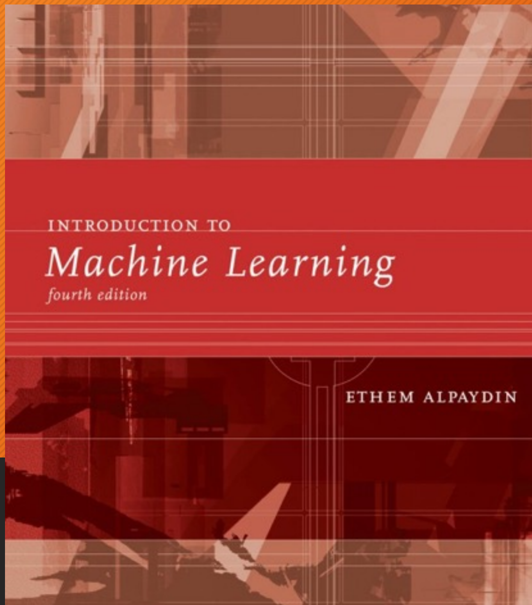


Lecture Slides for

INTRODUCTION TO Machine Learning 4e

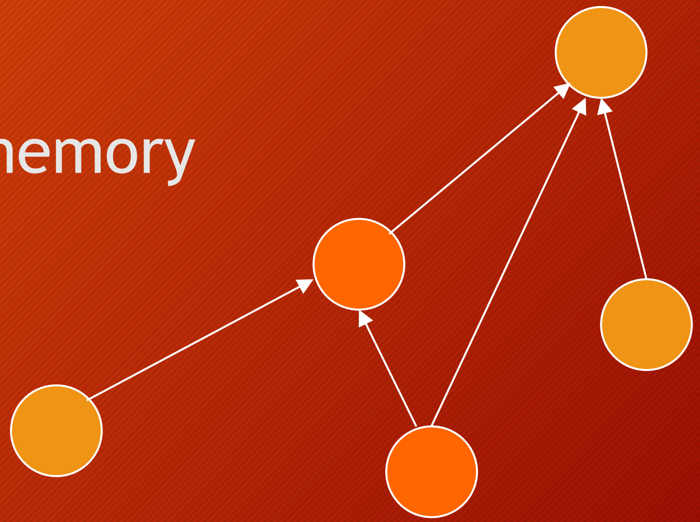
ETHEM ALPAYDIN
The MIT Press, 2020

ethem.alpaydin@gmail.com

CHAPTER 11:
Multilayer Perceptrons

Neural Networks

- Networks of processing units (neurons) with connections (synapses) between them
- Large number of neurons: 10^{10}
- Large connectitivity: 10^5
- Parallel processing
- Distributed computation/memory
- Robust to noise, failures



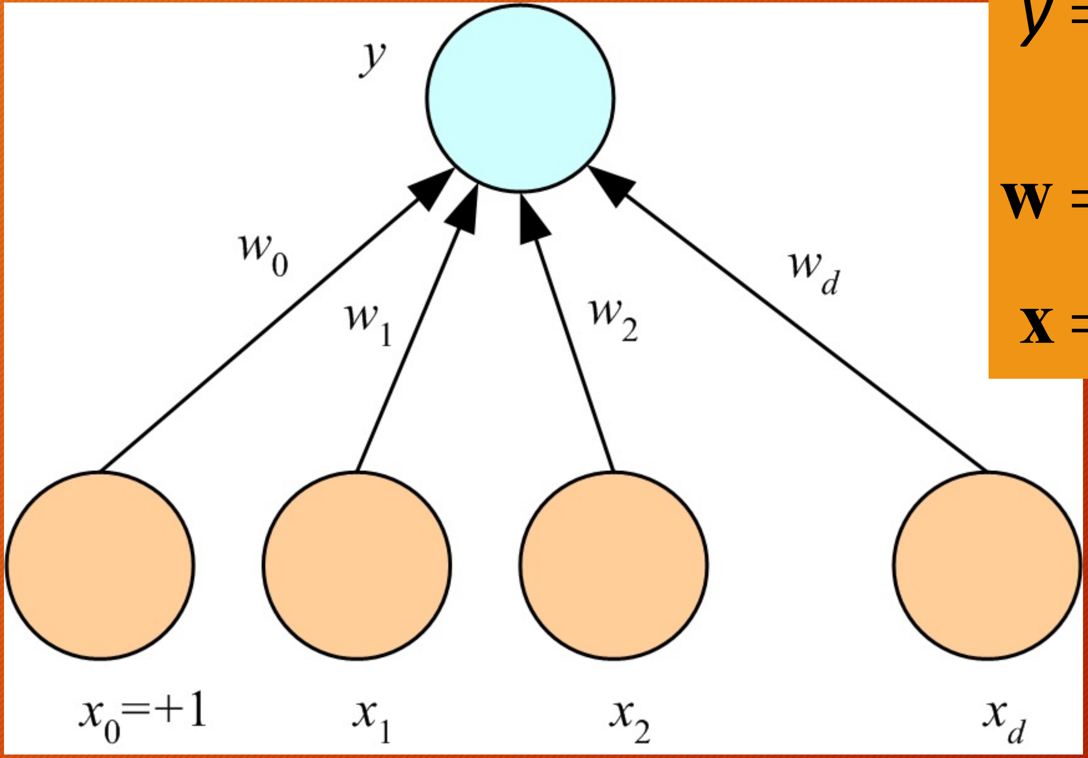
Understanding the Brain

- Levels of analysis (Marr, 1982)
  1. Computational theory
  2. Representation and algorithm
  3. Hardware implementation
- Reverse engineering: From hardware to theory
- Parallel processing: SIMD vs MIMD

