On Projection-Based Algorithms for Model-Order Reduction of Interconnects

Janet Meiling Wang, Chia-Chi Chu, *Member, IEEE*, Qingjian Yu, and Ernest S. Kuh, *Life Fellow, IEEE*

***Abstract*—Model-order reduction is a key technique to do fast simulation of interconnect networks. Among many model-order reduction algorithms, those based on projection methods work quite well. In this paper, we review the projection-based algorithms in two categories. The first one is the coefficient matching algorithms. We generalize the Krylov subspace method on moment matching at a single point, to multipoint moment-matching methods with matching points located anywhere in the closed right-hand side (RHS). of the complex plane, and we provide algorithms matching the coefficients of series expansion-based on orthonormal polynomials and generalized orthonormal basis functions in Hilbert and Hardy space. The second category belongs to the grammian-based algorithms, where we provide efficient algorithm for the computation of grammians and new approximate grammian-based approaches. We summarize some important properties of projection-based algorithms so that they may be used more flexibly.**

***Index Terms*—Coefficient matching, congruence transform, generalized orthonormal basis function, grammian, interconnect, model order reduction, projection–based algorithsms, multipoint moment matching, orthonormal polynomials.**

I. Introduction

With the rapid increase of the signal frequency and decrease of the feature sizes of high-speed electronic circuits, interconnect has becoming a dominant factor in determining circuit performance and reliability in deep submicron designs. As interconnects are typically of very large size and high-order, model-order reduction is a necessity for efficient interconnect modeling, simulation, design, and optimization.

There are two kinds of model-order reduction algorithms. The first one only cares about the preservation of some characteristics of the model in frequency or time domain without consideration of mapping the state variables of the original to those of the reduced model. Asymptotic waveform evaluation (AWE) [1] is a typical example of such algorithms. Though working quite efficiently in some cases, they often meet the problem of stability and passivity of the reduced model, numerical stability in the formation of the model, and less flexibility in meeting different requirements of the reduced model. The second one is based on mapping the state–space of the original model with a larger size to a state–space of the reduced model with a smaller size. This kind of algorithm is the projection-based one. Compared with the algorithms of the first kind, its implementation is generally a little bit more complicated, but it is easier to deal with the stability and passivity problem, with much better numerical stability in the formation of the model, and more flexible when multiple requirements are needed.

There have been many papers published on the projection-based model-order reduction algorithms. A typical one is the Krylov subspace methods which provide moment matching to a given order between the transfer functions of the reduced order model and the original one. The *Padé via Lanczos* algorithm [2] uses an oblique projection with two bi-orthonormal projection matrices. While the oblique-projection algorithms often result in better matching of coefficients in an expansion on some basis, the stability of the reduced model is not guaranteed. The Krylov subspace-based congruence transform [3], [4] uses one projection matrix and has been successfully applied in passive model-order reduction. The Krylov subspace method is restricted in the finite-order system and is fixed in one matching point and the accuracy of the reduced model is obtained by a suitable order of moment matching. It has been found that multipoint moment matching is more efficient than a single-point one, and we have provided a multipoint moment-matching algorithm for multiport distributed interconnect networks, where an integrated congruence transform was developed to map a system of infinite order to a system of finite order [5]. So far, the matching points are limited on the positive real axis (including the origin) and imaginary axis on the complex plane.

Another projection-based algorithm is the balanced truncation (BT) method [10], [11]. Such algorithms also belong to the oblique-projection methods. By using BT, a major submatrix of the grammian of the balanced system is preserved at the reduced model. Hankel norm error bounds are given for the choice of the order of the reduced model [12], [13]. Experiments on the use of this method show that it results in better approximation of the performance of the reduced model in a wide frequency range, than the moment-matching method with single matching point at $s = 0$, but the comparisons with multipoint moment-matching method has not been seen. The greatest disadvantage of this method is its high computational cost. Though many efforts have been made to reduce the cost [14], [15], it is still much less efficient than the moment-matching method.

All the above methods focus on the approximation of the characteristics of the reduced model in the frequency domain,

Manuscript received January 17, 2001; revised November 11, 2001. This work was supported by Semiconductor Research Corporation (SRC) under Contract 866.001. The work of C.-C. Chu was supported in part by the National Science Council, Taiwan, R.O.C. under Grant NSC37032F. This paper was recommended by Associate Editor K. Thulasiraman.

J. M. Wang is with the Electrical Computing Engineering Department, University of Arizona, Tucson, AR 85721 USA.

C.-C. Chu is with the Department of Electrical Engineering, Chang Gung University, Tao-Yuan 333, Taiwan, R.O.C.

Q. Yu is with Celestry Design Technologies, Inc., San Jose, CA 95134 USA (qjyu@celestry.com).

E. S. Kuh is with the Department of Electrical Engineering and Computer Sciences, University of California, Berkeley, CA 94720 USA.

Publisher Item Identifier 10.1109/TCSI.2002.804542.

and can be categorized as frequency domain methods. When the inductive effect becomes more and more serious in today's technology, the waveform of the impulse response of interconnects may be very complicated [17]. It is quite hard to predict the accuracy of the time-domain response of the reduced model, based on the accuracy of the frequency-domain response, and it is needed to do model-order reduction directly in the time domain. This work has now begun. In [16], the Krylov subspace approach was used to do time-domain model-order reduction and the derivatives of the circuit response are kept to a given order. This approach is not efficient in dealing with model-order reduction of linear interconnects as the derivatives are functions of time, when simulation advances, the circuit needs to be remodeled repeatedly.

In this paper, we review some of the key concepts and methods on the projection-based model-order reduction algorithms. We extend the coefficient matching method from matching the coefficients of the power series expansion in complex frequency s to matching the coefficients of the series expansion based on other basis functions, which include orthonormal polynomials in the frequency/time interval of interest and the generalized orthonormal basis functions in Hilbert space (time domain) and Hardy space (frequency domain). To reduce the computational cost of the BT method, we give an efficient algorithm for the computation of the controllability and observability grammians, and model-order reduction algorithms using approximate grammians. All these algorithms improve the performance of existing algorithms either in the order of reduction or in the computational cost. Also, we summarize some important properties of the projection-based model-order reduction algorithm, which will provide some guidance to the use and development of such algorithms.

The rest of the paper is organized as follows. In Section II, the block form of the state and output equations of an *RLC* networks is given, which will be used throughout the paper. In Section III, we provide multipoint moment-matching algorithms with matching points anywhere in the closed RHS complex plane. In Section IV, we provide algorithms for matching the coefficients of expansion series based on orthonormal polynomials in both frequency and time domains. In Section V, we provide algorithms for matching the coefficients of expansion series based on generalized orthonormal functions in Hilbert and Hardy space. In Section VI, we provide our new result on the improvement of the BT methods. In Section VII, we summarize some general properties of the projection-based model-order reduction algorithms, which are useful in the implementation and development of algorithms. We give experimental results in Section VIII and conclusions in Section IX.

II. State and Output Equations

In the general case, a circuit model of an *RLC* interconnect may consist of purely capacitive branches, purely resistive branches, and serially connected *R–L* branches. Except for the internal nodes of the *R–L* branches, it can be assumed that each floating node is connected to a grounded capacitor, and let $v_n(t)$ be the node voltage vector of these nodes. Let $i_L(t)$ be the vector of inductance currents and it is assumed that there are no inductance cutsets in the network. Then

$$\overline{x}(t) = \begin{bmatrix} v_n(t) \\ i_L(t) \end{bmatrix} \tag{2.1}$$

is the state vector of the network.

Let C be the nodal capacitance matrix, G the nodal conductance matrix, L, R, and A_L the branch inductance, resistance, and the incidence matrix of the *R–L* branches, respectively. Let $\overline{i}_s(t)$ be the input current vector and A_s the incidence matrix of the current source branches with the currents flowing out of these branches. Then, the state equations of the network can be written in the following form:

$$M\,\frac{d\,\overline{x}(t)}{dt} + N\,\overline{x}(t) = B\overline{i}_s(t) \tag{2.2}$$

where

$$M = \begin{bmatrix} C & \\ & L \end{bmatrix} \tag{2.3}$$

$$N = \begin{bmatrix} G & A_L \\ -A_L^T & R \end{bmatrix} \tag{2.4}$$

and

$$B = \begin{bmatrix} A_s \\ 0 \end{bmatrix}. \tag{2.5}$$

Let $\overline{y}(t)$ be the output vector of interest. In most practical cases of interconnect analysis, the node or branch voltages or currents at purely resistive or *R–L* branches are of interest, and the output equations may be written in the following form:

$$\overline{y}(t) = C^T\,\overline{x}(t). \tag{2.6}$$

In the typical case, when an interconnect is among several subcircuits and its reduced model is needed, we regard it as a multiport, and the port impedance (or admittance) matrix is of interest. In this case

$$\overline{y}(t) = v_s(t) = A_s^T v_n(t) = B^T\,\overline{x}(t) \tag{2.7}$$

where $v_s(t)$ is the vector of the source voltage of the current sources, and the matrices C and B in the output and state equations are the same.

We assume that $M,\ N \in R^{n\times n}$, $B \in R^{n\times m}$ and $C \in R^{n\times p}$ throughout the paper.

