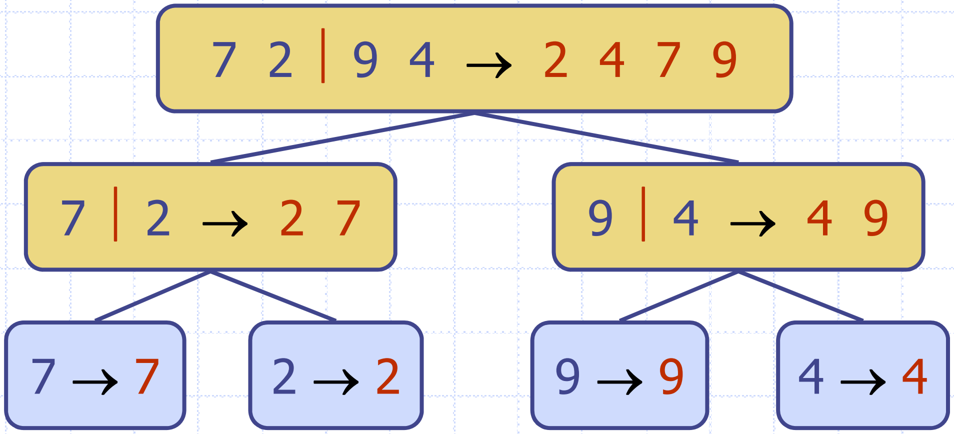
Lesson 5
Merge Sort: Collapsing Infinity To A Point



Divide-and-Conquer

- Divide-and conquer is a general algorithm design paradigm. Special case:
  - Divide: divide the input data S in two disjoint subsets S_1 and S_2
  - Conquer: solve the subproblems associated with S_1 and S_2
  - Combine: combine the solutions for S_1 and S_2 into a solution for S
- The base case for the recursion are subproblems of size 0 or 1
- Merge-sort is a sorting algorithm based on the divide-and-conquer paradigm

Merge-Sort

- Merge-sort on an input sequence S with n integers consists of three steps:
  - Divide: partition S into two sequences S_1 and S_2 of about $n/2$ elements each
  - Conquer: recursively sort S_1 and S_2
  - Combine: merge S_1 and S_2 into a single sorted sequence

Algorithm ***mergeSort***(S)
    Input sequence S with n
    Output sequence S sorted
    if $S.size() > 1$ **then**
        $(S_1, S_2) \leftarrow partition(S, n/2)$
        $mergeSort(S_1)$
        $mergeSort(S_2)$
        $S \leftarrow merge(S_1, S_2)$
    return S

Merging Two Sorted Sequences

- The *combine* step of merge-sort consists of merging two sorted sequences A and B into a sorted sequence S containing the union of the elements of A and B
- Merging two sorted arrays, each with $n/2$ elements takes $O(n)$ time

Algorithm *merge*(A, B)
    Input sequences A and B with
        $n/2$ integers each
    Output sorted sequence S of $A \cup B$

    $S \leftarrow$ empty sequence
    while $\neg A.isEmpty() \land \neg B.isEmpty()$ **do**
        if $A.first() <= B.first()$ **then**
            $S.insertLast(A.remove(A.first()))$
        else
            $S.insertLast(B.remove(B.first()))$
    while $\neg A.isEmpty()$ **do**
        $S.insertLast(A.remove(A.first()))$
    while $\neg B.isEmpty()$ **do**
        $S.insertLast(B.remove(B.first()))$
    return S

Implementation of Merge

```
public void merge(int[] tempStorage,
                  int lowerPointer,
                  int upperPointer,
                  int upperBound) {
   int j = 0; //tempStorage index
   int lowerBound = lowerPointer;
   //total number of elements to rearrange
   int n = upperBound - lowerBound + 1;
   //view the range [lowerBound,upperBound] as two arrays
   //[lowerBound, mid], [mid+1,upperBound] to be merged
   int mid = upperPointer -1;
   while(lowerPointer <= mid && upperPointer <= upperBound){
      if(theArray[lowerPointer] <= theArray[upperPointer]){
          tempStorage[j++] = theArray[lowerPointer++];
      }
      else {
          tempStorage[j++] = theArray[upperPointer++];
      }
   }
```

Merge (continued)

```
   //left array may still have elements
   while(lowerPointer <= mid) {
      tempStorage[j++] = theArray[lowerPointer++];
   }
   //right array may still have elements
   while(upperPointer <= upperBound){
      tempStorage[j++] = theArray[upperPointer++];
   }
   //replace the range [lowerBound,upperBound] in theArray with
   //the range [0,n-1] just created in tempStorage
   for(j=0; j<n; ++j) {
      theArray[lowerBound+j] = tempStorage[j];
   }
}
```

Implementation of MergeSort, In-Place

```
int[] theArray;

//public sorter
public int[] sort(int[] input){
    int n = input.length;
    int[] tempStorage = new int[n];
    theArray = input;
    mergeSort(tempStorage,0,n-1);
    return theArray;
}
```

(continued)

```
void mergeSort(int[] temp, int lower, int upper) {
        if(lower==upper){
                return;
        }
        else {
                int mid = (lower+upper)/2;
                mergeSort(temp,lower,mid);
                mergeSort(temp,mid+1, upper);
                merge(temp,lower,mid+1,upper);

        }
}
```

Worst-case Analysis

$T(n) = T(\lfloor n/2 \rfloor) + T(\lceil n/2 \rceil) + O(n)$

(or we could write $T(n) \leq T(\lfloor n/2 \rfloor) + T(\lceil n/2 \rceil) + c\,n$)