  Neural net: SIMD with modifiable local memory

  Learning: Update by training/experience

Perceptron



$$y = \sum_{j=1}^{d} w_j x_j + w_0 = \mathbf{w}^T \mathbf{x}$$

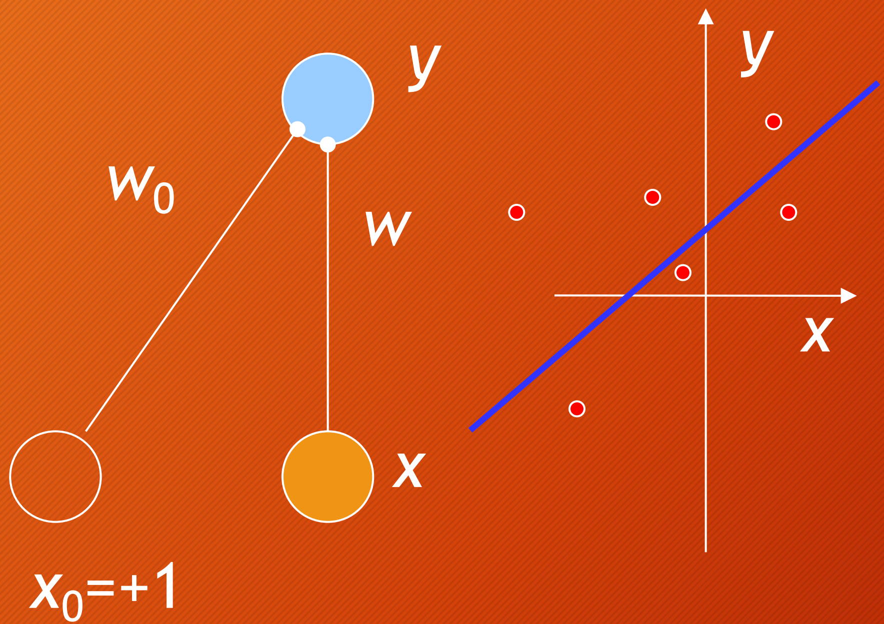
$$\mathbf{w} = [w_0, w_1, \ldots, w_d]^T$$

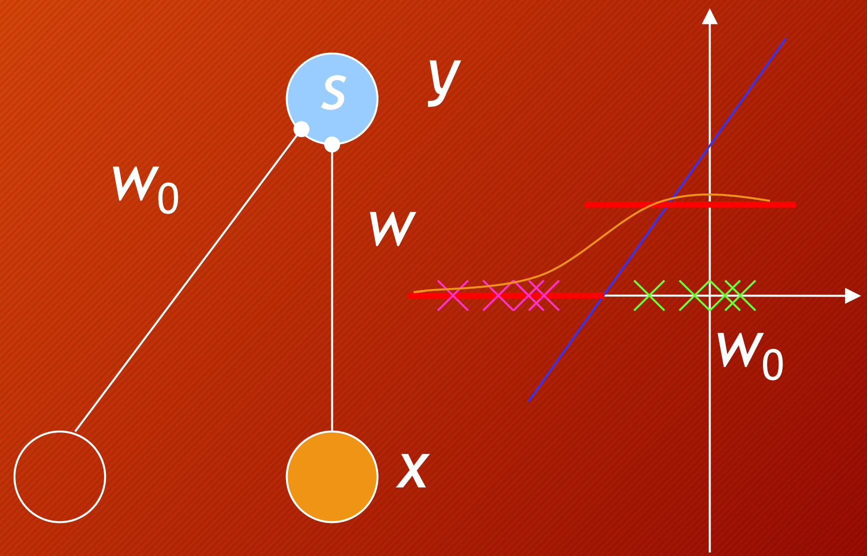
$$\mathbf{x} = [1, x_1, \ldots, x_d]^T$$

(Rosenblatt, 1962)

What a Perceptron Does

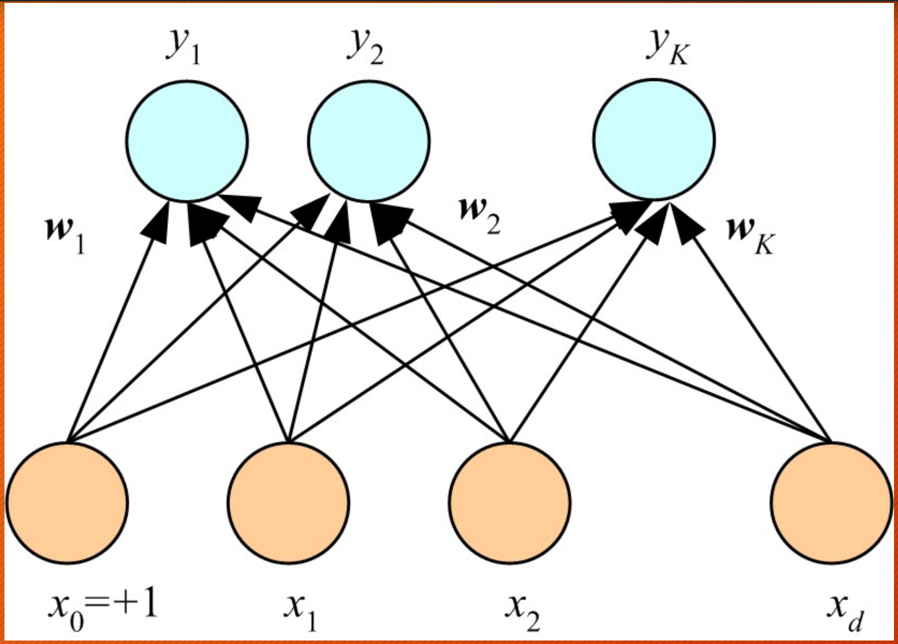
- Regression: $y = wx + w_0$
- Classification: $y = 1(wx + w_0 > 0)$





$$y = \text{sigmoid}(o) = \frac{1}{1 + \exp\left[-\mathbf{w}^T \mathbf{x}\right]}$$