The Laplace transform of (2.2) and (2.6) are the state and output equations in the frequency domain, and can be written as follows:

$$(sM + N)\,\overline{X}(s) = B\,\overline{I}_s(s) \tag{2.8}$$

and

$$\overline{Y}(s) = C^T\,\overline{X}(s). \tag{2.9}$$

When model-order reduction is concerned, the impulse response of the circuit is of the most interest, as the response due to any other input waveforms is uniquely determined by its impulse response. Let $i_s(t) = [\overline{i}_s^{\,1}(t),\, \overline{i}_s^{\,2}(t),\, \ldots,\, \overline{i}_s^{\,m}(t)]^T$, where

$\bar{i}_s^{\,i}(t) = e_i\delta(t)$, e_i is the ith unit vector, and $\delta(t)$ is the unit impulse function. Let the state and output vector corresponding to $\bar{i}_s^{\,i}(t)$ be $\overline{x}^{\,i}(t)$ and $\overline{y}^{\,i}(t)$, respectively. Let

$$x(t) = \left[\overline{x}^{\,1}(t), \overline{x}^{\,2}(t), \ldots, \overline{x}^{\,m}(t)\right] \tag{2.10}$$

and

$$y(t) = \left[\overline{y}^{\,1}(t), \overline{y}_2(t), \ldots, \overline{y}^{\,m}(t)\right]. \tag{2.11}$$

Then, from (2.2), (2.6), (2.8) and (2.9), we have

$$M\,\frac{dx(t)}{dt} + Nx(t) = B\delta(t) \tag{2.12}$$

$$y(t) = C^T x(t) \tag{2.13}$$

$$(sM + N)X(s) = B \tag{2.14}$$

$$Y(s) = C^T X(s). \tag{2.15}$$

These are the block forms of the state and output equations in the time and frequency domains, respectively, where $x(t) \in R^{n\times m}$ and $y(t) = R^{p\times m}$. When $C = B$, from (2.14) and (2.15)

$$Y(s) = B^T(sM + N)^{-1}B \tag{2.16}$$

and $Y(s)$ is the input impedance matrix of the network.

Now, we consider the projection-based model-order reduction. Let $V \in R^{n\times q}$ such that

$$x(t) = V\hat{x}(t) \tag{2.17}$$

where $\hat{x}(t) \in R^{q\times m}$. Then, V is the transformation matrix mapping the n-dimensional state–space to a q-dimensional space. Substitute (2.17) with (2.12), and premultiply V^T on both sides of the equations, we have

$$\hat{M}\,\frac{d\hat{x}(t)}{dt} + \hat{N}\hat{x}(t) = \hat{B}\delta(t) \tag{2.18}$$

where $\hat{M} = V^T MV$, $\hat{N} = V^T NV$ and $\hat{B} = V^T B$. This is the block form of the state equations of the reduced model. The output equations of the reduced model are in the form of

$$\hat{y}(t) = \hat{C}^T\hat{x}(t) \tag{2.19}$$

where $\hat{C}^T = C^T V$. The corresponding equations in the frequency domain are easily written and omitted.

III. Moment Matching Method

Expand $X(s)$ into a power series of variable $s - s_0$, where s_0 may be 0, positive real or a complex number with nonnegative real part

$$X(s) = \sum_{i=0}^{\infty} X^{(i)}(s_0)(s - s_0)^i \tag{3.1}$$

then

$$X^{(i)}(s_0) = \frac{1}{i!}\,\left.\frac{d^i X(s)}{ds^i}\right|_{s=s_0} \tag{3.2}$$

is called the ith moment of $X(s)$ at s_0. As for an RLC interconnect, $X(s)$ is analytic in the closed RHS of the s-plane, so that $X^{(i)}(s_0)$ is well defined for any s_0 in the region. On the other hand, let

$$X(s) = \sum_{i=1}^{\infty} X^{(i)}(\infty)s^{-i}. \tag{3.3}$$

Then, $X^{(i)}(\infty)$ is called the ith-order moment of $X(s)$ at $s = \infty$. The moments of $Y(s)$ are similarly defined, and

$$Y^{(i)}(s_0) = C^T X^{(i)}(s_0). \tag{3.4}$$

In the time domain, from

$$X(s) = \int_0^{\infty} x(t)e^{-st}\,dt \tag{3.5}$$

we have

$$X^{(i)}(s_0) = \frac{1}{i!}\int_0^{\infty}(-t)^i x(t)e^{-s_0 t}\,dt. \tag{3.6}$$

From

$$L\left[\frac{df(t)}{dt}\right] = sF(s) - f(0_+) \tag{3.7}$$

$$\lim_{s\to\infty} sF(s) = f(0+) \tag{3.8}$$

we have

$$X^{(i)}(\infty) = \left.\frac{d^{i-1}x(t)}{dt^{i-1}}\right|_{t=0_+}. \tag{3.9}$$

For $s_0 \neq \infty$, we have

$$X^{(0)}(s_0) = (s_0 M + N)^{-1}B \tag{3.10}$$

and

$$\begin{aligned} X^{(i)}(s_0) &= -(s_0 M + N)^{-1}MX^{(i-1)}(s_0) \\ &= \left[-(s_0M+N)^{-1}M\right]^i (s_0M+N)^{-1}B. \end{aligned} \tag{3.11}$$

For $s_0 = \infty$

$$X^{(1)}(\infty) = M^{-1}B \tag{3.12}$$

and for $i > 1$

$$X^{(i)}(\infty) = -M^{-1}NX^{(i-1)}(\infty) = \left[-M^{-1}N\right]^{i-1}M^{-1}B. \tag{3.13}$$

The moments can be computed very efficiently by using the above recursive formulas.

Now we are concerned about moment matching of the output vector of the reduced system with that of the original system.

Lemma 1: Let V be the transformation matrix for the generation of the reduced system. Suppose that

$$X^{(i)}(s_0) \in \text{colspan}(V) \tag{3.14}$$

where $i = 0 - m$ for a finite s_0 and $i = 1 - m$ with $s_0 = \infty$. Then

$$X^{(i)}(s_0) = V\hat{X}^{(i)}(s_0). \tag{3.15}$$

Proof: We give a proof for a finite s_0. The case where $s_0 = \infty$ is similar, and is omitted. We do similar treatments in the proof of the rest lemmas and theorems without declarations.

From the condition of the lemma, as $X^{(i)}(s_0) \in \text{colspan}(V)$ ($i = 0 - m$), there exist $\tilde{X}^{(i)}$s such that

$$X^{(i)}(s_0) = V\tilde{X}^{(i)}, \qquad i = 0 - m. \tag{3.16}$$

Now, we prove that $\tilde{X}^{(i)} = \hat{X}^{(i)}(s_0)$ for $i = 0 - m$.

Rewrite (3.10) and (3.11) in the following form:

$$\begin{bmatrix} s_0M+N & & & & \\ M & s_0M+N & & & \\ & M & s_0M+N & & \\ & & \cdots & \cdots & \\ & & & M & s_0M+N \end{bmatrix} \cdot \begin{bmatrix} X^{(0)}(s_0) \\ X^{(1)}(s_0) \\ X^{(2)}(s_0) \\ \cdots \\ X^{(m)}(s_0) \end{bmatrix} = \begin{bmatrix} B \\ 0 \\ 0 \\ \cdots \\ 0 \end{bmatrix}. \tag{3.17}$$

Substitute (3.16) with (3.17) and premultiply matrix W^T on both sides of the equations with

$$W = \text{diag}(\text{V}, \text{V}, \ldots, \text{V}) \tag{3.18}$$

then, we have the following equations:

$$\begin{bmatrix} s_0\hat{M}+\hat{N} & & & & \\ \hat{M} & s_0\hat{M}+\hat{N} & & & \\ & \hat{M} & s_0\hat{M}+\hat{N} & & \\ & & \cdots & \cdots & \\ & & & \hat{M} & s_0\hat{M}+\hat{N} \end{bmatrix} \cdot \begin{bmatrix} \tilde{X}^{(0)} \\ \tilde{X}^{(1)} \\ \tilde{X}^{(2)} \\ \cdots \\ \tilde{X}^{(m)} \end{bmatrix} = \begin{bmatrix} \hat{B} \\ 0 \\ 0 \\ \cdots \\ 0 \end{bmatrix}. \tag{3.19}$$

Note that the matrices on both sides of the above equations are just the same for the equations with the moments $\hat{X}^{(i)}(s_0)$ for $i = 0 - m$. From the uniqueness of the solution of the equations, it is known that $\hat{X}^{(i)}(s_0) = \tilde{X}^{(i)}$ and the lemma exists. □

Theorem 1: Under the condition of Lemma 1

$$\hat{Y}^{(i)}(s_0) = Y^{(i)}(s_0), \qquad i = 0 - m. \tag{3.20}$$

Proof:

$$\begin{aligned}\hat{Y}^{(i)}(s_0) &= \hat{C^T}\hat{X}^{(i)}(s_0) = C^TV\hat{X}^{(i)}(s_0) \\ &= C^TX^{(i)}(s_0) = Y^{(i)}(s_0).\end{aligned}$$

Theorem 2: In the case that matrices M and N in the state equations are symmetric, which happens in an *RC* interconnect, then under the condition of Lemma 1

$$\hat{Y}^{(i)}(s_0) = Y^{(i)}(s_0). \tag{3.21}$$

When s_0 is finite, (3.21) exists for $i = 0 - (2m + 1)$, and when $s_0 = \infty$, it exists for $i = 1 - 2m$.

Proof: Theorem 2 is a special case of Theorem 11 in Section V-B, and the detailed proof is omitted.

