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How to condense GSMs

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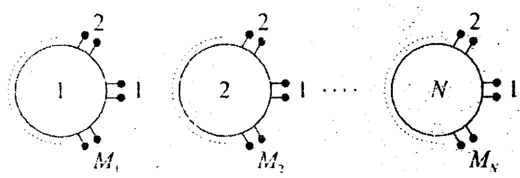

Fig. 1. Generic set of devices.

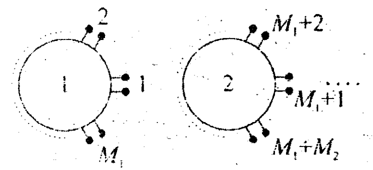

Fig. 2. Generic set of devices with global port numbering.

Abstract— **Abstract?**

Index Terms— **finite-element methods, sensitivity analysis, CAD, waveguide filters**

I. Introduction

GIVEN N separate devices, each exhibiting M_d $d = 1,\dots,N$ ports, each of them characterized by its own generalized scattering matrix (GSM) $[S]^{(d)}$ (Fig. 1). These GSMs are block matrices:

$$[S]^{(d)} = \begin{bmatrix} [S_{11}]^{(d)} & [S_{12}]^{(d)} & \dots & [S_{1M_d}]^{(d)} \\ [S_{21}]^{(d)} & [S_{22}]^{(d)} & \dots & [S_{2M_d}]^{(d)} \\ \vdots & \vdots & \ddots & \vdots \\ [S_{M_d1}]^{(d)} & [S_{M_d2}]^{(d)} & \dots & [S_{M_dM_d}]^{(d)} \end{bmatrix} \tag{1}$$

a single sparse GSM can be assembled from the local ones once a suitable global numbering scheme is devised for the ports. If, for example, the global numbers of the ports of the first device are chosen equal to the local ones, those of the second device equal to the local one plus M_1, those of the third equal to the local number plus $N_1 + N_2$ and so on (Fig. 2) the *expanded* block-diagonal GSM matrix is: ck.

$$[\bar{S}] = \begin{bmatrix} [S]^{(1)} & 0 & \dots & 0 \\ 0 & [S]^{(2)} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & [S]^{(N)} \end{bmatrix} \tag{2}$$

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The dimensions of (2) are $\bar{M} \times \bar{M}$, with $\bar{M} = \sum_{d=1}^{N} M_d$. Matrix (2) can itself been thought as made by smaller blocks, which are analogous to those in (1):

$$[\bar{S}] = \begin{bmatrix} [\bar{S}_{11}] & [\bar{S}_{12}] & \dots & [\bar{S}_{1\bar{M}}] \\ [\bar{S}_{21}] & [\bar{S}_{22}] & \dots & [\bar{S}_{2\bar{M}}] \\ \vdots & \vdots & \ddots & \vdots \\ [\bar{S}_{\bar{M}1}] & [\bar{S}_{\bar{M}2}] & \dots & [\bar{S}_{\bar{M}\bar{M}}] \end{bmatrix} \tag{3}$$

where most of the $\bar{S}_{ij}$ blocks are zero since if global port i and port j belong to different devices then there is no cross-talk.

II. Connecting two ports

Let's consider that port m and port n are to be mutually interconnected.

This implies that the GSM looses two ports reducing from a $\bar{M} \times \bar{M}$ matrix to a $\bar{M} - 2 \times \bar{M} - 2$ one. This condensation of two ports in one can be performed as in the following:

$$\begin{bmatrix} [b]_1 \\ \vdots \\ [b]_m \\ \vdots \\ [b]_n \\ \vdots \\ [b]_{\bar{M}} \end{bmatrix} = \begin{bmatrix} [\bar{S}_{11}] & \dots & [\bar{S}_{1m}] & \dots & [\bar{S}_{1n}] & \dots & [\bar{S}_{1\bar{M}}] \\ \vdots & \ddots & \vdots & & \vdots & & \vdots \\ [\bar{S}_{m1}] & \dots & [\bar{S}_{mm}] & \dots & [\bar{S}_{mn}] & \dots & [\bar{S}_{m\bar{M}}] \\ \vdots & & \vdots & \ddots & \vdots & & \vdots \\ [\bar{S}_{n1}] & \dots & [\bar{S}_{nm}] & \dots & [\bar{S}_{nn}] & \dots & [\bar{S}_{n\bar{M}}] \\ \vdots & & \vdots & & \vdots & \ddots & \vdots \\ [\bar{S}_{\bar{M}1}] & \dots & [\bar{S}_{\bar{M}m}] & \dots & [\bar{S}_{\bar{M}n}] & \dots & [\bar{S}_{\bar{M}\bar{M}}] \end{bmatrix} \begin{bmatrix} [a]_1 \\ \vdots \\ [a]_m \\ \vdots \\ [a]_n \\ \vdots \\ [a]_{\bar{M}} \end{bmatrix} \tag{4}$$

