Empirical Interpolation Method (EIM)

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Outline

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- Consider *non-affine* parameter dependence function:

$$s(\cdot;\mu) \in L^{\infty}(\Omega),$$

where Ω = spatial domain,and $\mu \in \mathscr{D}$ = parameter domain

- Define:

$$\mathscr{M}^{s} \equiv \{s(\cdot;\mu) : \mu \in \mathscr{D}\}$$

  - $\mathscr{M}^{s}$ could be a low-dimensional space $\Rightarrow \mathscr{M}^{s} \simeq span\{q_1, \dots, q_m\}$ with *small m*.
  - variation of $\mathscr{M}^{s}$ in $\mu \in \mathscr{D}$ can be captured by a selected points $\{z_1, \dots, z_m\} \in \Omega$ with *small m*.

- **Goal**: Approximate s by $\hat{s}$ in the form:

$$\begin{aligned} \hat{s}(x;\mu) &= \sum_{\ell=1}^{m} q_\ell(x)\beta_\ell(\mu), && \text{with} \\ \hat{s}(z_i;\mu) &= s(z_i;\mu) && i = 1, \dots, m, \end{aligned}$$

for all $\mu \in \mathscr{D}$. Closed-form for $\beta_1, \dots, \beta_m$?

Coefficient Approximation: Matrix Form

- Given

  Basis: $Q(x) \equiv [q_1(x), \ldots, q_m(x)]$

  Interpolation Points: $\mathbf{z} \equiv [z_1, \ldots, z_m]^T \in \Omega^m$

- Let

  Coefficient: $\beta(\mu) \equiv [\beta_1(\mu), \ldots, \beta_m(\mu)]^T \in \mathbb{R}^m$

  Function at $\mathbf{z}$: $s(\mathbf{z};\mu) \equiv [s(z_1;\mu), \ldots, s(z_m;\mu)]^T \in \mathbb{R}^m$

$$Q(\mathbf{z}) = \begin{bmatrix} q_1(z_1) & q_2(z_1) & \ldots & q_m(z_1) \\ q_1(z_2) & q_2(z_2) & \ldots & q_m(z_2) \\ \vdots & \vdots & \ddots & \vdots \\ q_1(z_m) & q_2(z_m) & \ldots & q_m(z_m) \end{bmatrix} \in \mathbb{R}^{m \times m},$$

- $\left\{ \begin{array}{lcl} \hat{s}(\mathbf{x};\mu) & = & Q(\mathbf{x})\beta(\mu) \\ \hat{s}(\mathbf{z};\mu) & = & s(\mathbf{z};\mu) \end{array} \right\} \Rightarrow \boxed{\beta(\mu) = (Q(\mathbf{z}))^{-1} s(\mathbf{z};\mu)}$

$$\hat{s}(\mathbf{x};\mu) = Q(\mathbf{x})(Q(\mathbf{z}))^{-1} s(\mathbf{z};\mu)$$

GIVEN:

$s(\cdot;\mu) \in L^{\infty}(\Omega;\mathscr{D})$

$\mathscr{M}^s \equiv \{s(\cdot;\mu) : \mu \in \mathscr{D}\}$

(**I**) BASIS: $\{\xi_\ell\}_{\ell=1}^m$

s.t.*span*$\{\xi_1,\ldots,\xi_m\} \simeq \mathscr{M}^s$

$$\{q\}_{\ell=1}^m \Updownarrow Q(x)$$

$$\downarrow$$

$$\{\beta_\ell(\mu)\}_{\ell=1}^m \Updownarrow \beta(\mu) = (Q(\mathbf{z}))^{-1}s(\mathbf{z};\mu)$$

$$\longrightarrow$$

(**III**) **OUTPUT:**

$$\hat{s}(x;\mu) = \sum_{\ell=1}^m q_\ell(x)\beta_\ell(\mu)$$

$$\Updownarrow$$

$$\hat{s}(x;\mu) = Q(x)(Q(\mathbf{z}))^{-1}s(\mathbf{z};\mu)$$

(**II**) INTERPOLATION

POINTS: $\mathbf{z} = \{z_\ell\}_{\ell=1}^m$

s.t. $Q(\mathbf{z})$ invertible

EIM

Empirical Interpolation Method (EIM)[Patera; 2004]

EIM Algorithm: Given a set of basis functions $\{\xi_i\}_{i=1}^m$, the set of m EIM interpolation points $\mathbf{z} = [z_1, \dots, z_m]^T$ is constructed as follows.