K Outputs



Regression:

$$y_i = \sum_{j=1}^{d} w_{ij} x_j + w_{i0} = \mathbf{w}_i^T \mathbf{x}$$

$$\mathbf{y} = \mathbf{W}\mathbf{x}$$

Classification:

$$o_i = \mathbf{w}_i^T \mathbf{x}$$

$$y_i = \frac{\exp o_i}{\sum_k \exp o_k}$$

choose C_i

if $y_i = \max_k y_k$

Training

- Online (instances seen one by one) vs batch (whole sample) learning:
  - No need to store the whole sample
  - Problem may change in time
  - Wear and degradation in system components
- Stochastic gradient-descent: Update after a single pattern
- Generic update rule (LMS rule):

$$\Delta w_{ij}^t = \eta \left(r_i^t - y_i^t \right) x_j^t$$

$$\text{Update} = \text{LearningFactor} \cdot \left(\text{DesiredOutput} - \text{ActualOutput} \right) \cdot \text{Input}$$

Training a Perceptron: Regression

- Regression (Linear output):

$$E^t\left(\mathbf{w} \mid \mathbf{x}^t, r^t\right) = \frac{1}{2}\left(r^t - y^t\right)^2 = \frac{1}{2}\left[r^t - \left(\mathbf{w}^T \mathbf{x}^t\right)\right]^2$$

$$\Delta w_j^t = \eta\left(r^t - y^t\right)x_j^t$$

Classification

- Single sigmoid output

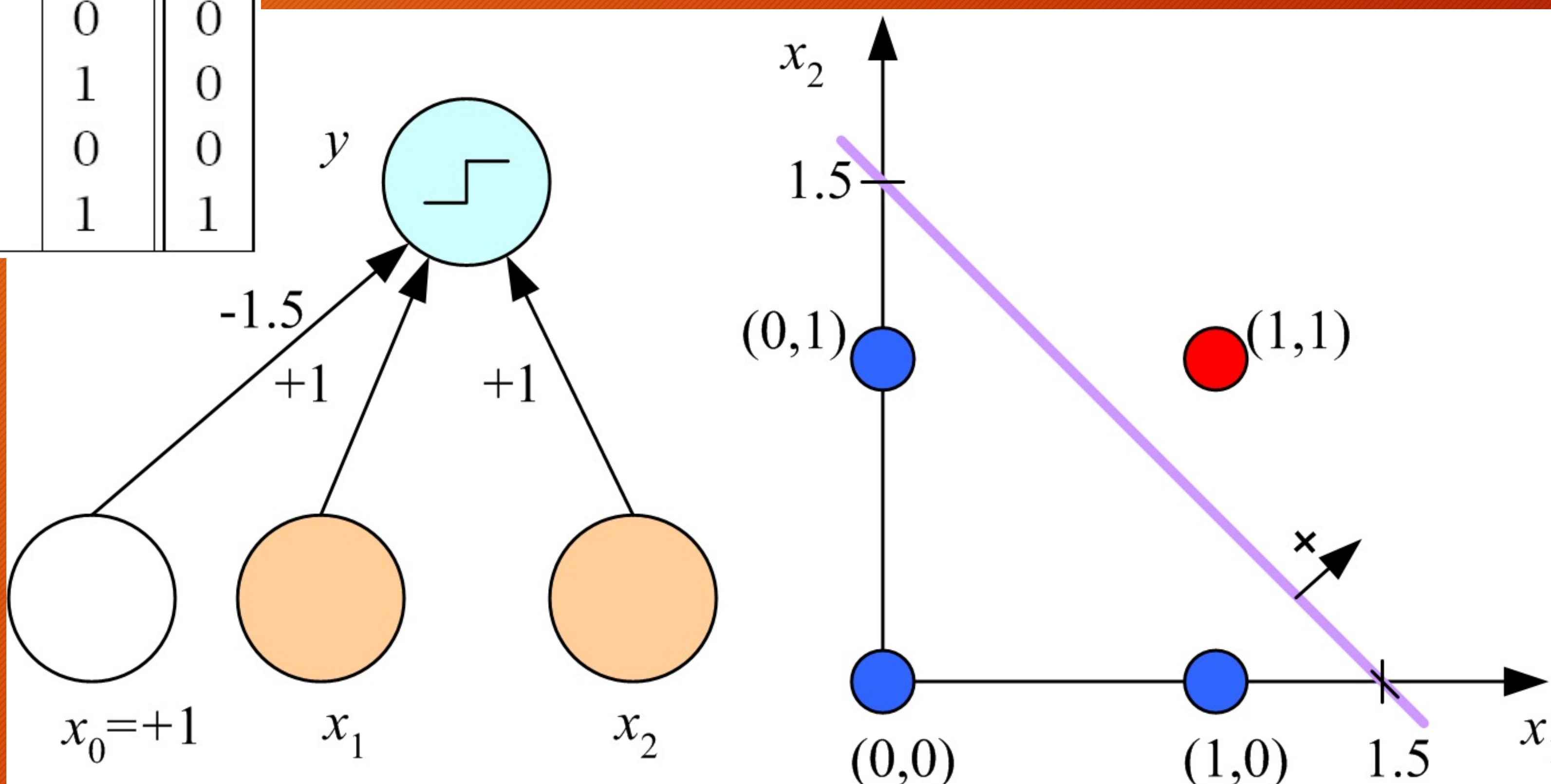
$$y^t = \text{sigmoid}\left(\mathbf{w}^T\mathbf{x}^t\right)$$
$$E^t\left(\mathbf{w} \mid \mathbf{x}^t, \mathbf{r}^t\right) = -r^t \log y^t - \left(1 - r^t\right)\log\left(1 - y^t\right)$$
$$\Delta w_j^t = \eta\left(r^t - y^t\right)x_j^t$$

