Parameter-preserving model reduction by an interpolatory balanced truncation approach

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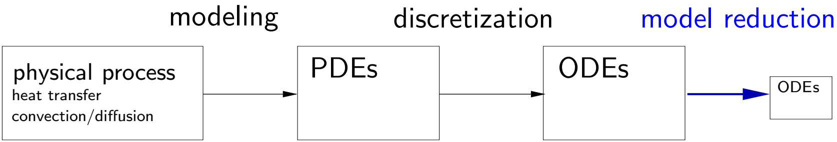
Outline

Parameter-preserving model reduction
by an interpolatory balanced truncation approach

- Model order reduction for LTI systems
  - Balanced truncation
- Model order reduction for parametric systems
  - Balanced truncation/interpolatory MOR
  - Example with 1 parameter
  - Use of sparse grids
  - Numerical results
- Conclusions/Outlook

Model order reduction for LTI systems



Typical requirements on methods for model reduction:

- preserve important system properties (like stability);
- small approximation error,
  existence of global, computable error bound;
- efficient implementation,
  apply to original large-scale system.

Balanced truncation [*Moore 81, Mullis/Roberts 76*]

Balanced system realization

For controllability Gramian $\mathcal{P}$ and observability Gramian $\mathcal{Q}$

$$\mathcal{P} = \mathcal{Q} = \begin{pmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_n \end{pmatrix} \quad \sigma_1 \geq \cdots \geq \sigma_n > 0$$

$\sigma_1, \cdots, \sigma_n$ Hankel singular values

Global error bound [*Enns, Glover 84*]:

$$\|y - \hat{y}\|_2 \leq \|G - \hat{G}\|_{\mathcal{H}_\infty} \|u\|_2, \quad \|G - \hat{G}\|_{\mathcal{H}_\infty} \leq 2 \left(\sum_{j=r+1}^{n} \sigma_j \right) < \mathrm{tol}$$

Compute $\mathcal{P}$ and $\mathcal{Q}$ as solutions of Lyapunov equations:

$$A\mathcal{P} + \mathcal{P}A^T + BB^T = 0, \qquad A^T\mathcal{Q} + \mathcal{Q}A + C^TC = 0$$

Low-rank solutions of Lyapunov equations

$$A\mathcal{P} + \mathcal{P}A^T + BB^T = 0, \qquad A^T\mathcal{Q} + \mathcal{Q}A + C^TC = 0$$

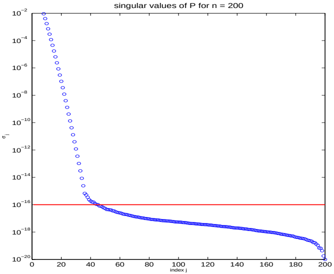
In many applications: [*Penzl 99, Ant./Sor./Zhou 02, Grasedyck 04*]

$$\operatorname{rank}(\mathcal{P}, \tau) = k_P \ll n, \quad \operatorname{rank}(\mathcal{Q}, \tau) = k_Q \ll n$$

$\Rightarrow$ Compute low-rank solution factors $\tilde{S} \in \mathbb{R}^{n \times k_P}$, $\tilde{R} \in \mathbb{R}^{n \times k_Q}$:

$$\mathcal{P} \approx \tilde{S}\tilde{S}^T, \qquad \mathcal{Q} \approx \tilde{R}\tilde{R}^T$$

- LR-ADI [*Penzl 00, Li/White 02*]
- Smith [*Penzl 00, Antoulas/Gugercin/Sor. 03*]
- Krylov [*Jaimoukha/Kasenally 94, Saad 90, Simoncini 07*]
- Sign function method [*Benner/Quintana-Ortí 99, B. 08*]



Parametric model order reduction

modeling — discretization — parametric MOR

physical process
heat transfer
convection/diffusion → parametric PDEs → parametric ODEs → param. ODEs

- preserve parameters as physical quantities
  - geometry, topology, material parameters
  - damping factors
  - film, heat exchange coefficients in boundary conditions
- applications
  - optimization and design
  - repeated simulation for varying parameter configurations

Important requirement on parametric MOR:

preserve parameters in reduced-order system!