Let $S = \{s_1, s_2, \ldots, s_k\}$ be the set of matching points with $s_i \neq s_j$ when $i \neq j$, and $D = \{d_1, d_2, \ldots, d_k\}$ be the set of matching orders (when s_i is finite, moment matching is from order 0 to d_i; and when $s_i = \infty$, moment matching is from order 1 to $d_i + 1$). Suppose that the first q s_is are real or ∞, and the rest ones are complex. We give the following algorithm for generation of a real transformation matrix V for the model-order reduction with moment matching at the matching point set S with the matching order set D.

Algorithm 1: Moment-Matching Method

```
{Input: M, N, B, S, D, k;q
  Output: Transformation matrix V;
  nn = 0; W = φ;
  for i = 1 to k do
    {if (s_i ≠ ∞)
      {A = (s_i M + N); C = −M;}
    else
      {A = M; C = −N;}
    solve Ar = B for r; /* r has m columns*/
    nn = orthonormal(r, W, nn, m);
    for j = 1 to d_i do
      {solve Ar = CW(;nn − m + 1: nn)¹ for r;
      nn = orthonormal(r, W, nn, m);
      }
  }
  for i = 1 to m * q do
    v_i = w_i;
  if (q ≠ k)
    {ii = m * q + 1;
    for i = m * q + 1 to nn do
      {v_ii = real(w_i); v_ii+1 = imag(w_i); ii = ii + 2;}
    }
}
```

The function $orthonormal()$ uses the modified Gram–Shmidt algorithm and can be described as follows, where $\langle r, u\rangle = (r^*)^T u$ stands for the inner product of two vectors r and u.

```
orthonormal(r, W, nn, m)
{for i = 1 to m do
{for j = 1 to nn do
  {a = <w_j, r_i>;
    r_i = r_i − a * w_j;
  }
  a = sqrt(<r_i, r_i>);
  nn = nn + 1;
  w_nn = r_i/a;
}
return(nn);}
```

[1] $W(;cola:\ colb)$ consists of the column vectors of W from column $cola$ to column $colb$.

There is a typical case that for each complex s_i, $d_i = 0$. In such a case, *Algorithm 1* may be simplified to *Algorithm 1.1*, where the operation with complex numbers in $orthonormal()$ is avoided and the algorithm becomes more efficient.

```
Algorithm 1.1: Simplified Moment-Matching
Method
{Input: M, N, B, S, D, k;
  Output: Transformation matrix V;
  nn = 0; V = φ;
  for i = 1 to k do
    {if (s_i ≠ ∞)
      {A = (s_i M + N); C = −M;}
    else
      {A = M; C = −N;}
    solve Ar = B for r;
    if s_i is real or ∞
    {nn = orthonormal(r, V, nn, m);
      for j = 1 to d_i do
      {solve Ar = CV(; nn − m + 1: nn) for r;
      nn = orthonormal(r, V, nn, m);
      }
    }
    else /* s_i is complex with d_i = 0 */
      {nn = orthonormal(real(r), V, nn, m);
      nn = orthonormal(imag(r), V, nn, m);
      }
  }
}
```

We now show that $X^{(j_i)}(s_i)$s [$i = 1 - k$, $j_i = 0 - d_i$ when s_i is finite, and $j_i = 1 - (d_i + 1)$ when $s_i = \infty$] are covered by colspan(V) so that *Algorithm 1* guarantees moment matching with matching point set S and order set D. To prove it, we need the following two lemmas.

Lemma 2: In the case that s_i is finite, let $A_i = (s_i M + N)^{-1} M$ and $q_i = (s_i M + N)^{-1} B$; and in the case that $s_i = \infty$, let $A_i = M^{-1} N$ and $q_i = M^{-1} B$. Denote

$$K(i, j) = \{q_i, A_i q_i, \ldots, A_i^j q_i\}. \tag{3.22}$$

Then, for $1 \le l < i$ and $0 \le j \le d_l$,

$$A_i A_l^j q_l \in K(i, 0) \bigcup_{m=1}^{i-1} K(m, d_m). \tag{3.23}$$

Lemma 3: For $1 \le i \le k$ and $0 \le j \le d_i$

$$K(i, j) \in \text{colspan}(W(i, j)) \tag{3.24}$$

where $W(i, j) = \{w_1, w_2, \ldots, w_n\}$ and $n = m(\sum_{t=1}^{i-1}(d_t + 1) + j + 1)$.

The proof of Lemmas 2 and 3 follows a way similar to the proof of Lemma 1, and [9, Theorem 2] and is omitted.

Theorem 3: For $i = 1 - k$ and $j = 0 - d_i$ for finite s_i and $j = 1 - (d_i + 1)$ when $s_i = \infty$,

$$X^{(j)}(s_i) \in \text{colspan}(V). \tag{3.25}$$

Proof: From Lemma 3, $X^{(j)}(s_i) \in \text{colspan}(W)$. It is obvious from *Algorithm 1* that colspan(W) = colspan(V), so Theorem 3 exists. □

Note that for a complex s_i, when $X^{(j)}(s_i) \in \text{colspan}(V)$, then its real part and imaginary part are in colspan(V), respectively. Therefore, its complex conjugate $\overline{X}^{(j)}(s_i) = X^{(j)}(\overline{s}_i) \in \text{colspan}(V)$. This means when s_i is a matching point with an order up to d_i, $\overline{s}_i$ is also a matching point with the same order.

The moment-matching model-order reduction preserves the moments at given points with given order from the original network. As the power-series expansion at a point will generate a good approximation of the frequency response near this point, so a single-point moment-matching method often needs a very high order to make the frequency response of the reduced model approximate that of the original system well in a wide range of frequencies. On the other hand, when multipoint moment-matching method is used, some lower order model may be generated to obtain the same degree of approximation over the same frequency range.

IV. Coefficient Matching Method With Series Expansion Based on Orthonormal Polynomials

The moment-matching method is based on matching the coefficients of the power series expansion of complex frequency s at some specified points. In approximation theory, there are better series expansions to approximate a function in certain interval, among which the orthonormal polynomials are widely used. In this section, we introduce the use of Chebyshev polynomials to do model-order reduction in both frequency and time domains [6], [7]. The series expansion of Chebyshev polynomials has the property of exponential convergence rate, so that lower order approximation may be obtained with high accuracy. Also, among the same degree of polynomials approximating a given function in a finite interval, the truncated Chebyshev series is very close to the minimax approximate polynomial, so that the truncated Chebyshev series expansion is nearly optimal in the minimax sense.

A. Frequency-Domain Approximation

In the case that the frequency response of a system is of interest, let $s = j\omega$ in (2.14), then, the state equations in the frequency domain are expressed as

$$(j\omega M + N)X(j\omega) = B \tag{4.1}$$

and the output equations are expressed as

$$Y(j\omega) = C^T X(j\omega). \tag{4.2}$$

Suppose that the maximum frequency of interest is $\omega_{\max}$. For $X(j\omega)$, the whole range of interest is $\omega \in [-\omega_{\max}, \omega_{\max}]$. We normalize the frequency by letting

$$\overline{\omega} = \frac{\omega}{\omega_{\max}} \tag{4.3}$$

so that $\overline{\omega} \in [-1, 1]$, which is the range where Chebyshev polynomials are defined.

Let $\overline{M} = \omega_{\max} M$, $\overline{N} = N$, $\overline{X}(j\varpi) = X(j\omega)$, $\overline{Y}(j\varpi) = Y(j\omega)$, then (4.1) and (4.2) become

$$\left(j\,\varpi\,\overline{M} + \overline{N}\right)\,\overline{X}(j\,\varpi) = B \tag{4.4}$$

and

$$\overline{Y}(j\,\varpi) = C^T\,\overline{X}(j\,\varpi). \tag{4.5}$$

Let $\overline{X}(j\,\varpi)$ be expanded into a series expansion of Chebyshev polynomials $T_i(\varpi)$ up to an order of K

$$\overline{X}(j\,\varpi) = \sum_{i=0}^{K} C_i T_i(\varpi). \tag{4.6}$$

We call the coefficient matrix C_i the ith Chebyshev coefficient matrix. Note that $T_i(\varpi)$ is a polynomial of variable ϖ with real coefficients, and the Chebyshev coefficient matrices are complex matrices in the general case.

To compute the Chebyshev coefficient matrices, note that

$$\varpi\,T_0(\varpi) = T_1(\varpi) \tag{4.7}$$

and for $i > 0$

$$\varpi\,T_i(\varpi) = \tfrac{1}{2}\left(T_{i-1}(\varpi) + T_{i+1}(\varpi)\right). \tag{4.8}$$

Then

$$\begin{aligned}\varpi\,\overline{X}(j\,\varpi) &= \tfrac{1}{2}\,\overline{C}_1 T_0(\varpi) + \left(\overline{C}_0 + \tfrac{1}{2}\,\overline{C}_2\right) T_1(\varpi)\\ &\quad + \sum_{i=2}^{K-1} \tfrac{1}{2}\cdot(\overline{C}_{i-1} + \overline{C}_{i+1})T_i(\varpi) + \tfrac{1}{2}\,\overline{C}_{K-1}T_K(\varpi).\end{aligned} \tag{4.9}$$

Substitute (4.6) and (4.9) into (4.4), and let the coefficient matrices of the same $T_i(\varpi)$ be equal, we have

$$\begin{bmatrix} \overline{N} & 0.5j\,\overline{M} & & & & \\ j\,\overline{M} & \overline{N} & 0.5j\,\overline{M} & & & \\ & 0.5j\,\overline{M} & \overline{N} & 0.5j\,\overline{M} & & \\ & & \cdots & \cdots & & \\ & & & & 0.5j\,\overline{M} & \overline{N} \end{bmatrix} \cdot \begin{bmatrix} C_0 \\ C_1 \\ C_2 \\ \cdots \\ C_K \end{bmatrix} = \begin{bmatrix} B \\ 0 \\ 0 \\ \cdots \\ 0 \end{bmatrix}. \tag{4.10}$$

This set of equations is of the block tridiagonal form, and can be solved efficiently by using LU decomposition. The degree K may be chosen such that

$$\frac{\|C^T C_K\|_\infty}{\max\limits_{0\le i<K} \|C^T C_i\|_\infty} < \epsilon \tag{4.11}$$

with a given ϵ. As $|T_i(\varpi)| \le 1$, such a condition can guarantee a good approximation of the truncated Chebyshev series expansion.

