
     mid ← (low + high)/2
     if  k < key(S.elemAtRank(mid))  then // one key comparison per iteration
        high ← mid - 1
      else
        low ← mid + 1
  if  k = key(S.elemAtRank(mid))   then  // done once outside the loop now
     return value(S.elemAtRank(mid))
  else
     return NO_SUCH_KEY
```

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Errors

- Does not handle a Sequence with 0 elements
  - mid is not initialized since loop is not entered and, further, it cannot be initialized to handle an empty Sequence
- Does not handle a Sequence with 1 element that matches k
  - The else eliminates mid when it hasn't yet been eliminated, so delete the + 1 from the else branch

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Binary Search Algorithm (better, but what else is wrong?)

```
Algorithm BinarySearch(S, k):
  Input: An ordered Sequence S storing n items, accessed by keys()
  Output: An element of S with key k.
  low ← 0
  high ← S.size() - 1
  while low ≤ high do
     mid ← (low + high)/2
     if  k < key(S.elemAtRank(mid))   then // one key comparison per iteration
        high ← mid - 1
      else
        low ← mid    // eliminate + 1 because mid has not been eliminated yet
  if S.size() > 0  Λ  k = key(S.elemAtRank(mid))   then   // handles empty S
     return value(S.elemAtRank(mid))
  else
     return NO_SUCH_KEY
```

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Error: loop does not terminate

- Does not handle a Sequence with 1 item (or a segment with 1 item) when its key matches k
  - The loop does not terminate
    - Modify the loop condition from ≤ to < so the loop terminates when high = low since low does not change when the key of the item equals k
- The rank mid may not contain the item with the key after fixing the loop's terminating condition
  - Either low or high will contain the key if it is in the Sequence
  - Fixing this eliminates the need to initialize mid before the loop since mid will only used inside the loop now

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Binary Search Algorithm (red shows corrections)

```
Algorithm BinarySearch(S, k):
  Input: An ordered Sequence S storing n items, accessed by keys()
  Output: An element of S with key k.
  low ← 0
  high ← S.size() - 1
  while low < high do                      // needs to be < to terminate
     mid ← (low + high + 1)/2              // needs to be the ceiling to terminate
     if k < key(S.elemAtRank(mid))  then // one key comparison per iteration
        high ← mid - 1
      else
        low ← mid  // + 1 because mid has not been eliminated yet
  if S.size() > 0 ∧ k = key(S.elemAtRank(high))  then // handles empty S
     return value(S.elemAtRank(high))   // high or low contain matching key
  else
     return NO_SUCH_KEY
```

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Binary Search Algorithm (change < to ≤ in the loop)

```
Algorithm BinarySearch(S, k):
  Input: An ordered Sequence S storing n items, accessed by keys()
  Output: An element of S with key k.
  low ← 0
  high ← S.size() - 1
  while low < high do                      // needs to be < to terminate
     mid ← (low + high + 1)/2              // needs to be the ceiling to terminate
     if k ≤ key(S.elemAtRank(mid))  then // change to ≤ instead of <
        high ← mid          // changed due to change of condition
      else
        low ← mid + 1       // changed due to change of condition
  if S.size() > 0 ∧ k = key(S.elemAtRank(high))  then // handles empty S
     return value(S.elemAtRank(high))   // high or low contain matching key
  else
     return NO_SUCH_KEY
```

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Errors

- The loop does not always terminate
  - mid needs to be the floor of the expression otherwise mid and high do not/cannot change which causes non-termination

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Binary Search Algorithm (loop now terminates)