Parametric model order reduction

Parametric system

$$\begin{aligned}\tfrac{d}{dt}x(t,p) &= A(p)\,x(t,p) + B(p)\,u(t)\\ y(t,p) &= C(p)^T x(t,p)\end{aligned}$$

with

- parameter vector $p \in \mathbb{R}^d$
- $x(t,p) \in \mathbb{R}^n$, input $u(t) \in \mathbb{R}^m$, output $y(t,p) \in \mathbb{R}^q$, $m, q \ll n$
- stability: $\lambda(A(p)) \subset \mathbb{C}^-$ for all p

Reduced parametric system

$$\begin{aligned}\tfrac{d}{dt}\hat{x}(t,p) &= \hat{A}(p)\,\hat{x}(t,p) + \hat{B}(p)\,u(t)\\ \hat{y}(t,p) &= \hat{C}(p)^T \hat{x}(t,p)\end{aligned}$$

- state $x(t,p) \in \mathbb{R}^r$ and $r \ll n$
- $\|y - \hat{y}\|$ bounded

Balanced truncation/interpolatory MOR for $d = 1$

1. Choose interpolation points $p_0, \ldots, p_k \in [a,\ b]$

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$$\hat{G}_j(s) = \hat{C}_j^T (sI_{r_j} - \hat{A}_j)^{-1} \hat{B}_j \quad \mathit{for} \ j = 0, \dots, k$$

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3. Parametric reduced-order system by Lagrange interpolation:

$$\begin{aligned} \hat{G}(s,p) &= \sum_{j=0}^{k} l_j(p) \hat{G}_j(s) \\ &= \sum_{j=0}^{k} \left(\prod_{i=0, i\neq j}^{k} \frac{p - p_i}{p_j - p_i} \right) \hat{C}_j^T (sI_{r_j} - \hat{A}_j)^{-1} \hat{B}_j = \end{aligned}$$

$$\begin{bmatrix} \hat{C}_0(p) \\ \vdots \\ \hat{C}_k(p) \end{bmatrix}^T \begin{bmatrix} (sI_{r_0} - \hat{A}_0)^{-1} & & \\ & \ddots & \\ & & (sI_{r_k} - \hat{A}_k)^{-1} \end{bmatrix} \begin{bmatrix} \hat{B}_0 \\ \vdots \\ \hat{B}_k \end{bmatrix}$$

Balanced truncation/interpolatory MOR for $d = 1$

- reduced complexity in numerical simulation:

  costs for evaluation of transfer function reduced from $\mathcal{O}(n^3)$ for $G(s,p)$ to $\mathcal{O}((k+1)\max(r_j)^3)$ for $\hat{G}(s,p)$

- reduced storage requirements from $\mathcal{O}(n^2)$ for original system to $\mathcal{O}((k+1)\max(r_j)^2)$ for reduced-order system

- global error bound by combination of BT error bound, i.e.

$$\|G_j(\cdot) - \hat{G}_j(\cdot)\|_{\mathcal{H}_\infty} \leq 2\left(\sum_{i=r_j+1}^{n} \sigma_i\right) < \mathrm{tol} \qquad (1)$$

  and error estimates for interpolation if

$$G(s,\cdot) \in C^{k+1}([a,\ b] \to \mathbb{C}^{q\times m}) \quad \forall s \in \mathbb{C}^+$$

$$\sup_{\substack{s\in\mathbb{C}^+\\ p\in[a,b]}} \|G(s,p)-\hat{G}(s,p)\| = \sup_{\substack{s\in\mathbb{C}^+\\ p\in[a,b]}} \|G(s,p)-\sum_{j=0}^{k} l_j(p)\hat{G}_j(s)\|$$

$$\leq \sup_{\substack{s\in\mathbb{C}^+\\ p\in[a,b]}} \|G(s,p)-\sum_{j=0}^{k} l_j(p)G_j(s)\| +$$