If it is found that K is not big enough to meet the condition of (4.11), we may increase K one by one until the condition is met. When K is increased by 1, in the RHS matrix of (4.10), n rows with zero elements are added. In the l.h.s. coefficient matrix, n columns and n rows are added, with matrix $0.5j\,\overline{M}$ inserted into the positions of $(nK+1\colon n(K+1),\, n(K+1)+1\colon n(K+2))$ and $(n(K+1)+1\colon n(K+2),\, nK+1\colon n(K+1))$,[2] and matrix $\overline{N}$ inserted into the positions of $(n(K+1)+1\colon n(K+2),\, n(K+1)+1\colon n(K+2))$. The LU decomposition for the old blocks remains unchanged, only the operation for the additional three blocks is needed. Also, the first $n(K+1)$ solutions to the equations $Lz = [B^T,\, 0,\, \ldots,\, 0]^T$ remain unchanged. These properties make the computation very efficient when the order of approximation is increased.

Another way for the computation of the Chebyshev coefficient matrices is by pseudospectral Chebyshev method [18]. When degree K is determined, let

$$\varpi_k = \cos\frac{\pi k}{K}, \qquad k = 0,\, 1,\, \ldots,\, K \tag{4.12}$$

which are the extrema points of $T_K(\varpi)$. Let $a_0 = a_K = 2$ and $a_k = 1$ for $0 < k < K$. Then

$$C_k = \frac{2}{Ka_k}\sum_{i=0}^{K}\frac{\overline{X}(j\,\varpi_i)}{a_i}\cos\frac{\pi ik}{K}. \tag{4.13}$$

As $\overline{X}(j\,\varpi_i) = X(j\omega_{\max}\varpi_i)$, the computation of the Chebyshev coefficient matrices amounts to the moment computation together with a fast Fourier transform (FFT), which can be done more efficiently than the previous method. Another advantage of the pseudospectral method is that

$$\sum_{k=0}^{K} C_k T_k(\varpi_i) = \overline{X}(j\,\varpi_i), \qquad i = 0,\, 1,\, \ldots,\, K \tag{4.14}$$

i.e., the truncated Chebyshev expansion is exact at the collocation points specified by (4.12). The disadvantage of this method is that when degree K is changed, the computation cannot be done incrementally but should be started from the very beginning.

The model-order reduction based on matching the Chebyshev coefficient matrices is provided by the following lemma and theorem.

Lemma 4: Suppose that $\overline{X}(j\,\varpi)$ and $\hat{X}(j\,\varpi)$ are the matrices of the state variables of the original system and reduced system, respectively, and V is the transformation matrix. Let C_k and $\hat{C}_k$ be the kth Chebyshev coefficient matrices of the Chebyshev series expansions of $\overline{X}(j\,\varpi)$ and $\hat{X}(j\,\varpi)$, respectively. If

$$C_k \in \text{colspan}(V), \qquad k = 0,\, 1,\, \ldots,\, K. \tag{4.15}$$

Then,

$$C_k = V\hat{C}_k, \qquad k = 0,\, 1,\, \ldots,\, K. \tag{4.16}$$

Theorem 4: Let $\overline{Y}(j\,\varpi)$ and $\hat{Y}(j\,\varpi)$ be the output functions of the original and reduced system, respectively. Let C_{Yk} and $\hat{C}_{Yk}$ be the kth Chebyshev coefficient matrices of the Chebyshev series expansions of these two functions, respectively. Then, under the condition of Lemma 4

$$\hat{C}_{Yk} = C_{Yk}, \qquad k = 0,\, 1,\, \ldots,\, K \tag{4.17}$$

[2] $(row1 : row2,\, col1 : col2)$ refers to a block in the rows between $row1$ and $row2$ and in the columns between $col1$ and $col2$.

i.e., the reduced order model preserves the Chebyshev coefficient matrices for the output functions from order 0 up to the order of K.

The proof of Lemma 4 can be done by using (4.10) and following the same way as used in the proof of Lemma 1, and the proof of Theorem 4 follows the same way as in the proof of Theorem 1.

The model-order reduction algorithm with the preservation of Chebyshev coefficient matrices can be stated as follows, where K_0 is the initial guess of the approximation order.

```
Algorithm 2: Frequency-domain model-order
reduction with preservation of Chebyshev
coefficient matrices
{Input: M, N, B, C, ω_max, ε, K_0;
  Output: Transformation matrix V;
  Let M = ω_max * M;
  for [K = K_0; if inequality (4.11) is not
  met; K++]
    Compute C_i, i = 0, 1, ..., K;
  nn = 0; V = φ;
  for i = 0 to K do
  {nn = orthonormal(real(C_i), V, nn, m);
    nn = orthonormal(imag(C_i), V, nn, m);
  }
}.
```

B. Time-Domain Approximation

When Chebyshev polynomials are used to approximate $x(t)$, we should first normalize the time variable to the interval $\bar{t} \in [-1, 1]$. Let $t \in [0, t_{\max}]$ be the time interval of interest. $t_{\max}$ may be chosen as the simulation time; and in most practical cases, when the input signal is a pulse, $t_{\max}$ may be chosen as the sum of pulse width, rising, and falling time. Let $\bar{t} = 2(t/t_{\max} - 0.5)$, $\tilde{M} = (2/t_{\max})M$, $\tilde{N} = N$, $\tilde{x}(\bar{t}) = x(t)$ and $\tilde{y}(\bar{t}) = y(t)$. Then, the state equations of the system equation (2.12) becomes

$$\tilde{M}\,\frac{d\tilde{x}(\bar{t})}{d\bar{t}} + \tilde{N}\tilde{x}(\bar{t}) = B\delta(\bar{t}+1) \tag{4.18}$$

and the output equations become

$$\tilde{y}(\bar{t}) = C^T\tilde{x}(\bar{t}). \tag{4.19}$$

We integrate both sides of (4.18) from -1_{-0} to some $\bar{t}$, and have

$$\tilde{M}\tilde{x}(\bar{t}) + \tilde{N}\int_{-1}^{\bar{t}} \tilde{x}(\tau)\,d\tau = B. \tag{4.20}$$

Let

$$\tilde{x}(\bar{t}) = \sum_{k=0}^{K} Q_k T_k(\bar{t}). \tag{4.21}$$

From the formulas of the integrals of the Chebyshev polynomials

$$\int_{-1}^{\bar{t}} T_0(\tau)\,d\tau = T_0(\bar{t}) + T_1(\bar{t}) \tag{4.22}$$

$$\int_{-1}^{\bar{t}} T_1(\tau)\,d\tau = \tfrac{1}{4}\left(T_2(\bar{t}) - T_0(\bar{t})\right). \tag{4.23}$$

and for $k > 1$

$$\int_{-1}^{\bar{t}} T_k(\tau)\,d\tau = \frac{1}{2}\left(\frac{T_{k+1}(\bar{t})}{k+1} - \frac{T_{k-1}(\bar{t})}{k-1}\right) + \frac{(-1)^{k+1}}{k^2-1} \tag{4.24}$$

we have

$$\begin{aligned}
\int_{-1}^{\bar{t}} &\tilde{x}(\tau)\,d\tau \\
&= \sum_{k=0}^{K} Q_k \int_{-1}^{\bar{t}} T_k(\tau)\,d\tau \\
&= \left(Q_0 - \frac{1}{4}\,Q_1 + \sum_{k=2}^{K} \frac{(-1)^{k+1}Q_k}{k^2-1}\right) T_0(\bar{t}) \\
&\quad + \left(Q_0 - \frac{1}{2}\,Q_2\right) T_1(\bar{t}) + \sum_{k=2}^{K-1} \frac{1}{2k}\,(Q_{k-1} - Q_{k+1})T_k(\bar{t}) \\
&\quad + \frac{1}{2K}\,Q_{K-1}T_K(\bar{t}).
\end{aligned} \tag{4.25}$$

Substitute (4.21) and (4.25) into (4.20), and let the coefficient matrices with the same $T_k(\bar{t})$ $(k = 0 - K)$ on the both sides of the equations be equal, we have the equations of the Chebyshev coefficient matrices Q_ks as shown in (4.26) at the bottom of the page.

