IMPEDANCE MATCHING

for
High-Frequency Circuit Design Elective

by
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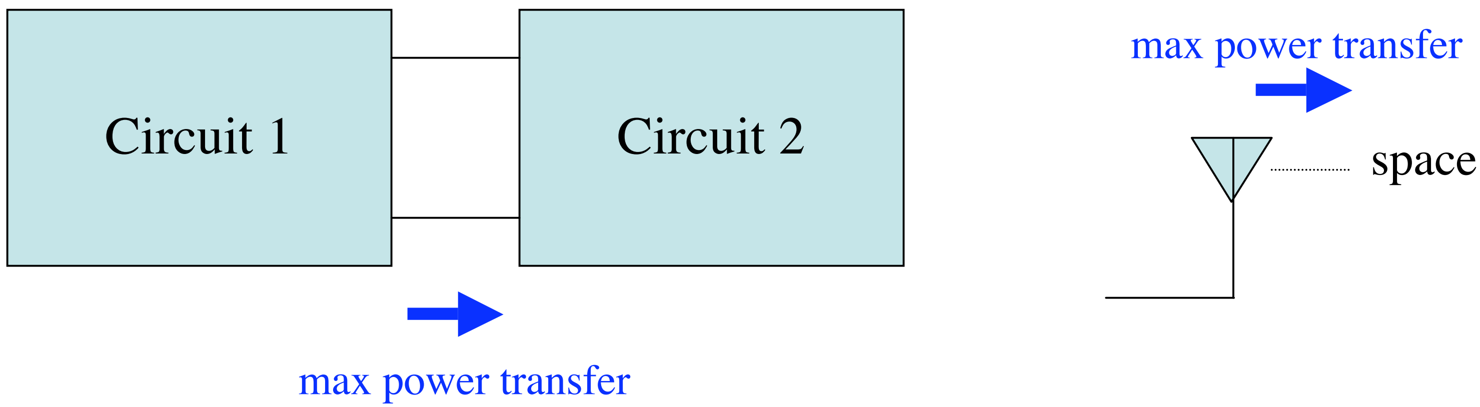
Contents

Impedance Matching

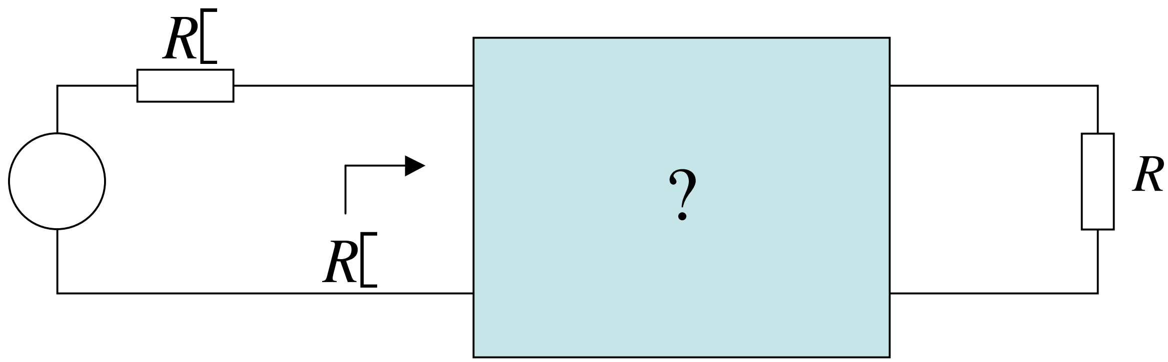
- Impedance matching is a major problem in high-frequency circuit design.

- It is concerned with matching one part of a circuit to another in order to achieve *maximum power transfer* between the two parts.



The problem

Given a load R, find a circuit that can match the driving resistance R' at frequency ω_0.



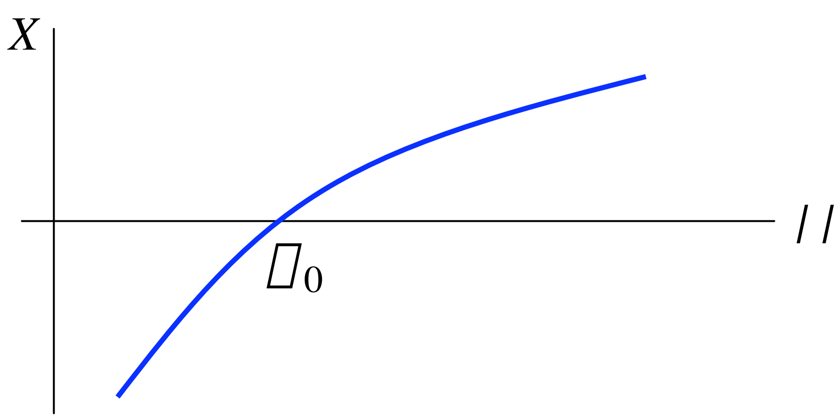
Obviously, the matching circuit must contain L and C in order to specify the matching frequency.

The Q factor approach to matching

The Q factor is defined as the ratio of stored to dissipated power

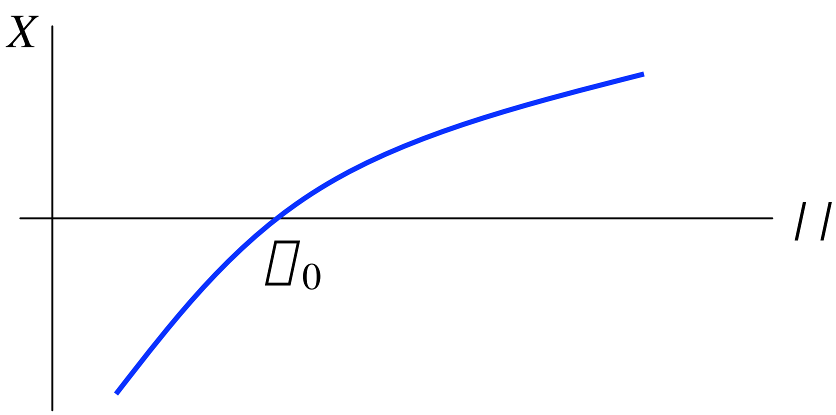
$$Q = \frac{2\pi \cdot (\text{max instantaneous energy stored})}{\text{energy dissipated per cycle}}$$

In general, a circuit's reactance is a function of frequency and the Q factor is defined at the resonance frequency ω_0.

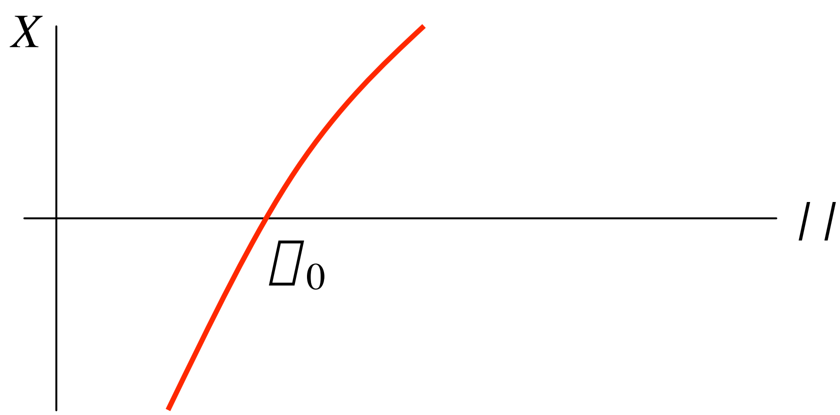


As we will see later, the Q factor can be used to modify the overall resistance of a circuit at some selected frequency, thus achieving a matching condition.

Low Q circuit



High Q circuit

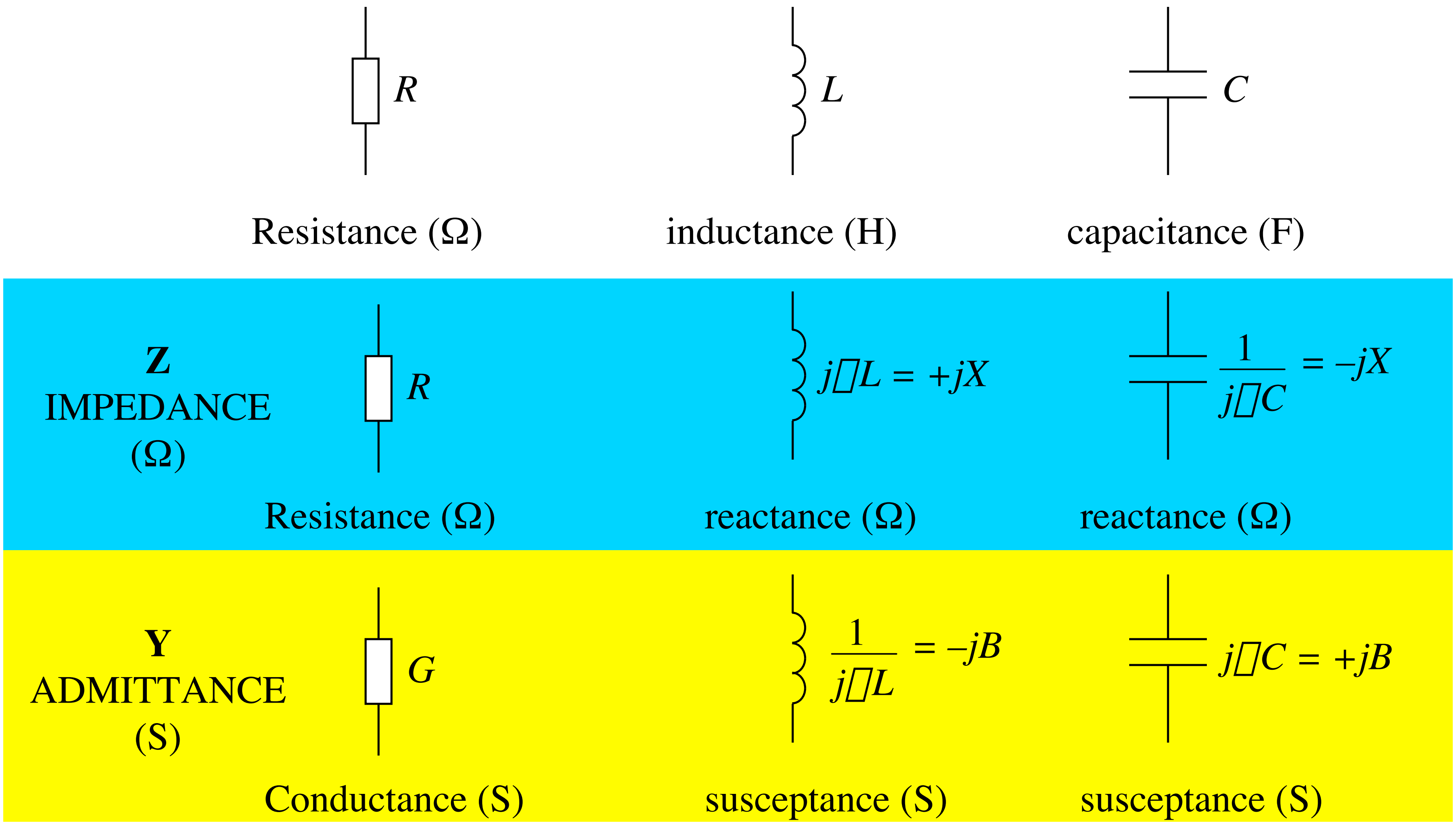


Definition: $$Q = \frac{\omega_0}{2G} \frac{dB}{d\omega}\bigg|_{\omega=\omega_0} = \frac{\omega_0}{2R} \frac{dX}{d\omega}\bigg|_{\omega=\omega_0}$$