- K>2 softmax outputs

$$y^t = \frac{\exp \mathbf{w}_i^T\mathbf{x}^t}{\sum_k \exp \mathbf{w}_k^T\mathbf{x}^t} \qquad E^t\left(\left\{\mathbf{w}_i\right\}_i \mid \mathbf{x}^t, \mathbf{r}^t\right) = -\sum_i r_i^t \log y_i^t$$
$$\Delta w_{ij}^t = \eta\left(r_i^t - y_i^t\right)x_j^t$$

Learning Boolean AND

x_1	x_2	r
0	0	0
0	1	0
1	0	0
1	1	1

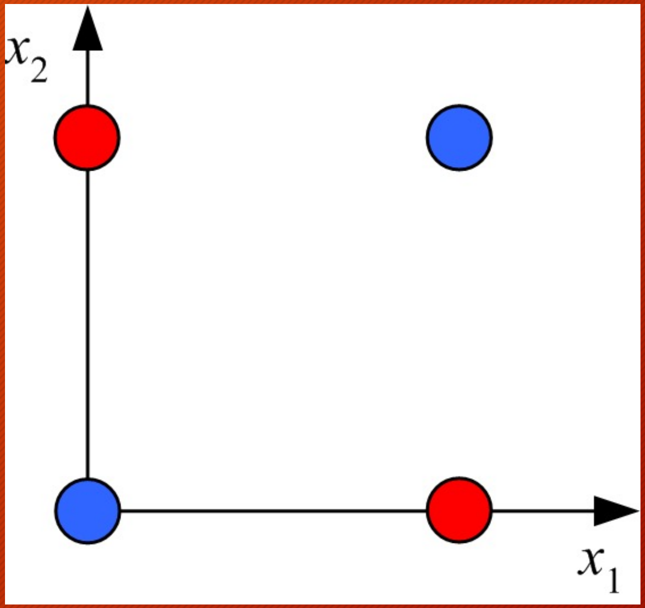


XOR

x_1	x_2	r
0	0	0
0	1	1
1	0	1
1	1	0

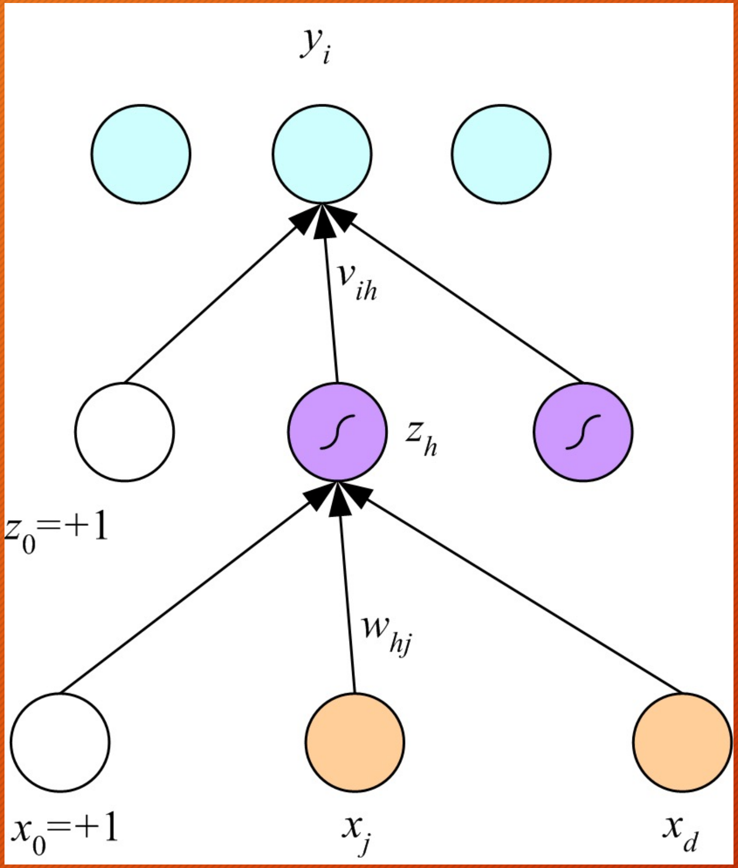
- No w_0, w_1, w_2 satisfy:

$$
\begin{aligned}
& & w_0 &\le 0 \\
& w_2 + & w_0 &> 0 \\
w_1 + & & w_0 &> 0 \\
w_1 + & w_2 + & w_0 &\le 0
\end{aligned}
$$