EIM steps

- Set $z_1 = \arg \text{ess } sup_{x\in\Omega}|\xi_1(x)|$.
  $q_1(x) = \frac{\xi_1(x)}{\xi_1(z_1)}$;
  $\mathbf{Q}^1(\mathbf{Z}^1) = q_1(z_1) = 1$.
- For $L = 2, \dots, m$,
  1. Solve ρ^{L-1} from :
  $$\mathbf{Q}^{L-1}(\mathbf{z}^{L-1})\rho^{L-1} = \xi_L(\mathbf{z}^{L-1}),$$
  2. Define
  $$r_L(x) = \xi_L(x) - \mathbf{Q}^{L-1}(x)\rho^{L-1},$$
  3. Set
  $$\boxed{z_L = \arg \text{ess } sup_{x\in\Omega}|r_L(x)|}$$
  4. Set $\boxed{q_L(x) = \frac{r_L(x)}{r_L(z_L)}}$

Note:

$$\begin{aligned}
[(\mathbf{Q}^{L-1}(\mathbf{z}^{L-1}))_{ij}] &= [q_j(z_i)] \in \mathbb{R}^{(L-1)\times(L-1)};\\
\xi_L(\mathbf{z}^{L-1}) &= [\xi_L(z_1), \dots, \xi_L(z_{L-1})]^T \in \mathbb{R}^{L-1};\\
\mathbf{z}^{L-1} &= [z_1, \dots, z_{L-1}]^T \in \Omega^{L-1};\\
\mathbf{Q}^{L-1}(x) &= [q_1(x), \dots, q_{L-1}(x)].
\end{aligned}$$

Final Output

Basis:	$\{q_1, \dots, q_m\}$
EIM points:	$\{z_1, \dots, z_m\}$

Set $\mathbf{Q}(x) = \mathbf{Q}^m(x)$
Set $\mathbf{Q}(\mathbf{z}) = \mathbf{Q}^m(\mathbf{z}^m)$
$s(x;\mu) \simeq \hat{s}_m(x;\mu) = Q(x)(Q(\mathbf{z}))^{-1}s(\mathbf{z};\mu)$

EIM points and bases

Ex: Input $\{\xi_L\}_{L=1}^{m}$ are eigenvectors of discrete Laplacian (Dr. Sorensen)

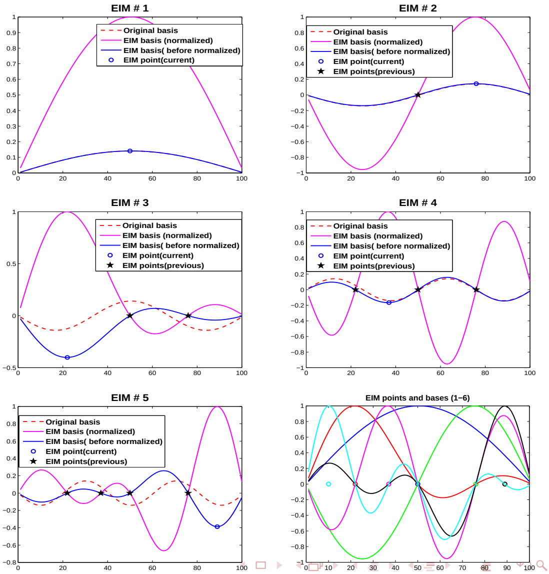
EIM steps

- Set $z_1 = \text{arg ess sup}_{x\in\Omega}|\xi_1(x)|$.
  $q_1(x) = \frac{\xi_1(x)}{\xi_1(z_1)}$;
  $\mathbf{Q}^1(\mathbf{z}^1) = q_1(z_1) = 1$.
- For $L = 2, \ldots, m$,
  1. Solve ρ^{L-1} from :
     $$\mathbf{Q}^{L-1}(\mathbf{z}^{L-1})\rho^{L-1} = \xi_L(\mathbf{z}^{L-1}),$$
  2. Define
     $$r_L(x) = \xi_L(x) - \mathbf{Q}^{L-1}(x)\rho^{L-1},$$
  3. Set $\boxed{z_L = \text{arg ess sup}_{x\in\Omega}|r_L(x)|}$
  4. Set $\boxed{q_L(x) = \frac{r_L(x)}{r_L(z_L)}}$