Condensing two ports effectively means that $[b]_n = [a]_m$ and $[b]_m = [a]_n$. This is possible if the expansion on the two ports is done over the same number and kind of modes. otherwise vectors $[b]_n$ $[a]_m$ do not have the same size and their entries do not have the same meaning. check. ck

This allows for the determination of $[a]_m$ and $[a]_n$ from the equations defined by the $n-$th and $m-$th row of matrix $[\bar{S}]$. Actually:

$$[b]_m = [a]_n = \sum_{i=1}^{\bar{M}} [\bar{S}_{mi}]\,[a]_i \tag{5}$$

$$[b]_n = [a]_m = \sum_{i=1}^{\bar{M}} [\bar{S}_{ni}]\,[a]_i \tag{6}$$

these can be rewritten as

$$\sum_{i=1}^{\tilde{M}}\Bigg|_{i\neq m,n} \left[\tilde{S}_{mi}\right][a]_i + \left[\tilde{S}_{mm}\right][a]_m + \left\{\left[\tilde{S}_{mn}\right] - [I]\right\}[a]_n = 0 \quad (7)$$

$$\sum_{i=1}^{\tilde{M}}\Bigg|_{i\neq m,n} \left[\tilde{S}_{ni}\right][a]_i + \left\{\left[\tilde{S}_{nm}\right] - [I]\right\}[a]_m + \left[\tilde{S}_{nn}\right][a]_n = 0 \quad (8)$$

or either

$$\left[\tilde{S}_{mm}\right][a]_m + \left\{\left[\tilde{S}_{mn}\right] - [I]\right\}[a]_n = -\sum_{i=1}^{\tilde{M}}\Bigg|_{i\neq m,n} \left[\tilde{S}_{mi}\right][a]_i; \quad (9)$$

$$\left\{\left[\tilde{S}_{nm}\right] - [I]\right\}[a]_m + \left[\tilde{S}_{nn}\right][a]_n = -\sum_{i=1}^{\tilde{M}}\Bigg|_{i\neq m,n} \left[\tilde{S}_{ni}\right][a]_i \quad (10)$$

This is a system in the form

$$\begin{bmatrix}[C] & [D]\\ [E] & [F]\end{bmatrix}\begin{bmatrix}[a]_m\\ [a]_n\end{bmatrix} = \begin{bmatrix}[h]\\ [k]\end{bmatrix} \quad (11)$$

having defined

$$[C] = \left[\tilde{S}_{mm}\right]; \quad (12)$$
$$[D] = \left\{\left[\tilde{S}_{mn}\right] - [I]\right\}; \quad (13)$$
$$[E] = \left\{\left[\tilde{S}_{nm}\right] - [I]\right\}; \quad (14)$$
$$[F] = \left[\tilde{S}_{nn}\right] \quad (15)$$

and

$$[h] = -\sum_{i=1}^{\tilde{M}}\Bigg|_{i\neq m,n} \left[\tilde{S}_{mi}\right][a]_i; \quad (16)$$

$$[k] = -\sum_{i=1}^{\tilde{M}}\Bigg|_{i\neq m,n} \left[\tilde{S}_{ni}\right][a]_i; \quad (17)$$

The solution of (11 can be written as

$$\begin{bmatrix}[a]_m\\ [a]_n\end{bmatrix} = \begin{bmatrix}[\Psi]^{(mm)} & [\Psi]^{(mn)}\\ [\Psi]^{(nm)} & [\Psi]^{(nn)}\end{bmatrix}\begin{bmatrix}[h]\\ [k]\end{bmatrix} \quad (18)$$

The evaluation of matrices $[\Psi]$ requires the inverse of at least two of sub-matrices $[C]$, $[F]$, $[E]$ or $[D]$ this is guaranteed to be possible because $[S]$ as a whole is non-singular, so the matrix of system (11) is non-siungular too.

Hence, if $[C]^{-1}$ exist, we define $[\Xi] = [E][C]^{-1}$ and it is

$$[\Phi]^{(nm)} = \left[[\Xi][D] - [F]\right]^{-1}[\Xi] \quad (19)$$
$$[\Phi]^{(nn)} = -\left[[\Xi][D] - [F]\right]^{-1} \quad (20)$$

otherwise, if $[C]$ is singular, $[E]$ must be non-singular! Hence, by naming $[\Xi] = [C][E]^{-1}$

$$[\Phi]^{(nm)} = \left[[D] - [\Xi][F]\right]^{-1} \quad (21)$$
$$[\Phi]^{(nn)} = -\left[[D] - [\Xi][F]\right]^{-1}[\Xi] \quad (22)$$