$$\sup_{\substack{s\in\mathbb{C}^+\\ p\in[a,b]}} \|\sum_{j=0}^{k} l_j(p)(G_j(s)-\hat{G}_j(s))\|$$

$$\overset{(1)}{\leq} \sup_{\substack{s\in\mathbb{C}^+\\ p\in[a,b]}} \|R_k(G,s,p)\| + \mathrm{tol}\cdot \sup_{p\in[a,b]} |\sum_{j=0}^{k} l_j(p)|$$

with remainder $R_k(G,s,p) = G(s,p)-\hat{G}(s,p)$

$$R_k(G,s,p) = \frac{1}{(k+1)!}\left(\frac{\partial^{k+1}}{\partial p^{k+1}}G(s,\xi(p))\right)\prod_{i=0}^{k}(p-p_i)$$

Numerical results - convection-diffusion equation $d = 1$

$$
\begin{aligned}
\frac{\partial T}{\partial t}(t,\xi) &= \Delta T(t,\xi) + p \cdot \nabla T(t,\xi) + b(\xi)u(t) && \xi \in (0,1)^2 \\
&\Downarrow \quad \textit{using FDM with } n = 400 \\
\frac{d}{dt}x(t) &= (A + p\,A_1)\,x(t) + b\,u(t) && b = e_1 \\
y(t) &= c^T x(t) && c^T = [1, 1, \cdots, 1]
\end{aligned}
$$

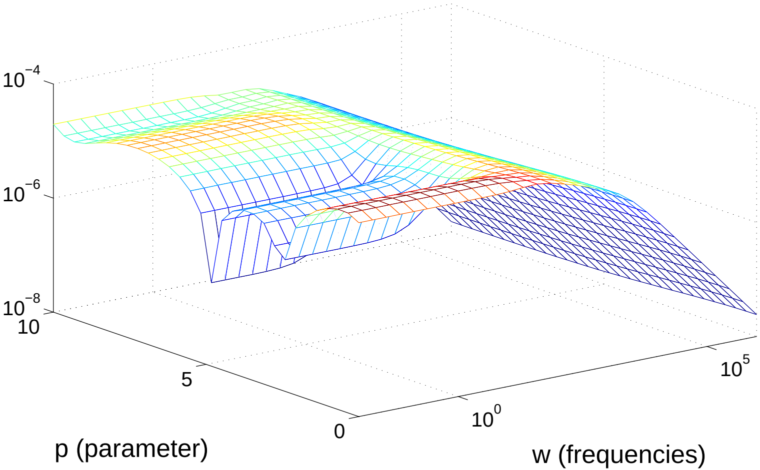
1. choose $p_0, \cdots, p_5 \in [0,\ 10]$ as Chebyshev points
2. prescribe BT error bound for $\hat{G}(s, p_j)$ by tol$=10^{-4}$

   $\Rightarrow$ systems of reduced order $r_j \in \{3, 4\}$
3. $\hat{G}(s, p)$ by Lagrange interpolation with error estimate

$$
\sup_{\substack{s \in [\jmath 10^{-2}, \jmath 10^{6}] \\ p \in [0,10]}} \| G(s,p) - \hat{G}(s,p) \| \leq 2.6 \times 10^{-5}
$$

Numerical results - convection-diffusion equation $d = 1$

$$\|G(s,p) - \hat{G}(s,p)\|_{\infty}$$



Balanced truncation/interpolatory MOR

But:

for higher dimensional parameter spaces $p \in [0,1]^d$ with $d \geq 3$ we need many interpolation points $\Rightarrow$ many times BT,

i.e. very high complexity!