```
Algorithm BinarySearch(S, k):
  Input: An ordered Sequence S storing n items, accessed by keys()
  Output: An element of S with key k.
  low ← 0
  high ← S.size() - 1
  while low < high do                      // needs to be < to terminate
     mid ← (low + high)/2                  // needs to be the floor to terminate
     if k ≤ key(S.elemAtRank(mid))  then // changed to ≤ instead of <
        high ← mid          // removed – 1 (since mid is not eliminated)
      else
        low ← mid + 1       // changed
  if S.size() > 0 ∧ k = key(S.elemAtRank(low))  then // handles empty S
     return value(S.elemAtRank(low))   // high or low contain matching key
  else
     return NO_SUCH_KEY
```

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Why a third version?

- Depends on the purpose
- The third version is an improvement in the binary search used by the Lookup Table

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Errors (none)

- Handles a Sequence with 0 elements
- Handles a Sequence with 1 element that matches the key k
- We do not want the ceiling((high+low)/2) this time
- The loop terminates
  - mid is initialized correctly with the floor of the expression (does not add 1)
- Handles a Sequence with 2 elements (or a segment with 2 elements) with one matching the key k
  - Two cases: first and second element
- Finds the key when it is in the Sequence by using rank low although could have left it as high

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The loop invariant of the loop in function BinarySearch

if the key k is in the Sequence S, then
S.elemAtRank(low) ≤ k ≤ S.elemAtRank(high)

- Informally, if key k is in the Sequence S, then k is the key of an item in S at a rank between low and high

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Is it worth exiting early from the loop?

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Binary Search Algorithm (Two comparisons per iteration)

```
Algorithm BinarySearch(S, k):
  Input: An ordered Sequence S storing n items, accessed by keys()
  Output: An element of S with key k.
  low ← 0
  high ← S.size() - 1
  while low ≤ high do
      mid ← (low + high)/2
      if k = key(S.elemAtRank(mid)) then {exit early from the loop}
         return value(S.elemAtRank(mid))
      else if k < key(S.elemAtRank(mid)) then
         high ← mid - 1
      else
         low ← mid + 1
  return NO_SUCH_KEY
```

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Binary Search Algorithm (One comparison per iteration)

```
Algorithm BinarySearch( S, k ):
  Input: An ordered Sequence S storing n items, sorted by keys()
  Output: An item of S with key k and rank between low & high.
  low ← 0
  high ← S.size() - 1
  while low < high do
     mid ← (low + high + 1)/2
     if k < key(S.elemAtRank(mid)) then {always does log n comparisons}
        high ← mid - 1
     else
        low ← mid   // + 1  does not yet eliminate mid
  if S.size() > 0  Λ  k = key(S.elemAtRank(low)) then
     return value(S.elemAtRank(low))
  else
     return NO_SUCH_KEY
```

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Homework

- Both algorithms make $O(\log n)$ key comparisons
- Which algorithm makes fewer actual key comparisons when the key is not in S?
- Which makes fewer comparisons, **on average**, when the key is in S, assuming all keys are equally likely to be searched?

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What's Wrong with this In Place Version of Partition

```
Algorithm inPlacePartition(S, lo, hi)
    Input Sequence S and ranks lo and hi, 0 ≤ lo ≤ hi < S.size()
    Output the pivot is now stored at its sorted rank

    p ← a random integer between lo and hi
    S.swapElements(S.atRank( lo ), S.atRank( p ))
    pivot ← S.elemAtRank(lo)
    j ← lo + 1
    k ← hi
    while  j ≤ k  do
        while  k > j   ∧  S.elemAtRank( k ) ≥ pivot   do
            k ←  k – 1
        while  j <  k   ∧  S.elemAtRank( j ) ≤ pivot   do
            j ←  j + 1

        if  j < k  then
            S.swapElements(S.atRank( j ), S.atRank( k ))

    S.swapElements(S.atRank( lo ), S.atRank( k )) {move pivot to sorted rank}
    return k
```

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Error

- Does not terminate!
- Some swaps could incorrectly or unnecessarily move elements/items

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Corrected In Place Version of Partition

```
Algorithm inPlacePartition(S, lo, hi)
    Input Sequence S and ranks lo and hi, 0 ≤ lo ≤ hi < S.size()
    Output the pivot is now stored at its sorted rank

    p ← a random integer between lo and hi
    S.swapElements(S.atRank( lo ), S.atRank( p ))
    pivot ← S.elemAtRank(lo)
    j ← lo + 1
    k ← hi
    while  j ≤ k  do
        while  k ≥ j  ∧ S.elemAtRank( k ) ≥ pivot  do
            k ← k − 1
        while  j ≤ k  ∧ S.elemAtRank( j ) ≤ pivot  do
            j ← j + 1

        if  j < k  then
            S.swapElements(S.atRank( j ), S.atRank( k ))

    S.swapElements(S.atRank( lo ), S.atRank( k )) {move pivot to sorted rank}
    return k
```

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