B = susceptance
X = reactance
R = resistance
G = conductance

It is easily shown that for linear parallel RLC circuits:
$Q = \omega_0 CR = R/(\omega_0 L)$

Essential revision (basic circuit theory)

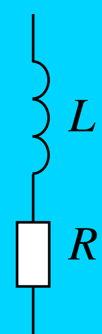


Essential revision (basic circuit theory)

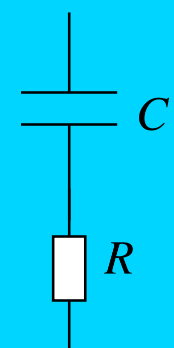
Quality factor (Q factor)

Series:

$$Q = \frac{X}{R} = \frac{1}{RB} = \frac{G}{B}$$



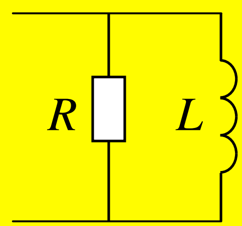
$$Q = \frac{\omega L}{R}$$



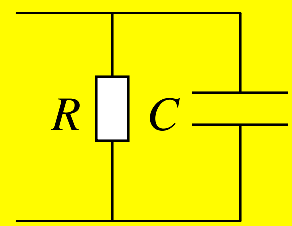
$$Q = \frac{1}{\omega CR}$$

Parallel:

$$Q = \frac{R}{X} = RB = \frac{B}{G}$$



$$Q = \frac{R}{\omega L}$$

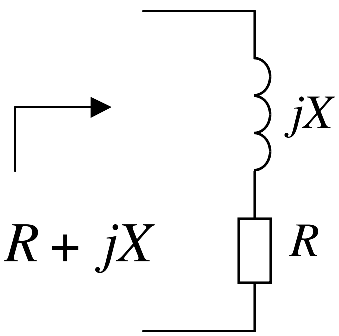


$$Q = \omega CR$$

Higher Q means that it is closer to the ideal L or C.

Essential revision (basic circuit theory)

Series to parallel conversion

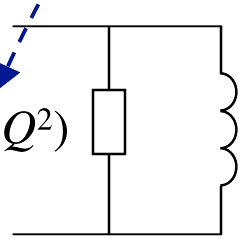


$$Z = R + jX$$

$$Y = \frac{1}{Z} = \frac{1}{R + jX} = \frac{R - jX}{R^2 + X^2} = \frac{R}{R^2 + X^2} - j\frac{X}{R^2 + X^2}$$

$$= \frac{\frac{1}{R}}{1 + \left(\frac{X}{R}\right)^2} + \frac{\frac{1}{X}}{j\left(\left(\frac{R}{X}\right)^2 + 1\right)}$$

$$= \frac{1}{R\left(1 + Q^2\right)} + \frac{1}{jX\left(\frac{1}{Q^2} + 1\right)}$$



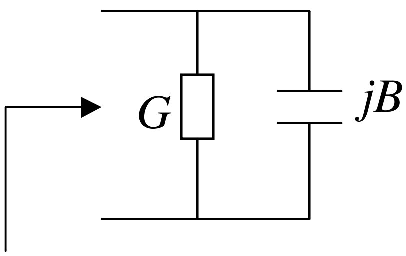
$jX\left(1 + \frac{1}{Q^2}\right)$

or $j\frac{R'}{Q}$

$$j\frac{R}{Q}\left(1 + Q^2\right) = jRQ\left(\frac{1}{Q^2} + 1\right)$$

Essential revision (basic circuit theory)

Parallel to series conversion

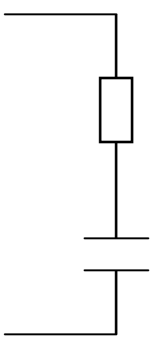


$$Y = G + jB$$

$$Z = \frac{1}{Y} = \frac{1}{G + jB} = \frac{G - jB}{G^2 + B^2} = \frac{G}{G^2 + B^2} - j\frac{B}{G^2 + B^2}$$

$$= \frac{\frac{1}{G}}{1 + \left(\frac{B}{G}\right)^2} + \frac{\frac{1}{B}}{j\left(\left(\frac{G}{B}\right)^2 + 1\right)}$$

$$= \frac{1}{G\left(1 + Q^2\right)} + \frac{1}{jB\left(\frac{1}{Q^2} + 1\right)}$$

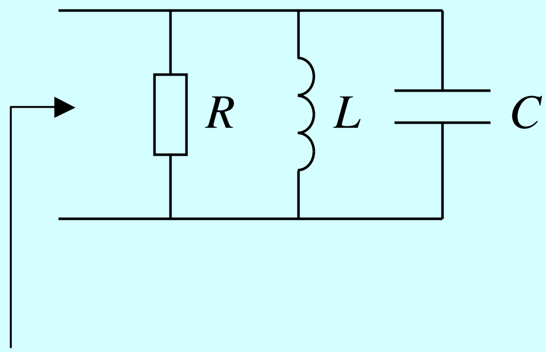


$G\left(1 + Q^2\right)$ conductance (S)

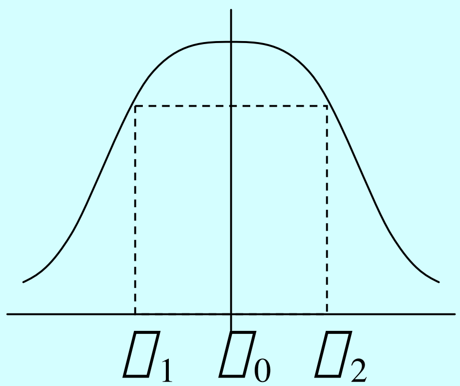
$jB\left(1 + \frac{1}{Q^2}\right)$ susceptance (S)

or $j\frac{G'}{Q} = j\frac{G}{Q}\left(1 + Q^2\right) = jGQ\left(\frac{1}{Q^2} + 1\right)$

Example: RLC circuit (Recall Year 1 material)



$$Z = \frac{1}{(1/R) + j\omega C - (j/\omega L)}$$



Resonant frequency is $\boxed{\omega_0 = \frac{1}{\sqrt{LC}}}$

Q factor is $\boxed{Q = R\sqrt{\frac{C}{L}}}$

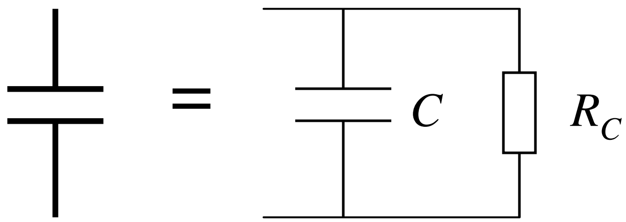
Z drops by $\sqrt{2}$ (3 dB) at ω_1 and ω_2.

$$\omega_{1,2} = \omega_0\left(\sqrt{1 + \frac{1}{4Q^2}} \pm \frac{1}{2Q}\right)$$

Bandwidth is $\boxed{\Delta\omega = \omega_2 - \omega_1 = \frac{1}{RC}}$

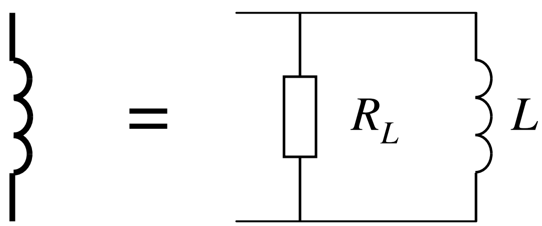
Note: ω_1 and ω_2 are called *3dB corner frequencies*. Their geometric mean is ω_0. For narrowband cases, their arithmetic mean is close to ω_0.

Practical components are lossy!



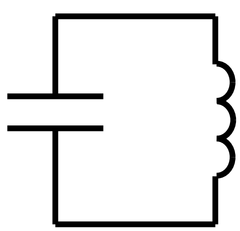
Q factor = $Q_C = \omega_0 C R_C$

(unloaded Q factor)



Q factor = $Q_L = R_L / \omega_0 L$

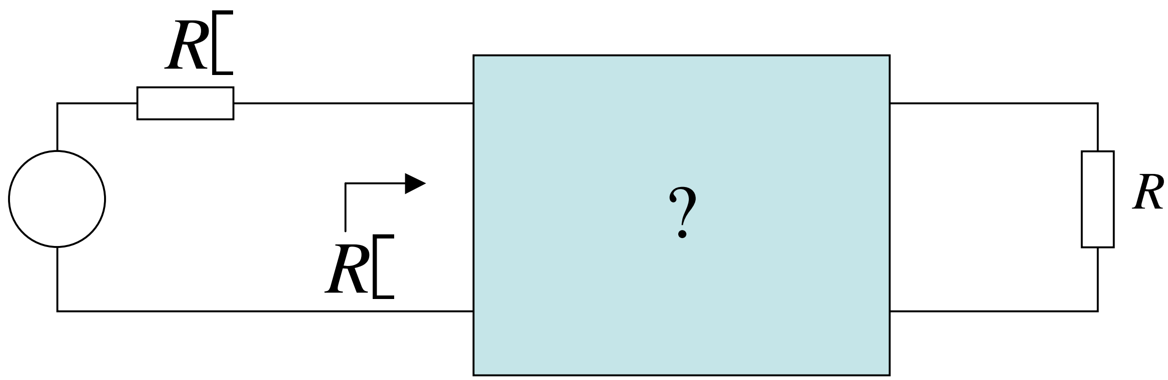
(unloaded Q factor)



Q_{LC} = unloaded Q factor for the paralleled LC components

$$\boxed{\frac{1}{Q_{LC}} = \frac{1}{Q_C} + \frac{1}{Q_L} \qquad \text{(easily shown)}}$$