Thus:

employ sparse grid interpolation [*Zenger 91, Griebel 91, Bungartz 92*]

main advantages:

- requires significantly fewer grid points
- preserves asymptotic error decay with increasing grid resolution (up to logarithmic factor)

Hierarchical basis decomposition in $d = 1$

On $[0,\ 1]$ construct (equidistant) grid with mesh size $h_\ell = 2^{-\ell}$ and associated $(2^\ell - 1)$-dim. space of piecewise linear functions S_ℓ

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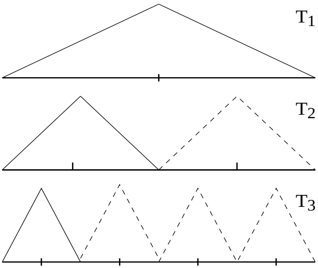
Hierarchical basis decomposition:

$$S_\ell = T_1 \oplus \cdots \oplus T_\ell$$

subspaces of S_3:



basis of $T_1,\ T_2,\ T_3$

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$$S_\ell = T_1 \oplus \cdots \oplus T_\ell$$

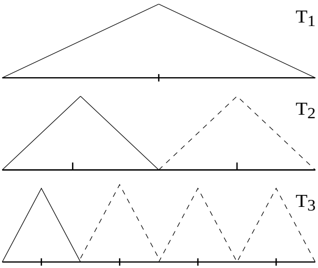
For $f \in C^2([0,\ 1])$ and interpolant $f_{\mathrm{I}} \in S_\ell$

$$f_{\mathrm{I}} = \sum_{i=1}^{\ell} f_i, \qquad f_i \in T_i$$

the interpolation error is bounded

- $\|f - f_{\mathrm{I}}\|_\infty \le \mathcal{O}(h_\ell^2)$
- $\|f_i\|_\infty \le \frac{1}{2} 4^{-i} \|\frac{\partial^2 f}{\partial x^2}\|_\infty$

subspaces of S_3:



basis of $T_1,\ T_2,\ T_3$

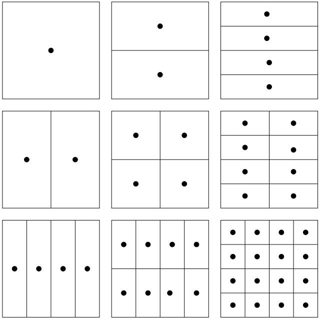
Hierarchical basis decomposition in $d = 2$

On $[0,\ 1]^2$ construct rectangular grid with mesh size $h_{\ell_1} = 2^{-\ell_1}, h_{\ell_2} = 2^{-\ell_2}$ and $(2^\ell - 1)^2$-dim. space of piecewise bilinear functions $S_{\underline{\ell}}$ $\quad (\underline{\ell} := (\ell, \ell))$

hierarchical basis decomposition:

$$S_{\underline{\ell}} = \bigoplus_{i_1=1}^{\ell} \bigoplus_{i_2=1}^{\ell} T_{\underline{i}}, \quad \underline{i} = (i_1, i_2)$$

subspaces of S_{33}:



supports of basis of $T_{11} \cdots$

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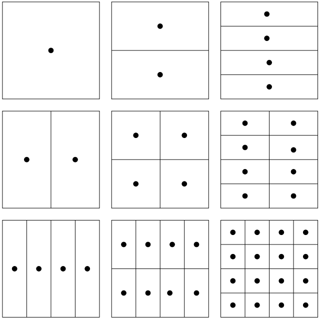
the interpolation error is bounded

- $\|f - f_{\mathrm{I}}\|_\infty \leq \mathcal{O}(h_\ell^2)$
- $\|f_{\underline{i}}\|_\infty \leq \frac{1}{4} 4^{-i_1 - i_2} \| \frac{\partial^4 f}{\partial x_1^2 \partial x_2^2} \|_\infty$

subspaces of S_{33}:



supports of basis of $T_{11} \cdots$

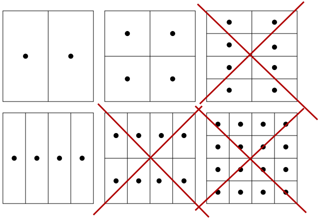
Sparse grids in $d = 2$ [*Zenger 91, Griebel 91, Bungartz 92*]

sparse decomposition: $\tilde{S}_{\underline{\ell}} = \bigoplus_{i_1+i_2 \leq \ell+1} T_{\underline{i}}, \quad \underline{i} = (i_1, i_2)$

with reduced dimension:

$$\dim \tilde{S}_{\underline{\ell}} = 2^{\ell}(\ell - 1) + 1$$

subspaces of S_{33}:



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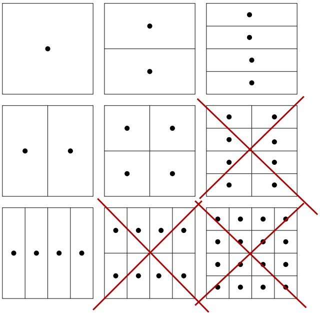
For $f : [0,\ 1]^2 \to \mathbb{R}$, $\frac{\partial^4 f}{\partial x_1^2 \partial x_2^2} \in C^0([0,\ 1]^2)$

$$\tilde{f}_{\mathrm{I}} = \sum_{i_1+i_2 \le \ell+1} f_{\underline{i}}, \qquad f_{\underline{i}} \in T_{\underline{i}}$$

the interpolation error is bounded

$$\|f - \tilde{f}_{\mathrm{I}}\|_{\infty} \le \mathcal{O}(h_{\ell}^2 log(h_{\ell}^{-1}))$$

subspaces of S_{33}:



supports of basis of $T_{11} \cdots$

Sparse grids [Zenger 91, Griebel 91, Bungartz 92]

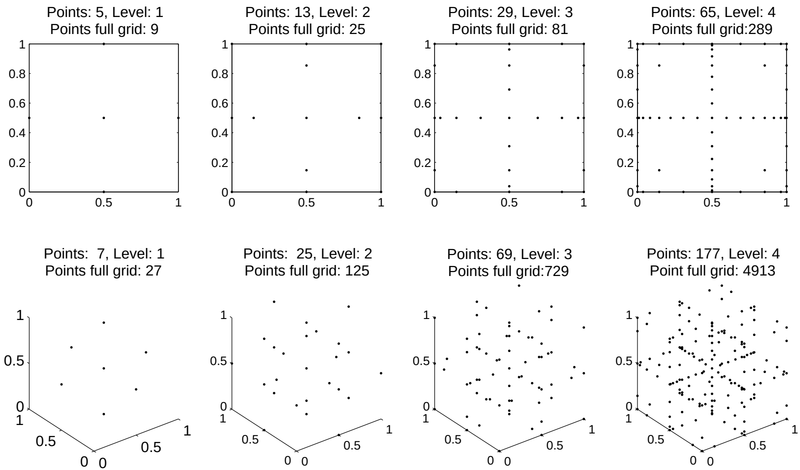
On $[0,\ 1]^d$ construct rectangular grid with mesh size $h_{\underline{\ell}}$.

For $f : [0,\ 1]^d \to \mathbb{R}$, $\frac{\partial^{2d} f}{\partial x_1^2 \dots \partial x_d^2} \in C^0([0,\ 1]^d)$ search

interpolant f_{I} in space of piecewise d-linear functions:

	full grid space	sparse grid space
	$S_{\underline{\ell}} = \bigoplus_{i_1=1}^{\ell} \cdots \bigoplus_{i_d=1}^{\ell} T_{\underline{i}}$	$\tilde{S}_{\underline{\ell}} = \bigoplus_{\lvert\underline{i}\rvert_1 \le \ell+d-1} T_{\underline{i}}$
dimension	$\mathcal{O}(h_\ell^{-d})$	$\mathcal{O}(h_\ell^{-1}(log(h_\ell^{-1}))^{d-1})$
$\lVert f - f_{\mathrm{I}} \rVert_\infty$	$\mathcal{O}(h_\ell^2)$	$\mathcal{O}(h_\ell^2(log(h_\ell^{-1}))^{d-1})$

MATLAB Sparse Grid Interpolation Toolbox:



We employ sparse grids for high-dim. parameter space $p \in \mathcal{I}^d$.

