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ESTIMATION OF THE ERROR IN THE REDUCED BASIS METHOD SOLUTION OF DIFFERENTIAL ALGEBRAIC EQUATION SYSTEMS*

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Abstract. In this paper the reduced basis method is described as it applies to the solution of the differential equations with algebraic constraints. The approximate solution is defined in terms of a sequence of reduced basis curves, and error bounds are derived. These are used to produce order estimates for direct sum Taylor, Lagrange, and Hermite subspaces.

Key words. differential algebraic equation systems, projection methods

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1. Introduction. The reduced basis method has been used to generate solution curve segments of large parametrized systems. The scheme approximates a solution segment by a reduced curve lying in a lower-dimensional manifold which is close to the true solution curve. This results in solving a smaller system. The basic idea of finding such an approximation is by projection and was proposed in [7] for the analysis of trusses. Although there is some overhead required to get to the reduced system, the time saved in solving the reduced system hopefully will compensate for it under proper implemention.

The method has been applied to structural analysis [2], [8], [9], steady fluid flow [10] and recently to the initial value problem of ordinary differential equations [12]. Mathematical analyses of the method in various formulations were presented in [3]-[6], [11], [12]. The results of the analyses demonstrate the potential of the reduced basis method.

In this paper we formulate the reduced basis method for the numerical solutions of initial value problems for differential algebraic equation systems. In § 2 we give a general scheme for approximating the solution curve by a sequence of reduced basis curves. We then derive bounds for the local and global errors, respectively, in §§ 3 and 4. Finally, in § 5 we examine the error estimates for the direct sum Taylor, Lagrange, and Hermite subspaces.

2. Reduced basis approximation and reduced system. In this paper we will deal with the following initial value differential algebraic equation system:

$$\dot{y}_1 = F_1(y_1, y_2), \tag{1}$$

$$F_2(y_1, y_2) = c_0, \tag{2}$$

$$(y_1(a), y_2(a)) = (\eta_1, \eta_2), \tag{3}$$

where

$$y_1 \in \mathfrak{R}^{n_1}, \quad y_2 \in \mathfrak{R}^{n_2} \quad \text{and} \quad F_1 : \mathfrak{R}^{n_1} \times \mathfrak{R}^{n_2} \to \mathfrak{R}^{n_1},$$
$$F_2 : \mathfrak{R}^{n_1} \times \mathfrak{R}^{n_2} \to \mathfrak{R}^{n_2} \quad \text{are} \quad C^r-\text{maps}, \qquad r \geqq 2.$$

The regularity set $R(F_2) = \{(y_1, y_2) \in \mathfrak{R}^{n_1} \times \mathfrak{R}^{n_2} \mid D_{y_2}F_2(y_1, y_2) \text{ is nonsingular}\}$ is open in $\mathfrak{R}^{n_1} \times \mathfrak{R}^{n_2}$. Let $M_0(F_2) = \{(y_1, y_2) \in R(F_2) \mid F_2(y_1, y_2) = c_0\}$ be nonempty. $M_0(F_2)$ is a manifold of $\mathfrak{R}^{n_1} \times \mathfrak{R}^{n_2}$ with dimension n_1 and of class C^r. (See [14].)

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In our setting, the solution of (1) and (2) through (η_1, η_2) exists and is unique if $(\eta_1, \eta_2) \in M_0(F_2)$. Moreover, it is of class C^{r-1} and has no endpoint in $M_0(F_2)$. (See [14].) Henceforth, we will assume that the initial point is on $M_0(F_2)$.

Let $(y_1, y_2): J \subset \mathfrak{R}^1 \to \mathfrak{R}^{n_1} \times \mathfrak{R}^{n_2}$ be the maximally extended C^{r-1} solution curve of (1), (2), and (3), and we are interested in solving for (y_1, y_2) on some interval $[a, b] \subset J$. To determine the approximate solution, we consider the so-called translated problems which define a sequence of problems having the same form as (1) and (2).

Let a partition of $[a, b]$, $a = t_0 < t_1 < \cdots < t_N = b$, be suitably made as the reduced basis algorithm proceeds. And let $h = \max_{1 \leqq j \leqq N} h_j$, where $h_j = t_j - t_{j-1}$.

For $j = 0, \cdots, N-1$, let $(y_{1_j}(t), y_{2_j}(t))$ solve the jth translated initial valued differential algebraic problem

$$\dot{y}_{1_j} = F_1(y_{1_j}, y_{2_j}), \tag{4}$$

$$F_2(y_{1_j}, y_{2_j}) = c_j, \tag{5}$$

$$(y_{1_j}(t_j), y_{2_j}(t_j)) = (\eta_{1_j}, \eta_{2_j}), \tag{6}$$

where

$$c_j = F_2(\eta_{1_j}, \eta_{2_j})$$

and

$$(\eta_{1_j}, \eta_{2_j}) = \begin{cases} (\eta_1, \eta_2), & j = 0, \\ (y_{1R,j-1}(t_j), y_{2R,j-1}(t_j)), & j = 1, \cdots, N-1. \end{cases}$$

Here $(y_{1R,j-1}, y_{2R,j-1}): [t_{j-1}, t_j] \to \mathfrak{R}^{n_1} \times \mathfrak{R}^{n_2}$ denotes the reduced basis approximation of $(y_{1,j-1}, y_{2,j-1})$ on $[t_{j-1}, t_j]$.

The reduced basis approximation of $y(t) \equiv (y_1(t), y_2(t))^T$ is defined as the composite of reduced basis curves

$$y_R(t) \equiv (y_{1R}(t), y_{2R}(t))^T = \begin{cases} (\eta_1, \eta_2)^T, & t = t_0, \\ (y_{1R,j}(t), y_{2R,j}(t))^T, & t_j \leqq t \leqq t_{j+1}, \quad j = 0, \cdots, N-1. \end{cases}$$

For $j = 1, \cdots, N-1$, the solution of the translated problems exists and is unique if $(\eta_{1_j}, \eta_{2_j}) \in M_j(F_2) = \{(y_1, y_2) \in R(F_2) \mid F_2(y_1, y_2) = c_j\}$. Note that $(\eta_{10}, \eta_{20}) = (\eta_1, \eta_2) \in M_0(F_2)$. Assume that $(\eta_{1k}, \eta_{2k}) \in M_k(F_2)$, $k = 1, \cdots, j-1$. Since

$$(y_{1R,j-1}(t_{j-1}), y_{2R,j-1}(t_{j-1})) = (y_{1,j-1}(t_{j-1}), y_{2,j-1}(t_{j-1})),$$

$(y_{1R,j-1}, y_{2R,j-1})$ and $(y_{1,j-1}, y_{2,j-1})$ are both continuous, and

$$(y_{1,j-1}(t), y_{2,j-1}(t)) \in M_{j-1}(F_2), \quad t_j \leqq t \leqq b.$$

Hence $(\eta_{1j}, \eta_{2j}) = (y_{1R,j-1}(t_j), y_{2R,j-1}(t_j))$ will be in $M_j(F_2)$ if $t_j - t_{j-1}$ is small enough.

Henceforth we will assume that the partition can be made in finitely many steps such that $(\eta_{1j}, \eta_{2j}) \in M_j(F_2)$ and $[t_j, b]$ is in the domain of the definition of (y_{1j}, y_{2j}), for $j = 1, \cdots, N-1$.

Now we can give the definition of the reduced basis curves. For each translated problem, the reduced basis curve $(y_{1R,j}, y_{2R,j})$ is defined to approximate the solution (y_{1j}, y_{2j}) in a lower-dimensional translated subspace (a subspace which is translated so that the origin is at (η_{1j}, η_{2j})). We denote it by S_j^t, where

$$S_j^t = S_j + (\eta_{1j}, \eta_{2j})$$

for some subspace $S_j \subset \mathfrak{R}^{n_1} \times \mathfrak{R}^{n_2}$. In order for the reduced basis problem to have the same structure as the translated problem, we will consider S_j to have the special form $S_j = S_{1j} \oplus S_{2j}$, where

$$S_{1j} = \operatorname{span} \langle (y_{1j}^1, 0)^T, \cdots, (y_{1j}^{m_{1j}}, 0)^T \rangle,$$
$$S_{2j} = \operatorname{span} \langle (0, y_{2j}^1)^T, \cdots, (0, y_{2j}^{m_{2j}})^T \rangle,$$

for some linearly independent sets $\{y_{1j}^1, \cdots, y_{1j}^{m_{1j}}\} \subset \mathfrak{R}^{n_1}$ and $\{y_{2j}^1, \cdots, y_{2j}^{m_{2j}}\} \subset \mathfrak{R}^{n_2}$. Then by a technique similar to that of Galerkin approximation, the reduced-basis approximation

$$(y_{1R,j}, y_{2R,j}) : [t_j, t_{j+1}] \to S_j^t$$

is defined to satisfy the following reduced basis problem:

$$(y_{1j}^k)^T[\dot{y}_{1R,j}(t) - F_1(y_{1R,j}(t), y_{2R,j}(t))] = 0, \qquad k = 1, \cdots, m_{1j}, \tag{7}$$

$$(y_{2j}^k)^T(F_2(y_{1R,j}(t), y_{2R,j}(t)) - c_j) = 0, \qquad k = 1, \cdots, m_{2j}, \tag{8}$$

$$(y_{1R,j}(t_j), y_{2R,j}(t_j)) = (\eta_{1j}, \eta_{2j}). \tag{9}$$

Let $Y_{1j} = [y_{1j}^1 \cdots y_{1j}^{m_{1j}}]$ and $Y_{2j} = [y_{2j}^1 \cdots y_{2j}^{m_{2j}}]$. Since $(y_{1R,j}(t), y_{2R,j}(t)) \in S_j^t$, we can write them as $y_{1R,j}(t) = Y_{1j}z_{1j}(t) + \eta_{1j}$ and $y_{2R,j}(t) = Y_{2j}z_{2j}(t) + \eta_{2j}$, for some $z_{1j}(t) \in \mathfrak{R}^{m_{1j}}$ and $z_{2j}(t) \in \mathfrak{R}^{m_{2j}}$. Then (7)–(9) can be written as

$$Y_{1j}^T Y_{1j} \dot{z}_{1j}(y) = Y_{1j}^T F_1(Y_{1j}z_{1j}(t) + \eta_{1j}, Y_{2j}z_{2j}(t) + \eta_{2j}),$$
$$Y_{2j}^T(F_2(Y_{1j}z_{1j}(t) + \eta_{1j}, Y_{2j}z_{2j}(t) + \eta_{2j}) - c_j) = 0,$$
$$(z_{1j}(t_j), z_{2j}(t_j)) = 0.$$