$$\begin{bmatrix}
\tilde{M} & \frac{1}{2K}\tilde{N} & & & & & \\
-\frac{1}{2(K-1)}\tilde{N} & \tilde{M} & \frac{1}{2(K-1)}\tilde{N} & & & & \\
 & -\frac{1}{2(K-2)}\tilde{N} & \tilde{M} & \frac{1}{2(K-2)}\tilde{N} & & & \\
 & & \cdots & \cdots & \cdots & & \\
 & & & & -\frac{1}{2}\tilde{N} & \tilde{M} & \tilde{N} \\
\frac{(-1)^{K+1}}{K^2-1}\tilde{N} & \frac{(-1)^{K}}{(K-1)^2-1}\tilde{N} & \frac{(-1)^{K-1}}{(K-2)^2-1}\tilde{N} & \cdots & -\frac{1}{3}\tilde{N} & -\frac{1}{4}\tilde{N} & \tilde{M}+\tilde{N}
\end{bmatrix}
\begin{bmatrix} Q_K \\ Q_{K-1} \\ \cdots \\ Q_2 \\ Q_1 \\ Q_0 \end{bmatrix}
=
\begin{bmatrix} 0 \\ 0 \\ 0 \\ \cdots \\ 0 \\ B \end{bmatrix} \tag{4.26}$$

The arrangement of the equations in reverse order (from K to 0) aims at avoiding the "block" fill-ins (an original empty block is filled by a nonempty block) during the LU decomposition of the coefficient matrix of the equations. By using such an arrangement, the block structure of the coefficient matrix remains unchanged after LU decomposition; otherwise, if the equations are arranged in a forward order from order 0 to order K, then after LU decomposition, the block structure will be upper_Hassenburg. Note that in the coefficient matrix, each submatrix $\tilde{N}$ has a coefficient depending on the value of K. When the order of approximation K is changed, the formation of and solution to (4.26) should be restarted. Therefore, the implementation of the time-domain model-order reduction based on the Chebyshev approximation is not as efficient as that of the frequency-domain method.

The pseudospectral Chebyshev method can also be used here to compute the Chebyshev coefficient matrices. When degree K is determined, let

$$\bar{t}_k = \cos\frac{\pi k}{K}, \qquad k = 0, 1, \ldots, K \tag{4.27}$$

which are the extrema points of $T_K(\bar{t})$. Let $a_0 = a_K = 2$ and $a_k = 1$ for $0 < k < K$. Then

$$Q_k = \frac{2}{Ka_k}\sum_{i=0}^{K}\frac{\tilde{x}(\bar{t}_i)}{a_i}\cos\frac{\pi ik}{K}. \tag{4.28}$$

When $\tilde{x}(\bar{t}) = \sum_{i=0}^{K} Q_i T_i(\bar{t})$, the derivatives of $\tilde{x}(\bar{t})$ at the collocation points can be expressed as follows [18]:

$$\left.\frac{d\tilde{x}(\bar{t})}{d\bar{t}}\right|_{\bar{t}=\bar{t}_i} = \sum_{k=0}^{K} d_{ik}\tilde{x}(\bar{t}_k) \tag{4.29}$$

where

$$d_{00} = \frac{2K^2+1}{6} = -d_{KK} \tag{4.30}$$

$$d_{kk} = -\frac{\bar{t}_k}{2\left(1-\bar{t}_k^{\,2}\right)}, \qquad k \neq 0, K \tag{4.31}$$

$$d_{ik} = \frac{(-1)^{i+k}a_i}{a_k(\bar{t}_i - \bar{t}_k)} \qquad i \neq k. \tag{4.32}$$

From (4.18), we have

$$\tilde{M}\tilde{x}(\bar{t}_K) = \tilde{M}\tilde{x}(-1) = B \tag{4.33}$$

and

$$\tilde{M}\frac{d\tilde{x}(\bar{t})}{d\bar{t}} + \tilde{N}\tilde{x}(\bar{t}) = 0, \qquad \bar{t} > -1. \tag{4.34}$$

We list (4.34) at the first K collocation points $\bar{t}_i$ $(0 \le i < K)$. From (4.29), we have

$$\tilde{M}\sum_{k=0}^{K} d_{ik}\tilde{x}(\bar{t}_k) + \tilde{N}\tilde{x}(\bar{t}_i) = 0. \tag{4.35}$$

By combining (4.33) and (4.35) for $i = 0 - K - 1$, we have the following set of equations:

$$\begin{bmatrix} d_{00}\tilde{M}+\tilde{N} & d_{01}\tilde{M} & \cdots & d_{0,K-1}\tilde{M} \\ d_{10}\tilde{M} & d_{11}\tilde{M}+\tilde{N} & \cdots & d_{1,K-1}\tilde{M} \\ & \cdots & \cdots & \\ d_{K-1,0}\tilde{M} & d_{K-1,1}\tilde{M} & \cdots & d_{K-1,K-1}\tilde{M}+\tilde{N} \end{bmatrix} \cdot \begin{bmatrix} \tilde{x}(\bar{t}_0) \\ \tilde{x}(\bar{t}_1) \\ \cdots \\ \tilde{x}(\bar{t}_{K-1}) \end{bmatrix} = \begin{bmatrix} -d_{0K}B \\ -d_{1K}B \\ \cdots \\ -d_{K-1,K}B \end{bmatrix}. \tag{4.36}$$

The coefficient matrix of the above equations is full in the block form, which makes it inefficient to find the solution.

The preservation of the Chebyshev coefficient matrices during the model-order reduction is based on Lemma 5 and Theorem 5. The proof of Lemma 5 and Theorem 5 is similar to that of Lemma 1 and Theorem 1, and is omitted.

Lemma 5: Suppose that $\tilde{x}(\bar{t})$ and $\hat{x}(\bar{t})$ are the matrices of the state variables of the original system and reduced system, respectively, and V is the transformation matrix. Let Q_k and $\hat{Q}_k$ be the kth Chebyshev coefficient matrices of the Chebyshev series expansions of $\tilde{x}(\bar{t})$ and $\hat{x}(\bar{t})$, respectively. If

$$Q_k \in \text{colspan}(V), \qquad k = 0, 1, \ldots, K. \tag{4.37}$$

Then,

$$Q_k = V\hat{Q}_k \qquad k = 0, 1, \ldots, K. \tag{4.38}$$

Theorem 5: Let $\tilde{y}(\bar{t})$ and $\hat{y}(\bar{t})$ be the output functions of the original and reduced system, respectively. Let Q_{Yk} and $\hat{Q}_{Yk}$ be the kth Chebyshev coefficient matrices of the Chebyshev series expansions of these two functions, respectively. Then, under the condition of Lemma 5

$$\hat{Q}_{Yk} = Q_{Yk} \qquad k = 0, 1, \ldots, K \tag{4.39}$$

i.e., the reduced order model preserves the Chebyshev coefficient matrices for the output functions from order 0 up to the order of K.

The selection of the order K can also be based on the inequality of (4.11) with C_K and C_i replaced by Q_K and Q_i, respectively. As mentioned above, to increase K one by one and compute the Q_ks each time until (4.11) is satisfied is not efficient. Therefore, we use an approximate method. After Q_k $(k = 0, 1, \ldots, K)$ is obtained for an order K, when the order is increased to $K+1$, let Q_k' $(k = 0, 1, \ldots, K+1)$ be the coefficients under the new order. Then, we simply let $Q_k' = Q_k$ for $k = 0, 1, \ldots, K$, and solve the equation

$$2(K+1)\tilde{M}Q_{K+1} + \tilde{N}Q_K = 0 \tag{4.40}$$

for Q_{K+1}. After K is determined, then we solve (4.26) for the Q_ks.

The time-domain model-order reduction algorithm with the preservation of Chebyshev coefficient matrices can be stated as follows, where K_0 is the initial guess of the approximation order.

```
Algorithm 3: Time-domain model-order
reduction with preservation of Chebyshev
coefficient matrices
{Input: M, N, B, C, t_max, ε, K_0;
  Output: Transformation matrix V;
  Let M = 2/t_max * M;
  Start from K_0, use approximate method to
  determine K;
  Compute Q_i, i = 0, 1, ..., K;
  nn = 0; V = φ;
  for i = 0 to K do
    nn = orthonormal(Q_i, V, nn, m);
}.
```

V. Coefficient Matching Method With Series Expansion Based on Generalized Orthonormal Basis Functions in Hilbert and Hardy Space

In this section, we will use the generalized orthonormal basis functions in Hilbert and Hardy space for model reduction. In general, the work corresponds to the classical Caratheodory interpolation, and has been found useful in recent years in the fields of signal processing and system and control [23]–[26]. The main advantage of the basis function over Chebyshev polynomials is that the function parameters can be selected to fit the given system well and the series expansion of the state vector of the system can converge very quickly. In [27], the Laguerre function, which is a one-parameter basis function, is used for model reduction. We will use the multi-parameter function here, which works better than the one-parameter function.

A. Orthonormal Rational and Exponential Functions

The orthonormal rational functions in the frequency domain and the orthonormal exponential functions in the time domain are related to each other. They were provided by the pioneer work of W. H. Kautz [19] and extended later [20]. We review the case of real parameters functions here. For complex parameter functions, see the details in [19].

Let α_i, $i = 1, 2, \ldots, n, \ldots$ be positive real numbers. Then, the set of orthonormal basis functions are defined by

$$\Phi_n(s) = \frac{\sqrt{2\alpha_n}}{s+\alpha_n} \prod_{i=1}^{n-1} \frac{s-\alpha_i}{s+\alpha_i}. \tag{5.1}$$

The α_is in (5.1) may be different or identical.

The orthonormal set $\{\Phi_i(s)\}$ in the frequency domain corresponds to a set $\{\phi_i(t)\}$ in the time domain. In the case that all α_is are different

$$\phi_n(t) = \sum_{i=1}^{n} C_{ni} e^{-\alpha_i t} \tag{5.2}$$

where C_{ni} is the residue of $\Phi_n(s)$ at $s = -\alpha_i$. In the case that all α_is are the same, $\phi_n(t)$ is the $n-1$th-order Laguerre function with parameter α.