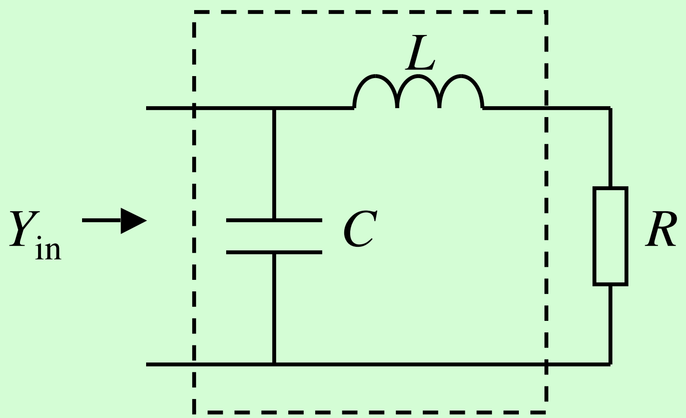
Simple matching circuits



L matching circuit (single LC section)
π matching circuit
T matching circuit

Design of *L* matching circuits

Series *L* circuit:



Objective: match Y_{in} to R' at ω_0

Begin with

$$Y_{\text{in}} = j\omega C + \frac{1}{R + j\omega L}$$

$$= \frac{R}{R^2 + (\omega L)^2} + j\left[\omega C - \frac{\omega L}{R^2 + (\omega L)^2}\right]$$

Obviously, the reactive part is cancelled if we have

$$\boxed{C = \frac{L}{R^2 + \omega_0^2 L^2} \quad \text{where} \quad \omega_0 = \sqrt{\frac{1}{LC} - \frac{R^2}{L^2}} \qquad (\#)}$$

Thus, at $\omega = \omega_0$, we have a resistance for Y_{in}, which should be set to R'.

$$\boxed{R' = \frac{R^2 + \omega_0^2 L^2}{R} = R\left(1 + Q^2\right) \qquad (*)}$$

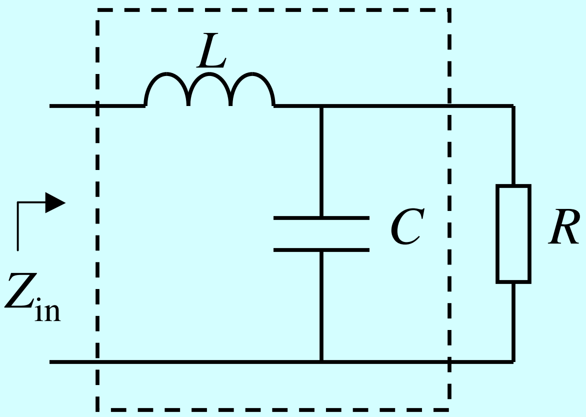
Here, Q is the Q-factor, which is equal to $\omega_0 L/R$ (for series L and R).

So, we can see clearly that Q is modifying R to achieve the matching condition.

Design procedure:

-Given R and R', find the required Q from (*).
-Given ω_0, find the required L from $Q = \omega_0 L/R$.
-From (#), find the required C to give the selected resonant frequency ω_0.

Shunt L circuit:



Begin with

$$Z_{in} = j\omega L + \frac{1}{G + j\omega C}$$

$$= \frac{G}{G^2 + \omega^2 C^2} + j\left[\omega L - \frac{\omega C}{G^2 + \omega^2 C^2}\right]$$

Reactive part is cancelled when

$$L = \frac{C}{G^2 + \omega_0^2 C^2} \quad \text{where } \omega_0 = \sqrt{\frac{1}{LC} - \frac{G^2}{C^2}} \quad (\#)$$

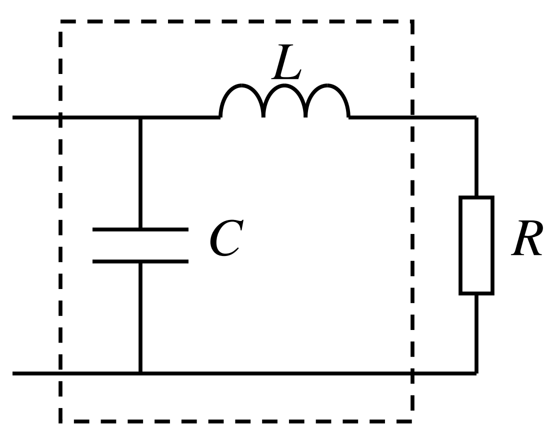
Finally, the matching condition requires that

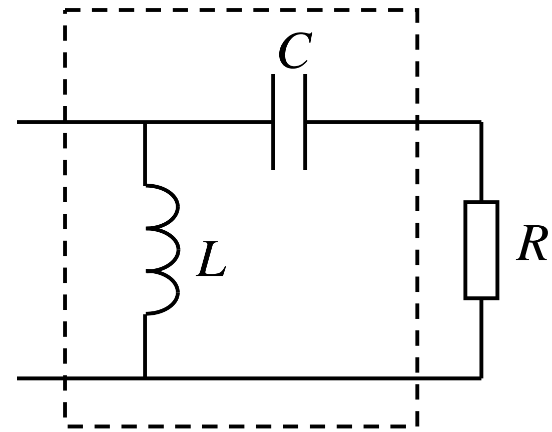
$$R' = \frac{1/G}{1 + (\omega_0 C/G)^2} = \frac{R}{1 + Q^2} \qquad (*)$$

Design procedure is similar to the series case.

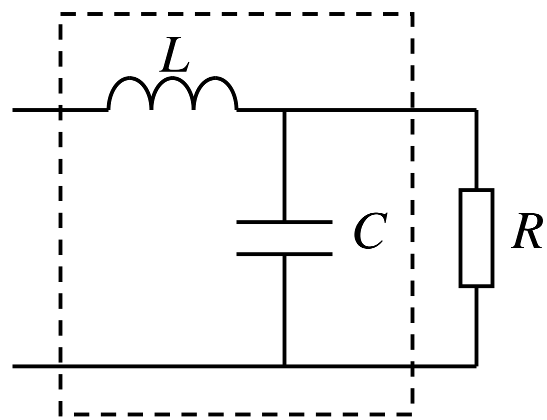
Other *L* circuit variations

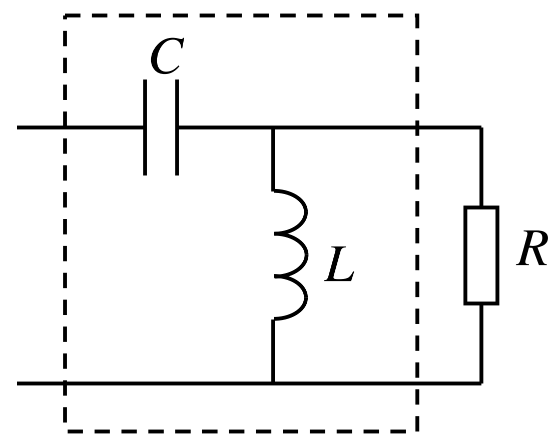
Series:





Shunt:

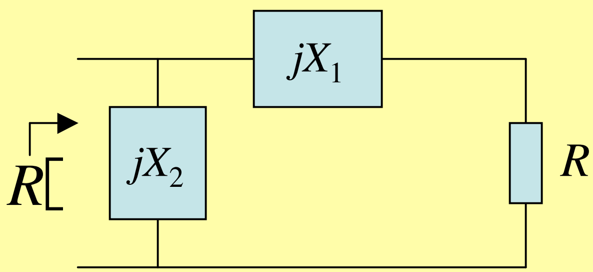




Exercise: derive design procedure for all other *L* circuits.

General procedure for designing *L* circuits

Series *L* circuit (suitable for *R'*>*R*) :

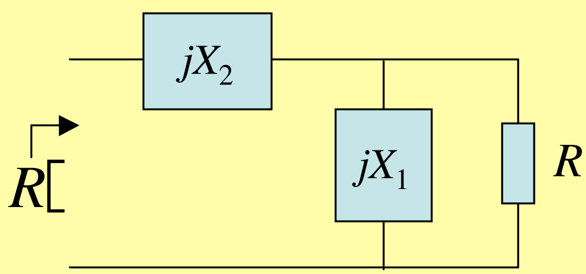


$$R' = R(1+Q^2)$$

$$jX_2 = -jX_1\left(1+\frac{1}{Q^2}\right) = -\frac{jR'}{Q}$$

$$Q = \frac{X_1}{R}$$

Shunt *L* circuit (suitable for *R'*<*R*) :



$$R' = \frac{R}{1+Q^2}$$

$$jX_2 = -\frac{jX_1}{1+\frac{1}{Q^2}} = -jR'Q$$

$$Q = \frac{B_1}{G} = \frac{R}{X_1}$$

Advantages of *L* circuits:

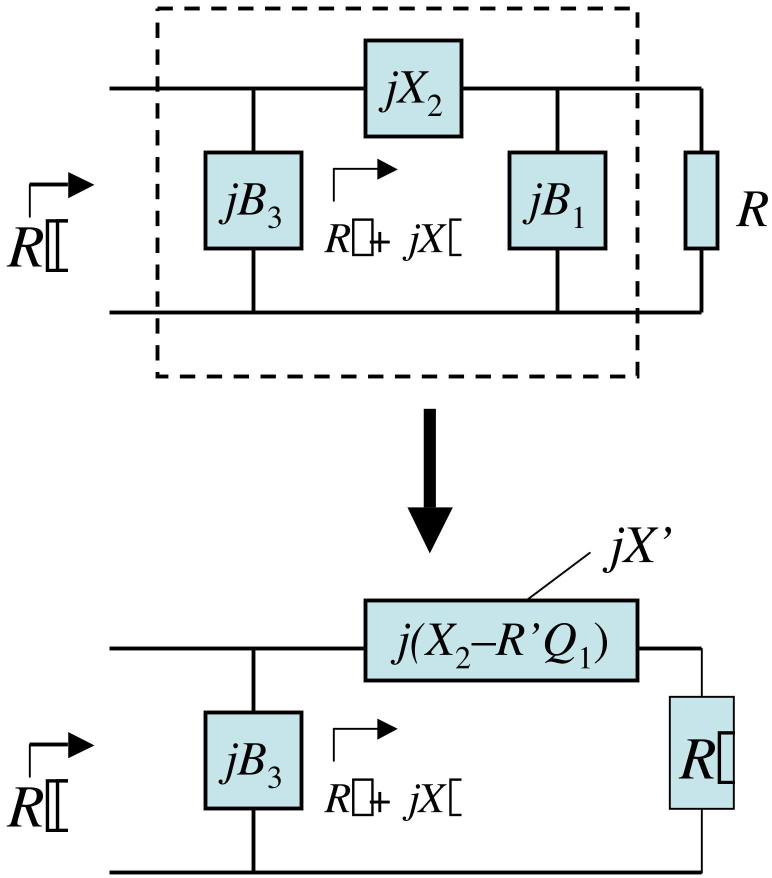
- Simple
- Low cost
- Easy to design

Disadvantages of *L* circuits:

- The value of *Q* is determined by the ratio of *R*/*R’*. Hence,
    - there is no control over the value of *Q*.
    - the bandwidth is also not controllable.