Or equivalently,

$$\dot{z}_{1j}(t) = G_{1j}(z_{1j}(t), z_{2j}(t)), \tag{10}$$

$$G_{2j}(z_{1j}(t), z_{2j}(t)) = d_j, \tag{11}$$

$$(z_{1j}(t_j), z_{2j}(t_j)) = 0, \tag{12}$$

where

$$G_{1j}(z_{1j}, z_{2j}) = (Y_{1j}^T Y_{1j})^{-1} Y_{1j}^T F_1(Y_{1j}z_{1j} + \eta_{1j}, Y_{2j}z_{2j} + \eta_{2j}),$$
$$G_{2j}(z_{2j}, z_{2j}) = Y_{2j}^T F_2(Y_{1j}z_{1j} + \eta_{1j}, Y_{2j}z_{2j} + \eta_{2j}), \qquad d_j = Y_{2j}^T c_j.$$

We shall call (10) and (11) the jth reduced system. Let

$M_j(G_{2j}) = \{(z_{1j}, z_{2j}) \in \mathfrak{R}^{m_{1j}} \times \mathfrak{R}^{m_{2j}} \mid G_{2j}(z_{1j}, z_{2j}) = d_j$ and $D_{z_2}G_{2j}(z_{1j}, z_{2j})$ is nonsingular$\}$.

Note that G_{1j} and G_{2j} are again C^r-maps on $\mathfrak{R}^{m_{1j}} \times \mathfrak{R}^{m_{2j}}$. Moreover, if $Y_{2j}^T D_{y_2} F_2(\eta_{1j}, \eta_{2j}) Y_{2j}$ is nonsingular, then $0 \in M_j(G_{2j})$. It follows from [14] that the jth reduced system has a unique solution $(z_{1j}(t), z_{2j}(t))$ through zero. Henceforth we shall assume that $Y_{2j}^T D_{y_2} F_2(\eta_{1j}, \eta_{2j}) Y_{2j}$ is nonsingular for the chosen Y_{2j}, and $[t_j, t_{j+1}]$ is in the domain of definition of $(z_{1j}(t), z_{2j}(t))$, for $j = 0, \cdots, N-1$.

3. Local error estimate. In order to obtain an estimate on the error of the reduced basis approximation, let us first examine the local errors

$$d_j \equiv (d_{1j}, d_{2j})^T = (y_{1j} - y_{1R,j}, y_{2j} - y_{2R,j})^T \quad \text{for } t \in [t_j, t_{j+1}], \quad j = 0, \cdots, N-1.$$

We begin by relating the local error in y_{2j} to its projection error. Then the theorem for the local error estimate can be developed after a similar result for y_{1j} is established.

Let

$$Y_j = \begin{pmatrix} Y_{1j} & 0 \\ 0 & Y_{2j} \end{pmatrix},$$

$$P_j = Y_j(Y_j^T Y_j)^{-1} Y_j^T = \begin{pmatrix} P_{1j} & 0 \\ 0 & P_{2j} \end{pmatrix},$$

where

$$P_{1j} = Y_{1j}(Y_{1j}^T Y_{1j})^{-1} Y_{1j} \quad \text{and} \quad P_{2j} = Y_{2j}(Y_{2j}^T Y_{2j})^{-1} Y_{2j}.$$

Note that P_j, P_{1j}, and P_{2j} are, respectively, projectors onto S_j and the column spaces of Y_{1j} and Y_{2j}. Without loss of generality, we will assume henceforth for convenience that the columns of Y_j are orthonormal. Let

$$y_{ij}(t) = x_{ij}(t) + \eta_{ij} \quad \text{and} \quad y_{iR,j}(t) = x_{iR,j}(t) + \eta_{ij} \quad \text{for } i = 1, 2, \quad j = 0, \cdots, N-1.$$

Let $\|\cdot\|$ be the Euclidean norm. Then the first part of the result is given by the following lemma.

LEMMA 1. *If $t_{j+1} - t_j$ is small enough, then there exist constants k_{1j} and k_{2j} such that*

$$\|x_{2j}(t) - x_{2R,j}(t)\| \leqq k_{1j}\|P_{2j}x_{2,j}(t) - x_{2,j}(t)\| + k_{2j}\|x_{1j}(t) - x_{1R,j}(t)\|$$

for $t \in [t_j, t_{j+1}]$.

Proof. Since F_2 is r times continuously differentiable, $F_2(\eta_{1j}, \eta_{2j}) = c_j$, and $D_{y_2}F_2(\eta_{1j}, \eta_{2j})$ is nonsingular (by assumption), we know by the implicit function theorem that there exists an r times continuously differentiable function

$$f_j\colon B_{\delta_{1j}} = \{\lambda \in \Re^{n_1} \mid \|\lambda\| \leqq \delta_{1j}\} \to \Re^{n_2} \tag{13}$$

such that

$$f_j(0) = 0,$$

$$F_2(\lambda + \eta_{1j}, f_j(\lambda) + \eta_{2j}) = c_j \quad \forall \lambda \in B_{\delta_{1j}}, \tag{14}$$

$\{(\lambda + \eta_{1j}, f_j(\lambda) + \eta_{2j}) \mid \lambda \in B_{\delta_{1j}}\} \subset M_j(F_2)$, by the continuity of $D_{y_2}F_2$ at (η_{1j}, η_{2j}).

Similarly, since $D_{z_2}G_2(0, 0)$ is nonsingular (by assumption), $G_{2j}(0, 0) = d_j$, and G_{2j} is a C^r-map, there exists an r times continuously differentiable function

$$\tilde{f}_{Rj}\colon \tilde{B}_{\delta_{Rj}} \to \Re^{m_{2j}}, \quad \text{where } \tilde{B}_{\delta_{Rj}} = \{\xi \in \Re^{m_{1j}} \mid \|\xi\| \leqq \delta_{Rj}\} \tag{15}$$

such that

$$\begin{gathered} \tilde{f}_{Rj}(0) = 0, \\ G_{2j}(\xi, \tilde{f}_{Rj}(\xi)) = d_j \quad \forall \xi \in \tilde{B}_{\delta_{Rj}}, \\ \{(\xi, \tilde{f}_{Rj}(\xi)) \mid \xi \in \tilde{B}_{\delta_{Rj}}\} \subset M_j(G_{2j}). \end{gathered} \tag{16}$$

Let

$$\lambda_R = Y_{1j}\xi, \quad f_{Rj}(\lambda_R) = Y_{2j}\tilde{f}_{Rj}(Y_{1j}^T\lambda_R), \quad \tilde{S}_1 = \text{span}\,\langle y_{1j}^1, \cdot, y_{1j}^{m_{1j}}\rangle \subset \Re^{n_1}. \tag{17}$$

Then by (16), $f_{Rj}(\lambda_R)$ satisfies

$$Y_{2j}^T F_2(\lambda_R + \eta_{1j}, f_{Rj}(\lambda_R) + \eta_{2j}) = Y_{2j}^T c_j \quad \text{for any } \lambda_R \in B_{\delta_{Rj}} = \{\lambda_R \in \tilde{S}_1 \mid \|\lambda_R\| \leqq \delta_{Rj}\}. \tag{18}$$

It follows from Lemma 4 (Appendix A) that there is a ball

$$B_{\delta_j} = \{\lambda \in \Re^{n_1} \mid \|\lambda\| \leqq \delta_j\} \quad \text{for some } 0 < \delta_j \leqq \min(\delta_{1j}, \delta_{Rj}),$$

such that for any $\lambda \in B_{\delta_j}$ and $\lambda_R \in B_{\delta_j} \cap \tilde{S}_1$

$$\|P_{2j}f_j(\lambda) - f_{Rj}(\lambda_R)\| \leqq \alpha_{1j}\alpha_{3j}\|P_{2j}f_j(\lambda) - f_j(\lambda)\| + \alpha_{2j}\|\lambda - \lambda_R\| \tag{19}$$

for some constants α_{1j}, α_{2j}, and α_{3j}. Since $y_{1j}(t_j)=y_{1R,j}(t_j)=\eta_{1j}$ and $y_{1j}(t)$ and $y_{1R,j}(t)$ are both continuous, there exists an interval $[t_j, t_j+\tau_j]$ such that

$$\|y_{1j}(t)-\eta_{1j}\|,\ \|y_{1R,j}(t)-\eta_{1j}\|\leqq\delta_j\quad\forall t\in[t_j, t_j+\tau_j].$$

Moreover, since f_j and $f_{R,j}$ are uniquely defined functions, by (14) and (18)

$$y_{2j}(t)-\eta_{2j}\equiv f_j(y_{1j}(t)-\eta_{1j})\quad\forall t\in[t_j, t_j+\tau_j],$$
$$y_{2R,j}(t)-\eta_{2j}\equiv f_{R,j}(y_{1R,j}(t)-\eta_{1j})\quad\forall t\in[t_j, t_j+\tau_j].$$

Hence, by (19),

$$\begin{aligned}&\|P_{2j}(y_{2j}(t)-\eta_{2j})-(y_{2R,j}(t)-\eta_{2j})\|\\&\quad\leqq\alpha_{1j}\alpha_{3j}\|P_{2j}(y_{2j}(t)-\eta_{2j})-(y_{2j}(t)-\eta_{2j})\|+\alpha_{2j}\|y_{1j}(t)-y_{1R,j}(t)\|,\end{aligned}$$

i.e.,

$$\|P_{2j}x_{2j}(t)-x_{2R,j}(t)\|\leqq\alpha_{1j}\alpha_{3j}\|P_{2j}x_{2j}(t)-x_{2j}(t)\|+\alpha_{2j}\|x_{1j}(t)-x_{1R,j}(t)\|$$

for any $t\in[t_j, t_j+\tau_j]$. Let

$$k_{1j}=1+\alpha_{1j}\alpha_{3j}\quad\text{and}\quad k_{2j}=\alpha_{2j}. \tag{20}$$

