Thermal Johnson Noise Generated by a Resistor

REFERENCES

Reif, *Fundamentals of Statistical and Thermal Physics*, pp. 589 - 594
Kittel, *Thermal Physics* , pp. 98-102
Kittel, *Elementary Statistical Physics,* pp. 141-149.

THEORY OF THERMAL JOHNSON NOISE

Thermal agitation of electrons in a resistor gives rise to a random fluctuation in the voltage across its terminals, known as Johnson noise. In Problem 1, you are to show that in a narrow band of frequencies, Δf, the contribution to the mean-squared noise voltage from this thermal agitation is,

$$\left\langle V(t)^2 \right\rangle_{time} = 4Rk_B T\, \Delta f \tag{1}$$

where R is the resistance in ohms and T is the temperature in degrees Kelvin for the resistor. k_B is Boltzmann constant (1.38×10^{-23} J/K).

This voltage is usually too small to be detected without amplification. If the resistor is connected across the input of a high-gain amplifier whose voltage gain as a function of frequency is $G(f)$, the mean square of the voltage output of the amplifier will be,

$$\left\langle V(t)^2 \right\rangle_{time} = 4Rk_B T \int_0^\infty [G(f)]^2 df + \left\langle V_N^2(t) \right\rangle_{time} \tag{2}$$

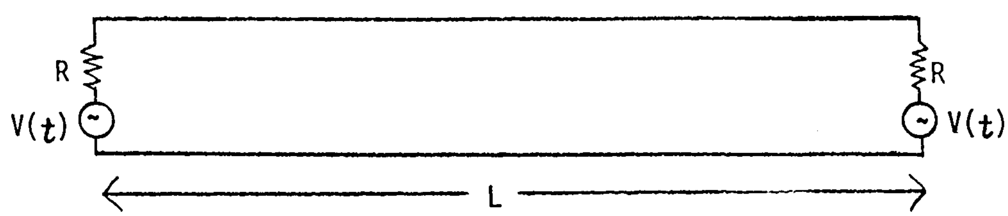
where $\left\langle V_N^2 \right\rangle_{time}$ is the output noise generated by the amplifier itself.

Thus by measuring $\left\langle V(t)^2 \right\rangle_{time}$ as a function of R and making a plot, one obtains $4kT\int_0^\infty [G(f)]^2 df$ from the slope, while the abscissa gives $\left\langle V_N^2 \right\rangle_{time}$. But the amplifier gain $G(f)$ can be independently measured and the gain integral

$\int_0^\infty [G(f)]^2 df$ evaluated. The slope will then give a value for Boltzmann constant k_B.

This is in outline the first part of the experiment. The second part involves measuring the noise voltage as a function of the temperature, to verify the expected temperature dependence.

Problem 1. Derivation of Eq. (1).
An electrical transmission line connected at one end to a resistor R and at the other end by an "equivalent" resistor R may be treated as a one-dimensional example of black body radiation.



At finite temperature T, the resistor R generates a noise voltage V(t) which will propagate down the line. If the characteristic impedance of the transmission line is made equal to R, the radiation incident on the "equivalent" resistor R from the first resistor R should be completely absorbed.

The permitted standing wave modes in the line have $\lambda = 2L/n$ and $f = (c/2L)n$, where n =1,2,3, etc., and v is the wave velocity in the line. The separation of the modes in frequency is $v/2L$ and the number of modes between f and $f + \Delta f$ is

$$\sigma(f)\, \Delta f = (2L/c)\, \Delta f \qquad (3)$$

From the Planck distribution or the equipartition theorem, the mean thermal energy contained in each electromagnetic mode or photon state in the line is,

$$\langle E(f)\rangle = \frac{hf}{e^{hf/k_BT}-1} \sim k_BT \qquad (4)$$

From Eq. (3) and (4) find the electromagnetic energy $\langle E(f)\rangle\,\sigma(f)\Delta f$ in a frequency interval Δf. One half of this energy is generated by the first resistor of R and propagating towards the "equivalent" resistor R. Knowing the propagation time from the generating resistor to the absorbing resistor $\Delta t = L/c$, show that the absorbed power by the "equivalent" resistor R equals