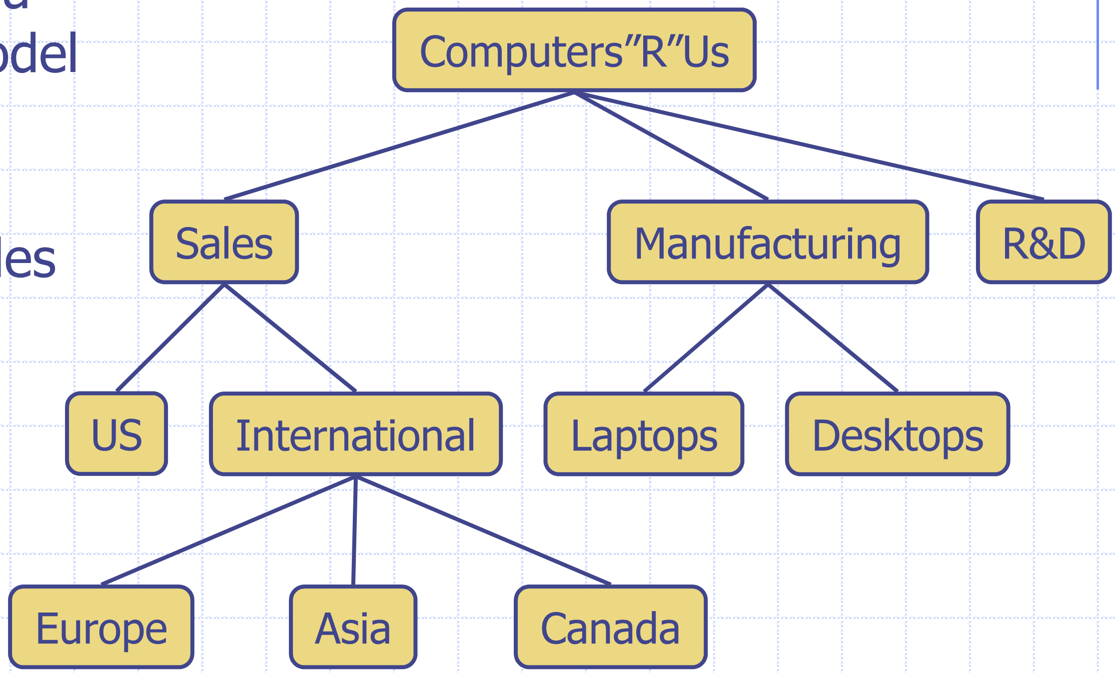
1. Assuming n a power of 2, this becomes
   $T(n) = 2T(n/2) + O(n)$

2. By the Guessing Method, we conclude
   $T(n)$ is $O(n\log n)$ (n a power of 2)

3. Verify that $n\log n$ and $T(n)$ are nondecreasing and argue that
   $T(n)$ is $O(n\log n)$ (all n)

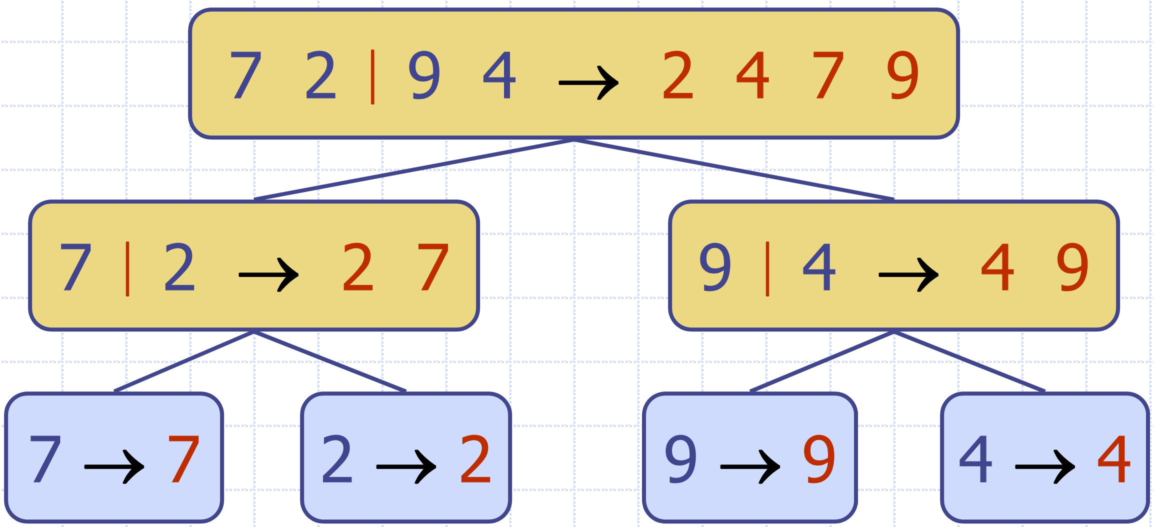
What is a Tree

- In computer science, a tree is an abstract model of a hierarchical structure
- A tree consists of nodes with a parent-child relation
- Applications:
  - Organization charts
  - File systems
  - Programming environments



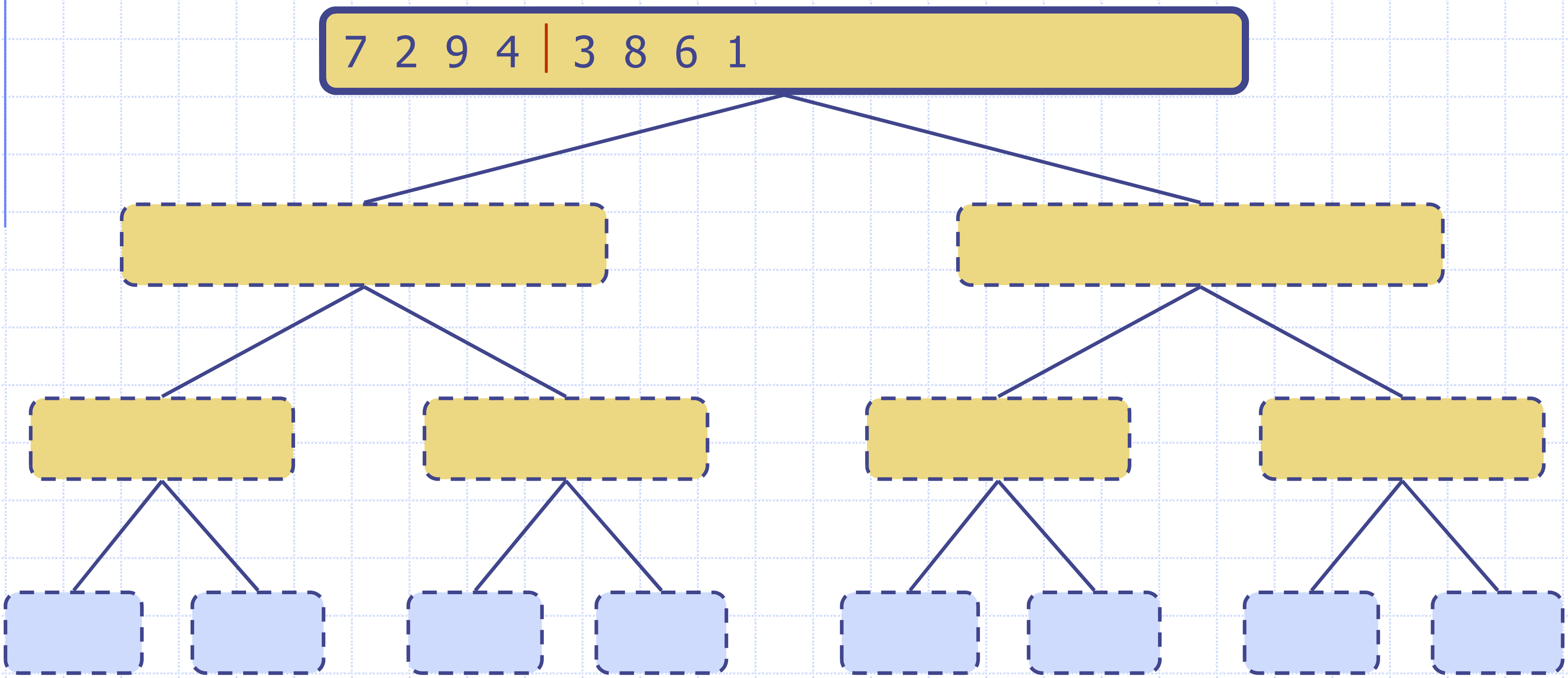
Merge-Sort Tree

- An execution of merge-sort may be depicted by a binary tree
  - each node represents a recursive call of merge-sort and stores
    - unsorted sequence before the execution and its partition
    - sorted sequence at the end of the execution
  - the root is the initial call
  - the leaves are calls on subsequences of size 0 or 1