Notice: $r_L(z_i) = q_L(z_i) = 0, \forall i = 1, \ldots, L-1$.

I.e., EIM points $z_1, \ldots, z_{L-1}$ are the zeros of $q_L(x)$

and $r_L(x)$



Remarks on EIM

- EIM constructs an approximation for *non-affine* parameter dependence function by using a method of greedy selection.

Given *linear independent set* $\{\xi_1, \dots, \xi_m\}$. Then, for $L = 1, \dots, m$,

- $r_L(z_L) \neq 0$, since $\{\xi_i\}_{i=1}^{L}$ are linearly independent.
- $\exists$ a transformation between $\left\{ \begin{array}{llll} \text{Input basis:} & \mathbf{W} & \equiv & [\xi_1, \dots, \xi_m] \\ \text{EIM basis:} & \mathbf{Q} & \equiv & [q_1, \dots, q_m] \end{array} \right\}$

$$\boxed{\mathbf{W} = \mathbf{QR}},$$

where

$$\mathbf{R} := \begin{bmatrix} \xi_1(z_1) & \rho_1^1 & \rho_1^2 & \dots & \rho_1^{m-1} \\ & r_2(z_2) & \rho_2^2 & \dots & \rho_2^{m-1} \\ & & r_2(z_2) & \dots & \rho_3^{m-1} \\ & & & \ddots & \vdots \\ & & & & \rho_{m-1}^{m-1} \\ & & & & r_m(z_m) \end{bmatrix} \in \mathbb{R}^{m \times m},$$

- $\Rightarrow span\{q_1, \dots, q_L\} = span\{\xi_1, \dots, \xi_L\}$

Remarks on EIM(cont')

- $r_L(z_i) = 0$ for all $i = 1, \ldots, L-1 \Rightarrow q_j(z_i) = 0$ for $0 \leq i < j \leq L$
- $\mathbf{Q}^L(\mathbf{z}^L) = [q_j(z_i)] \in \mathbb{R}^{L \times L}$ is lower triangular with:

$$\left\{ \begin{array}{ll} (\mathbf{Q}^L(\mathbf{z}^L))_{ij} = 0, & \text{if } i < j \\ (\mathbf{Q}^L(\mathbf{z}^L))_{ij} = 1, & \text{if } i = j \\ |(\mathbf{Q}^L(\mathbf{z}^L))_{ij}| \leq 1, & \text{if } i > j \end{array} \right\} \text{ for } i, j = 1, \ldots, L$$

- $\mathbf{Q}^L(\mathbf{z}^L)$ is invertible, since $det(\mathbf{Q}^L(\mathbf{z}^L)) = 1$.
- $\Rightarrow$ The EIM steps are well-defined.
- $\Rightarrow$ The EIM points will not repeat.
- There exists a unique EIM approx $\hat{s}_m$, since $\mathbf{Q}^m(\mathbf{z}^m)$ is invertible.
- The EIM is hierarchy:for $L = 1, \ldots, m-1$, small

$$\begin{array}{ccc} \{z_1, \ldots, z_L\} & \subsetneq & \{z_1, \ldots, z_{L+1}\} \\ \{q_1, \ldots, q_L\} & \subsetneq & \{q_1, \ldots, q_{L+1}\}. \end{array}$$

$\Rightarrow$ If given $span\{\xi_{m+1}\} \not\subseteq span\{\xi_1, \ldots, \xi_m\}$, we can extend EIM points: $\{z_L\}_{L=1}^{m+1} = \{z_L\}_{L=1}^{m} \cup \{z_{m+1}\}$;and bases $\{q_L\}_{L=1}^{m+1} = \{q_L\}_{L=1}^{m} \cup \{q_{m+1}\}$

