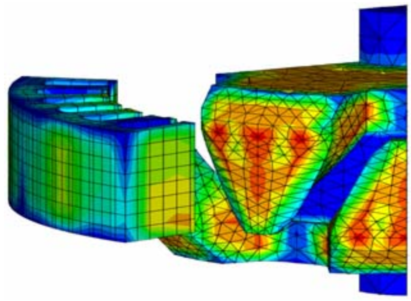
Parametric Model Reduction by Krylov Subspace Methods

Boris Lohmann and Rudy Eid,

GAMM-FA Dynamik und Regelungstheorie, München, 27.-28.03.09



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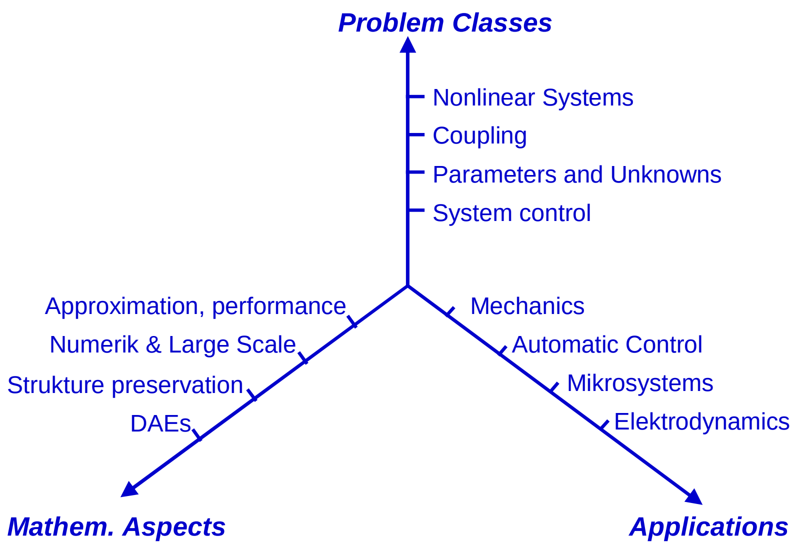
Challenges in 3 Dimensions



Challenge *Parametric Reduction*

Problem Classes

Goal: Preserve parameters (of material, geometry…) as symbolic quantities in the reduction process.

- Nonlinear Systems
- Coupling
- **Parameters** and Unknowns
- System control

Mathem. Aspects

- Approximation, performance
- Numerik & Large Scale
- Strukture preservation
- DAEs

Applications

- Mechanics
- Automatic Control
- Mikrosystems
- Elektrodynamics

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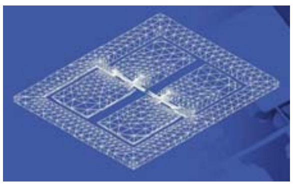
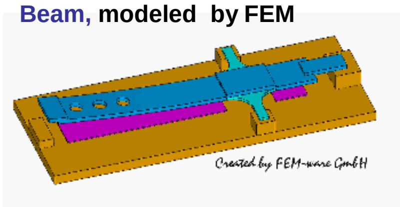
Parametric Systems

Butterfly Gyroscope
(Dag Billger, The Imego Institute, Sweden)

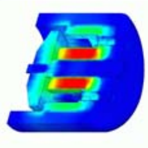


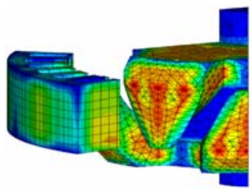
Linear

Beam, modeled by FEM



Electrical Drives (Co-oper. BOSCH)





Nonlinear

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Part 1:

Superposition of local models
as a *framework* for parametric
reduction

Assumptions

System:

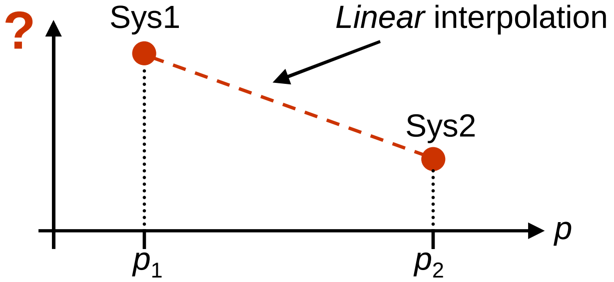
$$\dot{\boldsymbol{x}} = \boldsymbol{A}(p)\boldsymbol{x} + \boldsymbol{b}(p)u, \quad y = \boldsymbol{c}^T(p)\boldsymbol{x}$$