Solution: Add an element to provide added flexibility.
$\Rightarrow$ π circuits and *T* circuits

π matching circuits



Analysis by decomposing into two *L* circuit sections:

First section (from right):

$$R' = \frac{R}{1+Q_1^2} \qquad X' = X_2 - R'Q_1$$

$$Q_1 = \frac{B_1}{G} = B_1 R$$

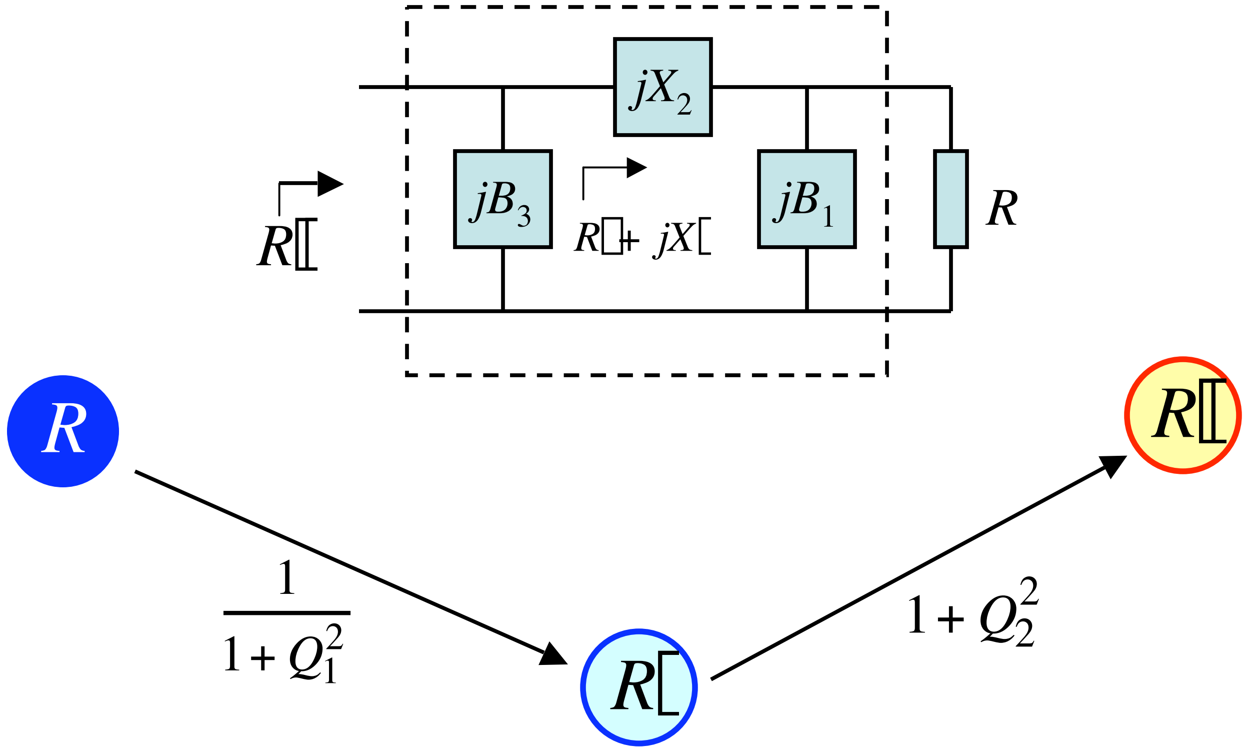
Second section:

$$Q_2 = \frac{X'}{R'} = \frac{X_2 - R'Q_1}{R'} \quad \Rightarrow \quad \frac{X_2}{R'} = Q_1 + Q_2$$

$$R'' = R'(1 + Q_2^2)$$

$$B'' = B_3 - \frac{Q_2}{R''} \quad \Rightarrow \quad B_3 = \frac{Q_2}{R''}$$

Impedance transformation in π matching circuits



Obviously, we have to set $Q_1 > Q_2$ if we want to have $R'' < R$. Likewise, we need $Q_1 < Q_2$ if we want to have $R'' > R$.

General procedure for designing π matching circuits

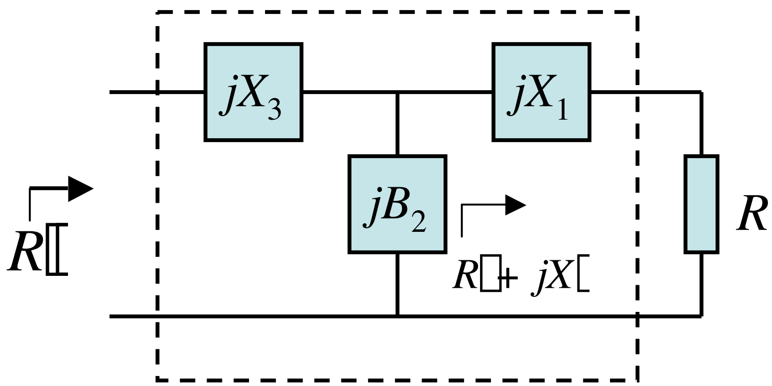
For $R'' < R$

1. Select Q_1 according to the max Q.
2. Find R' using $R' = R/(1+Q_1^2)$
3. Get Q_2 using $Q_2^2 = \frac{R''}{R'} - 1$
4. Obtain X_2 using $X_2 = R'(Q_1 + Q_2)$.
5. $B_1 = Q_1/R$
6. $B_3 = Q_2/R''$

For $R'' > R$

1. Select Q_2 according to the max Q.
2. Find R' using $R' = R''/(1+Q_2^2)$
3. Get Q_2 using $Q_1^2 = \frac{R}{R'} - 1$
4. Obtain X_2 using $X_2 = R'(Q_1 + Q_2)$.
5. $B_1 = Q_1/R$
6. $B_3 = Q_2/R''$

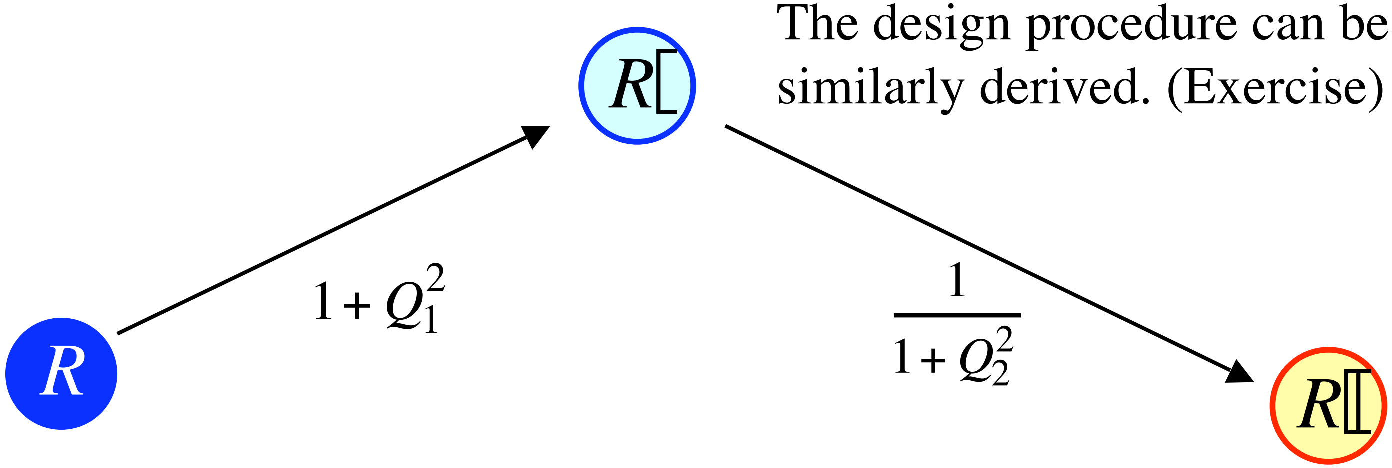
T matching circuits



The analysis is similar to the π case.

The difference is that R is first raised to R' by the series reactance, and then lowered to R'' by the shunt reactance.

The design procedure can be similarly derived. (Exercise)