Posteriori Error bound of EIM approx $\hat{s}_m$

Suppose ξ_{m+1} is not in $span\{\xi_1, \ldots, \xi_m\}$, and $s(\cdot;\mu) \in W^s_{m+1} := span\{\xi_1, \ldots, \xi_{m+1}\} \subset \mathscr{M}^s := \{s(\cdot;\mu) | \mu \in \mathscr{D}\}$. Define

$$\begin{aligned} \hat{\epsilon}(\mu) &:= |s(z_{m+1};\mu) - \hat{s}_m(z_{m+1};\mu)|, \\ \mathscr{E}(x;\mu) &:= \hat{\epsilon}(\mu) q_{m+1}(x), \end{aligned}$$

where z_{m+1} and $q_{m+1}(x)$ are the $(m+1)^{th}$ EIM point and basis. Then

1. $$s(x,\mu) - \hat{s}_m(x,\mu) = \pm \mathscr{E}(x;\mu),$$

2. $$\|s(\cdot,\mu) - \hat{s}_m(\cdot,\mu)\| \leq \hat{\epsilon}(\mu).$$

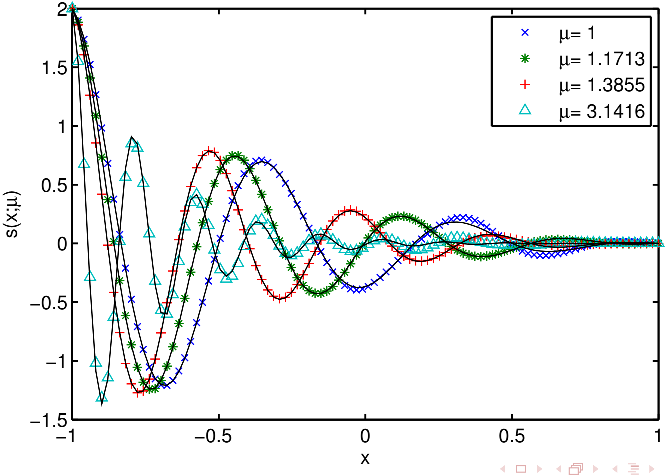
Note: in general, the assumption that $s(\cdot,\mu) \in W^s_{m+1}$ does not hold and the error bound above may not be exact (or applicable).

Ex: EIM for a nonlinear function

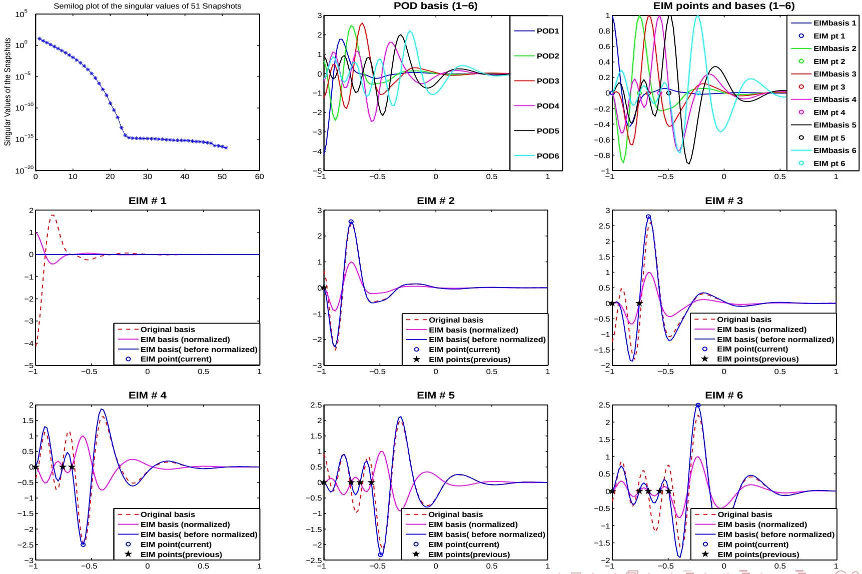
$$s(x;\mu) = (1-x)cos(3\pi\mu(x+1))e^{-(1+x)\mu},$$

where $x \in [-1, 1]$ and $\mu \in [1, \pi]$.