Matrices ***A,b,c*** only available at discrete values $p_1, p_2, \ldots$ of parameter *p:*

$\boldsymbol{A}(p_1)=\boldsymbol{A}_1$, $\boldsymbol{A}(p_2)=\boldsymbol{A}_2$, …

$\boldsymbol{b}(p_1)=\boldsymbol{b}_1$, $\boldsymbol{b}(p_2)=\boldsymbol{b}_2$, …

$\boldsymbol{c}(p_1)=\boldsymbol{c}_1$, $\boldsymbol{c}(p_2)=\boldsymbol{c}_2$, …



Some Weighted Superpositions

a) Superposition of ***coefficients*** (system matrices):

$$\dot{\boldsymbol{x}} = \left(\sum_{i=1}^{s} \omega_i(p)\boldsymbol{A}_i\right)\boldsymbol{x} + \left(\sum_{i=1}^{s} \omega_i(p)\boldsymbol{b}_i\right)u, \quad \boldsymbol{y} = \left(\sum_{i=1}^{s} \omega_i(p)\boldsymbol{c}_i^T\right)\boldsymbol{x}$$

= *exact* description if *p* affine: $\boldsymbol{A}=\boldsymbol{A}_0+\boldsymbol{A}_1 p$, $\boldsymbol{b}=\boldsymbol{b}_0+\boldsymbol{b}_1 p$, $\boldsymbol{c}=\boldsymbol{c}_0+\boldsymbol{c}_1 p$

b) Superposition of ***signals***:

$$y(t) = \sum_{i=1}^{s} \omega_i(p)y_i(t)\text{ , mit } \quad Y_i(s) = G_i(s)U(s)$$

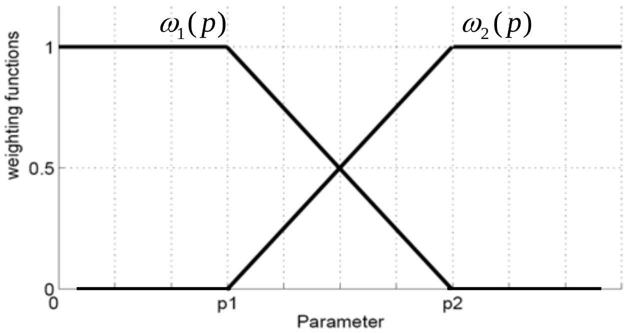
Leads to strange superpositions of resonance peaks, ☹,
like $y(t) = 0{,}5\underbrace{\sin 10t}_{y_1} + 0{,}5\underbrace{\sin 12t}_{y_2}$ instead of $y(t) = \sin 11t$

Weighting functions

a) Superposition of *coefficients* (system matrices):

$$\dot{\boldsymbol{x}} = \left(\sum_{i=1}^{s} \omega_i(p)\boldsymbol{A}_i\right)\boldsymbol{x} + \left(\sum_{i=1}^{s} \omega_i(p)\boldsymbol{b}_i\right)u, \quad \boldsymbol{y} = \left(\sum_{i=1}^{s} \omega_i(p)\boldsymbol{c}_i^T\right)\boldsymbol{x}$$

$$\sum \omega_i(p) \overset{!}{=} 1$$



Traditional Reduction

Traditionally: apply *one* common projector pair ***V*,*W***:

$$\underset{W^T V}{\dot{\boldsymbol{x}}} = \underset{W^T}{\left(\sum_{i=1}^{s} \omega_i(p)\boldsymbol{A}_i\right)} \underset{V}{\boldsymbol{x}} + \underset{W^T}{\left(\sum_{i=1}^{s} \omega_i(p)\boldsymbol{b}_i\right)} u, \quad \boldsymbol{y} = \left(\sum_{i=1}^{s} \omega_i(p)\boldsymbol{c}_i^T\right) \underset{V}{\boldsymbol{x}}$$

Problem: ***V*** (and ***W***) need *many* columns to well approximate *all s* local models! → large reduced order.