General procedure for designing *T* matching circuits

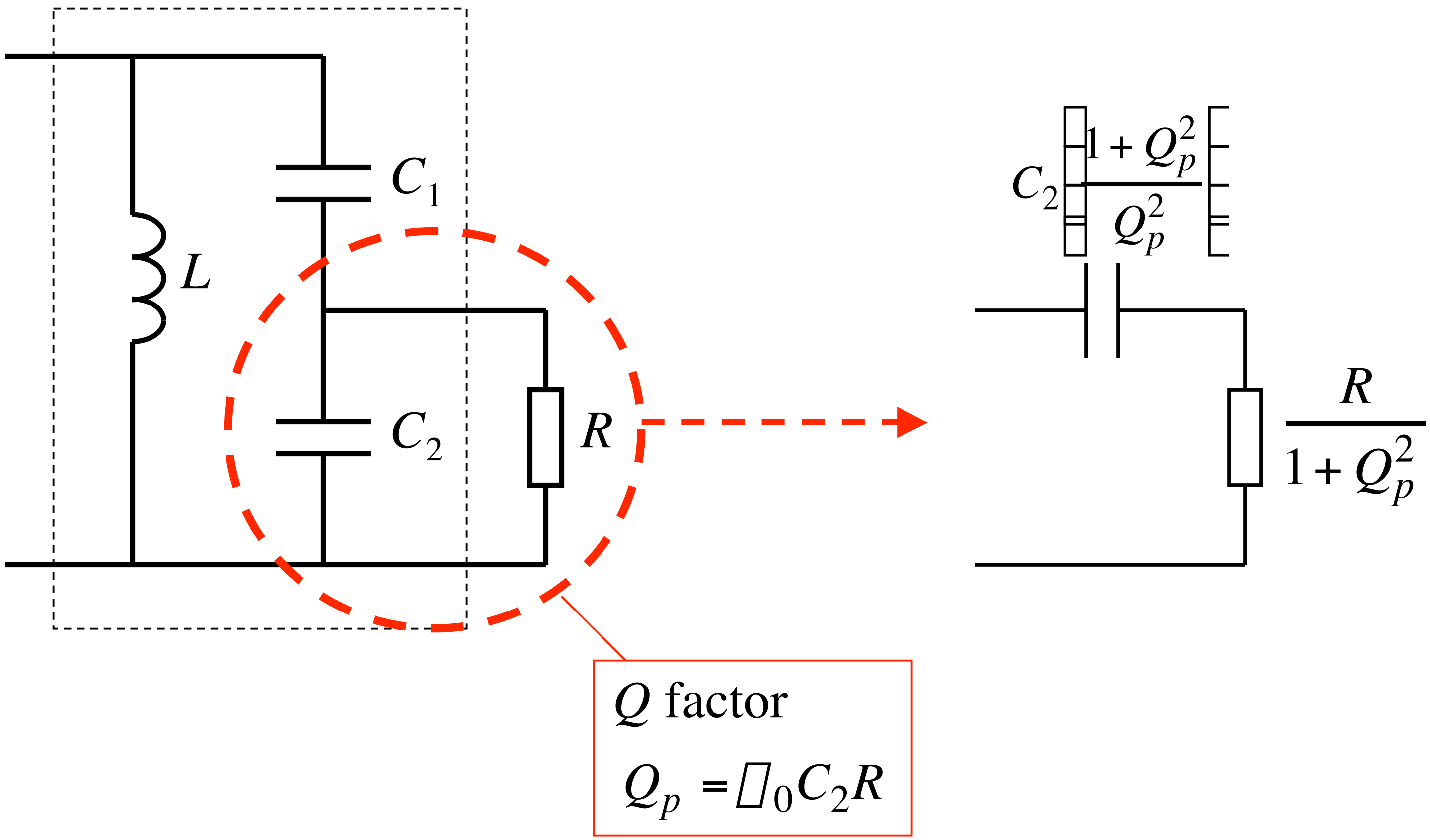
For $R'' > R$

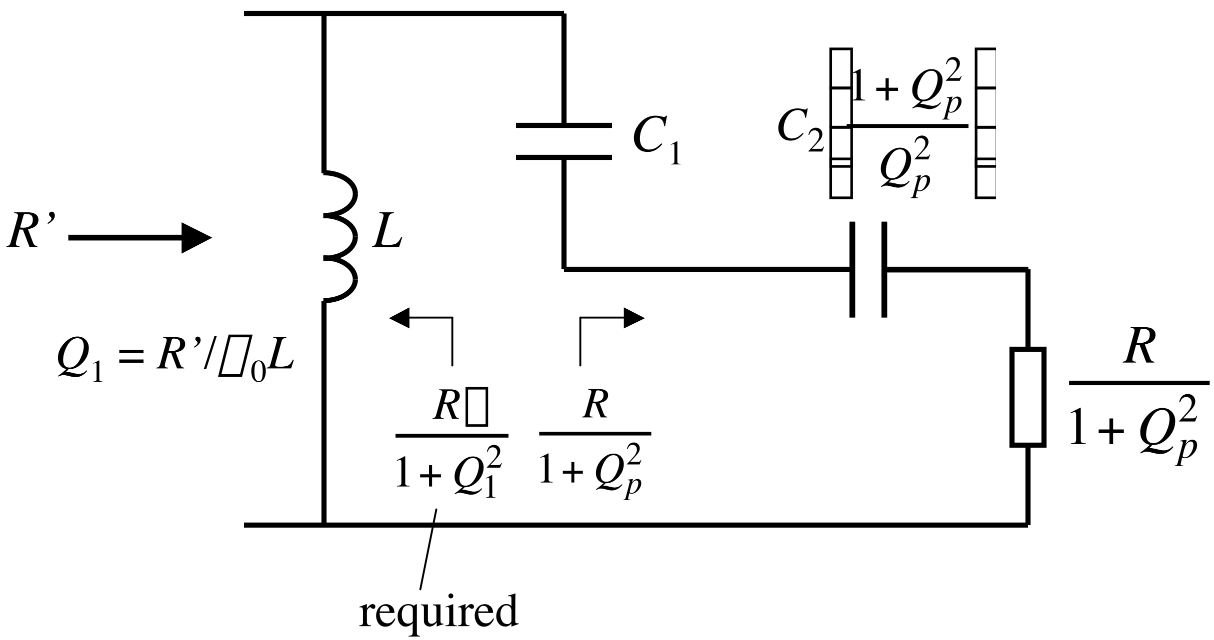
1. Select Q_1 according to the max Q.
2. Find R' using $R' = R(1+Q_1^2)$
3. Get Q_2 using $Q_2^2 = \dfrac{R'}{R''} - 1$
4. Obtain X_1 using $X_1 = Q_1 R$.
5. $B_2 = (Q_1 + Q_2)/R'$
6. $X_3 = Q_2 R''$

For $R'' < R$

1. Select Q_2 according to the max Q.
2. Find R' using $R' = R''(1+Q_2^2)$
3. Get Q_1 using $Q_1^2 = \dfrac{R'}{R} - 1$
4. Obtain X_1 using $X_1 = Q_1 R$.
5. $B_2 = (Q_1 + Q_2)/R'$
6. $X_3 = Q_2 R''$

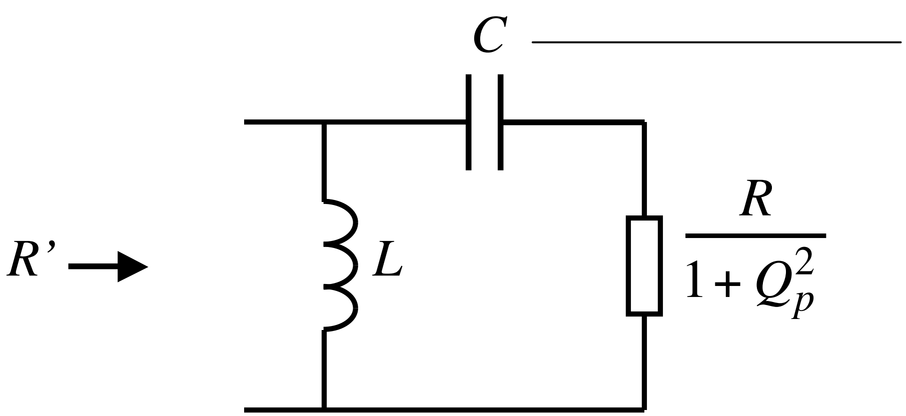
Tapped capacitor matching circuit





$$\frac{R'}{1+Q_1^2} = \frac{R}{1+Q_p^2} \quad \Rightarrow \quad Q_p = \sqrt{\frac{R}{R'}\left(1+Q_1^2\right) - 1}$$

$$\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2}\left(\frac{Q_p^2}{1+Q_p^2}\right)$$



For a high Q circuit, $\omega_0 \approx \sqrt{\frac{1}{LC}}$

Also, we have the alternative approximation for Q_1: $Q_1 \approx \omega_0 R' C$, which is set to $\omega_0 / \Delta\omega$.

Thus, we can go backward to find all the circuit parameters.

General procedure for designing tapped *C* circuits

1. Find Q_1 from $Q_1 = \omega_0 / \Delta\omega$
2. Given R', find C using $C = Q_1 / \omega_0 R' = 1 / 2\pi\, \Delta\omega R'$
3. Find L using $L = 1 / \omega_0^2 C$
4. Find Q_p using $Q_p = [\, (R/R')(1+Q_1^2) - 1 \,]^{1/2}$
5. Find C_2 from $C_2 = Q_p / \omega_0 R$
6. Find C_1 from $C_1 = C_{eq} C_2 / (C_{eq} - C_2)$ where $C_{eq} = C_2(1+ Q_p^2) / Q_p^2$

Advantages of π, T and tapped C circuits:

- specify Q factor (sharpness of cutoff)
- provide some control of the bandwidth

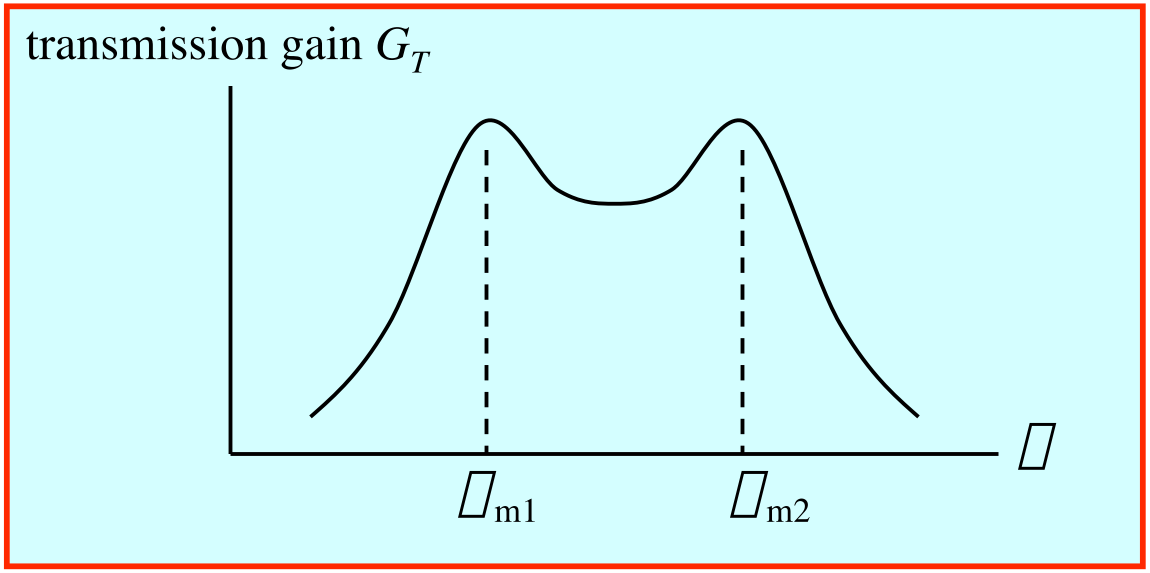
Disadvantage:

- no precise control of the bandwidth

For precise specification of bandwidth, use double-tuned matching circuits.