(Minsky and Papert, 1969)

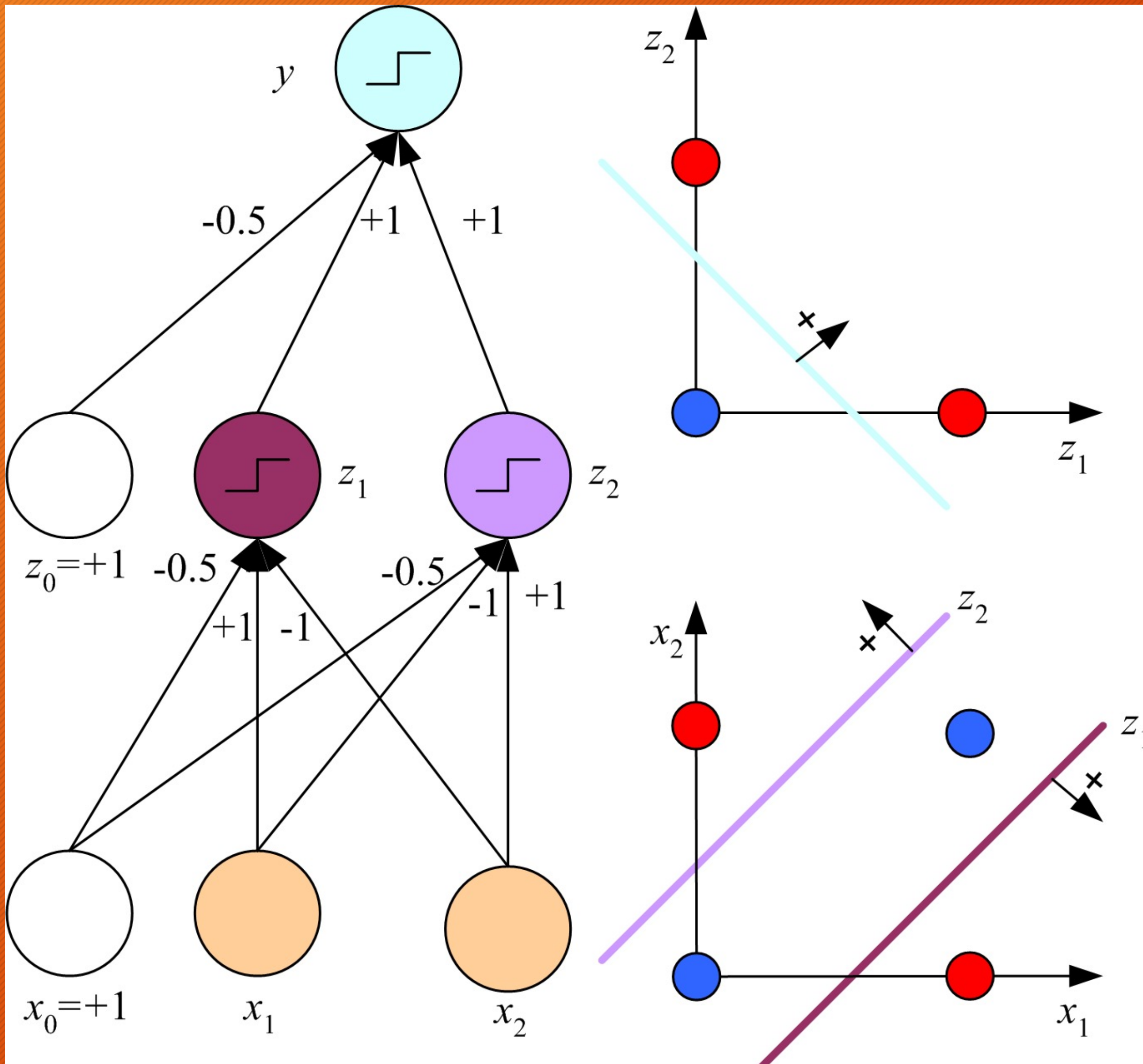
Multilayer Perceptrons



$$y_i = \mathbf{v}_i^T \mathbf{z} = \sum_{h=1}^{H} v_{ih} z_h + v_{i0}$$

$$z_h = \text{sigmoid}\left(\mathbf{w}_h^T \mathbf{x}\right) = \frac{1}{1 + \exp\left[-\left(\sum_{j=1}^{d} w_{hj} x_j + w_{h0}\right)\right]}$$

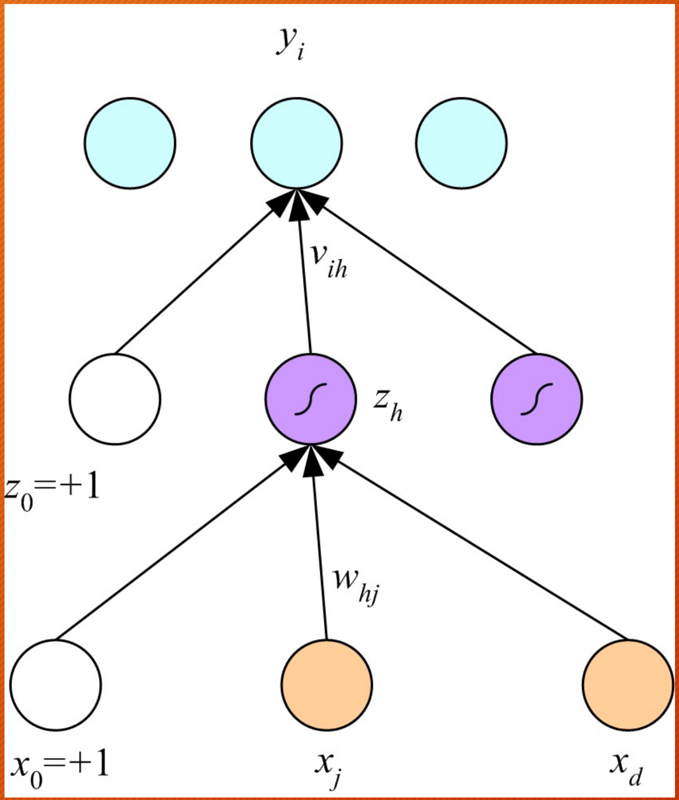
(Rumelhart et al., 1986)



