STK 210: Practical 7

Two caplets are selected at random in succession from a bottle containing 3 aspirin, 2 sedative and 4 laxative caplets.

- **Given:**

$$X = \text{ number of aspirin} \quad Y = \text{ number of sedative}$$

- The **joint probability distribution** of X and Y is

$$f(x,y) = P(X = x, Y = y) = \begin{cases} \dfrac{\binom{3}{x}\binom{2}{y}\binom{4}{2-x-y}}{\binom{9}{2}} & x = 0,1,2 \text{ and } y = 0,1,2 \\ 0 & \text{elsewhere} \end{cases}$$

- **Let:**
  - $F(x,y) = P(X \le x, Y \le y)$
    denote the **joint distribution function** of X and Y.
  - $g(x) = P(X = x)$ and $h(y) = P(Y = y)$
    denote the **marginal probability distributions** of X and Y respectively.
  - $G(x) = P(X \le x)$ and $H(y) = P(Y \le y)$
    denote the **marginal distribution functions** of X and Y respectively.
  - $v(x|Y = y)$ and $w(y|X = x)$
    denote the **conditional probability distributions** of X and Y respectively.

1. Create the (3×3) matrix $\mathbf{f}$ with the probabilities

$$f(x,y) = P(X = x, Y = y)$$

using the mathematical formula above.

Solution:

	$f(x,y)$	y: 0	1	2
	0	$6/36$	$8/36$	$1/36$
x	1	$12/36$	$6/36$	0
	2	$3/36$	0	0

$$\mathbf{f} = \begin{pmatrix} \frac{1}{6} & \frac{2}{9} & \frac{1}{36} \\ \frac{1}{3} & \frac{1}{6} & 0 \\ \frac{1}{12} & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0.16667 & 0.22222 & 2.7778 \times 10^{-2} \\ 0.33333 & 0.16667 & 0 \\ 8.3333 \times 10^{-2} & 0 & 0 \end{pmatrix}$$

SAS Program:

```
f=j(3,3,.);
do x=0 to 2;
do y=0 to 2;
row=x+1; col=y+1;
if x+y<=2 then
 f[row,col]=comb(3,x)*comb(2,y)*comb(4,2-x-y)/comb(9,2);
else f[row,col]=0;
end;
end;
```

2. Determine the following:

   Hint: Use the SUM function in PROC IML.

   (a) $P(0 \leq X \leq 1, 1 \leq Y \leq 2) =$ 0.4166667

   (b) $F(1,1) =$ 0.8888889

   (c) $F(1,2) =$ 0.9166667

   (d) $F(4,0) =$ 0.5833333

   (e) $F(2,2) =$ 1

3. Determine the (3×3) matrix $\mathbf{FF}$ with the following cumulative probabilities

$$\mathbf{FF} = \begin{pmatrix} F(0,0) & F(0,1) & F(0,2) \\ F(1,0) & F(1,1) & F(1,2) \\ F(2,0) & F(2,1) & F(2,2) \end{pmatrix}$$

   Hint: Use double DO-LOOP as in Question 1 and the SUM function.

   Solution:

	$F(x,y)$	y: 0	1	2
	0	6/36	14/36	15/36
x	1	18/36	32/36	33/36
	2	21/36	35/36	1

$$\mathbf{FF} = \begin{pmatrix} 0.16667 & 0.38889 & 0.41667 \\ 0.5 & 0.88889 & 0.91667 \\ 0.58333 & 0.97222 & 1.0 \end{pmatrix}$$

4. Determine the (3×1) column vector $\mathbf{g}$ with the following marginal probabilities

$$\mathbf{g} = \begin{pmatrix} g(0) \\ g(1) \\ g(2) \end{pmatrix}$$

   Solution:

x	$g(x)$
0	15/36
1	18/36
2	3/36

$$\mathbf{g} = \begin{pmatrix} 0.41667 \\ 0.5 \\ 8.3333 \times 10^{-2} \end{pmatrix}$$

   Make sure you understand the following PROC IML statements:

   - `g=f*J(3,1,1);`
   - `g=f[,+];`

5. Determine the (3×1) row vector $\mathbf{h}$ with the following marginal probabilities

$$\mathbf{h} = \begin{pmatrix} h(0) & h(1) & h(2) \end{pmatrix}$$

   Solution:

y	0	1	2
$h(y)$	21/36	14/36	1/36

$$\mathbf{h} = \begin{pmatrix} 0.58333 & 0.38889 & 2.7778 \times 10^{-2} \end{pmatrix}$$

   Make sure you understand the following PROC IML statements:

   - `h=J(1,3,1)*f;`
   - `h=f[+,];`

6. Determine the (3×1) column vector $\mathbf{GG}$ with the marginal distribution function of X i.e.

$$\mathbf{GG} = \begin{pmatrix} G(0) \\ G(1) \\ G(2) \end{pmatrix}$$

Solution:

x	$G(x)$
0	15/36
1	33/36
2	1

$$\mathbf{GG} = \begin{pmatrix} 0.416\,67 \\ 0.916\,67 \\ 1.0 \end{pmatrix}$$

Make sure you understand the following PROC IML statements:

- `GG=cusum(g);`
- `GG=FF[,3];`

Note: The last column of $\mathbf{FF}$ is equal to $\mathbf{GG}$.