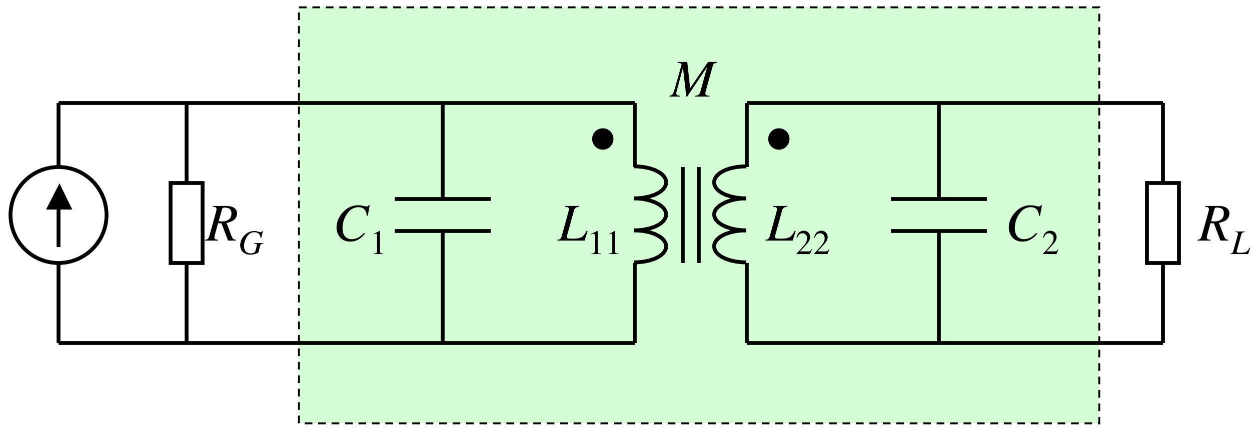
Double-tuned matching circuits

Specify the bandwidth by two frequencies ω_{m1} and ω_{m2} .



There is a mid-band dip, which can be made small if the pass band is narrow. Also, large difference in the impedances to be matched can be achieved by means of galvanic transformer.

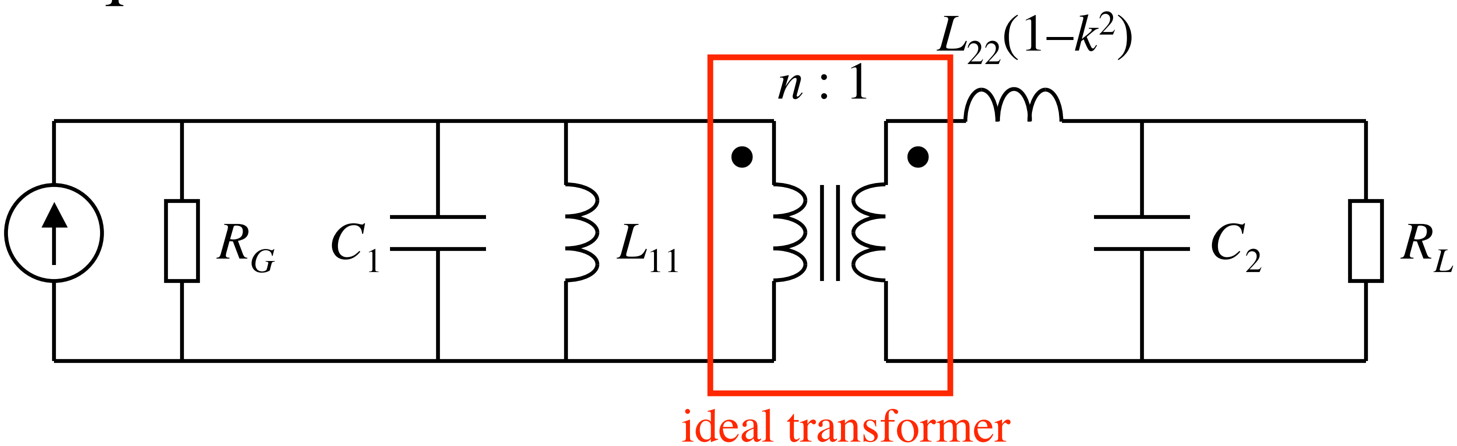
The construction of a double-tuned circuit typically includes a real transformer and two resonating capacitors.

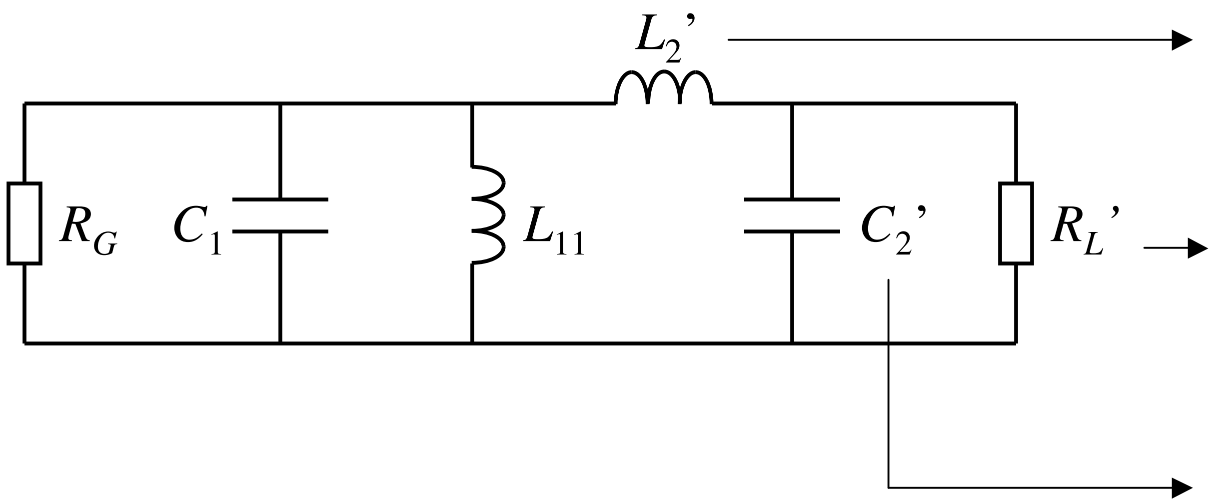


Transformer turn ratio n and coupling coefficient k are related by

$$n = \sqrt{\frac{L_{11}}{k^2 L_{22}}}$$

Equivalent models:



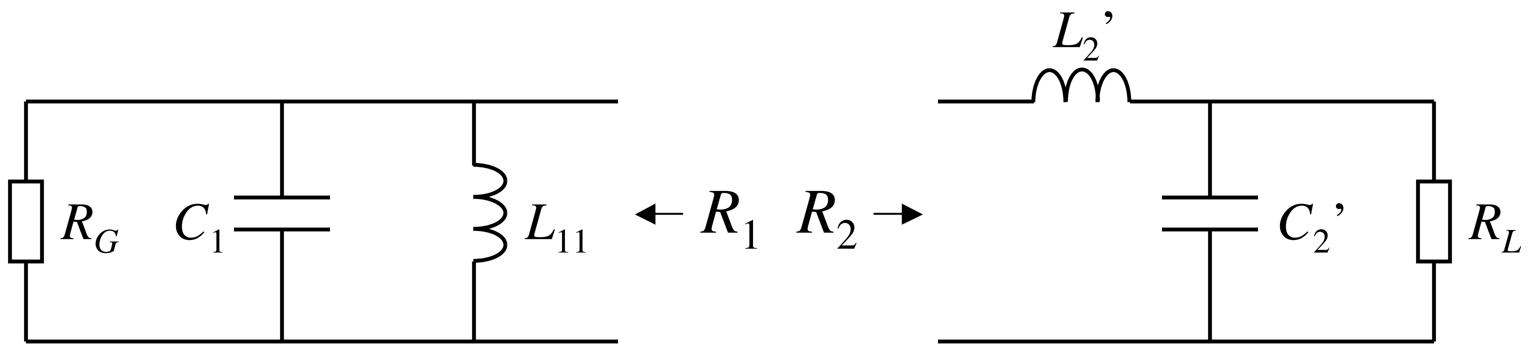


$$L_{11}\left(\frac{1}{k^2}-1\right)$$

$$\left(\frac{L_{11}}{k^2 L_{22}}\right) R_2$$

$$\left(\frac{L_{11}}{k^2 L_{22}}\right) C_2$$

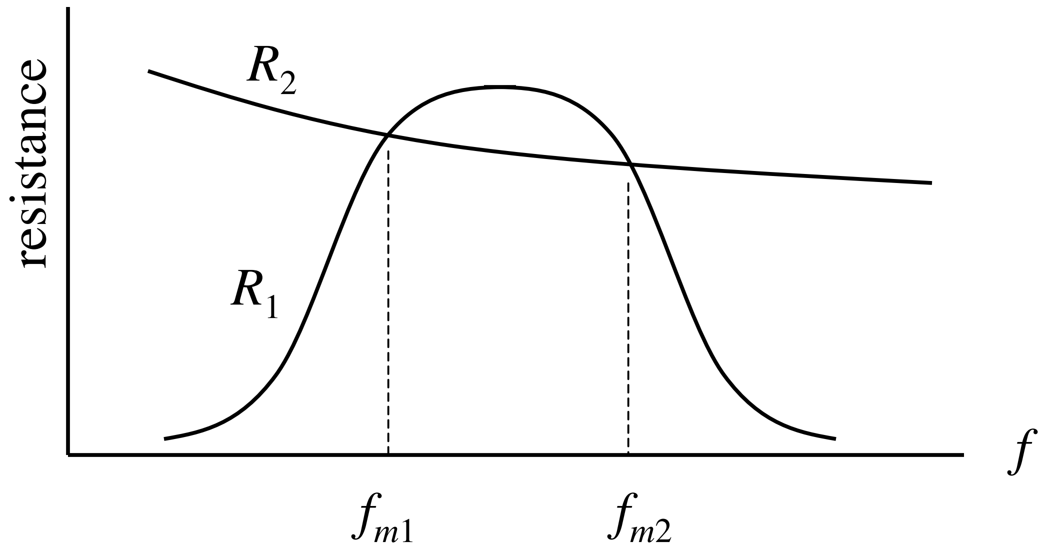
Exact match is to be achieved at two given frequencies: f_{m1} and f_{m2}.



Observe that:

- R_1 resonates at certain frequency, but is always less than R_G
- R_2 decreases monotonically with frequency

So, if R_L is sufficiently small, there will be two frequency values where $R_1 = R_2$.



Our objective here is to match R_G and R_L over a bandwidth Δf centered at f_o, usually with an allowable ripple in the pass band.

General Impedance Matching Based on Two-Port Parameters

Two-port models

Idea: we don't care what is inside, as long as it can be modelled in terms of *four parameters*.