Therefore, if $t_{j+1}-t_j\leqq\tau_j$, then for all $t\in[t_j, t_{j+1}]$ we have

$$\begin{aligned}\|x_{2j}(t)-x_{2R,j}(t)\|&\leqq\|x_{2j}(t)-P_{2j}x_{2j}(t)\|+\|P_{2j}x_{2j}(t)-x_{2R,j}(t)\|\\&\leqq k_{1j}\|P_{2j}x_{2j}(t)-x_{2j}(t)\|+k_{2j}\|x_{1j}(t)-x_{1R,j}(t)\|.\end{aligned}$$ □

Note that F_1 is a C^r-map on $\Re^{n_1}\times\Re^{n_2}$, so that there exists a constant M_j such that

$$\|(D_{y_1}F_1(y_1, y_2),\ D_{y_2}F_1(y_1, y_2))\|\leqq M_j\quad\forall(y_1-\eta_{1j}, y_2-\eta_{2j})\in B_{\delta_j}\times B_{\gamma_j}. \tag{21}$$

Here B_{γ_j} is the one defined in (30). Let

$$k_j=\sqrt{2}\max(1+k_{2j}, k_{1j}). \tag{22}$$

Then we have the following result.

THEOREM 1. *If $t_{j+1}-t_j$ is small enough, then*

$$\|d_j(t)\|\leqq k_j\exp[(1+k_{2j})M_j(t-t_j)]\sup_{s\in[t_j,t]}\|y_j(s)-w_j(s)\|$$

for any $w_j(t)\in S_j^t$.

Proof. Let τ_j, k_{1j}, k_{2j} be the ones defined in Lemma 1 and $t_{j+1}-t_j\leqq\tau_j$. Note that (10) is equivalent to

$$\frac{dy_{1R,j}(t)}{dt}=P_{1j}F_1(y_{1R,j}(t), y_{2R,j}(t)).$$

Then for any $t\in[t_j, t_{j+1}]$,

$$\begin{aligned}\|x_{1j}(t)-x_{1R,j}(t)\|&=\|y_{1j}(t)-y_{1R,j}(t)\|\\&=\left\|\int_{t_j}^t[F_1(y_{1j}(r), y_{2j}(r))-P_{1j}F_1(y_{1R,j}(r), y_{2R,j}(r))]\,dr\right\|\\&\leqq\left\|\int_{t_j}^t[F_1(y_{1j}(r), y_{2j}(r))-P_{1j}F_1(y_{1j}(r), y_{2j}(r))]\,dr\right\|\\&\quad+\left\|\int_{t_j}^t[P_{1j}F_1(y_{1j}(r), y_{2j}(r))-P_{1j}F_1(y_{1R,j}(r), y_{2R,j}(r))]\,dr\right\|\\&\leqq\|(y_{1j}(t)-\eta_{1j})-P_{1j}(y_{1j}(t)-\eta_{1j})\|+\|P_{1j}\|\cdot\int_{t_j}^t M_j\|y_j(r)-y_{R,j}(r)\|\,dr,\end{aligned}$$

where $y_{R,j}(r)=(y_{1R,j}(r), y_{2R,j}(r))^T$.

Let $x_j(t) \equiv (x_{1j}(t), x_{2j}(t))^T$ and $x_{R,j}(t) \equiv (x_{1R,j}(t), x_{2R,j}(t))^T$. Then

$$\|x_{1j}(t) - x_{1R,j}(t)\| \leqq \|x_{1j}(t) - P_{1j}x_{1j}(t)\| + M_j \int_{t_j}^{t} \|x_j(r) - x_{R,j}(r)\|\, dr$$

and

$$\begin{aligned}\|x_j(t) - x_{R,j}(t)\| &\leqq \|x_{1j}(t) - x_{1R,j}(t)\| + \|x_{2j}(t) - x_{2R,j}(t)\| \\ &\leqq (1 + k_{2j})\|x_{1j}(t) - x_{1R,j}(t)\| + k_{1j}\|P_{2j}x_{2j}(t) - x_{2j}(t)\| \\ &\leqq (1 + k_{2j})\|x_{1j}(t) - P_{1j}x_{1j}(t)\| + k_{1j}\|P_{2j}x_{2j}(t) - x_{2j}(t)\| \\ &\quad + (1 + k_{2j})M_j \int_{t_j}^{t} \|x_j(r) - x_{R,j}(r)\|\, dr \\ &\leqq k_j\|x_j(t) - P_jx_j(t)\| + (1 + k_{2j})M_j \int_{t_j}^{t} \|x_j(r) - x_{R,j}(r)\|\, dr.\end{aligned}$$

Hence by Gronwall's inequality, we have

$$\begin{aligned}\|x_j(t) - x_{R,j}(t)\| &\leqq k_j\|x_j(t) - P_jx_j(t)\| \\ &\quad + \int_{t_j}^{t} k_j\|x_j(r) - P_jx_j(r)\|(1 + k_{2j})M_j \exp\left[\int_r^t (1 + k_{2j})M_j\, ds\right] dr \\ &\leqq k_j \exp\left[(1 + k_{2j})M_j(t - t_j)\right] \sup_{s \in [t_j, t]} \|x_j(s) - P_jx_j(s)\|.\end{aligned}$$

Finally, since P_j is an orthogonal projector onto S_j, $\|x_j - P_jx_j\| \leqq \|x_j - u\|$ for all $u \in S_j$, so we have completed the proof. □

4. Global error estimate. Now we are ready to estimate the global error of the reduced basis approximation,

$$e(t) = (y_1(t) - y_{1R}(t), y_2(t) - y_{2R}(t))^T, \qquad t \in [a, b],$$

in terms of the size of the local errors $d_j(t)$.

First note that by (4) and (5), $(y_{1j}(t), y_{2j}(t))$ satisfies the following system for $t \in [t_j, b]$:

$$\begin{pmatrix}\dot{y}_{1j}(t) \\ \dot{y}_{2j}(t)\end{pmatrix} = V(y_{1j}, y_{2j}) \quad \text{on } M_j(F_2),$$

where

$$V(y_{1j}, y_{2j}) = \begin{pmatrix} I & 0 \\ D_{y_1}F_2(y_{1j}, y_{2j}) & D_{y_2}F_2(y_{1j}, y_{2j})\end{pmatrix}^{-1} \begin{pmatrix} F_1(y_{1j}, y_{2j}) \\ 0\end{pmatrix} \tag{23}$$

and

$$(y_{1j}(t_j), y_{2j}(t_j)) = (\eta_{1j}, \eta_{2j}).$$

Moreover, the vector fields $V(y_{1j}, y_{2j})$ are of class C^{r-1} on $M_j(F_2)$ for $r \geqq 2$. Hence we have the following results from [12, p. 1281].

THEOREM 2. *Suppose $V(y_1, y_2)$ is Lipschitz continuous on $R(F_2)$ with constant L. Then if $\|d_j(t)\| \leqq C$ for $t \in [t_j, t_{j+1}]$, $j = 0, \cdots, N-1$, and if $Nh \leqq B$, we have*

$$\|e(t)\| \leqq C\left(\frac{e^{LB} - 1}{Lh}\right).$$

Before we can apply the theorem for the error estimate, we must show that $\|d_j\|$ can be bounded independently of j. By Theorem 1, this is proved if we can show that k_j, M_j, and $\sup_{s\in[t_j,t]}\|y_j(s)-(\eta_{1j},\eta_{2j})^T\|$ (by choosing $w_j(t)=(\eta_{1j},\eta_{2j})^T$) are bounded independently of j. Let $x_j(t)\equiv(x_{1j}(t),x_{2j}(t))^T=y_j(t)-(\eta_{1j},\eta_{2j})^T$. Then $x_j(t)$ satisfies

$$\dot{x}_j(t)=V(x_{1j}(t)+\eta_{1j},x_{2j}(t)+\eta_{2j})\quad\text{on } M_j^t(F_2),$$
$$x_j(t_j)=0,$$

where

$$M_j^t(F_2)=\{x=(x_1,x_2)^T\,|\,(x_1,x_2)^T+(\eta_{1j},\eta_{2j})^T\in M_j(F_2)\}.$$

Therefore

$$\begin{aligned}\|x_j(t)\| &\leqq \int_{t_j}^t \|V(x_{1j}(s)+\eta_{1j},x_{2j}(s)+\eta_{2j})\|\,ds\\ &\leqq \int_{t_j}^t (\|V(x_{1j}(s)+\eta_{1j},x_{2j}(s)+\eta_{2j})-V(\eta_1,\eta_2)\|)\,ds+\int_{t_j}^t \|V(\eta_1,\eta_2)\|\,ds\\ &\leqq \int_{t_j}^t L\|x_j(s)\|\,ds+Lh\|(\eta_{1j},\eta_{2j})\|+k,\end{aligned}$$

where $k=h(L\|(\eta_1,\eta_2)\|+\|V(\eta_1,\eta_2)\|)$. By Gronwall's inequality, we have

$$\|x_j(t)\|\leqq(k+Lh\|(\eta_{1j},\eta_{2j})\|)e^{Lh}.$$

So it is sufficient to show the boundedness of (η_{1j},η_{2j}) to get that of $\sup_{s\in[t_j,t]}\|y_j(s)-(\eta_{1j},\eta_{2j})^T\|$. To show the boundedness of (η_{1j},η_{2j}), we need the following result.