7. Determine the (1×3) row vector $\mathbf{HH}$ with the marginal distribution function of Y i.e.

$$\mathbf{HH} = \begin{pmatrix} H(0) & H(1) & H(2) \end{pmatrix}$$

Solution:

y	0	1	2
$H(y)$	21/36	35/36	1

$$\mathbf{HH} = \begin{pmatrix} 0.5833333 & 0.9722222 & 1.0 \end{pmatrix}$$

Make sure you understand the following PROC IML statements:

- `HH=cusum(h);`
- `HH=FF[3,];`

Note: The last row of $\mathbf{FF}$ is equal to $\mathbf{HH}$.

8. Determine the (3×1) column vector $\mathbf{v}$ with the conditional distribution of X given $Y = 1$ i.e.

$$\mathbf{v} = \begin{pmatrix} v(0|Y=1) \\ v(1|Y=1) \\ v(2|Y=1) \end{pmatrix}$$

Solution:

x	$f(x|1)$
0	8/14
1	6/14
2	0

$$\mathbf{v} = \begin{pmatrix} 0.571\,43 \\ 0.428\,57 \\ 0 \end{pmatrix}$$

You can use any of the following PROC IML statements, **but** make sure you understand them all:

- `v=f[,2]/h[2];`
- `v=f[,2]/sum(f[,2]);`
- `v=f[,2]/f[+,2];`

9. Determine the (1×3) row vector $\mathbf{w}$ with the conditional distribution of Y given $X = 1$ i.e.

$$\mathbf{w} = \begin{pmatrix} w(0|X=1) & w(1|X=1) & w(2|X=1) \end{pmatrix}$$

Solution:

y	0	1	2
$w(y\|1)$	12/18	6/18	0

$$\mathbf{w} = \begin{pmatrix} 0.666\,67 & 0.333\,33 & 0 \end{pmatrix}$$

You can use any of the following PROC IML statements, **but** make sure you understand them all:

- `w=f[2,]/g[2];`
- `w=f[2,]/sum(f[2,]);`
- `w=f[,2]/f[2,+];`

10. If X and Y are independent it follows that $f(x,y) = g(x) \cdot h(y)\ \forall x, y$

(a) Calculate the matrix of probabilities under independence i.e.

$$\mathbf{p} = \begin{pmatrix} g(0)\cdot h(0) & g(0)\cdot h(1) & g(0)\cdot h(2) \\ g(1)\cdot h(0) & g(1)\cdot h(1) & g(1)\cdot h(2) \\ g(2)\cdot h(0) & g(2)\cdot h(1) & g(2)\cdot h(2) \end{pmatrix}$$

Solution:

$$\begin{aligned}
\mathbf{p} &= \mathbf{gh} \\
&= \begin{pmatrix} g(0) \\ g(1) \\ g(2) \end{pmatrix} \begin{pmatrix} h(0) & h(1) & h(2) \end{pmatrix} = \begin{pmatrix} g(0)\,h(0) & g(0)\,h(1) & g(0)\,h(2) \\ g(1)\,h(0) & g(1)\,h(1) & g(1)\,h(2) \\ g(2)\,h(0) & g(2)\,h(1) & g(2)\,h(2) \end{pmatrix} \\
&= \begin{pmatrix} 15/36 \\ 18/36 \\ 3/36 \end{pmatrix} \begin{pmatrix} 21/36 & 35/36 & 1 \end{pmatrix} = \begin{pmatrix} \frac{35}{144} & \frac{35}{216} & \frac{5}{432} \\ \frac{7}{24} & \frac{7}{36} & \frac{1}{72} \\ \frac{7}{144} & \frac{7}{216} & \frac{1}{432} \end{pmatrix} \\
&= \begin{pmatrix} 0.243\,06 & 0.162\,04 & 1.157\,4 \times 10^{-2} \\ 0.291\,67 & 0.194\,45 & 1.388\,9 \times 10^{-2} \\ 4.861\,1 \times 10^{-2} & 3.240\,7 \times 10^{-2} & 2.314\,8 \times 10^{-3} \end{pmatrix}
\end{aligned}$$

(b) Are X and Y independent? Explain.

Solution:

No, since

$$\mathbf{f} \neq \mathbf{p}$$

$$\begin{pmatrix} \frac{1}{6} & \frac{2}{9} & \frac{1}{36} \\ \frac{1}{3} & \frac{1}{6} & 0 \\ \frac{1}{12} & 0 & 0 \end{pmatrix} \neq \begin{pmatrix} \frac{35}{144} & \frac{35}{216} & \frac{5}{432} \\ \frac{7}{24} & \frac{7}{36} & \frac{1}{72} \\ \frac{7}{144} & \frac{7}{216} & \frac{1}{432} \end{pmatrix}$$

X and Y are not independent.