Two-port models

z-parameters (impedance matrix):

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \end{bmatrix}$$

$$\boxed{\begin{aligned} v_1 &= z_{11} i_1 + z_{12} i_2 \\ v_2 &= z_{21} i_1 + z_{22} i_2 \end{aligned}}$$

y-parameters (admittance matrix):

$$\begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = \begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

h-parameters (hybrid matrix):

$$\begin{bmatrix} v_1 \\ i_2 \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix} \begin{bmatrix} i_1 \\ v_2 \end{bmatrix}$$

g-parameters (hybrid matrix):

$$\begin{bmatrix} i_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} \begin{bmatrix} v_1 \\ i_2 \end{bmatrix}$$

Finding the parameters

e.g., z-parameters

$$v_1 = z_{11} i_1 + z_{12} i_2$$
$$v_2 = z_{21} i_1 + z_{22} i_2$$

$$z_{11} = \left.\frac{v_1}{i_1}\right|_{i_2 = 0} = \left.\frac{v_1}{i_1}\right|_{\text{port 2 open-circuited}}$$

$$z_{12} = \left.\frac{v_1}{i_2}\right|_{i_1 = 0} = \left.\frac{v_1}{i_2}\right|_{\text{port 1 open-circuited}}$$

$$z_{21} = \left.\frac{v_2}{i_1}\right|_{i_2 = 0} = \left.\frac{v_2}{i_1}\right|_{\text{port 2 open-circuited}}$$

$$z_{22} = \left.\frac{v_2}{i_2}\right|_{i_1 = 0} = \left.\frac{v_2}{i_2}\right|_{\text{port 1 open-circuited}}$$

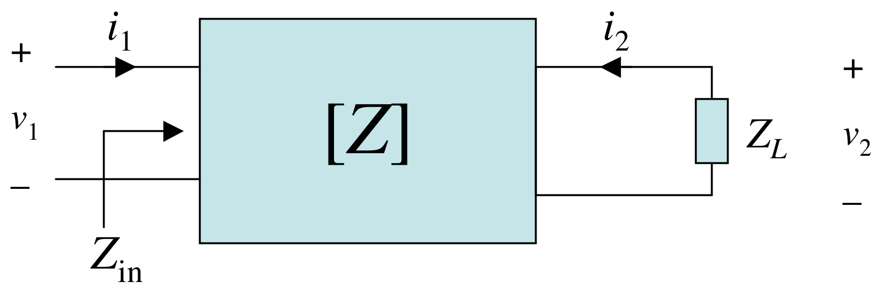
Finding the parameters

e.g., g-parameters

$$i_1 = g_{11}v_1 + g_{12}i_2$$
$$v_2 = g_{21}v_1 + g_{22}i_2$$

$$\boxed{\begin{aligned}
g_{11} &= \left.\frac{i_1}{v_1}\right|_{i_2=0} = \left.\frac{i_1}{v_1}\right|_{\text{port 2 open-circuited}} \\
g_{12} &= \left.\frac{i_1}{i_2}\right|_{v_1=0} = \left.\frac{i_1}{i_2}\right|_{\text{port 1 short-circuited}} \\
g_{21} &= \left.\frac{v_2}{v_1}\right|_{i_2=0} = \left.\frac{v_2}{v_1}\right|_{\text{port 2 open-circuited}} \\
g_{22} &= \left.\frac{v_2}{i_2}\right|_{v_1=0} = \left.\frac{v_2}{i_2}\right|_{\text{port 1 short-circuited}}
\end{aligned}}$$

Input impedance:



$$v_1 = z_{11} i_1 + z_{12} i_2$$
$$v_2 = z_{21} i_1 + z_{22} i_2$$

$\Rightarrow$

$$\frac{v_1}{i_1} = z_{11} + z_{12}\frac{i_2}{i_1}$$
$$\frac{v_2}{-i_2} = -z_{21}\frac{i_1}{i_2} - z_{22}$$

$\Rightarrow$

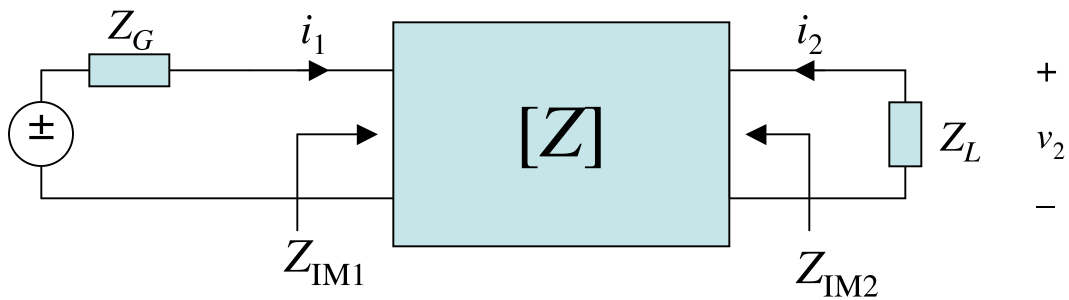
$$Z_{\text{in}} = z_{11} + z_{12}\frac{i_2}{i_1}$$
$$Z_L = -z_{21}\frac{i_1}{i_2} - z_{22}$$

$\Rightarrow$

$$\boxed{Z_{\text{in}} = z_{11} - \left[\frac{z_{12} z_{21}}{Z_L + z_{22}}\right]}$$

Similarly, we can find the input impedance at any port in terms of any of the two-port parameters, or even a combination of different two-port parameters.

We will see that the matching problem can be solved by making sure that both input and output ports are matched.



$$\text{matching:} \quad Z_G = Z_{IM1} \quad \text{and} \quad Z_{IM2} = Z_L$$

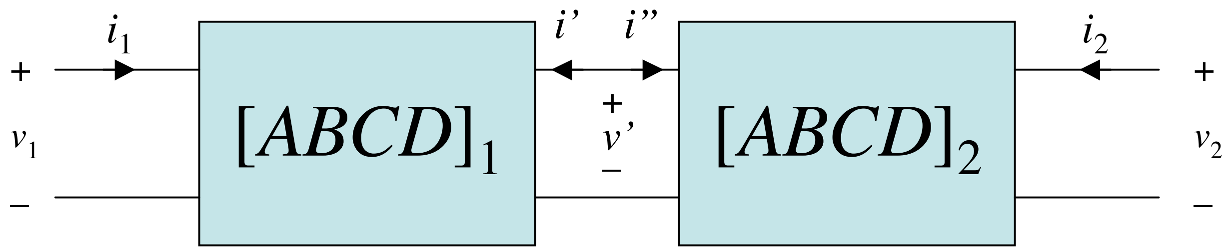
image impedances

The *ABCD* parameters (very useful form)

Here, voltage and current of port 1 are expressed in terms of those of port 2. So, this is neither an immittance matrix like *Z* and *Y*, nor a hybrid matrix like *G* and *H*.

$$\begin{bmatrix} v_1 \\ i_1 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} v_2 \\ -i_2 \end{bmatrix}$$

Note: the sign of i_2 in the above equation. This sign convention will make the *ABCD* matrix very useful for describing *cascade* circuits.



$$\begin{bmatrix} v_1 \\ i_1 \end{bmatrix} = \begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix} \begin{bmatrix} v' \\ -i' \end{bmatrix} \qquad \begin{bmatrix} v' \\ i'' \end{bmatrix} = \begin{bmatrix} A_2 & B_2 \\ C_2 & D_2 \end{bmatrix} \begin{bmatrix} v_2 \\ -i_2 \end{bmatrix}$$

Since $-i' = i''$, we have

$$\begin{bmatrix} v_1 \\ i_1 \end{bmatrix} = \begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix} \begin{bmatrix} A_2 & B_2 \\ C_2 & D_2 \end{bmatrix} \begin{bmatrix} v_2 \\ -i_2 \end{bmatrix}$$

So, if more two-ports are cascaded, the overall *ABCD* matrix is just the *product* of all the *ABCD* matrices.

To find the *ABCD* parameters, we may apply the same principle:

$$A = \left.\frac{v_1}{v_2}\right|_{i_2=0} = \left.\frac{v_1}{v_2}\right|_{\text{port 2 open-circuited}} = \frac{z_{11}}{z_{21}}$$

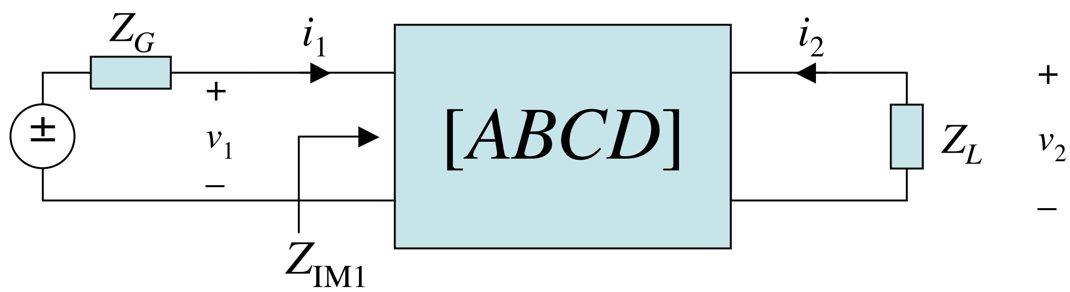
$$B = \left.\frac{-v_1}{i_2}\right|_{v_2=0} = \left.\frac{-v_1}{i_2}\right|_{\text{port 2 short-circuited}} = \frac{z_{11}z_{22} - z_{21}z_{12}}{z_{21}}$$

$$C = \left.\frac{i_1}{v_2}\right|_{i_2=0} = \left.\frac{i_1}{v_2}\right|_{\text{port 2 open-circuited}} = \frac{1}{z_{21}}$$

$$D = \left.\frac{-i_1}{i_2}\right|_{v_2=0} = \left.\frac{-i_1}{i_2}\right|_{\text{port 2 short-circuited}} = \frac{z_{22}}{z_{21}}$$

We can show easily that $\boxed{AD - BC = 1}$ if $z_{12} = z_{21}$, i.e., reciprocal circuit.