$$P(f)\Delta f = k_BT\Delta f. \qquad (5)$$

In thermal equilibrium, this power is simply the ohmic heating generated by a noise voltage source V(t) from the first resistor. Since V(t) is terminated by the absorbing resistor R and has an "internal" resistance R (the first resistor), it produces a current $I=V/(2R)$ in the line. Hence the power absorbed by the "equivalent" resistor R over the frequency interval Δf can also be calculated as

$$I^2R = \left(\frac{V}{2R}\right)^2 R = \frac{V^2}{4R} = \frac{V^2(f)\Delta f}{4R} \qquad (6)$$

By equating $P(f)\Delta f = k_BT\Delta f$ to $V^2(f)\Delta f/4R$, show that

$$V^2(f)\Delta f = 4k_BTR\Delta f \qquad (7)$$

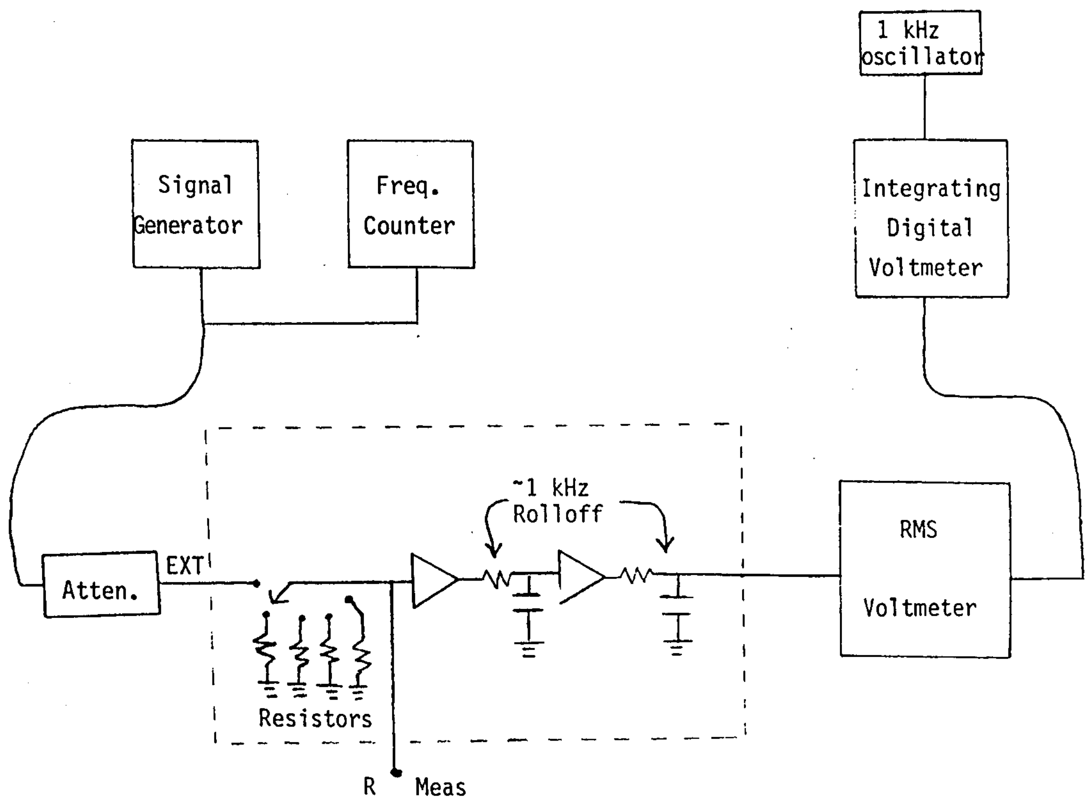
and

$$\left\langle V(t)^2\right\rangle_{\text{time}} = 4Rk_BT\,\Delta f \qquad (8)$$

This is known as Nyquist's theorem as shown in Eq. (1).