For instance, to match 2*q* moments at each local model, the reduced model's order will be *sq* (instead of *q* in non-parametric reduction)

New: Reduction by Local Projectors

Apply ***separate*** projectors $\boldsymbol{V}_i$, $\boldsymbol{W}_i$ to all local models:

$$\dot{\boldsymbol{x}} = \left(\sum_{i=1}^{s} \omega_i(p) \underset{(W_i^T V_i)^{-1} W_i^T}{} \boldsymbol{A}_i \underset{V_i}{}\right)\boldsymbol{x} + \left(\sum_{i=1}^{s} \omega_i(p) \underset{(W_i^T V_i)^{-1} W_i^T}{} \boldsymbol{b}_i\right) u, \quad \boldsymbol{y} = \left(\sum_{i=1}^{s} \omega_i(p)\boldsymbol{c}_i^T \underset{V_i}{}\right)\boldsymbol{x} \qquad (*)$$

+ Almost no additional numerical effort,
+ Much smaller reduced models (factor *s* when matching same number of moments).

Open question: are we allowed to sum up *physically different* reduced vectors $\dot{x}$? Answer:

Not at once, but after giving the local reduced models a ***common physical interpretation*** *of state variables*
(by applying state transformations $\boldsymbol{T}_i$)

State Transformations T_i (+ local projectors)

Define a linear combination $x^* = \underbrace{R}_{(q,n)} x$ of q state variables

and transform all local reduced models, to represent these state variables:

$$x_i^* = \underbrace{RV_i x_{i,red}}_{\hat{x}_i} \quad \Rightarrow \quad x_i^* = \underbrace{T_i}_{RV_i} x_{i,red}$$

In (*), substitute V_i by $V_i T_i^{-1}$ and W_i by $T_i W_i$

Remark: a well-conditioned choice of $\boldsymbol{R}$ is

$$[\boldsymbol{I} \quad \cdots \quad \boldsymbol{I}] \overset{!}{\approx} \boldsymbol{R}[\boldsymbol{V}_1 \quad \cdots \quad \boldsymbol{V}_s] \Rightarrow \quad \boldsymbol{R} = [\boldsymbol{I} \quad \cdots \quad \boldsymbol{I}][\boldsymbol{V}_1 \quad \cdots \quad \boldsymbol{V}_s]^+$$

Summary: Local Projectors

Full order model by superposition:

$$\left(\sum_{i=1}^{s} \omega_i(p)\boldsymbol{E}_i\right)\dot{\boldsymbol{x}} = \left(\sum_{i=1}^{s} \omega_i(p)\boldsymbol{A}_i\right)\boldsymbol{x} + \left(\sum_{i=1}^{s} \omega_i(p)\boldsymbol{b}_i\right)u \;, \quad \boldsymbol{y} = \left(\sum_{i=1}^{s} \omega_i(p)\boldsymbol{c}_i^T\right)\boldsymbol{x}$$

Reduced model by local projectors:

$$\left(\sum_{i=1}^{s} \omega_i(p)\boldsymbol{W}_i^T\boldsymbol{E}_i\boldsymbol{V}_i\right)\dot{\boldsymbol{x}} = \left(\sum_{i=1}^{s} \omega_i(p)\boldsymbol{W}_i^T\boldsymbol{A}_i\boldsymbol{V}_i\right)\boldsymbol{x} + \left(\sum_{i=1}^{s} \omega_i(p)\boldsymbol{W}_i^T\boldsymbol{b}_i\right)u, \qquad (*)$$

$$\boldsymbol{y} = \left(\sum_{i=1}^{s} \omega_i(p)\boldsymbol{c}_i^T\boldsymbol{V}_i\right)\boldsymbol{x}$$

Part 2:

Moment Matching

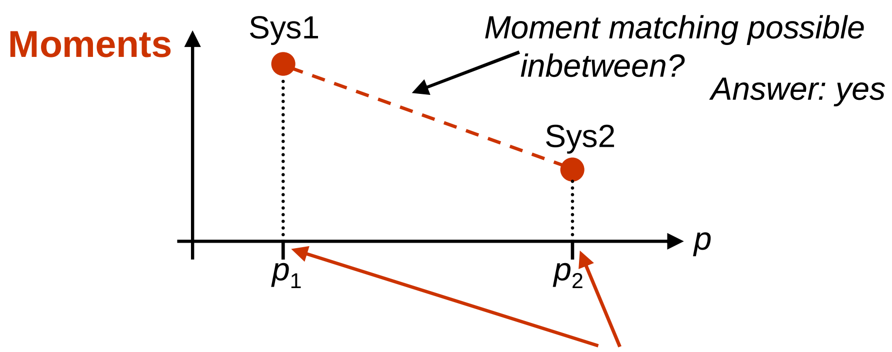
for any value of p

Moments m_j

Transfer function:

Taylor series at $s_0=0$

$$G(s) = \boldsymbol{c}^T(s\boldsymbol{E}-\boldsymbol{A})^{-1}\boldsymbol{b} = -\underbrace{\boldsymbol{c}^T\boldsymbol{A}^{-1}\boldsymbol{b}}_{m_0} - \underbrace{\boldsymbol{c}^T\boldsymbol{A}^{-1}\boldsymbol{E}\boldsymbol{A}^{-1}\boldsymbol{b}}_{m_1}s - \ldots - \underbrace{\boldsymbol{c}^T(\boldsymbol{A}^{-1}\boldsymbol{E})^j\boldsymbol{A}^{-1}\boldsymbol{b}}_{m_j}s^j \ldots$$



Standard reduction matches moments only *here, at p_1, p_2*

Krylov-Reduction matching interpolated moments at ***any*** *p*

System: $$\boldsymbol{E}(p)\dot{\boldsymbol{x}} = \boldsymbol{A}(p)\boldsymbol{x} + \boldsymbol{b}(p)u,\ \ y = \boldsymbol{c}^T(p)\boldsymbol{x}$$

Moments: $$m_j(p_1) = \boldsymbol{c}_1^T(\boldsymbol{A}_1^{-1}\boldsymbol{E}_1)^j\boldsymbol{A}_1^{-1}\boldsymbol{b}_1$$
$$m_j(p_2) = \boldsymbol{c}_2^T(\boldsymbol{A}_2^{-1}\boldsymbol{E}_2)^j\boldsymbol{A}_2^{-1}\boldsymbol{b}_2,...$$

Linear interpolation: $$m_j(p) = \omega_1(p)m_j(p_1) + \omega_2(p)m_j(p_2)...$$

Reduction steps: 1) Find locally reduced models

$$\boldsymbol{W}_i^T\boldsymbol{E}_i\boldsymbol{V}_i\dot{\boldsymbol{x}} = \boldsymbol{W}_i^T\boldsymbol{A}_i\boldsymbol{V}_i\boldsymbol{x} + \boldsymbol{W}_i^T\boldsymbol{b}_iu,\ \ y = \boldsymbol{c}_i^T\boldsymbol{V}_i\boldsymbol{x}\ \ \Leftrightarrow$$

$$(\boldsymbol{W}_i^T\boldsymbol{A}_i\boldsymbol{V}_i)^{-1}\boldsymbol{W}_i^T\boldsymbol{E}_i\boldsymbol{V}_i\dot{\boldsymbol{x}} = \boldsymbol{x} + (\boldsymbol{W}_i^T\boldsymbol{A}_i\boldsymbol{V}_i)^{-1}\boldsymbol{W}_i^T\boldsymbol{b}_iu,\ \ y = \boldsymbol{c}_i^T\boldsymbol{V}_i\boldsymbol{x}$$

where $$\boldsymbol{V}_i = [\boldsymbol{A}_i^{-1}\boldsymbol{b}_i, (\boldsymbol{A}_i^{-1}\boldsymbol{E}_i)\boldsymbol{A}_i^{-1}\boldsymbol{b}_i, ..., (\boldsymbol{A}_i^{-1}\boldsymbol{E}_i)^{q-1}\boldsymbol{A}_i^{-1}\boldsymbol{b}_i]$$
$$\boldsymbol{W}_i = [\boldsymbol{A}_i^{-1}\boldsymbol{c}_i, (\boldsymbol{A}_i^{-1}\boldsymbol{E}_i)\boldsymbol{A}_i^{-1}\boldsymbol{c}_i, ..., (\boldsymbol{A}_i^{-1}\boldsymbol{E}_i)^{q-1}\boldsymbol{A}_i^{-1}\boldsymbol{c}_i]$$

