Computing 38, 261–267 (1987)

Computing

Positive Interpolation with Rational Quadratic Splines

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Received December 22, 1986

Abstract — Zusammenfassung

Positive Interpolation with Rational Quadratic Splines. A necessary and sufficient criterion is presented under which the property of positivity carry over from the data set to rational quadratic spline interpolants. The criterion can always be satisfied if the occuring parameters are properly chosen.

AMS (MOS) Subject Classification: 65D07, 41A15.

Key words: Existence conditions, construction of positive splines, minimization of the curvature.

Positive Interpolation mit rational-quadratischen Splines. Unter positiver Interpolation wird die Aufgabenstellung verstanden, zu einer nichtnegativen Datenmenge nichtnegative Interpolierende zu konstruieren. Im Falle rational-quadratischer Splines wird eine notwendige und hinreichende Bedingung für die Durchführbarkeit positiver Interpolation hergeleitet, und es wird gezeigt, daß diese sich bei passender Wahl der vorkommenden Parameter stets erfüllen läßt.

1. Introduction

The problem of positive interpolation reads as follows: For given points

$$(x_0, y_0), (x_1, y_1), \ldots, (x_n, y_n)$$

with

$$x_0 = 0 < x_1 < \ldots < x_n = 1$$

and

$$y_0 \geqq 0, y_1 \geqq 0, \ldots, y_n \geqq 0 \tag{1.1}$$

construct, if possible at all, interpolants s which are nonnegative on the whole interval $[0, 1]$,

$$s(x) \geqq 0 \text{ for } 0 \leqq x \leqq 1. \tag{1.2}$$

Besides monotone and convex interpolation this is a further type of shape preserving interpolation being also of considerable practical interest.

In this note positive interpolation with quadratic as well as with rational quadratic splines will be discussed. Necessary and sufficient conditions are given under which positive interpolation is of success. It turns out that for quadratic splines these conditions may fail while for rational quadratic splines they can be satisfied if the parameters now occuring are properly chosen. Further, since positive interpolants are in general not uniquely determined a strategy for selecting one of them is in request. Here, following the papers [1] and [7] the mean curvature is taken as objective function.

Shape preserving interpolation has been considered by a large number of authors. The present note profits especially from the papers of Fritsch/Carlson [3] and Schmidt/Heß [7] though both are concerned with monotone and convex interpolation. Likewise for these problems several authors studied different types of rational splines, see above all Späth [9], Miroshnichenko [5] and Delbourgo/Gregory [2]. The rational splines now used are proposed by Schmidt [6], compare also with Gregory [4].

2. Positivity Condition for Quadratic Splines

In order to describe quadratic splines s on the grid

$$\Delta : x_0 = 0 < x_1 < \ldots < x_n = 1 \tag{2.1}$$

set $\tau_i = (y_i - y_{i-1})/h_i$, $h_i = x_i - x_{i-1}$. With $t = (x - x_{i-1})/h_i$ define for $x \in [x_{i-1}, x_i]$

$$s(x) = y_{i-1} + \tau_i h_i t + (m_{i-1} - \tau_i) h_i t(1-t), \; 0 \leqq t \leqq 1 \tag{2.2}$$

for $i = 1, \ldots, n$. Then s interpolates the given data set,

$$s(x_i) = y_i \; (i = 0, \ldots, n). \tag{2.3}$$

Further, $s \in C^1[0,1]$ if and only if

$$m_{i-1} + m_i = 2\tau_i \; (i = 1, \ldots, n-1), \tag{2.4}$$

and for the parameters $m_0, m_1, \ldots, m_{n-1}$

$$m_i = s'(x_i)$$

holds. Now, a condition is derived which assures

$$s(x) \geqq 0 \;\text{ for }\; x \in [x_{i-1}, x_i] \tag{2.5}$$

for the quadratic polynomial s. To this end $y_{i-1} \geqq 0$, $y_i \geqq 0$ is assumed to be valid.

Case 1: Let $m_{i-1} \geqq \tau_i$. Then (2.5) holds.