Plot of Approximate Functions (dim = 10) with Exact Functions (in black solid line)



Ex: EIM for a nonlinear function(cont'): $s(x;\mu) = (1-x)cos(3\pi\mu(x+1))e^{-(1+x)\mu}$



Ex: Application of the EIM on PDEs with nonlinearities

Ex. Unsteady 1D Burgers'Equation

$$\frac{\partial}{\partial t}y(x,t) - \nu \frac{\partial^2}{\partial x^2}y(x,t) + \frac{\partial}{\partial x}\left(s(y(x,t))\right) = 0, \qquad s(y(x,t)) = \frac{y(x,t)^2}{2},$$

$$y(0,t) = y(1,t) = 0,\ t \geq 0,\ \ y(x,0) = y_0(x),\ x \in \Omega \equiv [0,1],$$

Discretized system (Galerkin)

- Full-order system (dim = N): FE basis $\{\varphi_i\}_{i=1}^N$

$$\mathbf{M}_h \tfrac{d}{dt}\mathbf{y}(t) + \nu \mathbf{K}_h \mathbf{y}(t) - \mathbf{N}_h(\mathbf{y}(t)) = 0 \Rightarrow \boxed{y_h(x,t) = \textstyle\sum_{i=1}^N \varphi_i(x)\mathbf{y}_i(t)}$$

- Reduced-order system (dim = $k < N$): reduced basis $\{\phi_i\}_{i=1}^k$

$$\tilde{\mathbf{M}} \tfrac{d}{dt}\tilde{\mathbf{y}}(t) + \nu \tilde{\mathbf{K}}\tilde{\mathbf{y}}(t) - \tilde{\mathbf{N}}(\tilde{\mathbf{y}}(t)) = 0 \Rightarrow \boxed{\tilde{y}_h(x,t) = \textstyle\sum_{i=1}^k \phi_i(x)\tilde{\mathbf{y}}_i(t)},$$

E.g., $\{\phi_i\}_{i=1}^k$ can be constructed from POD.

Ex: Application of the EIM on PDEs with nonlinearities (cont')

Problem of reduced system from direct POD

Suppose

$$\Phi(x) = \Psi(x)\mathbf{U}_k,$$

where
$$\begin{cases} \text{FE basis:} & \Psi(x) \equiv [\psi_1(x), \dots, \psi_N(x)], \\ \text{Reduced basis:} & \Phi(x) \equiv [\phi_1(x), \dots, \phi_k(x)], \\ \text{Projection:} & \mathbf{U}_k \equiv [\mathbf{u}_1, \dots, \mathbf{u}_k] \end{cases} \in \mathbb{R}^{N\times k}.$$

Note:

$$\boxed{\tilde{\mathbf{N}}(\tilde{\mathbf{y}}(t)) = \underbrace{\mathbf{U}_k^T}_{k\times N} \underbrace{\mathbf{N}_h(\mathbf{U}_k\tilde{\mathbf{y}}(t))}_{N\times 1}} \Leftarrow \left\{ \begin{array}{lcl} \mathbf{N}_h(\mathbf{y}(t)) & = & \int_\Omega \Psi'(x)^T s(\Psi(x)\mathbf{y}(t))dx \\ \tilde{\mathbf{N}}(\tilde{\mathbf{y}}(t)) & = & \mathbf{U}_k^T \int_\Omega \Psi'(x)^T s(\Psi(x)\mathbf{U}_k\tilde{\mathbf{y}}(t))dx \end{array} \right\}$$

$\Rightarrow$ Computational Complexity still depends on N!!

WANT:

$$\boxed{\tilde{\mathbf{N}}(\tilde{\mathbf{y}}(t)) \leftarrow \underbrace{\mathbf{C}}_{k\times m} \underbrace{\hat{\mathbf{N}}(\tilde{\mathbf{y}}(t))}_{m\times 1}} \dashrightarrow \boxed{k, m \ll N} \dashrightarrow \text{Independent of } N$$

$\Rightarrow$ **C** and $\hat{\mathbf{N}}(\tilde{\mathbf{y}}(t))$ can be constructed from the **EIM**.