Krylov-Reduction matching interpolated moments at ***any*** *p*

2) Add (weighted) reduced models up to the result:

$$\boldsymbol{E}_r(p)\dot{\boldsymbol{x}} = \boldsymbol{x} + \boldsymbol{b}_r(p)u,\ \ y = \boldsymbol{c}_r^T(p)\boldsymbol{x}$$

where $$\boldsymbol{E}_r = \sum_{i=1}^{s}\omega_i(p)(\boldsymbol{W}_i^T\boldsymbol{A}_i\boldsymbol{V}_i)^{-1}\boldsymbol{W}_i^T\boldsymbol{E}_i\boldsymbol{V}_i$$
$$\boldsymbol{b}_r = \sum_{i=1}^{s}\omega_i(p)(\boldsymbol{W}_i^T\boldsymbol{A}_i\boldsymbol{V}_i)^{-1}\boldsymbol{W}_i^T\boldsymbol{b}_i$$
$$\boldsymbol{c}_r^T = \sum_{i=1}^{s}\omega_i(p)\boldsymbol{c}_i^T\boldsymbol{V}_i$$

This parametric reduced model matches exactly the linearly interpolated moments at *any* value of *p* !

Krylov-Reduction matching interpolated moments at ***any*** *p*

Proof (with $p \in [p_1, p_2]$):

$$m_{r0} = \boldsymbol{c}_r^T \boldsymbol{b}_r =$$
$$= [\omega_1 \boldsymbol{c}_1^T \boldsymbol{V}_1 + \omega_2 \boldsymbol{c}_2^T \boldsymbol{V}_2]\left[\omega_1 (\boldsymbol{W}_1^T \boldsymbol{A}_1 \boldsymbol{V}_1)^{-1} \boldsymbol{W}_1^T \boldsymbol{b}_1 + \omega_2 (\boldsymbol{W}_2^T \boldsymbol{A}_2 \boldsymbol{V}_2)^{-1} \boldsymbol{W}_2^T \boldsymbol{b}_2\right]$$
$$= [\omega_1 \boldsymbol{c}_1^T \boldsymbol{V}_1 + \omega_2 \boldsymbol{c}_2^T \boldsymbol{V}_2][\omega_1 \boldsymbol{r}_0 + \omega_2 \boldsymbol{r}_0] = \omega_1 \boldsymbol{c}_1^T \boldsymbol{A}_1^{-1} \boldsymbol{b}_1 + \omega_2 \boldsymbol{c}_2^T \boldsymbol{A}_2^{-1} \boldsymbol{b}_2 = m_0$$

where we used $\boldsymbol{b}_i = \boldsymbol{A}_i \boldsymbol{A}_i^{-1} \boldsymbol{b}_i = \boldsymbol{A}_i \boldsymbol{V}_i \boldsymbol{r}_0$ with $\boldsymbol{r}_0 = \boldsymbol{e}_1$

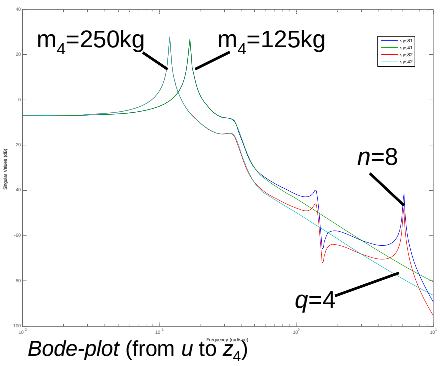
$$m_{r1} = \boldsymbol{c}_r^T \boldsymbol{E}_r \boldsymbol{b}_r = \boldsymbol{c}_r^T \boldsymbol{E}_r \boldsymbol{e}_1$$
$$= \ldots = m_1$$

Remark: *Arnoldi* can be used (instead of simple ***V, W*** used above) requiring a transformation ***T*** in low dimension, similar to part 1 (see appendix).
Other development points than s_0=0 can be used.