LEMMA 2. *If $Y_{2j}^T D_{y_2}F_2(\eta_1,\eta_2)Y_{2j}$ is bounded away from singularity for all j, then the norms of $Y_{2j}(Y_{2j}^T D_{y_2}F_2(\eta_1,\eta_2)Y_{2j})^{-1}Y_{2j}^T$ are bounded for all j.*

Proof. Let (λ_j,x_j) be an eigenvalue-vector pair of $Y_{2j}(Y_{2j}^T D_{y_2}F_2(\eta_1,\eta_2)Y_{2j})^{-1}Y_{2j}^T$. Since the columns of Y_{2j} are orthonormal (by assumption), it is not hard to see that $(1/\lambda_j, Y_{2j}^T x_j)$ is an eigenvalue-vector pair of $Y_{2j}^T D_{y_2}F_2(\eta_1,\eta_2)Y_{2j}$. Since $Y_{2j}^T D_{y_2}F_2(\eta_1,\eta_2)Y_{2j}$ are bounded away from singularity, there is some constant, say $\beta>0$, such that each eigenvalue of $Y_{2j}^T D_{y_2}F_2(\eta_1,\eta_2)Y_{2j}$ is bounded below by β in modulus. And, since we can find a consistent norm $N(\cdot)$ such that

$$N(Y_{2j}(Y_{2j}^T D_{y_2}F_2(\eta_1,\eta_2)Y_{2j})^{-1}Y_{2j}^T)\leqq\rho(Y_{2j}(Y_{2j}^T D_{y_2}F_2(\eta_1,\eta_2)Y_{2j})^{-1}Y_{2j}^T)+1,$$

by norm equivalence, there is a constant $\kappa>0$ such that

$$\begin{aligned}\|(Y_{2j}(Y_{2j}^T D_{y_2}F_2(\eta_1,\eta_2)Y_{2j})^{-1}Y_{2j}^T)\| &\leqq \kappa N(Y_{2j}(Y_{2j}^T D_{y_2}F_2(\eta_1,\eta_2)Y_{2j})^{-1}Y_{2j}^T)\\ &\leqq \kappa(\rho(Y_{2j}(Y_{2j}^T D_{y_2}F_2(\eta_1,\eta_2)Y_{2j})^{-1}Y_{2j}^T)+1)\\ &\leqq \kappa(1/\beta+1).\end{aligned}$$ □

Before we demonstrate the boundedness result of (η_{1j},η_{2j}), note that by (7) and (8) $(y_{1R,j},y_{2R,j})$ satisfies the following system for $t\in[t_j,t_{j+1}]$:

$$\begin{pmatrix}\dot{y}_{1R,j}(t)\\ \dot{y}_{2R,j}(t)\end{pmatrix}=V_{R,j}(y_{1R,j},y_{2R,j})\quad\text{on } M_{R,j},$$
$$(y_{1R,j}(t_j),y_{2R,j}(t_j))=(\eta_{1j},\eta_{2j}),$$

where

$$V_{R,j}(y_{1R,j}, y_{2R,j})$$
$$=\begin{pmatrix} P_{1j}F_1(y_{1R,j}, y_{2R,j}) \\ -Y_{2j}(Y_{2j}^TD_{y_2}F_2(y_{1R,j}, y_{2R,j})Y_{2j})^{-1}Y_{2j}^TD_{y_1}F_2(y_{1R,j}, y_{2R,j})P_{1j}F_1(y_{1R,j}, y_{2R,j})\end{pmatrix},$$

$$M_{R,j}=\{(y_1, y_2)\in\Re^{n_1}\times\Re^{n_2} \mid y_1=Y_{1j}z_1+\eta_{1j}, y_2=Y_{2j}z_2+\eta_{2j} \text{ for some } (z_1, z_2)\in M_j(G_{2j})\}.$$

Moreover, the vector field $V_{R,j}(y_{1R,j}, y_{2R,j})$ is of class C^{r-1} on $M_{R,j}$ for $r\geqq 2$. Let

$$R(Y_{2j}^TF_2Y_{2j})=\{(y_1, y_2)\in\Re^{n_1}\times\Re^{n_2} \mid Y_{2j}^TF_2(y_1, y_2)Y_{2j} \text{ is nonsingular}\}.$$

Then we have the following result for (η_{1j}, η_{2j}).

THEOREM 3. *Suppose $V_{R,j}(y_1, y_2)$ are Lipschitz continuous on $R(Y_{2j}^TF_2Y_{2j})$ with bounded Lipschitz constants. If $Y_{2j}^TD_{y_2}F_2(\eta_1, \eta_2)Y_{2j}$ are bounded away from singularity for all j, then the norms of (η_{1j}, η_{2j}) are bounded for all j.*

Proof. Consider the following initial value problems:

$$\begin{pmatrix}\dot\rho_{1j}\\ \dot\rho_{2j}\end{pmatrix}=V_{R,j}(\eta_1, \eta_2), \qquad t_j\leqq t\leqq b,$$

$$(\rho_{1j}(t_j), \rho_{2j}(t_j))=\begin{cases}(\eta_1, \eta_2), & j=0,\\ (\rho_{1,j-1}(t_j), \rho_{2,j-1}(t_j)), & j=1, 2, \cdots.\end{cases}$$

Since $Y_{2j}^TD_{y_2}F_2(\eta_1, \eta_2)Y_{2j}$ are bounded away from singularity for all j and P_{1j} are orthogonal, so it is clear from Lemma 2 that $\{(\rho_{1j}(t), \rho_{2j}(t)), j=0, 1, 2, \cdots, t\in[t_j, b]\}$ are uniformly bounded. That is, there is some constant K_1 such that $\|(\rho_{1j}(t), \rho_{2j}(t))\|\leqq K_1$, for all $t\in[t_j, b]$. Let

$$(x_{1j}(t), x_{2j}(t))=(y_{1R,j}(t), y_{1R,j}(t))-(\rho_{1j}(t), \rho_{2j}(t)).$$

We will complete the proof if we can show $\{(x_{1j}(t), x_{2j}(t))\}$ are also uniformly bounded. Note that $(x_{1j}(t), x_{2j}(t))$ satisfy the following system:

$$\begin{pmatrix}\dot x_{1j}\\ \dot x_{2j}\end{pmatrix}=V_{R,j}(y_{1R,j}, y_{2R,j})-V_{R,j}(\eta_1, \eta_2).$$

Since $V_{R,j}(y_1, y_2)$ are Lipschitz continuous on $R(Y_{2j}^TF_2Y_{2j})$ with bounded Lipschitz constants, say bounded by K_2, so we have

$$\begin{aligned}\|(\dot x_{1j}, \dot x_{2j})\| &\leqq K_2(\|(y_{1R,j}, y_{2R,j})\|+\|(\eta_1, \eta_2)\|)\\ &\leqq K_2(\|(x_{1j}(t), x_{2j}(t))+(\rho_{1j}(t), \rho_{2j}(t))\|+\|(\eta_1, \eta_2)\|).\end{aligned}$$

Let $\alpha\equiv\|(\eta_1, \eta_2)\|$. Then for $t_j\leqq t\leqq t_{j+1}$, we have

$$\begin{aligned}\|(x_{1j}(t), x_{2j}(t))\| &\leqq \|(x_{1j}(t_j), x_{2j}(t_j))\|+\int_{t_j}^t [K_2\|(x_{1j}(s), x_{2j}(s))+(\rho_{1j}(s), \rho_{2j}(s))\|+\alpha]\,ds\\ &\leqq \|(x_{1j}(t_j), x_{2j}(t_j))\|+K_3h+K_2\int_{t_j}^t \|(x_{1j}(s), x_{2j}(s))\|\,ds,\end{aligned}$$

where $K_3=K_1K_2+\alpha$.

It follows from the proof of [12, pp. 1282–1283] that $\|(x_{1j}(t), x_{2j}(t))\|$ are uniformly bounded. Hence $\{(\eta_{1j}, \eta_{2j}), j=0, 1, 2, \cdots\}$ are bounded. □

To show the boundedness results for k_j and M_j, by (20)–(22) we need to show that α_{1j}, α_{2j}, α_{3j}, and M_j are bounded for all j. It is shown in Appendix B that this is the case if $Y_{2j}^TD_{y_2}F_2(\eta_{1j}, \eta_{2j})Y_{2j}$ and $D_{y_2}F_2(\eta_{1j}, \eta_{2j})$ are bounded away from singularity.

5. Some reduced subspaces. From Theorem 2 we know that the order of the errors is one unit less than the smallest order of the local errors. Thus we can give the order estimate for the errors by estimating the local errors for the specific reduced subspaces. We will consider three kinds of the reduced subspaces. As we shall see, the dimension of the reduced subspace determines the order of the errors.

5.1. Direct sum Taylor subspace. Suppose that $(y_{1j}(t), y_{2j}(t))$ has $\max(m_{1j}, m_{2j})+1$ continuous derivatives on $[t_j, t_{j+1}]$. Then the direct sum Taylor subspace is given by

$$S_j = \operatorname{span}\{(u_{1j}^i, 0)^T, (0, u_{2j}^k)^T \mid u_{1j}^i = y_{1j}^{(i)}(t_j), i=1, \cdots, m_{1j}, \\ u_{2j}^k = y_{2j}^{(k)}(t_j), k=1, \cdots, m_{2j}\}. \tag{24}$$

To compute u_{1j}^i, u_{2j}^k, we need to differentiate (4) and (5) successively. For example,

$$\begin{aligned}
&u_{1j}^1 = F_1(\eta_{1j}, \eta_{2j}),\\
&D_{y_2}F_2(\eta_{1j}, \eta_{2j})u_{2j}^1 = -D_{y_1}F_2(\eta_{1j}, \eta_{2j})u_{1j}^1,\\
&u_{1j}^2 = D_{y_1}F_1(\eta_{1j}, \eta_{2j})u_{1j}^1 + D_{y_2}F_1(\eta_{1j}, \eta_{2j})u_{2j}^1,\\
&D_{y_2}F_2(\eta_{1j}, \eta_{2j})u_{2j}^2 = -D_{y_1y_1}F_2(\eta_{1j}, \eta_{2j})u_{1j}^1u_{1j}^1 - 2D_{y_1y_2}F_2(\eta_{1j}, \eta_{2j})u_{1j}^1u_{2j}^1\\
&\qquad\qquad -D_{y_2y_2}F_2(\eta_{1j}, \eta_{2j})u_{2j}^1u_{2j}^1 - D_{y_1}F_2(\eta_{1j}, \eta_{2j})u_{1j}^2.
\end{aligned}$$

Note that the unknowns u_{2j}^k, $k=1, \cdots, m_{2j}$, satisfy linear systems with the same coefficient matrix. So there is only one factorization to compute for all the u_{2j}^k, $k=1, \cdots, m_{2j}$. Clearly, S_j contains the function

$$(x_{1j}(t), x_{2j}(t)) = \sum_{l=1}^{\min(m_{1j}, m_{2j})} \frac{(t-t_j)^l}{l!}(u_{1j}^l, 0) + \sum_{l=1}^{\min(m_{1j}, m_{2j})} \frac{(t-t_j)^l}{l!}(0, u_{2j}^l).$$

Let $(w_{1j}(t), w_{2j}(t)) = (\eta_{1j}, \eta_{2j}) + (x_{1j}(t), x_{2j}(t))$. By Taylor's theorem we have that

$$\|y_j(t) - w_j(t)\| = O(h^{\min(m_{1j}, m_{2j})+1}).$$

Hence by Theorem 1, we have

$$\|d_j(t)\| = O(h^{\min(m_{1j}, m_{2j})+1}).$$

This proves the following corollary.