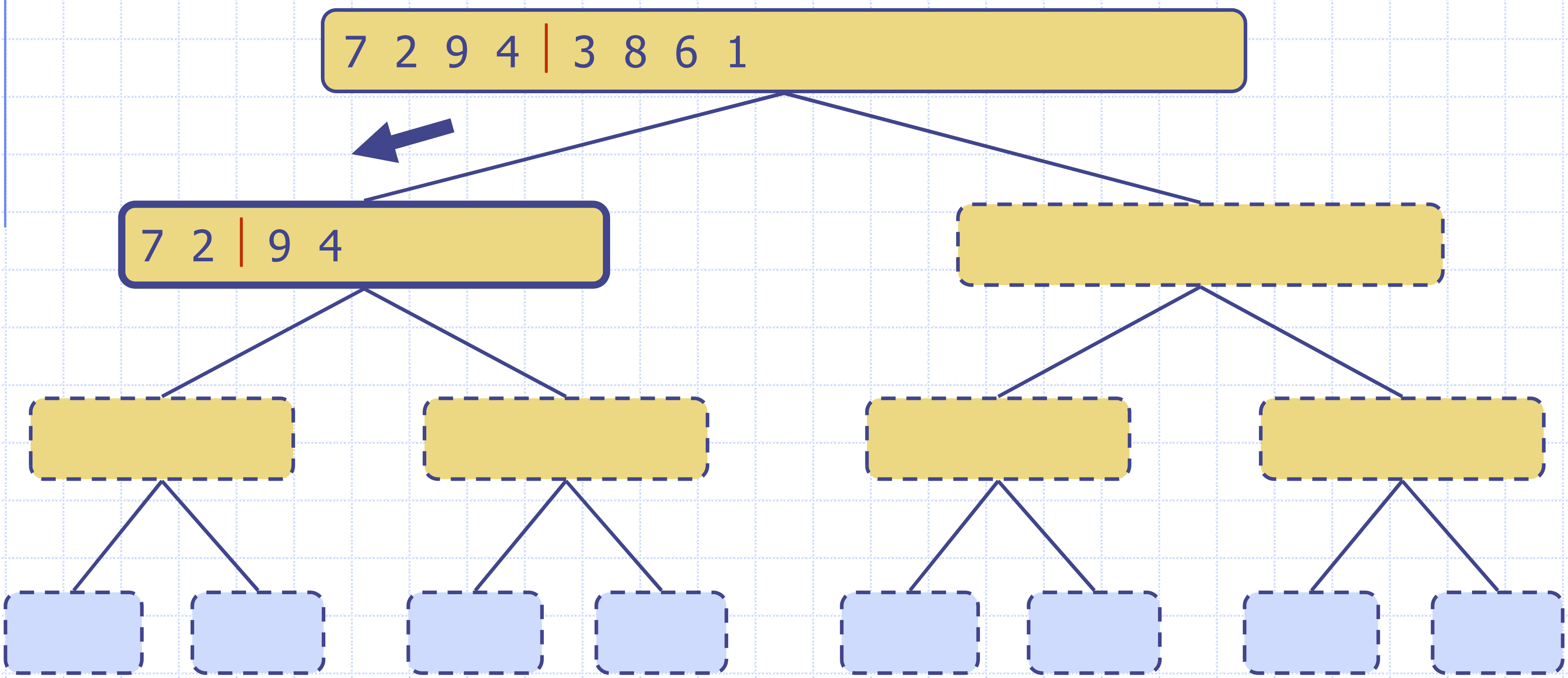
Execution Example

- Partition



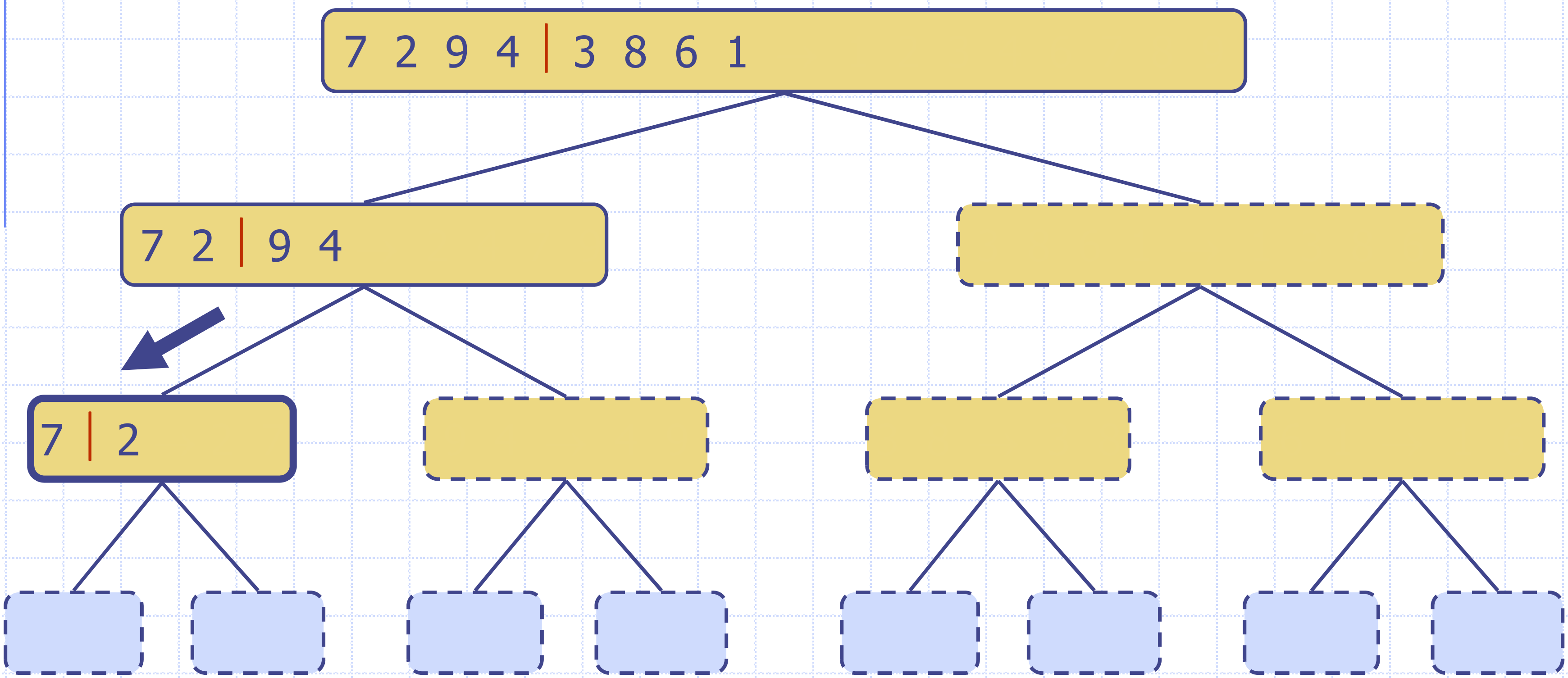
Execution Example (cont.)