Case 2: Let $m_{i-1} < \tau_i$. Denote by x^* the minimizer of s,

$$x^* = x_{i-1} + h_i t^*, \; t^* = -\frac{m_{i-1}}{2(\tau_i - m_{i-1})}.$$

Case 2.1: Let $t^* \leqq 0$, i.e. $m_{i-1} \geqq 0$. Then (2.5) follows.

Case 2.2: Let $t^* \geqq 1$, i.e. $m_{i-1} \geqq 2\tau_i$. Then (2.5) follows.

Case 2.3: Let $0 < t^* < 1$, i.e. $m_{i-1} \leqq u_i = \min\{0, 2\tau_i\}$. Then (2.5) follows if and only if $s(x^*) \geqq 0$. This inequality is equivalent with

$$h_i m_{i-1}^2 \leqq 4 y_{i-1}(\tau_i - m_{i-1})$$

leading to

$$m_{i-1} \geqq v_i = -\frac{2}{h_i}\{y_{i-1} + \sqrt{y_{i-1} y_i}\}. \tag{2.6}$$

Thus, because of $v_i \leqq u_i \leqq \tau_i$ the following assertion is proven.

Proposition 1: *The quadratic spline interpolant* (2.2) *is nonnegative on* $[0, 1]$ *if and only if*

$$m_{i-1} \geqq v_i \ (i=1, \ldots, n) \tag{2.7}$$

where the v_i*'s are defined by* (2.6).

Therefore, positive interpolation with quadratic C^1-splines is possible if and only if there is a set of numbers $m_0, \ldots, m_{n-1}$ which solves system (2.4), (2.7), and via formula (2.2) every solution set leads to a nonnegative spline interpolants.

3. Positivity Condition for Rational Quadratic Splines

The rational quadratic splines s used here depend on parameters $r_1 \geqq 0, \ldots, r_n \geqq 0$. For $x \in [x_{i-1}, x_i]$ with $t=(x-x_{i-1})/h_i$ set

$$s(x)=y_{i-1}+\tau_i h_i t+(m_{i-1}-\tau_i) h_i \frac{t(1-t)}{1+r_i t}, \quad 0 \leqq t \leqq 1. \tag{3.1}$$

For $r_i=0$ the quadratic splines (2.2) originate. These rational quadratic splines also satisfy interpolation condition (2.3) while the smoothness condition (2.4) now reads

$$m_{i-1}+(1+r_i) m_i=(2+r_i) \tau_i \,(i=1, \ldots, n-1). \tag{3.2}$$

The positivity condition can be derived as before by considering different cases. Another possibility is to use the identity

$$(1+r_i t) s(x)=A_i t(1-t)+B_i (t-C_i/B_i)^2$$

where

$$A_i=m_{i-1} h_i+(2+r_i) y_{i-1}+2\sqrt{(1+r_i) y_{i-1} y_i},$$

$$B_i=y_{i-1}+(1+r_i) y_i+2\sqrt{(1+r_i) y_{i-1} y_i}>0 \text{ for } y_{i-1}+y_i>0,$$

$$C_i=y_{i-1}+\sqrt{(1+r_i) y_{i-1} y_i} \geqq 0.$$

Because of $0 \leqq C_i/B_i \leqq 1$, $1+r_i t>0$ the positivity of s on $[x_{i-1}, x_i]$ is equivalent to $A_i \geqq 0$. Thus one gets

Proposition 2: *The rational quadratic spline interpolant* (3.1) *is nonnegative on* $[0, 1]$ *if and only if*

$$m_{i-1} \geqq v_i \ (i=1, \ldots, n) \tag{3.3}$$

where

$$v_i=-\frac{1}{h_i} \{(2+r_i) y_{i-1}+2\sqrt{(1+r_i) y_{i-1} y_i}\}. \tag{3.4}$$

In [6] C^1-spline interpolants (3.1) are shown to be convex on $[0, 1]$ if and only if

$$m_{i-1} \leqq \tau_i \ (i=1, \ldots, n), \tag{3.5}$$

while they are isotone on $[0, 1]$ if and only if

$$m_i \geqq 0 \ (i=0, \ldots, n). \tag{3.6}$$

Therefore, by adding (3.5) to (3.3) one gets a criterion under which rational splines are nonnegative and convex on $[0,1]$. Because of $v_i \leqq 0$, already condition (3.6) assures rational quadratic splines to be nonnegative and isotone on $[0,1]$.