x_1	x_2	z_1	z_2	y
0	0	0	0	0
0	1	0	1	1
1	0	1	0	1
1	1	0	0	0

x_1 XOR x_2 = (x_1 AND ~x_2) OR (~x_1 AND x_2)

Backpropagation



$$y_i = \mathbf{v}_i^T \mathbf{z} = \sum_{h=1}^{H} v_{ih} z_h + v_{i0}$$

$$z_h = \text{sigmoid}\left(\mathbf{w}_h^T \mathbf{x}\right)$$

$$= \frac{1}{1 + \exp\left[-\left(\sum_{j=1}^{d} w_{hj} x_j + w_{h0}\right)\right]}$$

$$\boxed{\frac{\partial E}{\partial w_{hj}} = \frac{\partial E}{\partial y_i} \frac{\partial y_i}{\partial z_h} \frac{\partial z_h}{\partial w_{hj}}}$$

Regression

$$E(\mathbf{W},\mathbf{v}\mid\mathcal{X})=\frac{1}{2}\sum_t\left(r^t-y^t\right)^2$$

Forward

$$y^t=\sum_{h=1}^{H}v_h z_h^t+v_0$$

$$z_h=\text{sigmoid}\left(\mathbf{w}_h^T\mathbf{x}\right)$$

$$\boldsymbol{x}$$

Backward

$$\Delta v_h=\sum_t\left(r^t-y^t\right)z_h^t$$

$$\begin{aligned}\Delta w_{hj}&=-\eta\frac{\partial E}{\partial w_{hj}}\\&=-\eta\sum_t\frac{\partial E}{\partial y^t}\frac{\partial y^t}{\partial z_h^t}\frac{\partial z_h^t}{\partial w_{hj}}\\&=-\eta\sum_t\boxed{-\left(r^t-y^t\right)}v_h z_h^t\left(1-z_h^t\right)x_j^t\\&=\eta\sum_t\left(r^t-y^t\right)v_h z_h^t\left(1-z_h^t\right)x_j^t\end{aligned}$$

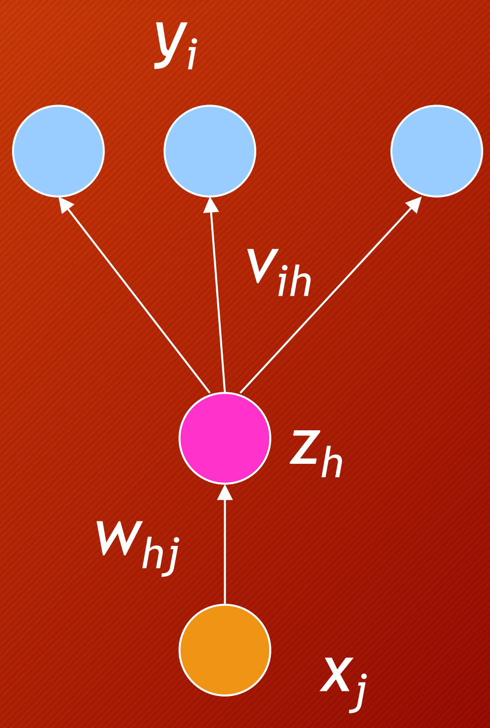
Regression with Multiple Outputs

$$E(\mathbf{W},\mathbf{V}|\mathcal{X}) = \frac{1}{2}\sum_t \sum_i \left(r_i^t - y_i^t\right)^2$$

$$y_i^t = \sum_{h=1}^{H} v_{ih} z_h^t + v_{i0}$$

$$\Delta v_{ih} = \eta \sum_t \left(r_i^t - y_i^t\right) z_h^t$$

$$\Delta w_{hj} = \eta \sum_t \left[\sum_i \left(r_i^t - y_i^t\right) v_{ih}\right] z_h^t \left(1 - z_h^t\right) x_j^t$$



Initialize all v_{ih} and w_{hj} to rand$(-0.01, 0.01)$

Repeat

For all $(\boldsymbol{x}^t, r^t) \in \mathcal{X}$ in random order

For $h = 1, \ldots, H$

$z_h \leftarrow \text{sigmoid}(\boldsymbol{w}_h^T \boldsymbol{x}^t)$

For $i = 1, \ldots, K$

$y_i = \boldsymbol{v}_i^T \boldsymbol{z}$

For $i = 1, \ldots, K$

$\Delta \boldsymbol{v}_i = \eta(r_i^t - y_i^t)\boldsymbol{z}$

For $h = 1, \ldots, H$

$\Delta \boldsymbol{w}_h = \eta(\sum_i (r_i^t - y_i^t) v_{ih}) z_h (1 - z_h) \boldsymbol{x}^t$