- Recursive call, partition



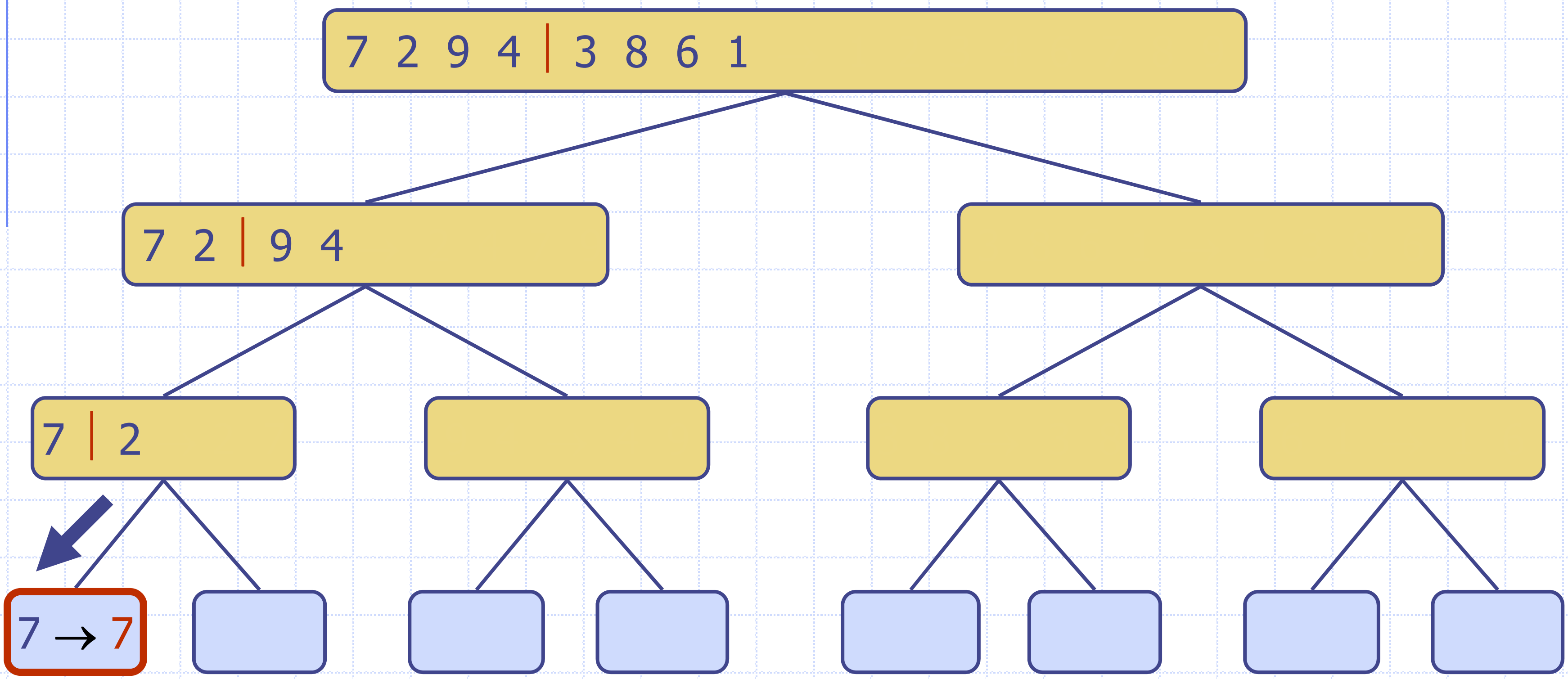
Execution Example (cont.)

- Recursive call, partition



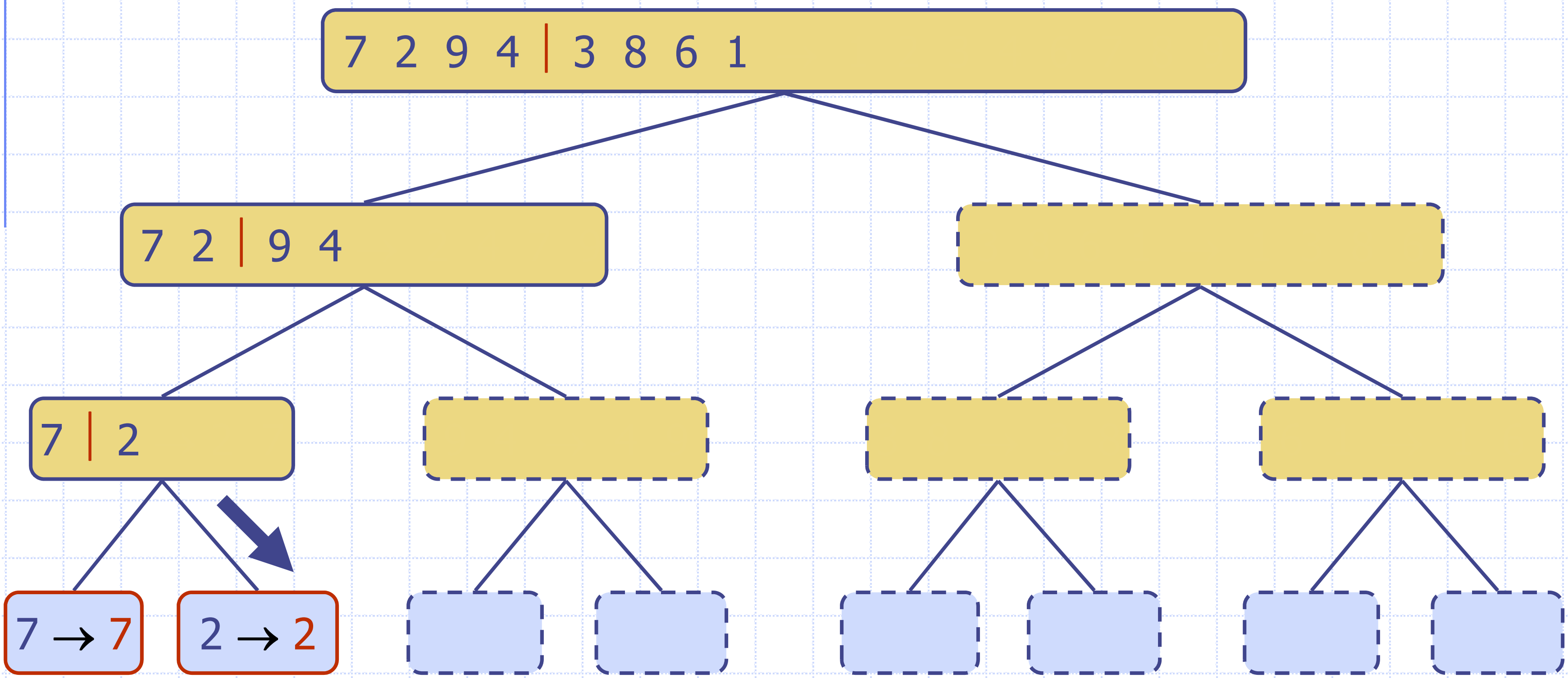
Execution Example (cont.)