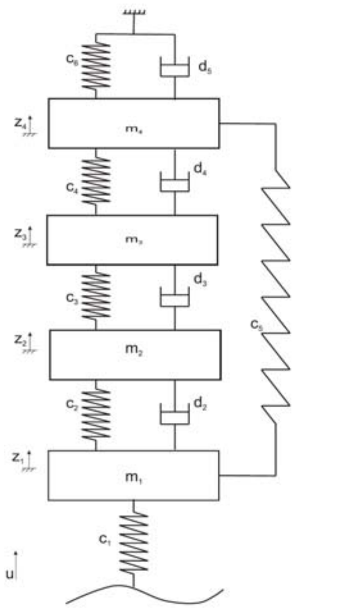
Example: suspension system of order 8

Experiment 1:

Mass m_4 changes between 125kg and 250kg



Bode-plot (from *u* to z_4)

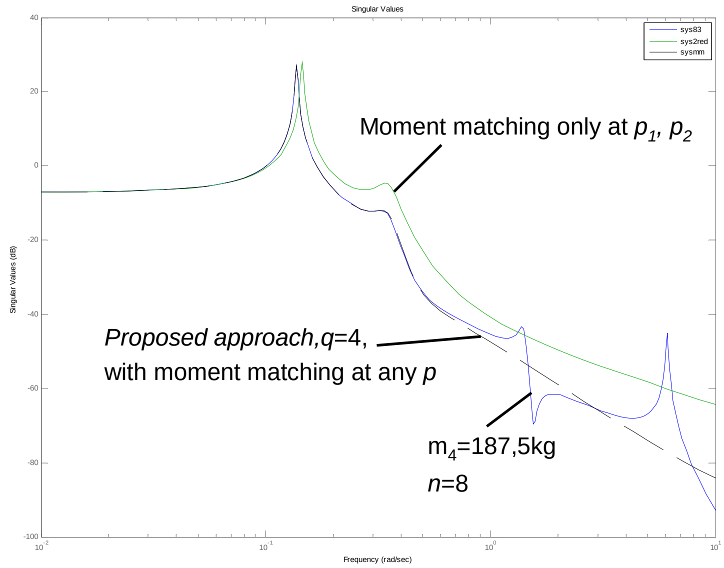


m_1= 1 Kg
m_2= 5 Kg
m_3= 25 Kg
m_4=125 Kg

c_1= 27 N/m
c_2= 9 N/m
c_3= 3 N/m
c_4= 1 N/m
c_5= [1, 3] N/m
c_6= [2, 4] N/m

d_2=0.1 Ns/m
d_3=0.4 Ns/m
d_4=1.6 Ns/m
d_5= [0,1] Ns/m

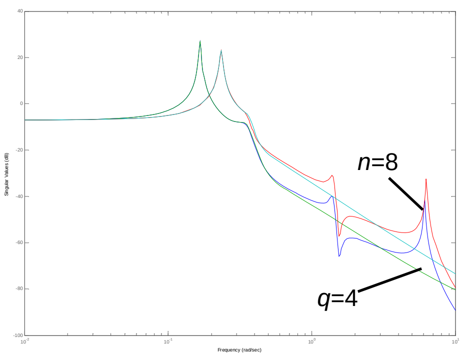
Example: suspension system of order 8



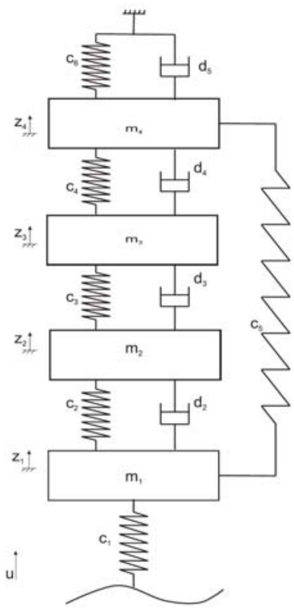
Example: suspension system of order 8

Experiment 2:

c_5,c_6,d_5 change simultaneously.