For $i = 1, \ldots, K$

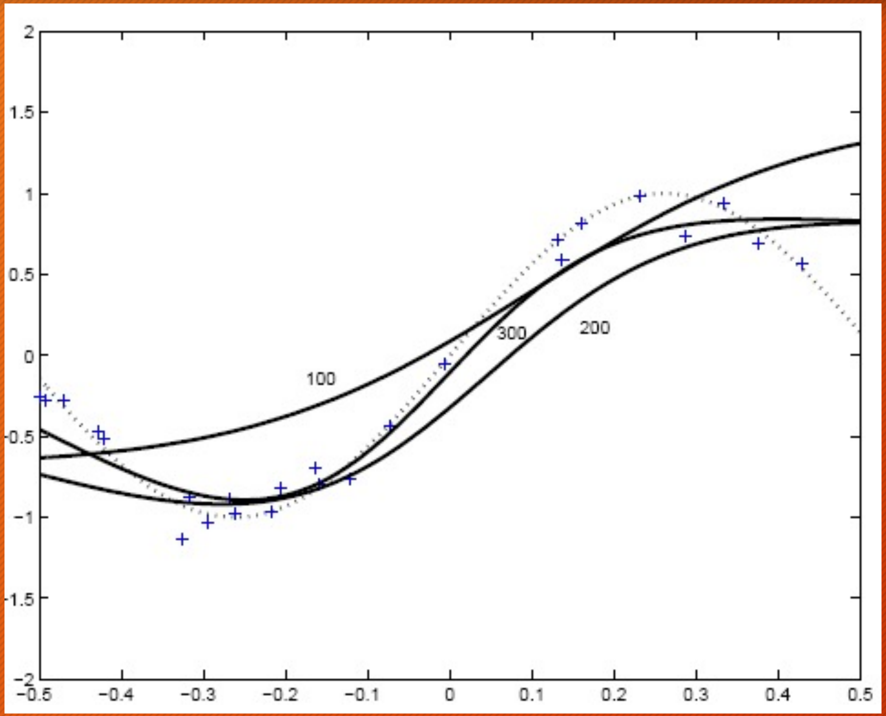
$\boldsymbol{v}_i \leftarrow \boldsymbol{v}_i + \Delta \boldsymbol{v}_i$

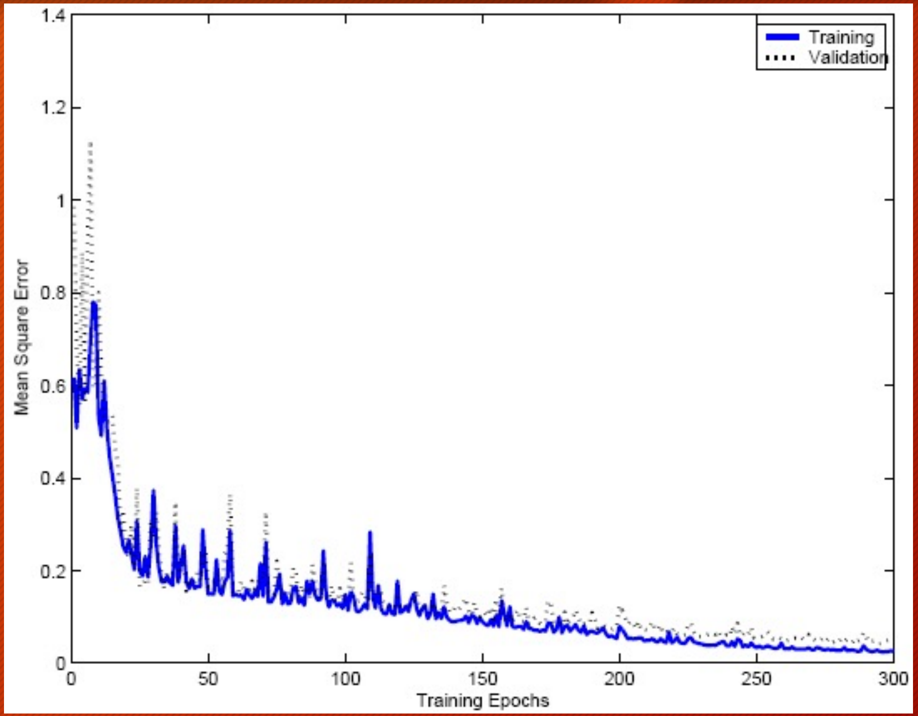
For $h = 1, \ldots, H$

$\boldsymbol{w}_h \leftarrow \boldsymbol{w}_h + \Delta \boldsymbol{w}_h$

Until convergence

1d Regression: Convergence





Learning Hidden Representations

- MLP is a generalized linear model where hidden units are the nonlinear basis functions:

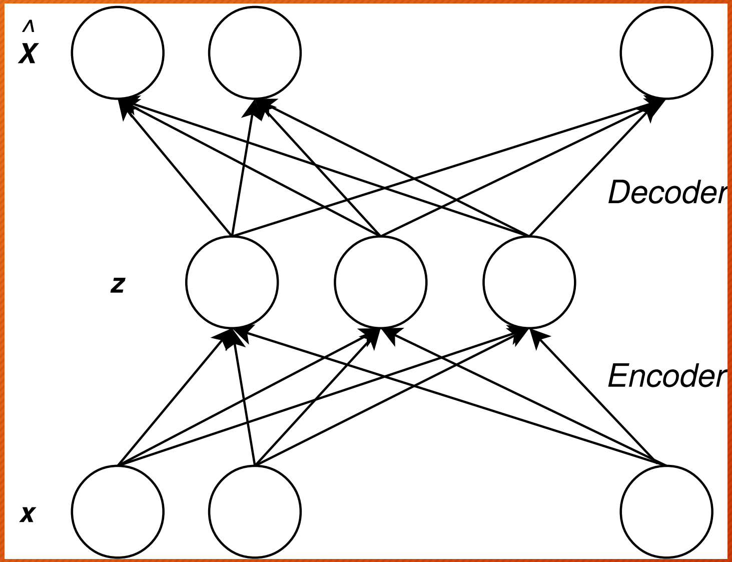
$$y = \sum_{h=1}^{H} v_h \phi(\boldsymbol{x}|\boldsymbol{w}_h)$$

where

$$\phi(\boldsymbol{x}|\boldsymbol{w}_h) \equiv \text{sigmoid}(\boldsymbol{w}_h^T \boldsymbol{x})$$

- The advantage is that the basis function parameters can also be learned from data.
- The hidden units, z_h, learn a *code/embedding*, a representation in the hidden space
- Transfer learning: Use code in another task
- Semisupervised learning: Transfer from an unsupervised to supervised problem

Autoencoders



$$\hat{\mathbf{x}}^t = \mathrm{Dec}(\mathbf{z}^t | \mathbf{V})$$

$$\mathbf{z}^t = \mathrm{Enc}(\mathbf{x}^t | \mathbf{W})$$

$$E(\mathbf{W}, \mathbf{V} | \mathcal{X}) = \sum_t \| \mathbf{x}^t - \hat{\mathbf{x}}^t \|^2 = \sum_t \| \mathbf{x}^t - \mathrm{Dec}(\mathrm{Enc}(\mathbf{x}^t | \mathbf{W}) | \mathbf{V}) \|^2$$

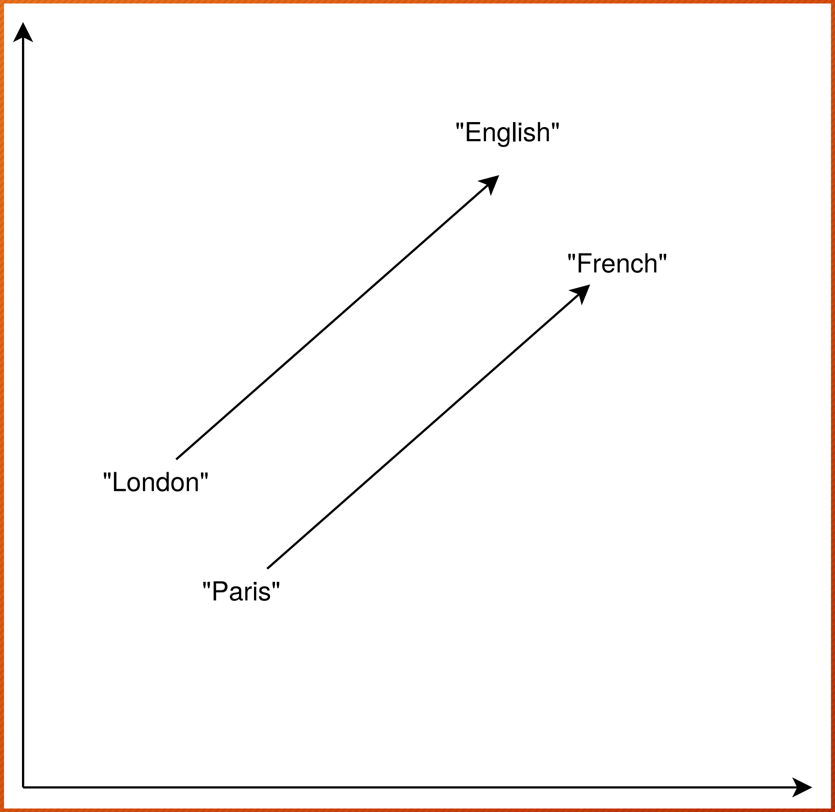
- Variants: Denoising, sparse autoencoders

Word2vec

- Learn an embedding for words for NLP
- Skipgram: An autencoder with linear encoder and decoder where input is the center word and output is a context word

- Similar words appear in similar contexts, so similar codes will be learned for them

Vector Algebra



Because they will always appear in similar contexts in pairs in a large corpus, we expect

vec("English")-vec("London")

to be similar to

vec("French")-vec("Paris")

so,

vec("Paris") + vec("English")-vec("London")

will be similar to

vec("French")

Two-Class Discrimination

- One sigmoid output y^t for $P(C_1|\boldsymbol{x}^t)$ and $P(C_2|\boldsymbol{x}^t) \equiv 1-y^t$

$$y^t = \text{sigmoid}\left(\sum_{h=1}^{H} v_h z_h^t + v_0\right)$$

$$E(\mathbf{W},\mathbf{v}|\mathcal{X}) = -\sum_t r^t \log y^t + \left(1-r^t\right)\log\left(1-y^t\right)$$

$$\Delta v_h = \eta \sum_t \left(r^t - y^t\right) z_h^t$$

$$\Delta w_{hj} = \eta \sum_t \left(r^t - y^t\right) v_h z_h^t \left(1 - z_h^t\right) x_j^t$$