- Recursive call, base case



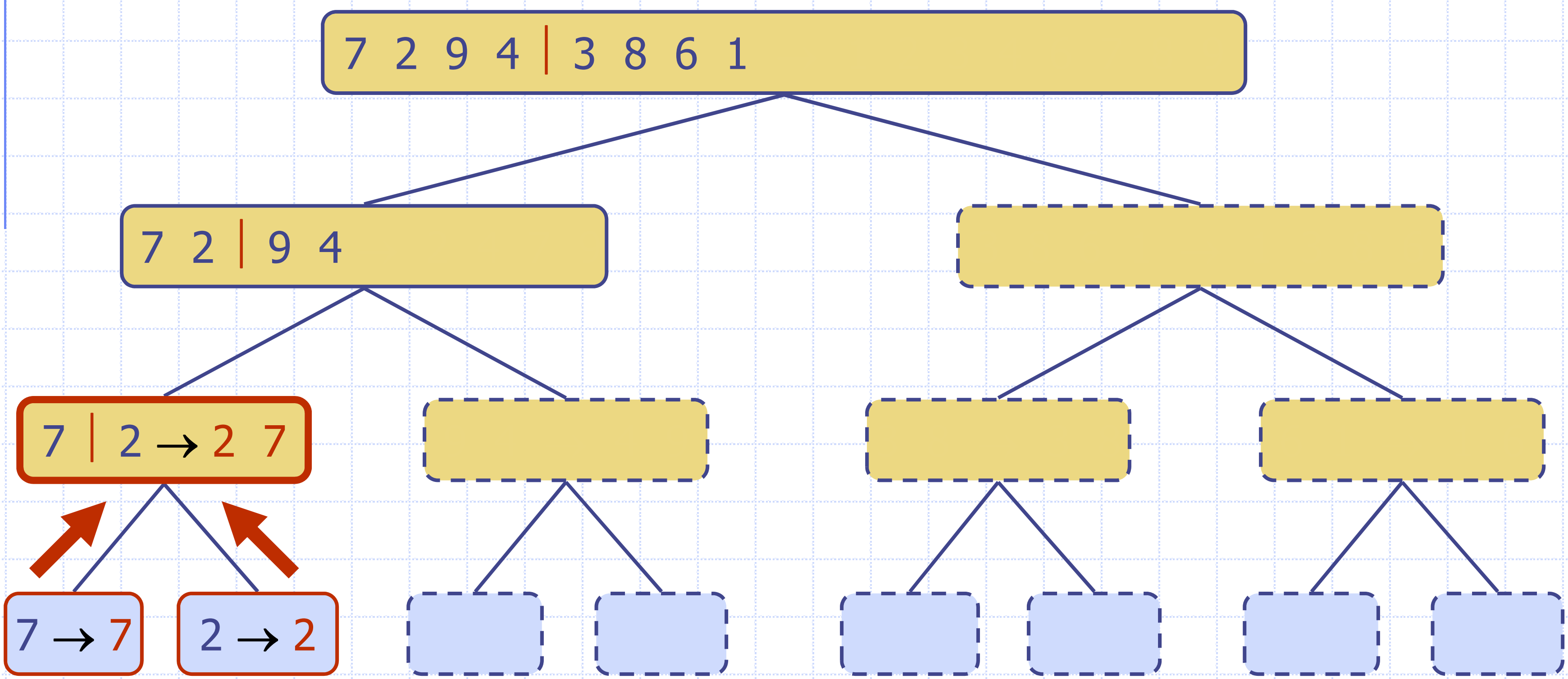
Execution Example (cont.)

- Recursive call, base case



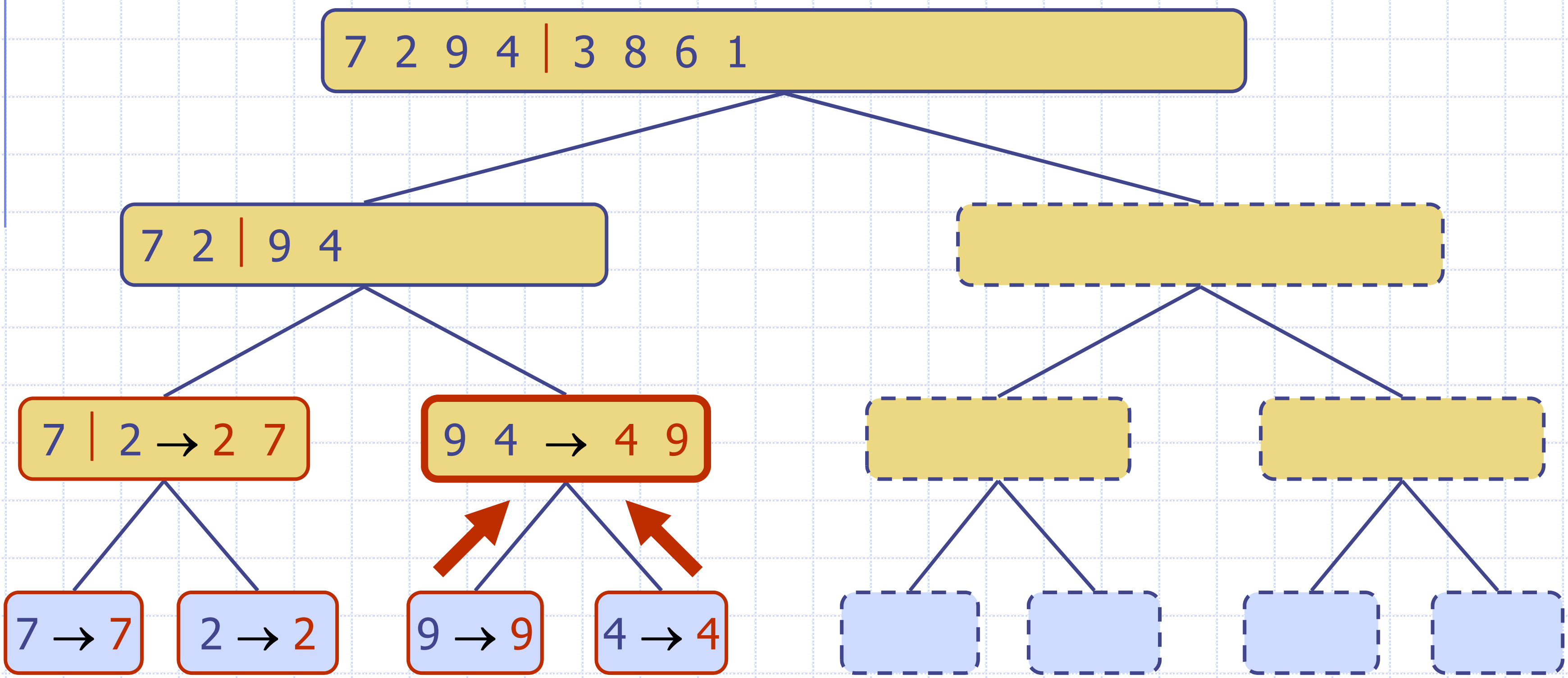
Execution Example (cont.)