Bode-plot (from u to z_4)

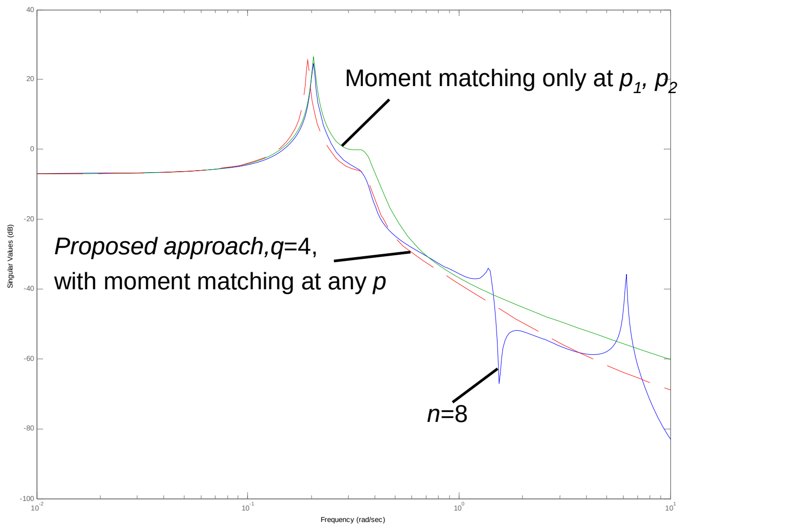


m_1= 1 Kg
m_2= 5 Kg
m_3= 25 Kg
m_4=125 Kg

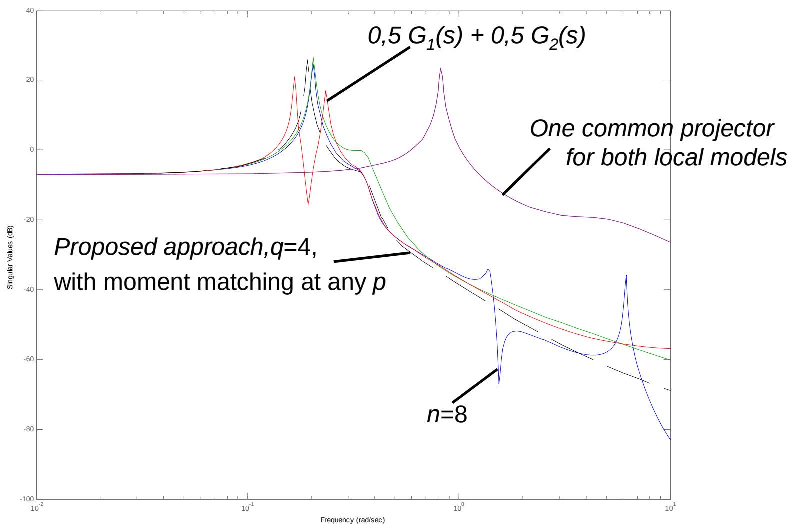
c_1= 27 N/m
c_2= 9 N/m
c_3= 3 N/m
c_4= 1 N/m
c_5= [1, 3] N/m
c_6= [2, 4] N/m

d_2=0.1 Ns/m
d_3=0.4 Ns/m
d_4=1.6 Ns/m
d_5= [0,1] Ns/m

Example: suspension system of order 8



Example: suspension system of order 8



Outlook

Nonlinear parametric reduction by superposition of *locally* reduced linear models

Given $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, p)$

Locally linear parametric representation (like in TPWL):

$$\dot{\boldsymbol{x}} = \sum_{i=1}^{s} \omega_i(\boldsymbol{x}, p)\left(\boldsymbol{f}_i + \boldsymbol{A}_i(\boldsymbol{x} - \boldsymbol{x}_i)\right) \qquad \boldsymbol{f}_i = \boldsymbol{f}(\boldsymbol{x}_i, p_i)$$

$$\boldsymbol{A}_i = \partial \boldsymbol{f} / \partial \boldsymbol{x}\,|_{\boldsymbol{x}_i, p_i}$$