Then, if $[F]^{-1}$ exist, by naming $[\Psi] = [D][F]^{-1}$ it is

$$[\Phi]^{(mm)} = -\left[[\Psi][E] - [C]\right]^{-1} \quad (23)$$
$$[\Phi]^{(mn)} = \left[[\Psi][E] - [C]\right]^{-1}[\Psi] \quad (24)$$

otherwise, if $[F]$ is singular, $[D]$ must be non-singular! Hence, by naming $[\Psi] = [F][D]^{-1}$

$$[\Phi]^{(mm)} = -\left[[E] - [\Psi][C]\right]^{-1}[\Psi] \quad (25)$$
$$[\Phi]^{(mn)} = \left[[E] - [\Psi][C]\right]^{-1} \quad (26)$$

By substituting back (16) into (18)

$$[a]_m = -[\Phi]^{(mm)}\sum_{i=1}^{\tilde{M}}\Bigg|_{i\neq m,n} \left[\tilde{S}_{mi}\right][a]_i - [\Phi]^{(mn)}\sum_{i=1}^{\tilde{M}}\Bigg|_{i\neq m,n} \left[\tilde{S}_{ni}\right][a]_i \quad (27)$$

$$[a]_n = -[\Phi]^{(nm)}\sum_{i=1}^{\tilde{M}}\Bigg|_{i\neq m,n} \left[\tilde{S}_{mi}\right][a]_i - [\Phi]^{(nn)}\sum_{i=1}^{\tilde{M}}\Bigg|_{i\neq m,n} \left[\tilde{S}_{ni}\right][a]_i \quad (28)$$

in a more compact form

$$[a]_m = -\left\{\sum_{i=1}^{\tilde{M}}\Bigg|_{i\neq m,n} \left[[\Phi]^{(mm)}\left[\tilde{S}_{mi}\right] + [\Phi]^{(mn)}\left[\tilde{S}_{ni}\right]\right][a]_i\right\} \quad (29)$$

$$[a]_n = -\left\{\sum_{i=1}^{\tilde{M}}\Bigg|_{i\neq m,n} \left[[\Phi]^{(nm)}\left[\tilde{S}_{mi}\right] + [\Phi]^{(nn)}\left[\tilde{S}_{ni}\right]\right][a]_i\right\} \quad (30)$$

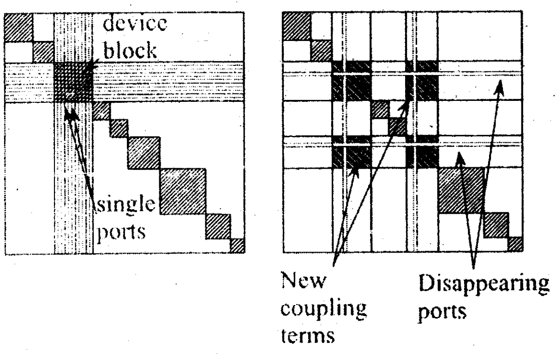

Fig. 3. Condensing two ports in one. GSM before (left) and after(right) condensatiom.

These latter can be substituted in all the *other* equations to eliminate a_m and a_n. In particular, the generic equation

$$[b]_i = \sum_{j=1}^{M} \left[\bar{S}_{i,j}\right] [a]_j \tag{31}$$

becomes

$$[b]_i = \sum_{j=1}^{M}\Bigg|_{j\neq m,n} \left[\tilde{S}_{i,j}\right] [a]_j \tag{32}$$

with

$$\begin{aligned}\left[\tilde{S}_{i,j}\right] = {} & \left[\bar{S}_{i,j}\right] - \\ & \left[\bar{S}_{i,m}\right]\left\{[\Phi]^{(mm)}\left[\bar{S}_{mj}\right] + [\Phi]^{(mn)}\left[\bar{S}_{nj}\right]\right\} - \\ & \left[\bar{S}_{i,n}\right]\left\{[\Phi]^{(nm)}\left[\bar{S}_{mj}\right] + [\Phi]^{(nn)}\left[\bar{S}_{nj}\right]\right\}\end{aligned} \tag{33}$$

It is worth noticing that matrix $[\bar{S}]$ can be updated by iteratively condensing couples of ports up to the point that only true external ports are left. It is also very important to note that there is no need, from the implementation point of view, to actually eliminate rows and columns of $[\bar{S}]$ as condensation takes place, but that blocks in rows and columns pertaining to condensed ports can be simply put to zero.

It is also to be remarked that the diagonal-block nature of $[\bar{S}]$ is spoiled by the condensation process inasmuch, if port n belongs to device i and port m to device j then, before condensation, devices i and j form two separate diagonal blocks in $[\bar{S}]$. After condensation all blocks on rows and columns pertaining to at least one of the two devices are now full (Fig. 3)

Once all the condensation operations took place a new GSM, of the correct dimension, can be obtained by selecting and reassembling the correct blocks from $[\bar{S}]$.