Matching problem

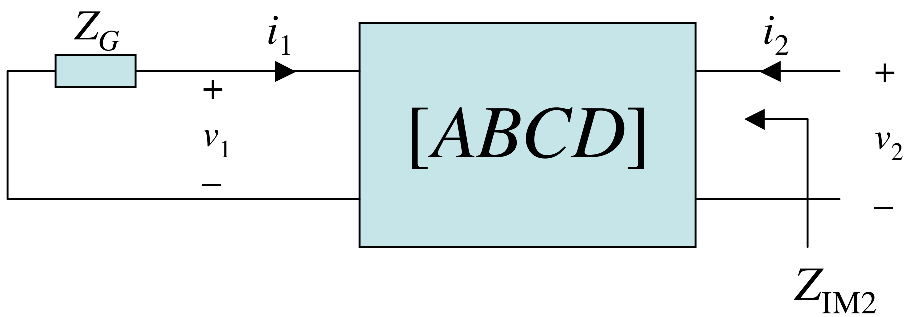


Input image impedance

$$v_1 = Av_2 - Bi_2$$
$$i_1 = Cv_2 - Di_2 \qquad \Rightarrow \qquad Z_{\mathrm{in}} = \frac{v_1}{i_1} = \frac{Av_2 - Bi_2}{Cv_2 - Di_2}$$

$$= \frac{A\dfrac{v_2}{-i_2} + B}{C\dfrac{v_2}{-i_2} + D}$$

$$= \frac{AZ_L + B}{CZ_L + D}$$



Output image impedance

$$\begin{aligned} v_1 &= Av_2 - Bi_2 \\ i_1 &= Cv_2 - Di_2 \end{aligned} \quad || \quad \begin{aligned} v_2 &= Dv_1 - Bi_1 \\ i_2 &= Cv_1 - Ai_1 \end{aligned} \quad || \quad Z_{\mathrm{IM2}} = \frac{v_2}{i_2} = \frac{Dv_1 - Bi_1}{Cv_1 - Ai_1}$$

because $AD - BC = 1$

$$= \frac{D\dfrac{v_1}{-i_1} + B}{C\dfrac{v_1}{-i_1} + A}$$

$$= \frac{DZ_G + B}{CZ_G + A}$$

Under matched conditions,

$$Z_G = Z_{\text{IM1}} \quad \text{and} \quad Z_L = Z_{\text{IM2}}$$

| |

$$Z_{\text{IM1}} = Z_G = \frac{AZ_L + B}{CZ_L + D} \quad \text{and} \quad Z_{\text{IM2}} = Z_L = \frac{DZ_G + B}{CZ_G + A}$$

| |

$$Z_{\text{IM1}} = \sqrt{\frac{AB}{CD}} \quad \text{and} \quad Z_{\text{IM2}} = \sqrt{\frac{DB}{AC}}$$

Alternatively, we have

$$Z_{\text{IM1}} = \sqrt{\frac{z_{11}}{y_{11}}} \quad \text{and} \quad Z_{\text{IM2}} = \sqrt{\frac{z_{22}}{y_{22}}}$$

Note: image impedances ***are different from*** input and output impedances.

1. Image impedances do not depend on the load impedance or the source impedance. They are purely dependent upon the circuit.

$$Z_{\mathrm{IM1}} = \sqrt{\frac{z_{11}}{y_{11}}} \quad \text{and} \quad Z_{\mathrm{IM2}} = \sqrt{\frac{z_{22}}{y_{22}}}$$

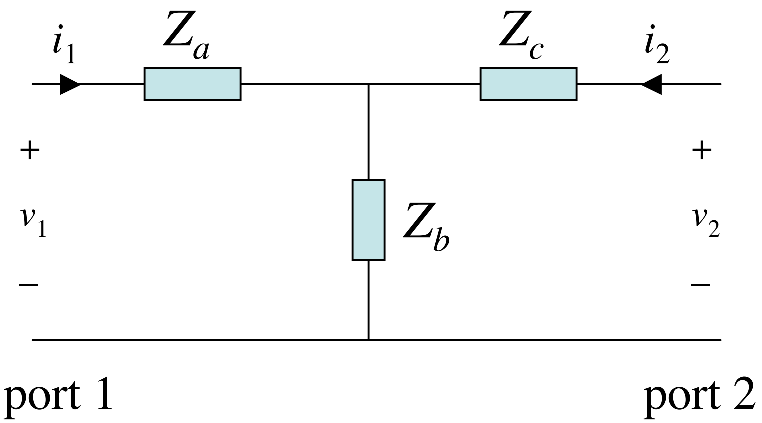
2. Input impedance (Z_{in}) depend on the load impedance. Output impedance (Z_{out}) depends on the source impedance. For example,

$$Z_{\mathrm{in}} = z_{11} - \left(\frac{z_{12} z_{21}}{Z_L + z_{22}} \right)$$

Matching conditions:

- Source impedance equals input image impedance
- Load impedance equals output image impedance

Example



We can easily see that

$$z_{11} = \left.\frac{v_1}{i_1}\right|_{\text{port 2 open-circuited}} = Z_a + Z_b$$

$$y_{11} = \left.\frac{i_1}{v_1}\right|_{\text{port 2 short-circuited}} = \frac{1}{Z_a + Z_b \| Z_c}$$

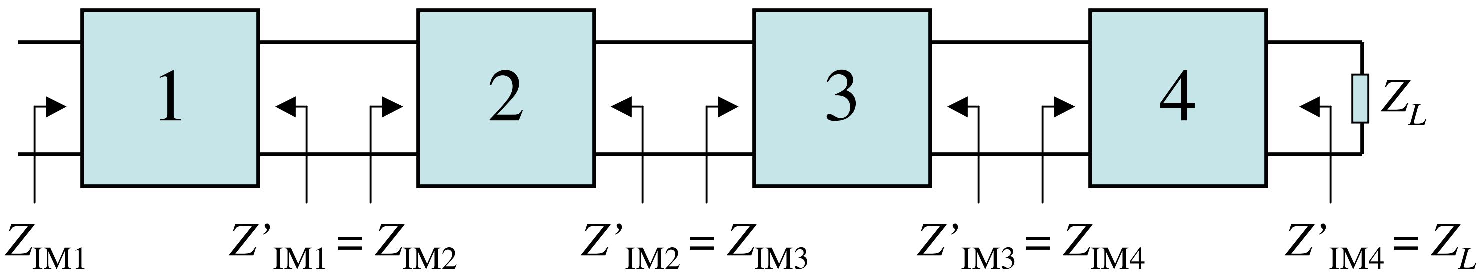
$$z_{22} = \left.\frac{v_2}{i_2}\right|_{\text{port 1 open-circuited}} = Z_b + Z_c$$

$$y_{22} = \left.\frac{i_2}{v_2}\right|_{\text{port 1 short-circuited}} = \frac{1}{Z_c + Z_a \| Z_b}$$

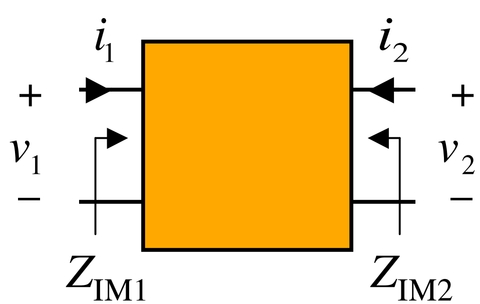
Thus, the ***image impedances*** are

$$Z_{\text{IM1}} = \sqrt{(Z_a + Z_b)(Z_a + Z_b \| Z_c)} \quad \text{and} \quad Z_{\text{IM2}} = \sqrt{(Z_c + Z_b)(Z_c + Z_a \| Z_b)}$$

Matching a cascade of circuits



Convention



A wave or signal entering into circuit 1 from left side will travel without reflection through the circuits if all ports are matched.

Propagation constant γ

$$e^{\gamma} = \sqrt{\frac{\text{input power}}{\text{output power}}} = \sqrt{\frac{v_1 i_1}{v_2(-i_2)}} = \frac{v_1}{v_2}\sqrt{\frac{Z_{\mathrm{IM2}}}{Z_{\mathrm{IM1}}}}$$

Propagation equations

$$\boxed{e^{\gamma} = \sqrt{\frac{v_1 i_1}{v_2(-i_2)}} = \frac{v_1}{v_2}\sqrt{\frac{Z_{\mathrm{IM2}}}{Z_{\mathrm{IM1}}}}} \quad || \quad e^{\gamma} = \frac{v_1}{v_2} \quad \text{if the 2-port circuit is symmetrical}$$

In general,

$$\frac{v_1}{v_2} = \frac{Av_2 - Bi_2}{v_2} = A + \frac{B}{Z_{\mathrm{IM2}}}$$

$$= A + B\sqrt{\frac{AC}{BD}} = \sqrt{\frac{A}{D}}\left(\sqrt{AD} + \sqrt{BC}\right)$$

$$\frac{i_1}{-i_2} = CZ_{\mathrm{IM2}} + D = \sqrt{\frac{D}{A}}\left(\sqrt{AD} + \sqrt{BC}\right)$$

Thus,

$$e^{\gamma} = \sqrt{\frac{v_1 i_1}{-v_2 i_2}} = \sqrt{AD} + \sqrt{BC}$$

$$e^{-\gamma} = \sqrt{AD} - \sqrt{BC}$$

Combining e^{γ} and $e^{-\gamma}$, we have

$$\cosh\gamma = \frac{e^{\gamma} + e^{-\gamma}}{2} = \sqrt{AD}$$

$$\sinh\gamma = \frac{e^{\gamma} - e^{-\gamma}}{2} = \sqrt{BC}$$

Define

$$n = \sqrt{\frac{Z_{\mathrm{IM1}}}{Z_{\mathrm{IM2}}}} = \sqrt{\frac{A}{D}}$$

We have

$$\boxed{\begin{aligned} A &= n\cosh\gamma \\ B &= nZ_{\mathrm{IM2}}\sinh\gamma \\ C &= \frac{\sinh\gamma}{nZ_{\mathrm{IM2}}} \\ D &= \frac{\cosh\gamma}{n} \end{aligned}}$$

From the *ABCD* equation, we have

$$v_1 = n v_2 \cosh\gamma + n i_2 Z_{\mathrm{IM2}} \sinh\gamma$$

$$i_1 = \frac{v_2}{n Z_{\mathrm{IM2}}} \sinh\gamma + \frac{i_2}{n} \cosh\gamma$$

Dividing gives

$$Z_{\mathrm{in}} = \frac{v_1}{i_1} = n^2 Z_{\mathrm{IM2}} \frac{Z_L + Z_{\mathrm{IM2}} \tanh\gamma}{Z_L \tanh\gamma + Z_{\mathrm{IM2}}}$$

For a transmission line, $Z_{\mathrm{IM1}} = Z_{\mathrm{IM2}} = Z_o$, where Z_o is usually called the ***characteristic impedance*** of the transmission line. Also, **for a lossless transmission line,** $\gamma = j\beta$ is pure imaginary, and thus tanh becomes tan, sinh becomes sin, cosh becomes cosh.

$$Z_{\mathrm{in}} = \frac{v_1}{i_1} = Z_o \frac{Z_L + jZ_o \tan\beta}{Z_o + jZ_L \tan\beta}$$

This is just the same transmission line equation. In communication, we usually express θ as electrical length, and is equal to

$$\theta = \omega l / v = 2\pi l / \lambda$$

wavelength

frequency in rad/s   length of transmission line   velocity of propagation

So, we can easily verify the following standard results:

1. If the transmission line length is $\lambda/2$ or λ, then the input impedance is just equal to the load impedance.
2. If the transmission line length is $\lambda/4$, then the input impedance is Z_o^2/Z_L.

Impedance value for other lengths can be found from the equation or conveniently by using a Smith chart.