1. With level ℓ choose $\mathcal{O}(h_\ell^{-1}(log(h_\ell^{-1}))^{d-1})$ sparse grid points

Balanced truncation/interpolatory MOR in $\mathcal{I}^d$

1. With level ℓ choose $\mathcal{O}(h_\ell^{-1}(log(h_\ell^{-1}))^{d-1})$ sparse grid points
2. Apply balanced truncation to $G_j(s) := G(s, p_j)$:

$$\hat{G}_j(s) = \hat{C}_j^T(sI_{r_j} - \hat{A}_j)^{-1}\hat{B}_j,$$

determine r_j by prescribed error tolerance:

$$\|G_j - \hat{G}_j\|_{\mathcal{H}_\infty} \leq tol$$

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   determine r_j by prescribed error tolerance:

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3. Parametric reduced-order system:

$$\hat{G}(s,p) = \sum_{|\underline{i}|_1 \leq \ell+d-1} \phi_{\underline{i}}(p)\, c_{\underline{i}}(\hat{G}_1(s), \hat{G}_2(s), \cdots)$$

   with interpolation error

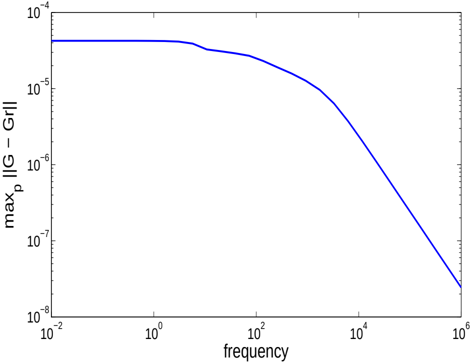
$$\|G - \hat{G}\|_\infty \leq \mathrm{tol} \cdot C \cdot \sup_{p \in \mathcal{I}^d} \sum_{|\underline{i}|_1 \leq \ell+d-1} |\phi_{\underline{i}}(p)| + \mathcal{O}(h_\ell^2(log(h_\ell^{-1}))^{d-1})$$

Numerical results - convection-diffusion equation $d = 2$

$$\begin{aligned}\frac{\partial T}{\partial t}(t,\xi) &= \Delta T(t,\xi) + \mathbf{p}\cdot\nabla T(t,\xi) + b(\xi)u(t) \quad \xi\in(0,1)^2\\ &\Downarrow \quad \textit{using FDM with } n = 400\\ \frac{d}{dt}x(t) &= (A + p_1A_1 + p_2A_2)\,x(t) + b\,u(t)\end{aligned}$$

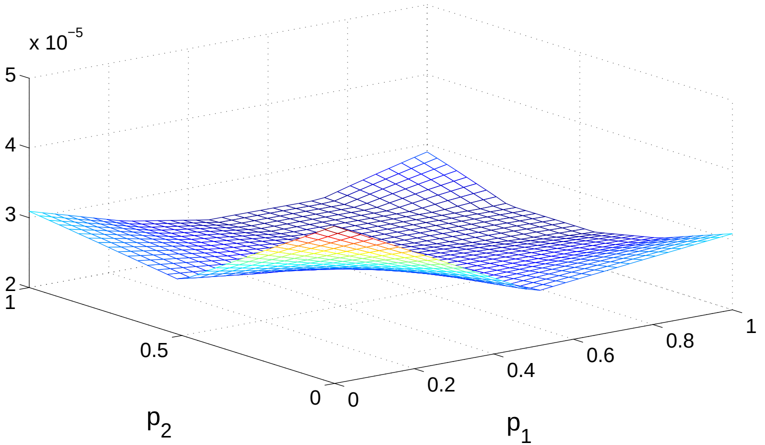
- $b = e_1,\ c^T = [1, 1, \cdots, 1]$
- MATLAB Sparse Grid Interpolation Toolbox [*Klimke/Wohlmuth 05, Klimke 07*]
- parameter space: $p_1, p_2 \in [0,\ 1]$
- Chebyshev-Gauss-Lobatto grid with polynomial interpolation