B. Convergence Property of Series Expansion of Exponential Function Via Orthonormal Basis Functions

An impulse response of a finite-order linear *RC* (*RL*) network consists of exponential functions with negative real exponents. Now, we consider one of its components $f(t) = e^{-\sigma t}$ with $\sigma > 0$. Let

$$f(t) = \sum_{i=0}^{\infty} F_i \phi_i(t). \tag{5.3}$$

Then

$$F_i = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(-j\omega)\Phi_i(j\omega)\, d\omega = \Phi_i(\sigma). \tag{5.4}$$

From this equation, we have the following results.

Theorem 6: If $\sigma = \alpha_i$. Then, for $k > i$

$$F_k = 0. \tag{5.5}$$

This theorem shows that when $\sigma = \alpha_i$, the expansion converges in finite terms.

Now, suppose that $\alpha_i < \alpha_{i+1}\ \forall\, i \ge 0$ and $\alpha_{p-1} < \sigma < \alpha_p$. For $k > p$

$$F_k = \frac{\sqrt{2\alpha_k}}{\sigma+\alpha_k} \prod_{q=1}^{k-1} \frac{\sigma-\alpha_k}{\sigma+\alpha_k}$$

and

$$F_{k+1} = \frac{\sqrt{2\alpha_{k+1}}}{\sigma+\alpha_{k+1}} \prod_{q=1}^{k} \frac{\sigma-\alpha_k}{\sigma+\alpha_k}.$$

Then

$$\left|\frac{F_{k+1}}{F_k}\right| = \left|\sqrt{\frac{\alpha_{k+1}}{\alpha_k}}\,\frac{\sigma-\alpha_k}{\sigma+\alpha_{k+1}}\right| = \sqrt{\frac{\alpha_{k+1}}{\alpha_k}}\,\frac{\alpha_k-\sigma}{\sigma+\alpha_{k+1}}$$
$$< \frac{\alpha_k}{\alpha_{k+1}}\sqrt{\frac{\alpha_{k+1}}{\alpha_k}} = \sqrt{\frac{\alpha_k}{\alpha_{k+1}}}. \tag{5.6}$$

From (5.6), we have the following theorem.

Theorem 7: If $\alpha_{p-1} < \sigma < \alpha_p$ and for $k \ge p$, $\alpha_k/\alpha_{k+1} \le \rho^2$, then

$$\left|\frac{F_{k+1}}{F_k}\right| < \rho. \tag{5.7}$$

In this case, the series expansion of $e^{-\sigma t}$ in terms of the orthonormal exponential basis functions converges exponentially.

In the case of Laguerre functions, If $\sigma = \alpha$, then $F_k = 0$ for $k > 1$, and the expansion converges at its first term. In the case that $\sigma \ne \alpha$, we have

$$|F_k| = \frac{\sqrt{2\alpha}}{\sigma+\alpha}\left|\frac{\sigma-\alpha}{\sigma+\alpha}\right|^{k-1}.$$

In this case, the expansion converges exponentially, and the convergence rate

$$\rho = \left|\frac{\sigma-\alpha}{\sigma+\alpha}\right| = \left|\frac{\beta-1}{\beta+1}\right| \tag{5.8}$$

where $\beta = \sigma/\alpha$. It can be seen that the closer β to 1, the more rapidly the expansion converges.

Now, we consider the general case that the whole impulse response is approximated by the Laguerre functions with parameter α. Suppose that the eigenvalues of the state matrix are $-\sigma_i$ $(i = 1 - n)$ with $\sigma_1 < \sigma_2 < \cdots < \sigma_n$. The convergence rate corresponding to the component $e^{-\sigma_i t}$ is denoted by

$$\rho(\sigma_i, \alpha) = \left| \frac{\sigma_i - \alpha}{\sigma_i + \alpha} \right|. \tag{5.9}$$

Given the σ_i's, it is needed to find an optimal value of $\alpha = \alpha_{\text{opt}}$ such that the maximum value of $\rho(\sigma_i, \alpha)$ w.r.t. all the σ_is is minimized. Let

$$\rho_{\text{opt}} = \min_{\alpha} \max_{\sigma_i} \rho(\sigma_i, \alpha) \tag{5.10}$$

and

$$\alpha_{\text{opt}} = \arg\min_{\alpha} \max_{\sigma_i} \rho(\sigma_i, \alpha). \tag{5.11}$$

We have the following theorem regarding the choice of α_{opt}.

Theorem 8: When

$$\alpha = \alpha_{\text{opt}} = \sqrt{\sigma_1 \sigma_n} \tag{5.12}$$

$$\rho_{\text{opt}} = \frac{\sqrt{\beta} - 1}{\sqrt{\beta} + 1} \tag{5.13}$$

where $\beta = \sigma_n / \sigma_1$. We omit the proof for brevity.

From the above theorems, we suggest to use the following method to select the parameters of the orthonormal basis functions for model reduction of *RC* and *RL* circuits. We first find the approximations of its largest and smallest eigenvalues $-\sigma_{\min}$ and $-\sigma_{\max}$ by using the Lanczos algorithm [28], [29]. Then, let $\alpha_1 = \sigma_{\min}$, $\alpha_p = \sigma_{\max}$,

$$q = \left(\frac{\sigma_{\max}}{\sigma_{\min}} \right)^{1/(p-1)} \tag{5.14}$$

and

$$\alpha_{i+1} = q\alpha_i, \qquad i = 1, 2, \ldots, p-2. \tag{5.15}$$

By using this selection, as $\alpha_1 = \sigma_{\min}$ and $\alpha_p = \sigma_{\max}$, the fastest and slowest components of the impulse response will be kept well in the impulse response of the reduced model and the expansion of other components converges exponentially, which will be benefit for a good approximation of the impulse response of the reduced model.

C. Preservation of Coefficients in Series Expansion of Orthonormal Basis Functions and Moment Matching

Now we consider an arbitrary impulse response $f(t)$, and expand it via the orthonormal exponential basis functions $\{\phi_i(t)\}$ such that

$$f(t) = \sum_{i=1}^{l} a_i \phi_i(t) \tag{5.16}$$

where

$$a_i = \int_0^\infty f(t)\phi_i(t)\, dt. \tag{5.17}$$

Suppose that the exponential basis functions are with parameters $\{s_i\}$s and suppose that the parameters are different. Then

$$a_i = \sum_{j=1}^{i} C_{ij} \int_0^\infty f(t) e^{-s_j t}\, dt = \sum_{j=1}^{i} C_{ij} F(s_j) \tag{5.18}$$

where $F(s)$ is the Laplace transform of $f(t)$, and $F(s_j)$ is the moment of $F(s)$ at $s = s_j$. Therefore, a_i is a linear combination of the moments of $F(s)$ at the points equal to the parameters of the basis functions. Note that the coefficients C_{ij}s are determined by the basis functions only, and have nothing to do with the function $f(t)$ itself. Therefore, it can be understood that the problem of forming a reduced order model with the preservation of the expansion coefficients of its impulse response can be transformed into an equivalent problem of model-order reduction with the corresponding moment matching. When the set $S = \{s_i\}$ of the parameters is given, *Algorithm 1* or *Algorithm 1.1* may be called to generate the transformation matrix V, and the new system can be formed by the congruence transform w.r.t. matrix V.

VI. Improvements in Grammian-Based Methods

A. Introduction

Another approach for model-order reduction is the grammian-based method.

Rewrite the state equations of (2.2) into the following form:

$$\frac{d\,\overline{x}(t)}{dt} = A\,\overline{x}(t) + Eu(t) \tag{6.1}$$

where $A = -M^{-1}N$, $E = M^{-1}B$ and $u(t) = \bar{i}_s(t)$, and the output equations

$$\overline{y}(t) = C^T \overline{x}(t). \tag{6.2}$$

The controllability grammian W_c and the observability grammian W_o are defined, respectively, as

$$W_c = \int_0^\infty e^{At} E E^T e^{A^T t}\, dt \tag{6.3}$$

and

$$W_o = \int_0^\infty e^{A^T t} C C^T e^{At}\, dt. \tag{6.4}$$

In (6.3), $h(t) \equiv e^{At}E$ is the impulse response of the state vector $x(t)$, and

$$W_c = \int_0^\infty h(t) h^T(t)\, dt. \tag{6.5}$$

Similarly, in (6.4), $\tilde{h}(t) \equiv e^{A^T t}C$ is the impulse response of the state vector of the dual system,[3] and

$$W_o = \int_0^\infty \tilde{h}(t) \tilde{h}^T(t)\, dt. \tag{6.6}$$

It can be seen from the above two equations that W_c and W_o are positive definite.

[3]See Section VII-B for the definition of the dual system.

The most commonly used approach to compute the grammians is to solve the following two Lyaponov equations:

$$AW_c + W_cA^T + EE^T = 0 \tag{6.7}$$

and

$$A^TW_o + W_oA + CC^T = 0 \tag{6.8}$$

which is the most expensive part in the computation cost for the grammian-based model-order reduction algorithms.

When a transformation $P \in R^{n\times n}$ is applied to the original system such that $\overline{x}(t) = P\hat{x}(t)$, we will have

$$\dot{\hat{x}}(t) = \hat{A}\hat{x} + \hat{E}u(t) \tag{6.9}$$

$$y(t) = \hat{C}^T\hat{x}(t) \tag{6.10}$$

where $\hat{A} = P^{-1}AP$, $\hat{E} = P^{-1}E$ and $\hat{C}^T = C^TP$, and the grammians in the new system become

$$\hat{W}_c = P^{-1}W_c(P^{-1})^T \tag{6.11}$$

and

$$\hat{W}_o = P^TW_oP. \tag{6.12}$$

The positive definiteness of $\hat{W}_c$ and $\hat{W}_o$ remains unchanged if P is of full rank.