COROLLARY 1. *Suppose $y_j(t)$ has $\max(m_{1j}, m_{2j})+1$ continuous derivatives on $[t_j, t_{j+1}]$. Then for the reduced subspace (24) we have*

$$\|d_j(t)\| = O(h^{\min(m_{1j}, m_{2j})+1}).$$

5.2. Direct sum Lagrange subspace. To define the Lagrange subspace at the jth reduction step, we assume that the solution $(y_{1j}(t), y_{2j}(t))$ is defined on $[t_{j-\max(m_{1j}, m_{2j})}, b]$, has $\min(m_{1j}, m_{2j})+1$ continuous derivatives on $[t_{j-\min(m_{1j}, m_{2j})}, t_{j+1}]$, and in addition to $(y_{1j}(t_j), y_{2j}(t_j)) = (\eta_{1j}, \eta_{2j})$, $(y_{1j}(t), y_{2j}(t))$ is known at previous $\max(m_{1j}, m_{2j})$ points, $t_{j-1}, \cdots, t_{j-\max(m_{1j}, m_{2j})}$. Then we define the direct sum Lagrange subspace as follows:

$$S_j = \operatorname{span}\{(u_{1j}^i, 0)^T, (0, u_{2j}^k)^T \mid u_{1j}^i = y_{1j}(t_{j-i}) - \eta_{1j}, i=1, \cdots, m_{1j}, \\ u_{2j}^k = y_{2j}(t_{j-k}) - \eta_{2j}, k=1, \cdots, m_{2j}\}.$$

Let $u_{1j}^0=0$ and $u_{2j}^0=0$; then the direct sum Lagrange subspace contains the function

$$x_j(t)=\sum_{l=0}^{\min(m_{1j},m_{2j})}\lambda_l(t)(u_{1j}^l,0)^T+\sum_{l=0}^{\min(m_{1j},m_{2j})}\lambda_l(t)(0,u_{2j}^l)^T,$$

where

$$\lambda_l(t)=\prod_{\substack{k=0\\k\neq l}}^{\min(m_{1j},m_{2j})}\frac{t-t_{j-k}}{t_{j-l}-t_{j-k}}. \tag{25}$$

Since $\sum_{l=0}^{\min(m_{1j},m_{2j})}\lambda_l(t)=1$, we have

$$x_j(t)=\sum_{l=0}^{\min(m_{1j},m_{2j})}\lambda_l(t)(y_{1j}(t_{j-l}),y_{2j}(t_{j-l}))^T-(\eta_{1j},\eta_{2j})^T.$$

Again by letting $(w_{1j}(t),w_{2j}(t))^T=(\eta_{1j},\eta_{2j})^T+x_j(t)$ and by the error formula for interpolation, we have

$$\|y_j(t)-w_j(t)\|=O(h^{\min(m_{1j},m_{2j})+1}),\quad\text{and so}$$
$$\|d_j(t)\|=O(h^{\min(m_{1j},m_{2j})+1}).$$

In fact, $y_j(t)$ is usually not known except at t_j. However, we do have approximations to $y_j(t_{j-1}),\cdots,y_j(t_{j-\max(m_{1j},m_{2})})$ if $t_{\max(m_{1j},m_{2j})}\geqq a$, namely, (η_{1i},η_{2i}), $i=j-1,\cdots,j-\max(m_{1j},m_{2j})$. Define $u_{1j}^i=\eta_{1,j-i}-\eta_{1j}$, $i=1,\cdots,m_{1j}$ and $u_{2j}^k=\eta_{2,j-k}-\eta_{2j}$, $k=1,\cdots,m_{2j}$. Then

$$x_j(t)=\sum_{l=0}^{\min(m_{1j},m_{2j})}\lambda_l(t)(\eta_{1,j-l},\eta_{2,j-l})^T-(\eta_{1j},\eta_{2j})^T.$$

Now suppose $\mu_1\leqq h_i/h_{i+1}\leqq\mu_2$, for some constants $\mu_1,\mu_2>0$, $i=1,2,\cdots$. Then it is clear from [13, pp. 64–66] that each factor of $\lambda_l(t)$ is bounded above by

$$\frac{\mu_2^{k+1}-1}{\mu_2-1}\bigg/\frac{\mu_1^{l+1}-\mu_1^{k+1}}{\mu_1-1}\quad\text{on }[t_j,t_{j+1}].$$

So $|\lambda_l(t)|\leqq\lambda$ on $[t_j,t_{j+1}]$, for some λ independent of h. Let $\eta_{j-l}\equiv(\eta_{1,j-l},\eta_{2,j-l})^T$. Then for $t\in[t_j,t_{j+1}]$,

$$\begin{aligned}&\|y_j(t)-w_j(t)\|\\&\leqq\left\|y_j(t)-\sum_{l=0}^{\min(m_{1j},m_{2j})}\lambda_l(t)y_j(t_{j-l})\right\|+\left\|\sum_{l=0}^{\min(m_{1j},m_{2j})}\lambda_l(t)[y_j(t_{j-l})-\eta_{j-l}]\right\|\\&\leqq O(h^{\min(m_{1j},m_{2j})+1})+\max_{l=0,\cdots,\min(m_{1j},m_{2j})}\|y_j(t_{j-l})-\eta_{j-l}\|\sum_{l=0}^{\min(m_{1j},m_{2j})}|\lambda_l(t)|\\&\leqq O(h^{\min(m_{1j},m_{2j})+1})+(\min(m_{1j},m_{2j})+1)\lambda\max_{l=0,\cdots,\min(m_{1j},m_{2j})}\|y_j(t_{j-l})-\eta_{j-l}\|.\end{aligned}$$

Note that $y_{j-l}(t_{j-l})=\eta_{j-l}$. Assume that $y_{j-1}(t),\cdots,y_{j-\min(m_{1j},m_{2j})}(t)$ are defined on $[t_{j-\min(m_{1j},m_{2j})},b]$ and $\|d_{j-i}(t)\|\leqq O(h^{m+1})$, $i=1,\cdots,\min(m_{1j},m_{2j})$, for some $m>0$. Using the fact that $e^x\geqq1+x$ and $e^x-1\leqq xe^x$ for all x, it is not difficult to see [12, p. 1281] that

$$\begin{aligned}\|y_j(t_{j-l})-\eta_{j-l}\|&\leqq\sum_{k=j}^{j-l+1}\|y_k(t_{j-l})-y_{k-1}(t_{j-l})\|\\&\leqq\sum_{k=j}^{j-l+1}O(h^{m+1})e^{L(t_k-t_{j-l})}\leqq O(h^{m+1})\sum_{k=j}^{j-l+1}e^{L(k-j+l)h}\\&\leqq O(h^{m+1})\sum_{k=1}^{l}e^{kLh}\leqq O(h^{m+1})le^{Lh(l+1)}=O(h^{m+1}).\end{aligned}$$

Hence we have

$$\|d_j(t)\| = O(h^{\min(m, m_{1j}, m_{2j})+1}).$$

This gives us the next corollary.

COROLLARY 2. *Suppose $y_j(t)$ is defined on $[t_{j-\max(m_{1j}, m_{2j})}, b]$ and has $\min(m_{1j}, m_{2j})+1$ continuous derivatives on $[t_{j-\min(m_{1j}, m_{2j})}, t_{j+1}]$. Suppose further that $y_{j-1}(t), \cdots, y_{j-\min(m_{1j}, m_{2j})}(t)$ are defined on $[t_{j-\min(m_{1j}, m_{2j})}, b]$ and $\|d_{j-i}(t)\| \leqq O(h^{m+1})$, $i=1, \cdots, \min(m_{1j}, m_{2j})$, for some $m>0$. Then for the reduced subspace*

$$S_j = \operatorname{span}\{(u_{1j}^i, 0)^T, (0, u_{2j}^k)^T \mid u_{1j}^i = \eta_{1,j-i} - \eta_{1j}, i=1, \cdots, m_{1j},$$
$$u_{2j}^k = \eta_{2,j-k} - \eta_{2j}, k=1, \cdots, m_{2j}\},$$

we have

$$\|d_j(t)\| = O(h^{\min(m, m_{1j}, m_{2j})+1}).$$

5.3. Direct sum Hermite subspace. With the same assumptions as in the case of the direct sum Lagrange subspace, but now supposing that the solution has $2\min(m_{1j}, m_{2j})+1$ continuous derivatives on $[t_{j-\min(m_{1j}, m_{2j})}, t_{j+1}]$, we define the direct sum Hermite subspace to be

$$S_j = \operatorname{span}\{(u_{1j}^{i,k}, 0)^T, (0, u_{2j}^{p,q})^T, i=1, \cdots, m_{1j}, p=1, \cdots, m_{2j}, \text{ for } k=q=1,$$
$$i=0, \cdots, m_{1j}, p=0, \cdots, m_{2j}, \text{for } k=q=2\},$$

where

$$u_{1j}^{i,1} = y_{1j}(t_{j-1}) - \eta_{1j}, \quad i=1, \cdots, m_{1j}, \quad u_{1j}^{i,2} = y_{1j}^{(1)}(t_{j-i}), \quad i=0, \cdots, m_{1j},$$
$$u_{2j}^{p,1} = y_{2j}(t_{j-p}) - \eta_{2j}, \quad p=1, \cdots, m_{2j}, \quad u_{2j}^{p,2} = y_{2j}^{(1)}(t_{j-p}), \quad p=0, \cdots, m_{2j}.$$