- Merge



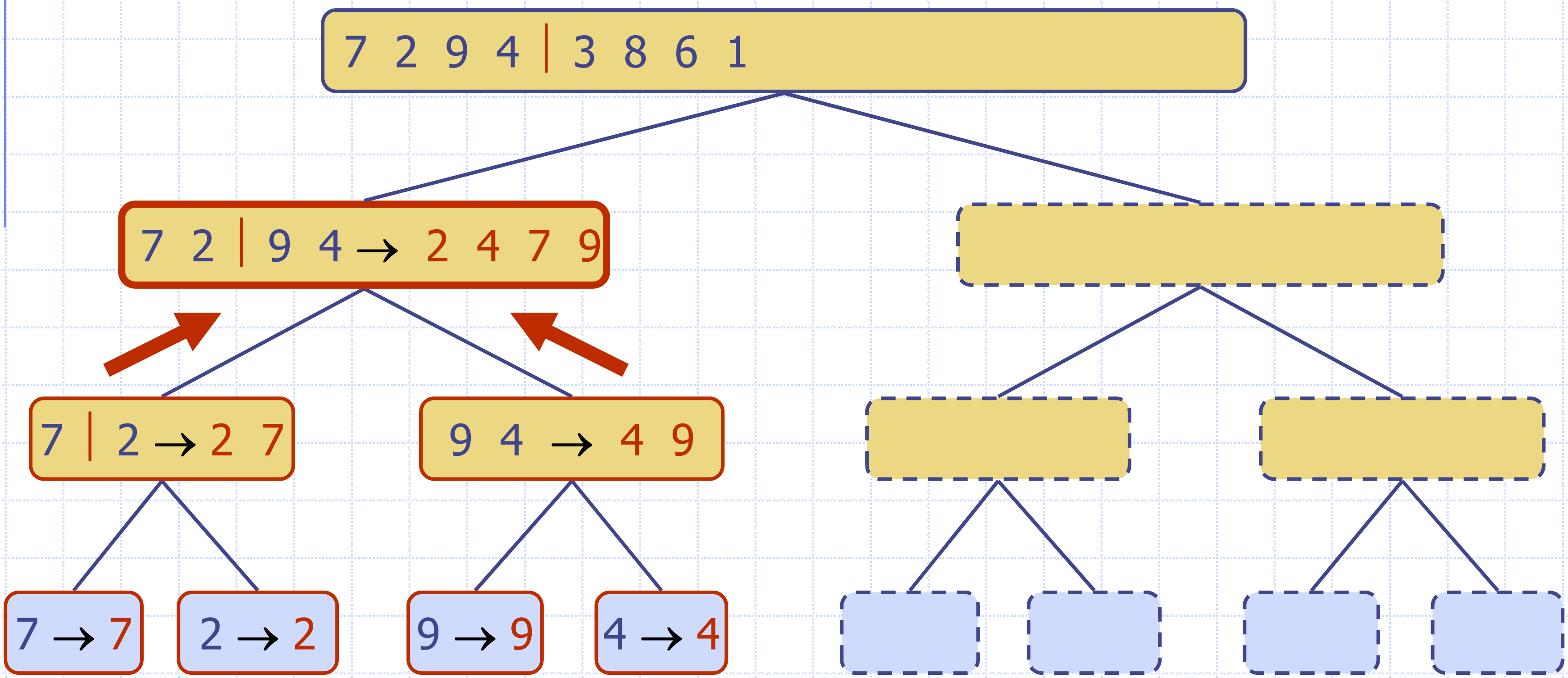
Execution Example (cont.)