With a proper choice of P, it can be made that

$$\hat{W}_c = \hat{W}_o = \Sigma^2 = \mathrm{diag}(\sigma_1^2, \sigma_2^2, \ldots, \sigma_n^2). \tag{6.13}$$

In such a case, the transformed system is called balanced. For a balanced system, if we partition the system matrices together with the grammians conformally such that

$$\hat{A} = \begin{bmatrix} \hat{A}_{11} & \hat{A}_{12} \\ \hat{A}_{21} & \hat{A}_{22} \end{bmatrix} \tag{6.14}$$

$$\hat{E} = \begin{bmatrix} \hat{E}_1 \\ \hat{E}_2 \end{bmatrix} \tag{6.15}$$

$$\hat{C}^T = \begin{bmatrix} \hat{C}_1^T & \hat{C}_2^T \end{bmatrix} \tag{6.16}$$

$$\hat{W}_c = \hat{W}_o = \begin{bmatrix} \Sigma_1^2 & \\ & \Sigma_2^2 \end{bmatrix} \tag{6.17}$$

where $\hat{A}_{11} \in R^{r\times r}$, $\hat{E}_1 \in R^{r\times m}$, $\hat{C}_1 \in R^{r\times p}$, and $\Sigma_1 \in R^{r\times r}$, then, the reduced system specified by the matrices $(\hat{A}_{11}, \hat{E}_1, \hat{C}_1^T)$ is stable, balanced, and has its controllability and observability grammians equal to Σ_1^2. Such a method is called the BT method, as the reduced model is formed by truncating the "unimportant" part of the state vector of a balanced system.

Let $Z(s) = C^T(sI - A)^{-1}E$ and $\hat{Z}(s) = \hat{C}_1^T(sI - \hat{A}_{11})^{-1}\hat{E}_1$. Then, it has been shown that

$$\left\| Z(s) - \hat{Z}(s) \right\|_\infty \le 2 \sum_{i=r+1}^{n} \sigma_i \tag{6.18}$$

where $\|.\|_\infty$ is the H_∞ norm defined by $\|Z(s)\|_\infty \equiv \sup_\omega \sigma_{\max}(Z(j\omega))$, $-\infty < \omega < \infty$, and $\sigma_{\max}(Z(j\omega))$ is the maximum singular value of $Z(j\omega)$ [12].

The most numerically stable method for implementing the BT is the so-called *square root* method. Let $W_c = Y_cY_c^T$ and $W_o = Y_oY_o^T$ be the Cholesky decomposition of W_c and W_o, and the singular value decomposition of $Y_o^TY_c = U\Gamma^2V^T$ with $\Gamma^2 = diag(\gamma_1^2, \gamma_2^2, \ldots, \gamma_n^2)$, and U and V are orthogonal matrices. Let U_r (V_r) be the submatrix of U (V) with its first r columns, and $\Gamma_r^2 = \mathrm{diag}(\gamma_1^2, \ldots, \gamma_r^2)$. Let

$$P_c = Y_cV_r\Gamma_r^{-1} \tag{6.19}$$

$$P_o = Y_oU_r\Gamma_r^{-1}. \tag{6.20}$$

Then, the BT system with order r is given by [32]

$$\hat{A}_{11} = P_o^TAP_c \quad \hat{E}_1 = P_o^TE \quad \hat{C}_1^T = C^TP_c. \tag{6.21}$$

The square root method indicates that the BT method is generally a two sided projection method with left projection matrix P_o and right projection matrix P_c such that $P_o^TP_c = I_r$. In the symmetric case that $A = A^T$ and $E = C$, $W_c = W_o$ and $P_c = P_o$, then it becomes a one sided projection method.

B. *Computation of Grammians for RLC Interconnect*

When an *RLC* interconnect is regarded as an m port network, the matrix B in the state equations (2.12) and matrix C in the output equation (2.13) are equal. We can take advantage of this equality to reduce the computation cost for the grammians.

Substitute $A = -M^{-1}N$ and $E = M^{-1}B$ into (6.7) and (6.8), we have

$$-M^{-1}NW_c - W_cN^TM^{-1} + M^{-1}BB^TM^{-1} = 0 \tag{6.22}$$

$$-N^TM^{-1}W_o - W_oM^{-1}N + BB^T = 0. \tag{6.23}$$

Equation (6.22) is equivalent to

$$-NW_cM - MW_cN^T + BB^T = 0. \tag{6.24}$$

Let

$$W_o = MX_oM. \tag{6.25}$$

Equation (6.23) becomes

$$-N^TX_oM - MX_oN + BB^T = 0. \tag{6.26}$$

Suppose that the submatrix $C \in M$ is of dimension $R^{nv\times nv}$ and $L \in M$ is of dimension $R^{ni\times ni}$. Let $S = \mathrm{diag}(\mathrm{I_{nv}}, -\mathrm{I_{ni}})$. Then, $S = S^{-1}$, $M = SMS$, $N = SN^TS$ and $N^T = SNS$. Also, from (2.5), $B = SB$ and $B^T = B^TS$. Then, from (6.24), we have

$$-SN^TSW_cSMS - SMSW_cSNS + SBB^TS = 0 \tag{6.27}$$

or

$$-N^TSW_cSM - MSW_cSN + BB^T = 0. \tag{6.28}$$

Let

$$X_c = SW_cS \tag{6.29}$$

we have

$$-N^TX_cM - MX_cN + BB^T = 0. \tag{6.30}$$

Comparing (6.30) with (6.26), it can be seen that $X_c = X_o$, and we have the following theorem.

Theorem 9: Let X_c be the solution to the generalized Lyaponov equation (6.30), then

$$W_c = SX_cS \tag{6.31}$$

and

$$W_o = MX_cM. \tag{6.32}$$

From (6.31) and (6.32), we have

$$W_o = MSW_cSM \tag{6.33}$$

and the Cholesky factors Y_c and Y_o are related by

$$Y_o = MSY_c. \tag{6.34}$$

By using Theorem 9, we only need to solve one generalized Lyaponov equation and do one Cholesky factorization. This will save the computation cost at about 50%. The solution to the generalized Lyaponov equation can be found by using the matrix sign function [33].

C. Passivity of the Reduced Model Via Grammian-Based Approach

We first consider the case of an *RC* interconnect. In such a case, the state equations and output equations are

$$C\dot{v}_n(t) + Gv_n(t) = A_si_s(t) \tag{6.35}$$

$$y(t) = v_s(t) = A_s^Tv_n(t). \tag{6.36}$$

Let $x(t) = C^{1/2}v_n(t)$, then we have

$$\dot{x}(t) = Ax(t) + Ei_s(t) \tag{6.37}$$

$$y(t) = E^Tx(t) \tag{6.38}$$

where $A = -C^{-(1/2)}GC^{-(1/2)}$ is symmetric and nonpositive definite, and $E = C^{-(1/2)}A_s$. In this case, $P_c = P_o$, and the BT will result in a passive model.

For an *RLC* interconnect, let $x(t)$ in (2.12) be $x(t) = M^{-(1/2)}z(t)$, then the state and output equations of the system become

$$\dot{z}(t) = Az(t) + Ei_s(t) \tag{6.39}$$

$$y(t) = E^Tz(t) \tag{6.40}$$

where $A = -M^{-(1/2)}NM^{-(1/2)}$ and $E = M^{-(1/2)}B$. Note that in this case, N is not symmetric, and in the general case, $W_c \neq W_o$. When $W_c \neq W_o$, the balancing transformation matrix P cannot be orthonormal. Otherwise, from (6.11) and (6.12), we have $\hat{W}_c = P^TW_cP = \Sigma^2$ and $\hat{W}_o = P^TW_oP = \Sigma^2$, and $W_c = W_o = P\Sigma^2P^T$, which violates the assumption.

As for an *RLC* interconnect, the balancing transformation matrix is not orthonormal in the general case, there is no guarantee for the passivity of the reduced model by the BT method. We have indeed found an example that the reduced model via BT is stable but not passive.

There are some other grammian-based algorithms which can guarantee the passivity of the reduced model [34], [35]. Though H_∞ norm error bounds have been derived for these algorithms, the high computation cost and numerical problem with the left and right factorization of $Z(s) + Z^T(-s)$ make them impractical for model-order reduction with large-scale interconnect networks.

Another way to guarantee the passivity of the reduced model via the grammian-based approach is to use approximate grammians. In [37], a dominant grammian eigenspace approach based on vector alternate-direction-implicit (ADI) algorithm is provided. Here, we provide two other methods based on the approximate grammians.