Let $u_{1j}^{0,1}=0$ and $u_{2j}^{0,1}=0$. Then S_j contains the function

$$\begin{aligned} x_j(t) = &\sum_{l=0}^{\min(m_{1j}, m_{2j})} [1-2\dot{\lambda}_l(t_{j-l})(t-t_{j-l})]\lambda_l(t)^2(u_{1j}^{l,1}, 0)^T \\ &+ \sum_{l=0}^{\min(m_{1j}, m_{2j})} [1-2\dot{\lambda}_l(t_{j-l})(t-t_{j-l})]\lambda_l(t)^2(0, u_{2j}^{l,1})^T \\ &+ \sum_{l=0}^{\min(m_{1j}, m_{2j})} (t-t_{j-l})\lambda_l(t)^2(u_{1j}^{l,2}, 0)^T \\ &+ \sum_{l=0}^{\min(m_{1j}, m_{2j})} (t-t_{j-l})\lambda_l(t)^2(0, u_{2j}^{l,2})^T, \end{aligned}$$

where $\lambda_l(t)$ is defined as in (25). Note that

$$\begin{aligned} x_j(t) = &\sum_{l=0}^{\min(m_{1j}, m_{2j})} [1-2\dot{\lambda}_l(t_{j-l})(t-t_{j-l})]\lambda_l(t)^2(y_{1j}(t_{j-l}), y_{2j}(t_{j-l}))^T \\ &+ \sum_{l=0}^{\min(m_{1j}, m_{2j})} (t-t_{j-l})\lambda_l(t)^2(y_{1j}^{(1)}(t_{j-l}), y_{2j}^{(1)}(t_{j-l}))^T - (\eta_{1j}, \eta_{2j})^T. \end{aligned}$$

Hence by the error formula we have

$$\|y_j(t) - w_j(t)\| = O(h^{2(\min(m_{1j}, m_{2j})+1)}),$$

where $(w_{1j}(t), w_{2j}(t))^T = (\eta_{1j}, \eta_{2j})^T + x_j(t)$. Thus

$$\|d_j(t)\| = O(h^{2(\min(m_{1l}, m_{2l})+1)}).$$

In practice we can approximate $y_j(t_{j-i})$ by η_{j-i} and $y_j^{(1)}(t_{j-i})$ by $V(\eta_{1,j-i}, \eta_{2,j-i})$ with V defined in (23). It is clear that $[1-2\dot{\lambda}_l(t_{j-l})(t-t_{j-l})]\lambda_l(t)^2 = O(1)$ and $(t-t_{j-l})\lambda_l(t)^2 = O(h)$. Suppose $V(y_1, y_2)$ is Lipschitz continuous on $R(F_2)$. Then, using a similar argument to that of § 5.2, we have the following corollary.

COROLLARY 3. *In addition to the assumptions in Corollary* 2, *we assume further that* $y_j(t)$ *has* $2(\min(m_{1j}, m_{2j})+1)$ *continuous derivatives on* $[t_{j-\min(m_{1j}, m_{2j})}, b]$ *and* $V(y_1, y_2)$ *is Lipschitz continuous on* $R(F_2)$. *Then for the reduced subspace*

$$S_j = \operatorname{span}\{(u_{1j}^{i,k}, 0)^T, (0, u_{2j}^{p,q})^T, i=1, \cdots, m_{1j}, p=1, \cdots, m_{2j}, \text{ for } k=q=1,$$
$$i=0, \cdots, m_{1j}, p=0, \cdots, m_{2j}, \text{ for } k=q=2\},$$

where

$$u_{1j}^{i,1} = \eta_{1,j-1} - \eta_{1j}, \quad i=1, \cdots, m_{1j}, \quad u_{2j}^{p,1} = \eta_{2,j-p} - \eta_{2j}, \quad p=1, \cdots, m_{2j},$$
$$u_{1j}^{i,2} = F_1(\eta_{1,j-i}, \eta_{2,j-i}), \qquad i=0, \cdots, m_{1j},$$

and $u_{2j}^{p,2}$ satisfies

$$D_{y2}F_2(\eta_{1,j-p}, \eta_{2,j-p})u_{2j}^{p,2} = -D_{y1}F_2(\eta_{1,j-p}, \eta_{2,j-p})F_1(\eta_{1,j-p}, \eta_{2,j-p}), \qquad p=0, \cdots, m_{2j},$$

we have

$$\|d_j(t)\| = O(h^{\min(m, 2m_{1j}+1, 2m_{2j}+1)+1}).$$

Appendix A. In this Appendix we prove a technical result (Lemma 4) that is used in the proof of Lemma 1 in the main body of this paper. We begin with a preliminary lemma. In order to simplify the notation, we suppress subscript j in this Appendix.

LEMMA 3. *For the functions* f, $\tilde{f}_R$, *and* f_R *defined in* (13), (15), *and* (17), *and balls* B_{δ_1}, B_{δ_R} *defined in* (13), (18), *respectively, we have*

$$\begin{aligned} &Y_2^T[F_2(\lambda+\eta_1, Y_2p(\lambda)+\eta_2) - F_2(\lambda_R+\eta_1, Y_2q(\lambda_R)+\eta_2)] \\ &\quad = Y_2^T[F_2(\lambda+\eta_1, P_2f(\lambda)+\eta_2) - F_2(\lambda+\eta_1, f(\lambda)+\eta_2)] \quad \forall \lambda \in B_{\delta_1}, \quad \lambda_R \in B_{\delta_R}, \end{aligned} \tag{26}$$

where $p(\lambda) = Y_2^T f(\lambda)$ *and* $q(\lambda_R) = \tilde{f}_R(Y_1^T\lambda_R)$.

Proof. By (14) and (18),

$$Y_2^T F_2(\lambda_R+\eta_1, f_R(\lambda_R)+\eta_2) = Y_2^T c = Y_2^T F_2(\lambda+\eta_1, f(\lambda)+\eta_2) \tag{27}$$

for any $\lambda_R \in B_{\delta_R}$ and $\lambda \in B_{\delta_1}$.

From (27), it follows that for any $\lambda_R \in B_{\delta_R}$ and $\lambda \in B_{\delta_1}$,

$$\begin{aligned} &Y_2^T[F_2(\lambda+\eta_1, P_2f(\lambda)+\eta_2) - F_2(\lambda_R+\eta_1, f_R(\lambda_R)+\eta_2)] \\ &\quad = Y_2^T[F_2(\lambda+\eta_1, P_2f(\lambda)+\eta_2) - F_2(\lambda+\eta_1, f(\lambda)+\eta_2)]. \end{aligned}$$

Equivalently,

$$\begin{aligned} &Y_2^T[F_2(\lambda+\eta_1, Y_2Y_2^Tf(\lambda)+\eta_2) - F_2(\lambda_R+\eta_1, f_R(\lambda R)+\eta_2)] \\ &\quad = Y_2^T[F_2(\lambda+\eta_1, P_2f(\lambda)+\eta_2) - F_2(\lambda+\eta_1, f(\lambda)+\eta_2)]. \end{aligned}$$

Then, by defining $p(\lambda) = Y_2^Tf(\lambda)$ and $q(\lambda_R) = \tilde{f}_R(Y_1^T\lambda_R)$, we complete the proof. □

LEMMA 4. *For the functions* f, f_R *defined in* (13), (17), *and the subspace* $\tilde{S}_1$ *defined in* (17), *we have*

$$\|P_2 f(\lambda) - f_R(\lambda_R)\| \leqq \alpha_1\alpha_3 \|P_2 f(\lambda) - f(\lambda)\| + \alpha_2\|\lambda - \lambda_R\| \quad \forall \lambda \in B_\delta, \quad \lambda_R \in B_\delta \cap \tilde{S}_1,$$

for some constants α_1, α_2, α_3, *and* $B_\delta = \{\lambda \in \mathfrak{R}^{n_1} | \, \|\lambda\| \leqq \delta, 0 < \delta \leqq \min(\delta_1, \delta_R)\}$.

Proof. Define

$$A(\lambda, v, \omega) = Y_2^T(F_2(\lambda + \eta_1, Y_2\omega + \eta_2) - c) - v,$$

which is a C^r-map from $\mathfrak{R}^{n_1} \times \mathfrak{R}^{m_2} \times \mathfrak{R}^{m_2}$ to $\mathfrak{R}^{m_2}$.