SAMPLE AND APPARATUS

A very low noise operational amplifier is used as the first stage of amplification for the Johnson noise. Roll-off filters limit the bandwidth on the low frequency side, while parasitic capacitance shunting the resistors will limit it on the high frequency side. As a result, the bandwidth is approximately 1 kHz. The apparatus is connected with shielded coaxial cables as shown to reduce pickup.



The sine wave oscillator is used to measure the gain of the amplifier. The oscillator output is put through an attenuator to reduce it to the level needed to be able to insert into the amplifier. The attenuation factor of the attenuator is accurately given by the controls and does not need to be calibrated. Its output should be directly connected to the amplifier EXT input to avoid voltage drops in connecting cables. The frequency f of the oscillator can be accurately set and determined with the Integrating Digital Voltmeter. Check with the Instructor or T.A. for how to set it up.

PROCEDURE FOR MEASURING THE GAIN INTEGRAL $\int_0^\infty [G(f)]^2 df$

You obtain $\int_0^\infty [G(f)]^2 df$ by measuring the amplifier gain $G(f)$ at a discrete and yet evenly spaced set of frequency values (f) and then evaluating the discrete sum $\sum [G(f)]^2 \Delta f$ numerically.

Let the amplifier warm up for at least half an hour before starting this process. The amplifier is powered by batteries. If the unit has not been used recently, batteries need to be checked. To measure $G(f)$, set the input switch to EXT, connect a precision, broadband voltage attenuator between the input of the amplifier. The attenuator is used to assist you in determining the amplifier gain $G(f)$ as follows.

Supply the input of the attenuator with a sinusoidal voltage signal of $V_i = 1$ volt at a frequency between zero and 4000 Hz. For the best accuracy, measure the applied voltage signal with either a digital voltmeter or an oscilloscope. Since at f = 1000 Hz, $G(f)$ is roughly 10000, you can set the attenuation parameter $N_{dB} \sim 80$ dB that gives a voltage attenuation factor of $G_A = 10^{N_{dB}/20} \sim 10000$). The output of the attenuator, V_i/G_A, is then fed into the input of the amplifier. You measure the amplifier output $V_o = (V_i/G_A)G(f)$ with the same voltmeter or the oscilloscope. The amplifier gain is given by

$$G(f) = (V_o/V_i)G_A = (V_o/V_i)10^{N_{dB}/20} \tag{9}$$

It is best that you adjust N_{dB} so that $V_o/V_i \sim 1$. Repeat the measurement at a series of frequencies, to obtain a discrete set of $G(f)$ values. From these results, you can numerically calculate $\int_0^\infty [G(f)]^2 df$.

JOHNSON NOISES FROM RESISTORS AT ROOM TEMPERATURE:

Disconnect the attenuator from the amplifier. Connect the shunt to the EXT connector to avoid having any stray voltage. Connect one of the resistors to the amplifier using the switch on the amplifier chassis. Use the HP3400A RMS Voltmeter to measure the rms voltage of the amplified Johnson noise signal. Since the rms voltage fluctuates on the time scale of a fraction of second, it is difficult to obtain an accurate reading of the mean of the rms voltage. To obtain the latter, you use an HP240IC Integrating Digital Voltmeter with the following procedure.

(1) Connect the DC output at the rear panel of the HP3400A RMS Voltmeter to "Hi" and "Lo" on the front panel of HP240IC Integrating Digital Voltmeter;
(2) On the front panel, set "Function" to "VOLT";
(3) Set "Range " to "10V";
(4) Set "SAMPLE PERIOD" to "1 SEC";
(5) Send a 1000-Hz sinusoidal signal from a HP200CD Wide Range Oscillator to "External Clock Input" at the rear of the Integrating Digital Voltmeter;
(6) Set the frequency standard (STD) next to "External Clock Input" to "EXT";
(7) Wait for 100 seconds before the voltage integration and average is complete. The displayed voltage value V_d is the 100-second average of the DC output multiplied by 100. To convert this value to the rms value of the amplified Johnson noise V_o, you need to divide V_d by 100 and then multiply the scale on the front panel of the HP3400A RMS Voltmeter.