- Recursive call, …, base case, merge



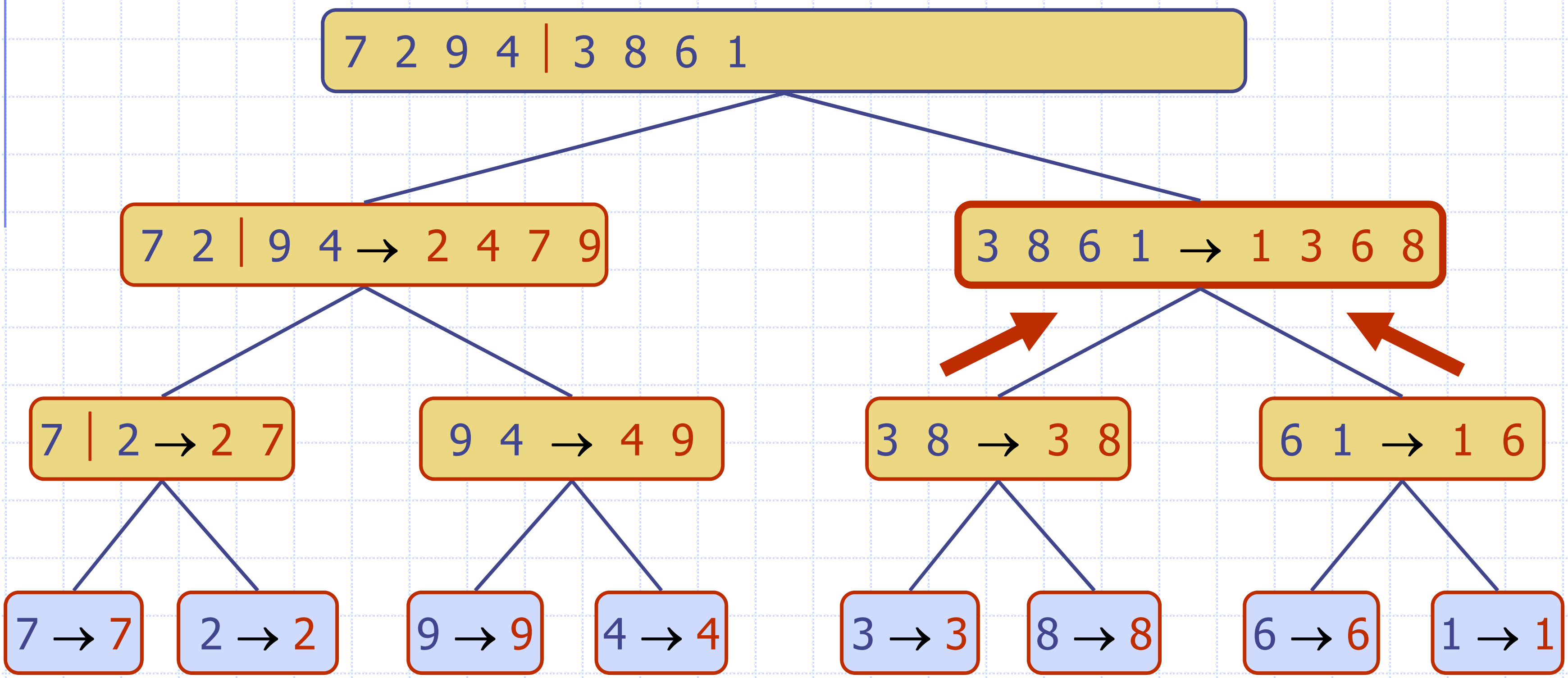
Execution Example (cont.)

- Merge



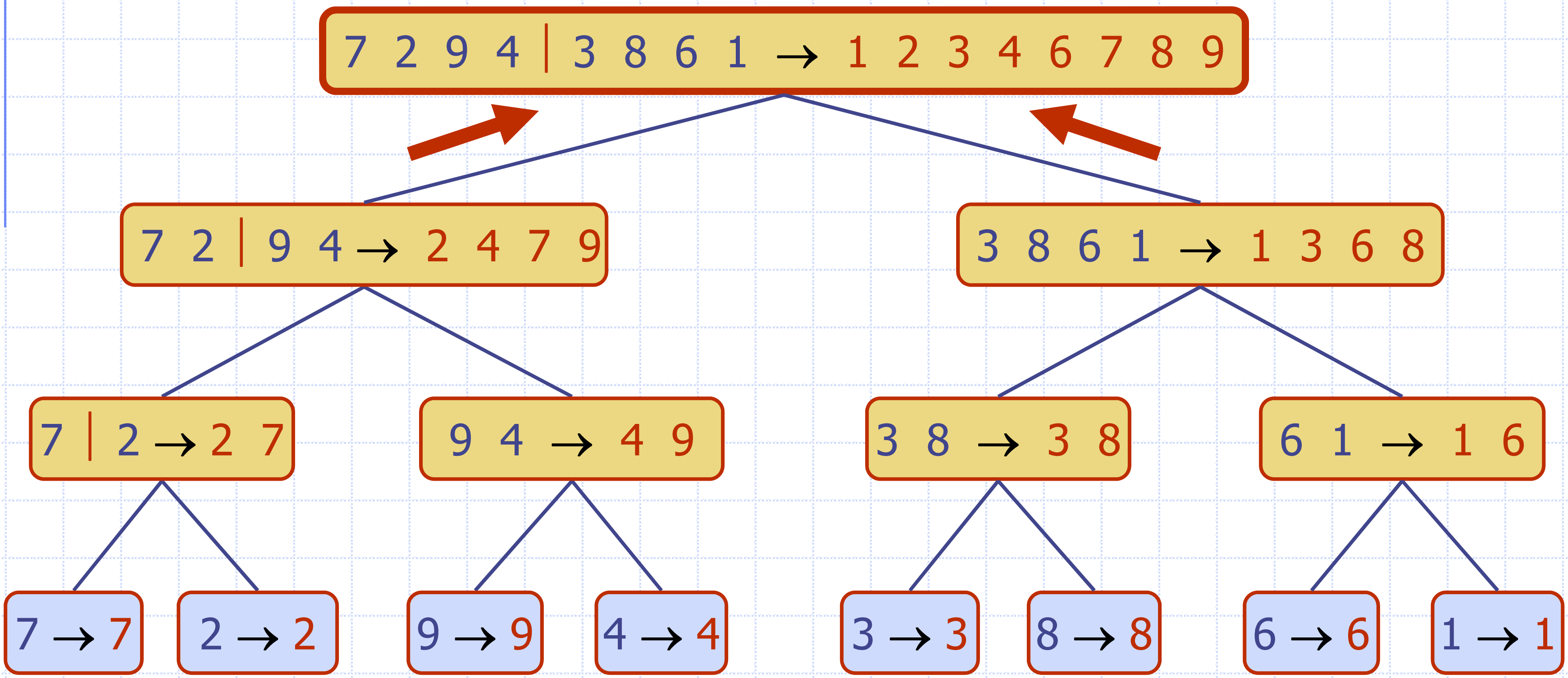
Execution Example (cont.)

- Recursive call, …, merge, merge



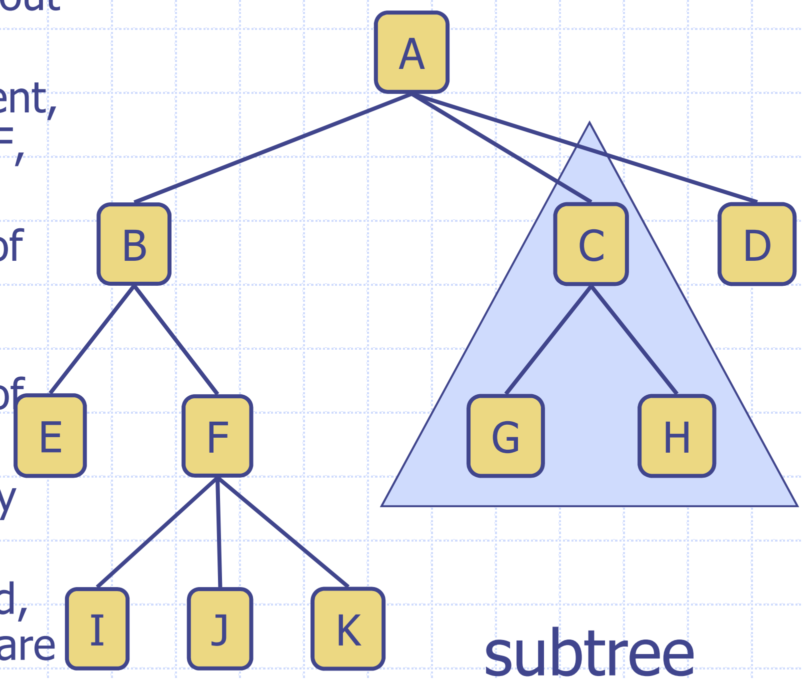
Execution Example (cont.)