4. Existence of Nonnegative Interpolants

As seen before the solvability of system (2.4), (2.7) and of the more general system (3.2), (3.3) is of interest. This question can be decided by means of the following

Algorithm: $d_1 = v_1$, $c_1 = 1$ and for $i = 1, \ldots, n-1$:

$$\begin{aligned} d_{i+1} &= d_i + (-1)^i c_i \{(1+r_i) v_{i+1} + v_i - (2+r_i)\tau_i\}, \\ c_{i+1} &= (1+r_i) c_i. \end{aligned} \tag{4.1}$$

Now, with (3.2) the assertion

$$m_{i-1} - v_i = (-1)^i \frac{d_i - m_0}{c_i} \quad (i=1, \ldots, n) \tag{4.2}$$

is easily verified. Hence, condition (3.3) reads

$$\begin{aligned} m_0 &\leqq d_i \text{ for } i \text{ even}, \\ m_0 &\geqq d_i \text{ for } i \text{ odd}. \end{aligned}$$

Thus, by setting

$$\begin{aligned} d^* &= \min \{d_i \colon i \text{ even}, \ 1 \leqq i \leqq n\}, \\ d_* &= \max \{d_i \colon i \text{ odd}, \ 1 \leqq i \leqq n\} \end{aligned} \tag{4.3}$$

an essential result of this note follows.

Proposition 3: *Nonnegative rational quadratic C^1-spline interpolants exist if and only if*

$$d_* \leqq d^*. \tag{4.4}$$

Such interpolants can be constructed via (3.2), (3.1) *by starting with any* $m_0 \in [d_*, d^*]$.

For quadratic splines, i.e. for $r_1 = \ldots = r_n = 0$ the test (4.4) may fail. On the other hand, since $v_i \to -\infty$ for $r_i \to +\infty$ if $y_{i-1} > 0$, for sufficiently large $r_2, r_3, r_4, \ldots$ it follows successively

$$\ldots d_3 \leqq d_1 < d_2 \leqq d_4 \ldots,$$

and therefore condition (4.4) is satisfied. Thus one gets

Proposition 4: *Let* $y_0 > 0$, $y_1 > 0, \ldots, y_n > 0$. *Then, for sufficiently large parameters* $r_1, \ldots, r_n$, *the rational quadratic C^1-spline interpolants* (3.1) *are nonnegative on* $[0,1]$.

5. Symmetric Positive Interpolation

Suppose that the data set

$$(x_{-n}, y_{-n}), \ldots, (x_0, y_0), \ldots, (x_n, y_n)$$

is positive and symmetric,

$$x_{-i} = -x_i, \ y_{-i} = y_i > 0 \ (i=0, \ldots, n), \tag{5.1}$$

and $x_n = 1$. In order to construct a symmetric nonnegative interpolant on $[-1, 1]$ it obviously suffices to have a nonnegative interpolant s on $[0, 1]$ with

$$s'(x_0) = m_0 = 0. \tag{5.2}$$

In this case, by means of the C^1-condition (3.2), all quantities $m_1, \ldots, m_n$ are uniquely defined. Because of $m_{i-1} = m_{i-1}(r_1, \ldots, r_{i-1})$ and $v_i \to -\infty$ for $r_i \to +\infty$ the inequality $v_i \leqq m_{i-1}$ is valid for r_i sufficiently large. Hence, it follows

Proposition 5: *Assume* (5.1). *Then, for sufficiently large parameters* $r_1, \ldots, r_n$ *the* C^1*-spline interpolant* (3.1) *is nonnegative and symmetric on* $[-1, 1]$.