Measure the noise voltage of each of the resistors. The value of the resistance of each of the resistors is written on the amplifier box. If you wish to check the resistance of these resistors, you may do so using the terminal provided, but **BE SURE TO TURN THE AMPLIFIER OFF** before doing so.

Use Eq. (2) to calculate Boltzmann constant k_B, taking into account the corrections mentioned below. Also, you should compare the value of the amplifier noise, $\langle V_N^2 \rangle_{time}$, obtained from your data of the noise voltage measured at the amplifier output when the input is shorted.

TEMPERATURE DEPENDENCE OF THE JOHNSON NOISE

A shielded resistor in a sealed $\frac{1}{2}$-inch diameter stainless steel tube is provided to explore the temperature dependence of Johnson noise. The interior of the tube is filled with helium gas for thermal contact between resistor and the outside. Connect this probe directly to the EXT connector on the amplifier (additional cable will only add capacitance and microphonic noise).

Record the RMS voltage produced by this resistor at room temperature (~ 300 K as measured with thermometer), and at liquid nitrogen (77 K) and liquid helium (4.2 K). For low temperature measurements, make sure that the probe is filled with helium gas before it is immersed in the containers of liquid nitrogen and liquid helium. The helium gas will not become liquified and help to cool the resistor to the final temperature by conducting the heat away from it.

Plot the rms. noise voltage as a function of the temperature. Also, measure the resistance of the resistor at each of the temperatures (since the resistance of most resistors is a strong function of the temperature).

If you find any discrepancy between the measurement and the theory, suggest what their source(s) might be.

MEASURING NOISE SPECTRA WITH A LabVIEW VIRTUAL INSTRUMENT

In this section of the laboratory, you

(1) learn how to use a computer-aided data acquisition method (LabVIEW virtual instrument) to perform voltage measurement;

(2) measure thermal Johnson noise power spectra or $V^2(f)$ using a Fast-Fourier-Transform program on LabVIEW and verify that thermal Johnson noise is indeed frequency-independent;

(3) determine the resistance and temperature dependence of the noise spectra and in turn calculate the Boltzmann constant k_B;

(4) (Optional) use Johnson noise spectra to determine the frequency response of an amplifier gain G(f).

The LabVIEW program for the thermal Johnson noise is called "Johnson_Noise _2002.vi." It is in the Physics 122 folder on the PC computer by the experiment. It is placed in the "Physics122Lab_folder" on the C Drive. It is ready to be used to measure the noise spectra for the various fixed resistors in the amplifier box.

The program measures the noise voltage V(t) by digitizing a voltage input using an analog-to-digital converter on a data acquisition board inside the computer. The board measures a user-set total number of samples N_s with a user-set sampling rate f_r. f_r is equal to twice of the maximum measurable frequency f_{max}. This is because that one needs to sample at least two time points on a sine wave to determine its frequency. This is known as the Nyquist sampling theorem.

The program then computes the Fourier transform V(f) of the measured voltage V(t) by using a computer algorithm called the Fast Fourier Transform (FFT), invented by Cooley and Tukey. This algorithm is much more efficient if the total number of samples $N_s = 2^n$, with n being an integer. Using this algorithm, the maximum number of data points obtained over the frequency range from 0 to f_{max} is $N_f = N_s/2$. The LabVIEW panel displays the real-time signal and the power spectral density $V^2(f)$ of each run.