- Merge

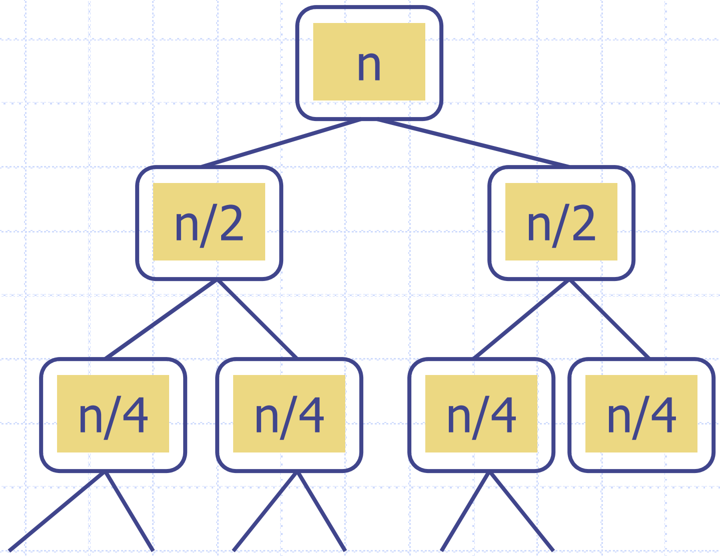


Tree Terminology

- ***Root***: node without parent (A)
- ***Internal node***: node with at least one child (A, B, C, F)
- ***Leaf*** (or "external") ***node*** is a node without children (E, I, J, K, G, H, D)
- ***Ancestors of a node***: parent, grandparent, grand-grandparent, etc. (ancestors of K: F, B, A)
- ***Depth of a node***: number of ancestors of the node (depth of K = 3)
- ***Levels of a tree***: Level n of a tree is the set of all nodes having depth n. (Level 1 of this tree is {B, C, D})
- ***Height of a tree***: maximum depth of any node (height of tree = 3)
- ***Descendant of a node***: child, grandchild, grand-grandchild, etc. (descendants of B are E, F, I, J, K)
- ***Subtree***: tree consisting of a node and its descendants



Tree Exercise



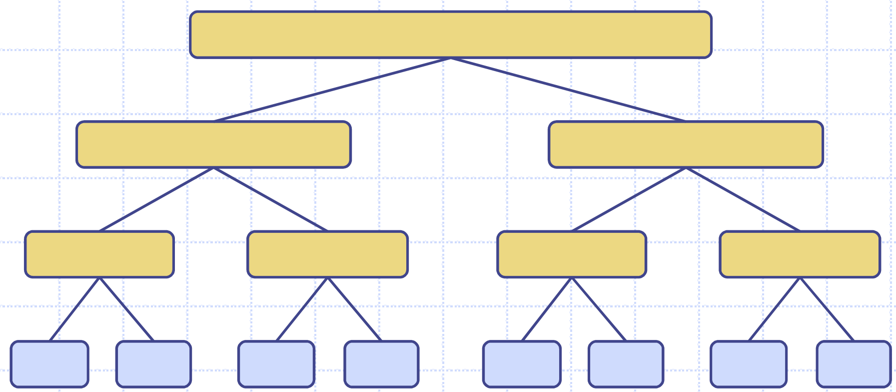
Continue building the tree above until each node at the bottom level contains "1", but no node at a previous level contains a "1"

1. Assuming n is a power of 2, what is the height of the tree?
2. Assuming n is any positive integer, what is the asymptotic height of the tree (assume division operations are now "integer division")?
3. Asymptotically, what is the sum of all values contained in the nodes in the tree?

Alternate Analysis of Merge-Sort

- The height h of the merge-sort tree is $O(\log n)$
  - at each recursive call we divide the sequence in half (See Exercise)
- The overall amount or work done at the nodes of depth i is $O(n)$
  - we partition and merge 2^i sequences of size $n/2^i$
  - we make 2^{i+1} recursive calls
- Thus, the total running time of merge-sort is $O(n \log n)$

depth	#seqs	size
0	1	n
1	2	$n/2$
i	2^i	$n/2^i$
…	…	…



Comparison With Other Sorting Algorithms

- Demo confirms that MergeSort's O(nlog n) estimated running time is truly much faster than those of the inversion-bound algorithms and LibrarySort

- Can see why MergeSort is not inversion bound by example: [4, 3, 2, 1]:

  #inversions = 6

  #comparisons = 4

Main Point

By using a Divide and Conquer strategy, MergeSort overcomes the limitations that prevent inversion-bound sorting algorithms from performing faster than n^2. An essential characteristic of this strategy is the relationship of whole to part – wholes are successively collapsed and the collapsed values are combined to produce a new whole. This is different from the incremental approach of inversion-bound algorithms. We see here an application of the MVS principle of *akshara:* Creation arises in the collapse of the unbounded value of wholeness to a point.

Handling Duplicates

- Issue arises during the merge step – if element in left half equals element in right half, insert element in left half first

		d		

		d		

- Stability

Name	Date Received
. . .	. . .
Dave	11/5/2003
Dave	12/1/2004
Dave	1/8/2005
Dave	4/2/2006
. . .	. . .

If you sort by date, then by name, you want date field to remain sorted.

Handling Duplicates (cont)

- Definition. Suppose

  $$S = <(k_0,e_0), (k_1,e_1),\ldots,(k_n,e_n)>$$

  is a list of pairs with keys k_0, k_1, ..., k_n. A sorting algorithm is *stable* if, whenever it is the case that (k_i, e_i) precedes (k_j, e_j) before sorting (so that $i < j$) and $k_i = k_j$, then it continues to be true after sorting by keys that the pair (k_i, e_i) precedes (k_j, e_j)

Stable sorting does not change the order of duplicates

Stability of Sorting Algorithms

- MergeSort is stable because of our strategy for handling duplicates during Merge
- Are InsertionSort, BubbleSort, SelectionSort stable?

Main Point

Stability of a sorting algorithm requires maintenance of nonchange in the midst of change. This is an example in the world of sorting routines of the inner dynamics of outward success, as described in SCI: The more the inner quality of awareness remains established in silence, the more outer dynamism is supported for success and fulfillment.