Since $A(0,0,0) = 0$ and $D_\omega A(0,0,0) = Y_2^T D_{y_2} F_2(\eta_1, \eta_2) Y_2$ is nonsingular by assumption, it follows from the implicit function theorem that there exists an r times continuously differentiable map $a: B_{\delta_2} \times B_\rho \to \mathfrak{R}^{m_2}$ such that

$$a(0,0) = 0 \quad \text{and} \quad A(\lambda, v, a(\lambda, v)) = 0 \quad \forall (\lambda, v) \in B_{\delta_2} \times B_\rho,$$

where

$$B_{\delta_2} = \{\lambda \in \mathfrak{R}^{n_1} | \, \|\lambda\| \leqq \delta_2\} \quad \text{and} \quad B_\rho = \{v \in \mathfrak{R}^{m_2} | \, \|v\| \leqq \rho\},$$

$$\text{for some } 0 < \delta_2 \leqq \min(\delta_1, \delta_R), \quad \text{and} \quad \rho > 0.$$

Since a is r times continuously differentiable, there exist constants α_1 and α_2 such that

$$\|D_v a(\lambda, v)\| \leqq \alpha_1 \quad \text{and} \quad \|D_\lambda a(\lambda, v)\| \leqq \alpha_2 \quad \forall (\lambda, v) \in B_{\delta_2} \times B_\rho. \tag{28}$$

Let

$$v(\lambda) = Y_2^T(F_2(\lambda + \eta_1, Y_2 p(\lambda) + \eta_2) - c),$$
$$v_R(\lambda_R) = Y_2^T(F_2(\lambda_R + \eta_1, Y_2 q(\lambda_R) + \eta_2) - c).$$

Since $p(0) = 0$, $q(0) = 0$, and $F_2(\eta_1, \eta_2) = c$, by the continuity of f, $\tilde{f}_R$, and F_2, there exists a nontrivial ball

$$B_\delta = \{\lambda \in \mathfrak{R}^{n_1} | \, \|\lambda\| \leqq \delta\} \subset B_{\delta_2}$$

such that

$$v(\lambda), v_R(\lambda_R) \in B_\rho \quad \forall \lambda \in B_\delta \quad \text{and} \quad \lambda_R \in B_\delta \cap \tilde{S}_1.$$

Then by the uniqueness of a, we have

$$\begin{aligned} &p(\lambda) \equiv a(\lambda, v(\lambda)) \quad \forall \lambda \in B_\delta, \\ &q(\lambda_R) \equiv a(\lambda_R, v_R(\lambda_R)) \quad \forall \lambda_R \in B_\delta \cap \tilde{S}_1. \end{aligned} \tag{29}$$

Furthermore, since $f(\lambda)$ is continuous and $f(0) = 0$, there exists a finite ball

$$B_\gamma = \{x \in \mathfrak{R}^{n_2} | \, \|x\| \leqq \gamma\} \tag{30}$$

such that

$$f(\lambda), P_2 f(\lambda) \in B_\gamma \quad \forall \lambda \in B_\delta.$$

Also, since F_2 is of class C^r, there exists a constant α_3 such that

$$\|D_{y_2} F_2(\lambda + \eta_1, y_2 + \eta_2)\| \leqq \alpha_3 \quad \forall (\lambda, y_2) \in B_\delta \times B_\gamma. \tag{31}$$

Note that (26) can be written as

$$\begin{aligned} v(\lambda) - v_R(\lambda_R) = Y_2^T[F_2(\lambda + \eta_1, P_2 f(\lambda) + \eta_2) - F_2(\lambda + \eta_1, f(\lambda) + \eta_2)] \\ \forall \lambda \in B_\delta, \lambda_R \in B_\delta \cap \tilde{S}_1. \end{aligned} \tag{32}$$

Then for any $\lambda \in B_\delta$, and $\lambda_R \in B_\delta \cap \tilde{\tilde{S}}_1$, (28)–(32) and the mean value theorem imply that

$$
\begin{aligned}
\|P_2 f(\lambda) - f_R(\lambda_R)\| &= \|Y_2(p(\lambda) - q(\lambda_R))\| \\
&\leqq \|Y_2\| \, \|a(\lambda, v(\lambda)) - a(\lambda_R, v_R(\lambda_R))\| \\
&\leqq \|a(\lambda, v(\lambda)) - a(\lambda, v_R(\lambda_R))\| \\
&\qquad + \|a(\lambda, v_R(\lambda_R)) - a(\lambda_R, v_R(\lambda_R))\| \\
&\leqq \alpha_1 \|v(\lambda) - v_R(\lambda_R)\| + \alpha_2 \|\lambda - \lambda_R\| \\
&\leqq \alpha_1 \|Y_2^T\| \, \|F(\lambda + \eta_1, P_2 f(\lambda) + \eta_2) - F_2(\lambda + \eta_1, f(\lambda) + \eta_2)\| \\
&\qquad + \alpha_2 \|\lambda - \lambda_R\| \\
&\leqq \alpha_1 \alpha_3 \|P_2 f(\lambda) - f(\lambda)\| + \alpha_2 \|\lambda - \lambda_R\|.
\end{aligned} \tag{33}
$$

□

Appendix B. In this Appendix we show that the quantities α_{1j}, α_{2j}, α_{3j}, and M_j appearing in (20) and (21) are bounded for all j.

THEOREM 4. *In addition to the assumptions in Theorem* 3, *if $Y_{2j}^T D_{y_2} F_2(\eta_{1j}, \eta_{2j}) Y_{2j}$ is bounded away from singularity for all j, then α_{1j} and α_{2j} are uniformly bounded.*

Proof. Since $Y_{2j}^T D_{y_2} F_2(\eta_{1j}, \eta_{2j}) Y_{2j}$ is bounded away from singularity for all j, there exists a constant $\beta_1 > 0$ such that

$$|\det(Y_{2j}^T D_{y_2} F_2(\eta_{1j}, \eta_{2j}) Y_{2j})| \geqq \beta_1.$$

Let

$$
\begin{aligned}
\hat{\mathcal{H}} = \{(y_1, y_2) \in \mathfrak{R}^{n_1} \times \mathfrak{R}^{n_2} \mid |\det(Y^T D_{y_2} F_2(y_1, y_2) Y)| \geqq \beta_1 > 0, \\
\text{for some } Y \in \mathfrak{R}^{n_2 \times m_2} \text{ satisfying } \|Y\| = 1\}.
\end{aligned}
$$

Clearly, $\hat{\mathcal{H}}$ is closed. Since the (η_{1j}, η_{2j}) are bounded by Theorem 3 and are all in $\hat{\mathcal{H}}$, they are contained in some compact set $\mathcal{H} \subset \hat{\mathcal{H}}$. Let us define

$$A(\lambda, v, \omega, \zeta_1, \zeta_2) = Y_{2j}^T(F_2(\lambda + \zeta_1, Y_{2j}\omega + \zeta_2) - F_2(\zeta_1, \zeta_2)) - v$$

from $\mathfrak{R}^{n_1} \times \mathfrak{R}^{m_{2j}} \times \mathfrak{R}^{m_{2j}} \times \mathcal{H}$ to $\mathfrak{R}^{m_{2j}}$, which is a C^r-map. Since for each $(\zeta_1, \zeta_2) \in \mathcal{H}$, $A(0, 0, 0, \zeta_1, \zeta_2) = 0$ and $D_\omega A(0, 0, 0, \zeta_1, \zeta_2) = Y_{2j}^T D_{y_2} F_2(\zeta_1, \zeta_2) Y_{2j}$ is nonsingular, by the implicit function theorem, for each $(\xi_1, \xi_2) \in \mathcal{H}$, there exists an r times continuously differentiable map $a_{(\xi_1, \xi_2)}: B_{\delta^{(\xi_1,\xi_2)}} \times B_{\rho^{(\xi_1,\xi_2)}} \times B_{\tau_1^{(\xi_1,\xi_2)}} \times B_{\tau_2^{(\xi_1,\xi_2)}} \to \mathfrak{R}^{m_{2j}}$ such that

$$a_{(\xi_1, \xi_2)}(0, 0, \xi_1, \xi_2) = 0$$

and

$$
\begin{aligned}
A(\lambda, v, a_{(\xi_1, \xi_2)}(\lambda, v, \zeta_1, \zeta_2), \zeta_1, \zeta_2) = 0 \\
\forall (\lambda, v, \zeta_1, \zeta_2) \in B_{\delta^{(\xi_1,\xi_2)}} \times B_{\rho^{(\xi_1,\xi_2)}} \times B_{\tau_1^{(\xi_1,\xi_2)}} \times B_{\tau_2^{(\xi_1,\xi_2)}}.
\end{aligned}
$$

Here $B_{\delta^{(\xi_1,\xi_2)}}$ and $B_{\rho^{(\xi_1,\xi_2)}}$ are balls centered at the origin with radii $\delta^{(\zeta_1,\zeta_2)}$ and $\rho^{(\zeta_1,\zeta_2)}$ in $\mathfrak{R}^{n_1}$ and $\mathfrak{R}^{m_{2j}}$, respectively, and

$$B_{\tau_i}^{(\xi_1,\xi_2)} = \{\zeta_i \in \mathfrak{R}^{n_i} \mid \|\zeta_i - \xi_i\| < \tau_i^{(\xi_1,\xi_2)}\}, \qquad i = 1, 2.$$

Since $\mathcal{H}$ is a compact set, we can extract a finite covering of $\mathcal{H}$ from $\{B_{\tau_1^{(\xi_1,\xi_2)}/2} \times B_{\tau_2^{(\xi_1,\xi_2)}/2}, (\xi_1, \xi_2) \in \mathcal{H}\}$, say $\bigcup_{l=1}^L B_{\tau_1^{(\xi_1^l,\xi_2^l)}/2} \times B_{\tau_2^{(\xi_1^l,\xi_2^l)}/2}$. Let

$$\alpha_1 = \max_{1 \leqq l \leqq L} \max \{\|D_v a_{(\xi_1^l, \xi_2^l)}(\lambda, v, \zeta_1, \zeta_2)\| \mid (\lambda, v, \zeta_1, \zeta_2) \in \bar{B}_\delta \times \bar{B}_\rho \times \bar{B}_{\tau_1^{(\xi_1^l,\xi_2^l)}/2} \bar{B}_{\tau_2^{(\xi_1^l,\xi_2^l)}/2}\}$$

and

$$\alpha_2 = \max_{1 \leq l \leq L} \max \{ \| D_\lambda a_{(\xi_1^l, \xi_2^l)}(\lambda, v, \zeta_1, \zeta_2) \| \mid (\lambda, v, \zeta_1, \zeta_2) \in \bar{B}_\delta \times \bar{B}_\rho \times \bar{B}_{\tau_1^{(\xi_1^l, \xi_2^l)}/2} \bar{B}_{\tau_2^{(\xi_1^l, \xi_2^l)}/2} \},$$

where

$$(34) \qquad B_\delta = \bigcap_{l=1}^{L} B_{\delta^{(\xi_1^l, \xi_2^l)}/2}, \; B_\rho = \bigcap_{l=1}^{L} B_{\rho^{(\xi_1^l, \xi_2^l)}/2},$$

and $\bar{B}_\delta$, $\bar{B}_\rho$, $\bar{B}_{\tau_1^{(\xi_1^l, \xi_2^l)}/2}$, and $\bar{B}_{\tau_2^{(\xi_1^l, \xi_2^l)}/2}$ are the closures of the unbarred sets.