Note that for each measurement of N_s samples, the power spectral density (PSD) is quite noisy. We improve the signal-to-noise ratio of the measured power spectral density by averaging N such spectra. The signal-to-noise ratio is improved by $\sqrt{N}$. In the LabVIEW panel, the averaged power spectrum $V^2(f)\Delta f$ where $\Delta f = f_{max}/N_f = f_r/N_s$ is the frequency interval between the data points.

Note that the time-averaged mean squared total noise $\langle V^2(t)\rangle$ equals the mean squared total noise in frequency space, i.e.,

$$\langle V^2(t)\rangle = \int_0^\infty V^2(f)\Delta f = \sum_{n=1}^{N_f} V_n^2(f)\Delta f. \tag{10}$$

The program calculates the sum $\sum_{n=1}^{N_f} V_n^2(f)\Delta f$ that yields $\langle V^2(t)\rangle$.

Finally, measurements of noise are very important to physics experiments, because the actual noise levels in the experiment can determine whether one can measure small signal levels in the experiment. Measurements of noise power spectra as described here are frequently performed to understand the sources of the noise in the experiment. If you understand the noise in your experiment, you can then work to reduce noise sources by, for example, choosing components with less noise, averaging longer to reduce the effects of noise on the signal, or working in frequency regions where the noise is lower. In fact, specialized frequency analyzers exist; these are instruments which can easily measure such noise spectra, and they work the same way your LabVIEW computer program does.

This noise power spectrum measurement by a computer and fast Fourier Transform is particularly useful for measuring the noise of the resistor in the separate probe as a function of temperature (room temperature (~300K), in liquid nitrogen (77K), and in liquid helium (4.2K). The noise from this resistor is particularly susceptible to microphonic noise. Microphonic noise is the noise voltage generated in electric wires due to their motion through capacitive effect or piezo-electric effect. Thus it can be generated from the probe being shaken, by people walking in the room causing vibrations in the probe, etc. Measuring the noise power spectrum allows you to distinguish the Johnson noise (which is not frequency dependent) from microphonic noise and line frequency noise that peak at specific frequencies such as multiples of 60Hz.

(1) **Learning a computer-aided data acquisition with a LabVIEW virtual instrument:**

Consult with the T.A. or an instructor of the Physics 122 Lab on how the LabVIEW works and quickly explain how "Johnson_Noise _2002.vi" operates. You should make an effort to familiarize yourself with the concept and strategy of a LabVIEW virtual instrument for computer-aided data acquisition.

For your experiment, you need to create your own folder to store the data files and any LabVIEW programs that you have created or saved in your own "name". You may want to bring a floppy diskette (you can buy them at the bookstore) and back up your files onto it, so that if anything should happen to the computer or its hard disk, your files will not be lost.

(2) **Measure thermal Johnson noise power spectra or** $V^2(f)\Delta f$ **using "Johnson noise power spectrum analyzer" and verify that thermal Johnson noise is indeed frequency-independent**

Question: How does the noise power spectrum $V^2(f)\Delta f$ compare to the square of the amplifier gain $G^2(f)$ that you measured ?

(3) **Determine the resistance and temperature dependence of the noise spectra and in turn calculate the Boltzmann constant k_B**

Question: How does the noise power spectrum $V^2(f)\Delta f$ (shape and magnitude) vary with the resistance R ?

Question: How does the noise power spectrum $V^2(f)\Delta f$ vary with temperature T ?

Question: Can you think of a way to use the noise power spectrum and the measured gain curve $G^2(f)$ to calculate the Boltzmann constant k_B without the AC Voltmeter and the Integrating Voltmeter ?

(4) (Optional) **Use Johnson noise spectra to determine the frequency response of an amplifier gain G(f)**

Since the thermal Johnson noise from a resistor is a broad frequency-band generator with a constant power spectral density $V^2(f) = 4k_B TR$, one can use the amplified thermal Johnson noise to measure the amplifier gain G(f) using a fast Fourier transform method with a LabVIEW program. Consult the instructor or T.A. for this option.

Johnson Noise Set-Up