6. Positive Splines with Minimal Curvature

If the existence condition (4.4) is strictly satisfied, i.e. if $d_* < d^*$ then there exists an infinite number of nonnegative spline interpolants s. For preferring one of them the geometric curvature

$$\int_0^1 \frac{s''(x)^2\, dx}{(1 + s'(x)^2)^3}$$

is taken as objective function. In order to get a more convenient function $s'(x)$ is approximated by τ_i for $x_{i-1} \leqq x \leqq x_i$. Thus, one is led to minimize

$$f_2(s) = \sum_{i=1}^{n} w_i \int_{x_{i-1}}^{x_i} s''(x)^2\, dx \tag{6.1}$$

where $w_i = 1/(1 + \tau_i^2)^3$, see [7]. Now in view of (3.1) and (4.2), the one-dimensional quadratic program

$$\begin{aligned} f_2(s) &= \sum_{i=1}^{n} R_i (\tau_i - m_{i-1})^2 \\ &= \sum_{i=1}^{n} \frac{R_i}{c_i^2} (m_0 - d_i + (-1)^i c_i (\tau_i - v_i))^2 \to \min! \end{aligned} \tag{6.2}$$

$$\text{subject to } d_* \leqq m_0 \leqq d^*$$

where

$$R_i = \frac{4 w_i ((1 + r_i)^5 - 1)}{5 h_i r_i (1 + r_i)^3}$$

arises. This problem is easily solved. Let $\tilde{m}_0$ be the unconstrained minimizer of (6.2),

$$\tilde{m}_0 = \sum_{i=1}^{n} \frac{R_i}{c_i^2} (d_i - (-1)^i c_i (\tau_i - v_i)) \Bigg/ \sum_{i=1}^{n} \frac{R_i}{c_i^2},$$

then the minimizer m_0^* of program (6.2) reads as follows:

$$m_0^* = \begin{cases} \tilde{m}_0 & \text{for } d_* \leqq \tilde{m}_0 \leqq d^* \\ d^* & \text{for } \tilde{m}_0 > d^* \\ d_* & \text{for } \tilde{m}_0 < d_*. \end{cases} \tag{6.3}$$

The objective function

$$f_\infty(s) = \max_{i=1,\ldots,n} \sqrt{w_i} \max_{x_{i-1} \leqq x \leqq x_i} |s''(x)| \tag{6.4}$$

can also be recommended. In this case the program

$$\begin{gathered} f_\infty(s) = \max_{i=1,\ldots,n} \frac{2\sqrt{w_i}(1+r_i)}{h_i} |m_0 - d_i + (-1)^i(\tau_i - v_i)| \to \min! \\ \text{subject to } d_* \leqq m_0 \leqq d^* \end{gathered} \tag{6.5}$$

is obtained. For an algorithm which solves programs of this type effectively it is referred to [6].

7. Numerical Tests

In order to demonstrate positive interpolation with rational quadratic C^1-splines as an example the data set

x_i	0	1	2	3	4
y_i	0	1	M	1	0

is chosen. Because these splines tend to a linear C^0-spline for $r_1 \to +\infty, \ldots, r_n \to +\infty$, see (3.1), the parameters $r_1, \ldots, r_n$ should be determined as small as possible such that the nonnegative interpolants occur. According to this the smallest integer $r = r_1 = \ldots = r_n$ is computed by a search procedure. Some results are given in the following table, together with the intervals $[d_*, d^*]$ and the optimal m_0^* defined by formula (6.3).

M	r	d_*	d^*	m_0^*
0.5	0	0.585786	5.41421	3
1	0	0	4	2
2	0	0	0	0
2.1	1	0	7.2	0.513255
2.3	1	0	3.6	0.516126
2.5	2	0	32.4317	0.82677
2.6	2	0	32.7521	0.837144
3	2	0	16	0.858628
4	3	0	25	0.936159
$M \geqq 5$	$M-1$	0	$(M+1)^2$	

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