From the uniqueness of the functions a_j in Lemma 1, we have

$$a_j(\lambda, v) \equiv a_{(\xi_1^l, \xi_2^l)}(\lambda, v, \eta_{1j}, \eta_{2j}) \quad \text{for some } l \in 1, \cdots, L,$$

where

$$(\eta_{1j}, \eta_{2j}) \in B_{\tau_1^{(\xi_1^l, \xi_2^l)}/2} B_{\tau_2^{(\xi_1^l, \xi_2^l)}/2}.$$

Hence by requiring that

$$B_{\delta_{2j}} \times B_{\rho_{1j}} \subset \bar{B}_\delta \times \bar{B}_\rho$$

in Lemma 1, the result of the lemma still holds, yet we have $\alpha_{1j} \leqq \alpha_1$ and $\alpha_{2j} \leqq \alpha_2$, for all j. □

THEOREM 5. *In addition to the assumptions in Theorem* 3, *if $D_{y_2}F_2(\eta_{1j}, \eta_{2j})$ is bounded away from singularity for all j, then α_{3j} and M_j are bounded for all j.*

Proof. Since the $D_{y_2}F_2(\eta_{1j}, \eta_{2j})$ are bounded away from singularity, there is a constant, say $\beta_3 > 0$, such that $|\det(D_{y_2}F_2(\eta_{1j}, \eta_{2j}))| \geqq \beta_3$. Let

$$\hat{\mathcal{G}} = \{(y_1, y_2) \mid |\det(D_{y_2}F_2(y_1, y_2))| \geqq \beta_3\}.$$

Clearly, $\hat{\mathcal{G}}$ is closed. Again, since the (η_{1j}, η_{2j}) are bounded by Theorem 3, and are all in $\hat{\mathcal{G}}$, (η_{1j}, η_{2j}) can be all contained in some compact set $\tilde{\mathcal{G}} \subset \hat{\mathcal{G}}$. Let us define

$$\check{\mathcal{G}} = \{(y_1, y_2) \mid \|(y_1, y_2) - (z_1, z_2)\| \leqq 1 \text{ for some } (z_1, z_2) \in \tilde{\mathcal{G}}\},$$

which is again compact. Now by the uniform continuity of $\det(D_{y_2}F_2(y_1, y_2))$ on $\check{\mathcal{G}}$, we have

$$|\det(D_{y_2}F_2(y_1, y_2)) - \det(D_{y_2}F_2(z_1, z_2))| < \beta_3/2 \qquad \text{for } \|(y_1, y_2) - (z_1, z_2)\| \lneqq \delta_1 \quad \text{for some } \delta_1 > 0.$$

Let

$$\mathcal{G} = \{(y_1, y_2) \mid \|y_1 - z_1\| \leqq \delta \text{ for some } (z_1, y_2) \in \tilde{\mathcal{G}}\},$$

where $\delta = \min(1, \delta_1)$. Then $|\det(D_{y_2}F_2(y_1, y_2))| \geqq \beta_3/2$ for any $(y_1, y_2) \in \mathcal{G}$.

Consider the C^r-map

$$S(y_1, y_2) = (y_1, F_2(y_1, y_2)) \text{ on } \mathcal{G}.$$

For each $(y_1, y_2) \in \mathcal{G}$, since $D_{y_2}F_2(y_1, y_2)$ is nonsingular, the Jacobian of S at (y_1, y_2) is nonzero. Therefore, by the inverse function theorem, there exist open sets $U_{(y_1, y_2)}$ and $V_{(y_1, y_2)}$ containing (y_1, y_2) and $(y_1, c = F_2(y_1, y_2))$, respectively, such that S is one to one from $U_{(y_1, y_2)}$ onto $V_{(y_1, y_2)}$ and $U_{(y_1, y_2)} = S^{-1}(V_{(y_1, y_2)})$. Also there is a local function $s_{(y_1, y_2)}$ on $V_{(y_1, y_2)}$ (see [1, p. 374–375]) such that

$$\begin{aligned} &s_{(y_1, y_2)}(y_1, c) = y_2, \\ &F_2(z_1, s_{(y_1, y_2)}(z_1, d)) = d \quad \text{for } (z_1, d) \in V_{(y_1, y_2)}, \\ &U_{(y_1, y_2)} = S^{-1}(V_{(y_1, y_2)}) = \{(z_1, s_{(y_1, y_2)}(z_1, d)) \mid (z_1, d) \in V_{(y_1, y_2)}\}. \end{aligned}$$

Let $X_{(y_1,y_2)}$ be a proper open subset of $V_{(y_1,y_2)}$ such that the closure of $X_{(y_1,y_2)}$ is contained in $V_{(y_1,y_2)}$. Now since $\mathscr{G}$ is a compact set and S is continuous, $S(\mathscr{G})$ is also compact. Thus we can extract a finite covering of $S(\mathscr{G})$ from $\{X_{(y_1,y_2)},\ (y_1,y_2)\in\mathscr{G}\}$, say $\bigcup_{i=1}^{I} X_{(y_1^i,y_2^i)}$. Let

$$\gamma=\max_{i=1,\cdots,I}\ \max\{\|s_{(y_1^i,y_2^i)}(z_1,d)\|\,|\,(z_1,d)\in\bar{X}_{(y_1^i,y_2^i)}\},$$

where $\bar{X}_{(y_1^i,y_2^i)}$ is the closure of the unbarred set.

Then by requiring $\delta_{3j}\leqq\delta$ and by the uniqueness of the function f_j in Lemma 1, for each $\lambda\in B_{\delta_{3j}}$ we have

$$f_j(\lambda)\equiv s_{(y_1^k,y_2^k)}(\lambda+\eta_{1j},c_j)-\eta_{2j}\quad\text{for some } 1\leqq k\leqq I,$$

where

$$(\lambda+\eta_{1j},c_j)\in X_{(y_1^k,y_2^k)}.$$

Thus

$$\|f_j(\lambda)+\eta_{2j}\|\leqq\gamma\quad\forall\lambda\in B_{\delta_{3j}}.$$

Therefore, we can choose

$$B_{\gamma_j}\subset\bar{B}_\gamma=\{y\in\Re^{n_2}|\,\|y\|\leqq\gamma\}.$$

Hence, by the continuity of $D_{y_2}F_2(y_1,y_2)$ and $(D_{y_1}F_1(y_1,y_2),\ D_{y_2}F_1(y_1,y_2))$, we have

$$\alpha_{3j}\leqq\max_{(y_1,y_2)\in\mathscr{N}}\|D_{y_2}F_2(y_1,y_2)\|,$$

$$M_j\leqq\max_{(y_1,y_2)\in\mathscr{N}}\|(D_{y_1}F_1(y_1,y_2),\ D_{y_2}F_1(y_1,y_2))\|,$$

where

$$\mathscr{N}=\{(y_1,y_2)\in\Re^{n_1}\times\Re^{n_2}|\,\|y_1-z_1\|\leqq\delta,\text{ and }\|y_2-z_2\|\leqq\gamma\text{ for some }(z_1,z_2)\in\tilde{\mathscr{G}}\}.\quad\square$$

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REFERENCES

[1] T. M. APOSTOL, *Mathematical Analysis*, Second edition, Addison-Wesley, Reading, MA, 1975.

[2] B. O. ALMROTH, P. STERN, AND F. A. BROGAN, *Automatic choice of global shape functions in structural analysis*, AIAA J., 16 (1978), pp. 525–528.

[3] J. P. FINK AND W. C. RHEINBOLDT, *On the error behavior of the reduced basis technique for nonlinear finite element approximations*, Z. Angew. Math. Mech., 63 (1983), pp. 21–28.

[4] ———, *On the discretization error of parametrized nonlinear equations*, SIAM J. Numer. Anal., 20 (1983), pp. 732–746.

[5] ———, *Local error estimates for parametrized nonlinear equations*, SIAM J. Numer. Anal., 20 (1985), pp. 729–735.

[6] M. LIN LEE, *The reduced basis method for differential algebraic equation systems*, Tech. Report ICMA-85-85, Institute for Computational Mathematics and Applications, University of Pittsburgh, Pittsburgh, PA, July, 1985.

[7] D. A. NAGY, *Model representation of geometrically nonlinear behavior by the finite element method*, Comput. Structures, 10 (1977), pp. 683–688.

[8] A. K. NOOR AND J. M. PETERS, *Reduced basis technique for nonlinear analysis of structures*, AIAA J., 18 (1980), pp. 455–462.

[9] A. K. NOOR, C. M. ANDERSEN, AND J. M. PETERS, *Reduced basis technique for collapse of shells*, AIAA J., 19 (1981), pp. 393–397.

[10] J. S. PETERSON, *High Reynold number solution for incompressible viscous flow using the reduced basis technique*, Tech. Report ICMA-83-49, Institute for Computational Mathematics and Applications, University of Pittsburgh, Pittsburgh, 1983.

[11] T. A. PORSCHING, *Estimation of the error in the reduced basis method solution of nonlinear equations*, Math. Comp., 45 (1985), pp. 487-496.

[12] T. A. PORSCHING AND M. LIN LEE, *The reduced basis method for initial value problems*, SIAM J. Numer. Anal., 24 (1987), pp. 1277-1287.

[13] L. F. SHAMPINE AND M. K. GORDON, *Computer Solution of Ordinary Differential Equations, the Initial Value Problem*, W. H. Freeman, San Francisco, 1974.

[14] W. C. RHEINBOLDT, *Differential-algebraic system as differential equations on manifolds*, Math. Comp., 43 (1984), pp. 473-482.