Numerical results - convection-diffusion equation $d = 2$

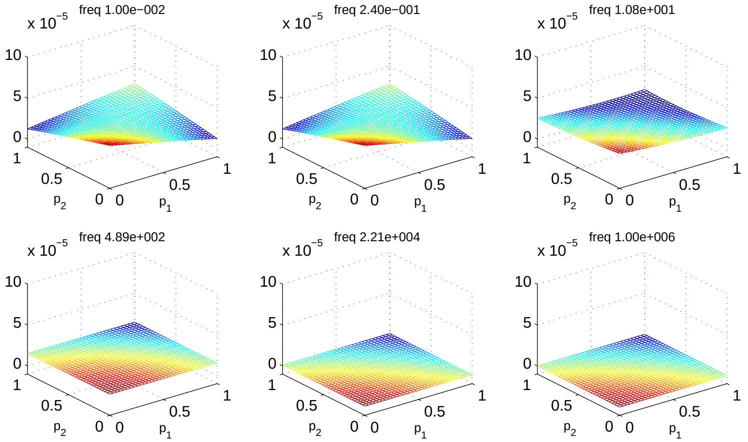


- we choose level for grid refinement: $\ell = 1$
  $\Rightarrow$ 5 sparse grid points
- error tol for BT applied to $G(s, p_j)$: 10^{-4}
  $\Rightarrow$ systems of reduced order $r_j = 3$, $j = 1, \cdots, 5$
- estimated interpolation error: 1.8×10^{-4}

$\mathcal{H}_\infty$ error of the transfer function

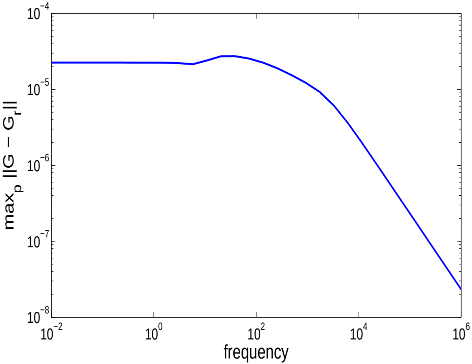


Absolute errors of the transfer function at selected frequencies



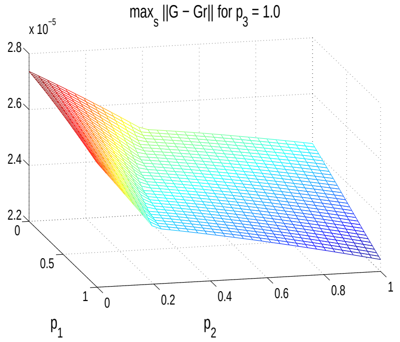
Numerical results - convection-diffusion equation $d = 3$

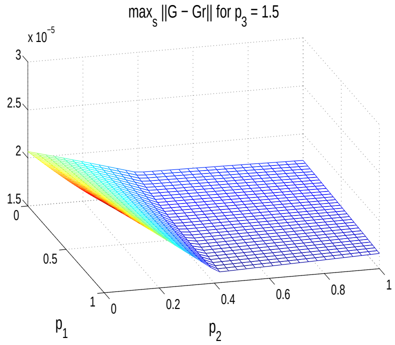
$$\dot{x}(t) = (p_3 A + p_1 A_1 + p_2 A_2)\, x(t) + B\, u(t), \quad p_1, p_2 \in [0,\ 1],\ p_3 \in [1,\ 1.5]$$



- choose abs. err. tol. for grid refinement: 10^{-4}
  $\Rightarrow$ level $\ell = 2$ and 25 sparse grid points
- error tol for BT applied to $G(s, p_j)$: 10^{-4}
  $\Rightarrow$ systems of reduced order $r_j = 3$
- estimated interpolation error: 6.6×10^{-4}

$\mathcal{H}_\infty$ error of the transfer function





Conclusions/Outlook

Summary:

- We have developed a balanced truncation/interpolatory method for parametric model reduction with reduced complexity in numerical simulations, error estimates.
- The method can be applied to higher ($d \leq 10$) dimensional parameter spaces.

Next steps:

- only function for evaluation of reduced-order system, search for explicit description of TFM, state-space model;
- use sparse grids also for other interpolatory methods as proposed in [*Beattie/Benner/Gugercin 09*];
- combine sparse grid interpolation with $\mathcal{H}_2$-optimal model reduction [*Gugercin/Antoulas/Beattie 08*].