Ex: Application of the EIM on PDEs with nonlinearities(cont')

How EIM works?

- Approximation from EIM:

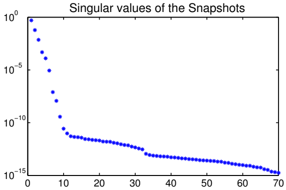
$$s(\Psi(x)\mathbf{U}_k\tilde{\mathbf{y}}(t)) \simeq \hat{s}(x,t) = Q(x)(Q(\mathbf{z}))^{-1}s(\mathbf{z},t),$$

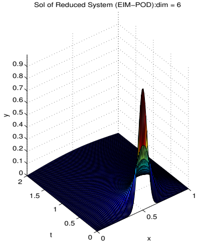
$$\begin{array}{llll} \text{EIM Basis:} & Q(x) & \equiv & [q_1(x),\ldots,q_m(x)] \\ \text{EIM Interpolation Points:} & \mathbf{z} & \equiv & [z_1,\ldots,z_m]^T \quad \in \Omega^m \end{array}$$

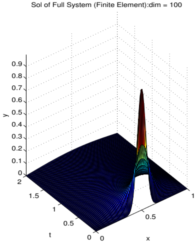
- Approximation for the Nonlinear term :

$$\begin{aligned} \tilde{\mathbf{N}}(\tilde{\mathbf{y}}(t)) &= \underbrace{\mathbf{U}_k^T}_{k\times N} \underbrace{\int_\Omega \Psi'(x)^T s(\Psi(x)\mathbf{U}_k\tilde{\mathbf{y}}(t))dx}_{N\times 1} \\ &\simeq \underbrace{\mathbf{U}_k^T \left(\int_\Omega \Psi'(x)^T Q(x)dx\right)(Q(\mathbf{z}))^{-1}}_{\underbrace{=\mathbf{C}}_{k\times m\text{: precomputed}}} \underbrace{\underbrace{s(\mathbf{z},t)}_{=\hat{\mathbf{N}}(\tilde{\mathbf{y}}(t))}}_{m\times 1} \end{aligned}$$

Plots of Numerical Solutions from EIM with POD basis







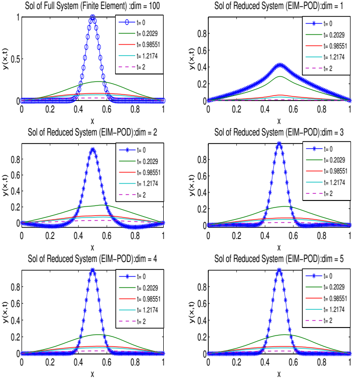
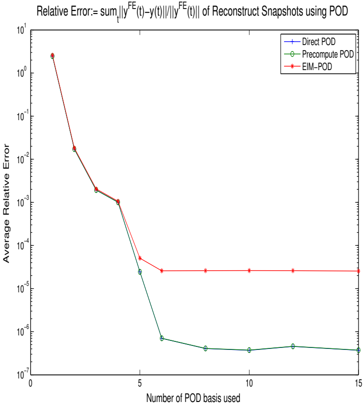


Figure: EIM-POD

ACCURACY vs. COMPLEXITY



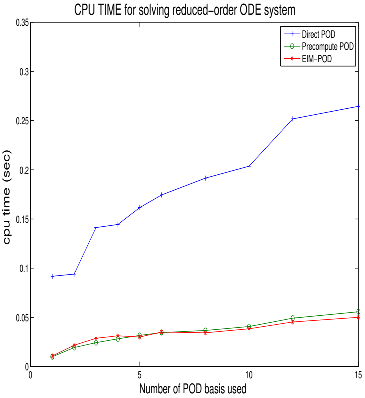


Figure: LEFT: Error $E_{avg} = \frac{1}{n_t}\sum_{i=1}^{n_t} \frac{\|\tilde{y}_h(\cdot,t_i)-y_h(\cdot,t_i)\|_X}{\|y_h(\cdot,t_i)\|_X}$ .RIGHT: CPU time (sec)