Reduced system:

$$\dot{\boldsymbol{x}}_r(t) = \sum_{i=1}^{s} \omega_i(\boldsymbol{V}_i \boldsymbol{x}_r(t), p) \cdot \boldsymbol{W}^T\left(\boldsymbol{f}_i + \boldsymbol{A}_i(\boldsymbol{V}_i \boldsymbol{x}_r(t) - \boldsymbol{x}_i)\right)$$

normalize ω_i to have $\sum \omega_i = 1$

References

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Appendix

For *numerical reasons*, the projection matrices $\mathbf{V}_i, \mathbf{W}_i$ are typically *orthogonalized* by the famous *Arnoldi algorithm* before use as projector. If we do so, vectors $\mathbf{r}_{0i}$ that solve $\mathbf{A}_i^{-1}\mathbf{b}_i = \mathbf{V}_i\mathbf{r}_{0i}$ are no longer the same for any i, and vectors $\mathbf{r}_{1i}$ that solve $\mathbf{A}_i^{-1}\mathbf{E}_i\mathbf{A}_i^{-1}\mathbf{b}_i = \mathbf{V}_i\mathbf{r}_{1i}$ are no longer the same for any i, which, however, was needed in the proof above. A remedy is the following:

- Calculate orthogonal projectors $\mathbf{V}_i^{\neg}, \mathbf{W}_i^{\neg}$ using the Arnoldi algorithm as in conventional (non-parametric) model reduction. As a byproduct, the algorithm also delivers upper triangular non-singular matrices [3] $\mathbf{H}_{Vi}$ and $\mathbf{H}_{Wi}$ satisfying

$$\mathbf{V}_i = \mathbf{V}_i^{\neg}\mathbf{H}_{Vi}\ , \quad \mathbf{W}_i = \mathbf{W}_i^{\neg}\mathbf{H}_{Wi}. \tag{37}$$

- Out of them, choose one pair of matrices $\mathbf{H}_{Vi}$ and $\mathbf{H}_{Wi}$ (preferably belonging to a "central" or "average" value of the parameter or parameter set) and denote these two matrices by $\overline{\mathbf{H}}_V$ and $\overline{\mathbf{H}}_W$.

- For the reduction of all the local models, use the *new projectors*

$$\mathbf{V}_{new,i} = \mathbf{V}_i^{\neg}\underbrace{\mathbf{H}_{Vi}\overline{\mathbf{H}}_V^{-1}}_{\mathbf{T}_{Vi}}\ , \quad \mathbf{W}_{new,i} = \mathbf{W}_i^{\neg}\underbrace{\mathbf{H}_{Wi}\overline{\mathbf{H}}_W^{-1}}_{\mathbf{T}_{Wi}}. \tag{38}$$

With this choice, all matrices $\mathbf{V}_i, \mathbf{W}_i$ can be expressed from their substitutes $\mathbf{V}_{new,i}, \mathbf{W}_{new,i}$ by

$$\begin{aligned}\mathbf{V}_i &= [\mathbf{A}_i^{-1}\mathbf{b}_i,\ \mathbf{A}_i^{-1}\mathbf{E}_i\mathbf{A}_i^{-1}\mathbf{b}_i, ...] = \mathbf{V}_i^{\neg}\mathbf{H}_{Vi}\overline{\mathbf{H}}_V^{-1}\overline{\mathbf{H}}_V = \mathbf{V}_{new,i}\overline{\mathbf{H}}_V\ ,\\ \mathbf{W}_i &= [\mathbf{A}_i^{-1}\mathbf{c}_i,\ \mathbf{A}_i^{-1}\mathbf{E}_i\mathbf{A}_i^{-1}\mathbf{c}_i, ...] = \mathbf{W}_i^{\neg}\mathbf{H}_{Wi}\overline{\mathbf{H}}_W^{-1}\overline{\mathbf{H}}_W = \mathbf{W}_{new,i}\overline{\mathbf{H}}_W\end{aligned}$$

i.e. by multiplying the new projector with one *common* matrix, $\overline{\mathbf{H}}_V$ or $\overline{\mathbf{H}}_W$. The above proof of moment matching can now be repeated without essential changes.