Let the Krylov subspace $K(A, E, k) = \{E, AE, \ldots, A^kE\}$, $(sI - A)^{-1} = \sum_{i=0}^{n-1} A^iF_i(s)$ and $f_i(t) = L^{-1}(F_i(s))$. Then, it is known that [36]

$$W_c = K(A, E, n-1)\Theta(n-1)K^T(A, E, n-1) \tag{6.41}$$

where

$$\Theta(k) = \int_0^\infty F_k(t)F_k^T(t)\,dt \tag{6.42}$$

with

$$F_k^T(t) = [f_0(t)I_m,\, f_1(t)I_m,\, \ldots,\, f_k(t)I_m]. \tag{6.43}$$

Let

$$W_c(k) = K(A, E, k)\Theta(k)K^T(A, E, k) \tag{6.44}$$

be an approximation of W_c. It can be seen that when $k = n-1$, $W_c(n-1) = W_c$. Let $W_c(k) = Y_c(k)Y_c^T(k)$, then $Y_c(k) = K(A, E, k)\Theta^{1/2}(k)$ is called the kth-order dominant controllability grammian eigenspace [37]. Similarly, we can define the kth-order dominant observability grammian eigenspace $Y_o(k)$, and we try to find an orthonormal basis V of $\{Y_c(k), Y_o(k)\}$ as the transformation matrix to form the reduced model.

To find an orthonormal basis for $Y_c(k)$ directly meets with the difficulty in finding $f_i(t)$, $i = 1, \ldots, k$. However, there is an indirect way to do it efficiently. Let $V_c(k)$ be an orthonormal basis of $K(A, E, k)$ so that for any $0 \le i \le k$, there exists G_i such that

$$A^iE = V_c(k)G_i, \qquad i = 0, \ldots, k. \tag{6.45}$$

Then

$$A^iE\Theta^{1/2}(k) = V_c(k)H_i(k), \qquad i = 0, \ldots, k \tag{6.46}$$

where $H_i(k) = G_i\Theta^{1/2}(k)$, and $V_c(k)$ is an orthonormal basis of $Y_c(k)$, too. Therefore, to find an orthonormal basis of the dominant grammian subspace $\{Y_c(k), Y_o(k)\}$ can be solved by finding an orthonormal basis of $\{K(A, E, k), K(A^T, E, k)\}$. Note that $K(A, E, k)$ is the subspace formed by the 0th to the kth-order moments at ∞ frequency of the original system and $K(A^T, E, k)$ is the subspace formed by the 0th to kth-order moments at ∞ frequency of the dual system. Therefore, the approximate grammian approach can be transformed to an equivalent moment-matching problem at ∞ frequency. This method is called the modified dominant grammian eigenspace method (scheme 1).

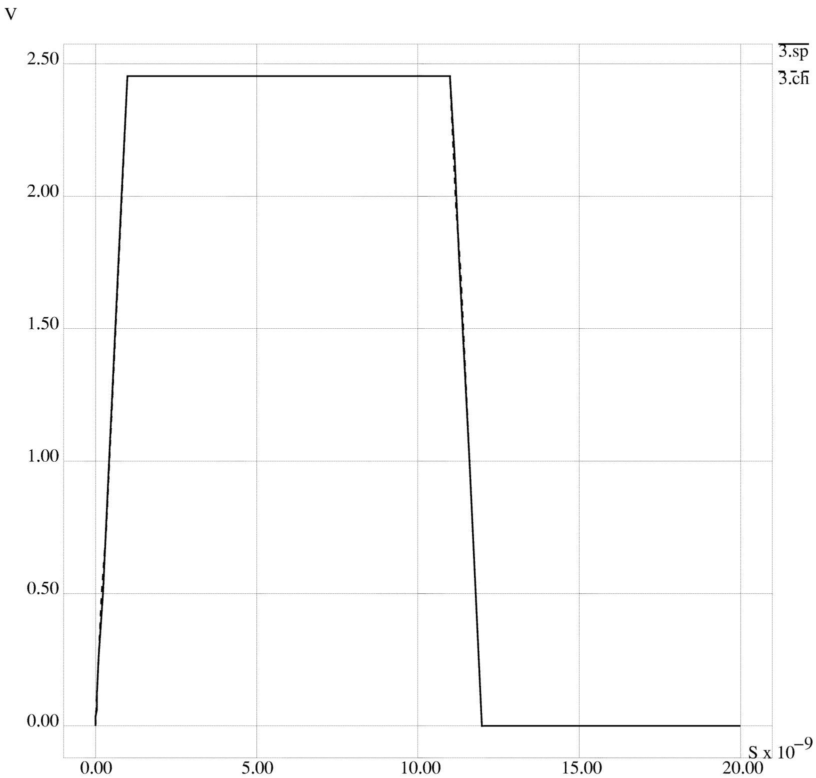


Fig. 1. Time-domain response of Example 1.

The approximate dominant grammian eigenspace method can also be implemented by using a system companion to the system described by (6.39) and (6.40). The companion system is defined by the following equations:

$$A\,\frac{d\,\overline{z}(t)}{dt} = \overline{z}(t) - Ei_s(t) \tag{6.47}$$

$$\overline{y}(t) = E^T A^{-1}\overline{z}(t). \tag{6.48}$$

Let $\overline{W}_c$ and $\overline{W}_o$ be the controllability and observability grammians of the companion system, then it can be proved that $\overline{W}_c = W_c$ and $\overline{W}_o = W_o$ [38]. By using the same argument as in the previous paragraph, to find an orthonormal basis of $Y_c(k)$ $(Y_o(k))$ can be transformed to find an orthonormal basis of $K(A^{-1}, A^{-1}E, k)$ $(K(A^{-T}, A^{-T}E, k))$. Note that these two bases are the bases of the Krylov subspace spanned by the moments of the original system and its dual at zero frequency and the approximate grammian approach can also be implemented by using moment-matching method at zero frequency. This method is called the modified dominant grammian eigenspace method (scheme 2).

Intuitively speaking, the approximate grammian method via moment matching at ∞ (0) frequency will result in better approximation at high (low) frequencies. As most interconnects behave as low pass, we are in favor of using scheme 2.

VII. General Properties of Projection-Based Algorithms

In the previous sections, we introduced some projection-based algorithms for model-order reduction of linear interconnects. In this section, we summarize some general properties of the projection-based algorithms, which are useful for the development and application of the algorithms.

A. Preservation of Passivity

Theorem 10: If the transformation matrix V is of full rank, the reduced order model formed by the congruence transform w.r.t. V is passive.

The proof can be found in [5], and is omitted here.

B. Dual System in Model-Order Reduction

In the algorithms stated in the previous sections, the formation of the matrix V for the generation of the reduced model is based on the information of the state equations only, and the output equations play no rule in the process. In this subsection, we will introduce a method to utilize the information of the output equations, which gives new options to the algorithms for model-order reduction.

Definition 1—Dual System: For a system described by its state equations

$$M\,\frac{d\,\overline{x}(t)}{dt} + N\,\overline{x}(t) = Bu(t) \tag{7.1}$$

and output equations

$$\overline{y}(t) = C^T\,\overline{x}(t) \tag{7.2}$$

its dual system is defined by the state equations

$$M\,\frac{d\,\overline{x}_d(t)}{dt} + N^T\,\overline{x}_d(t) = Cu_d(t) \tag{7.3}$$

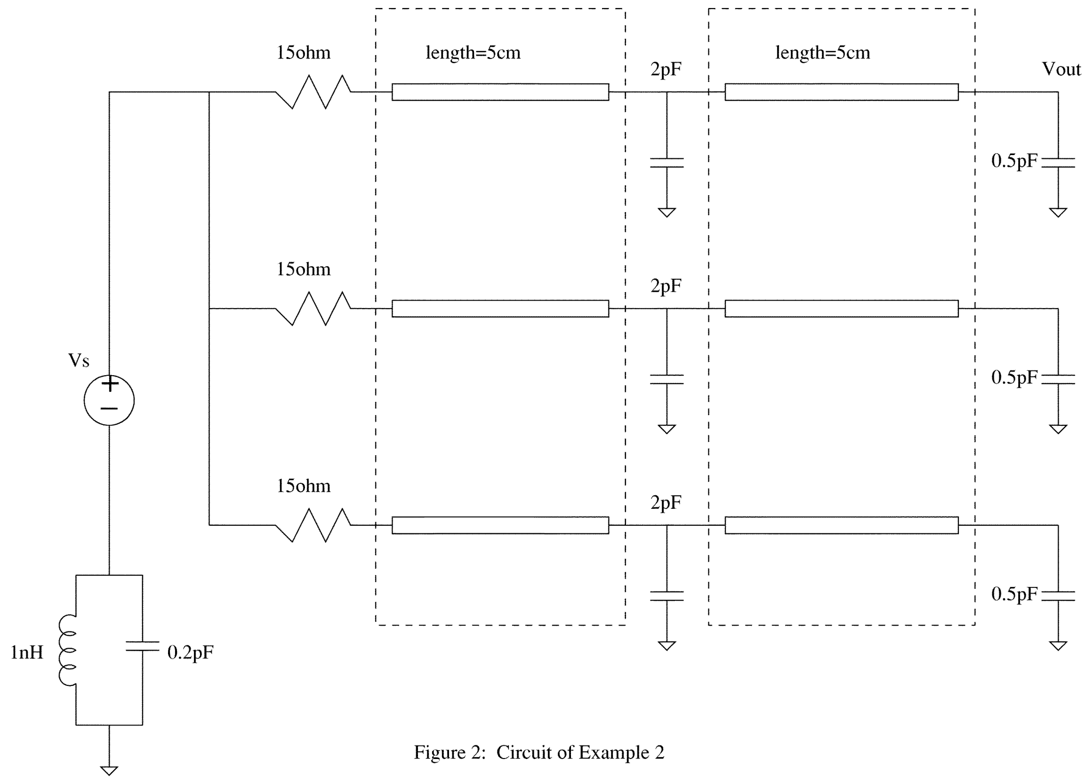


Fig. 2. Circuit of Example 2.

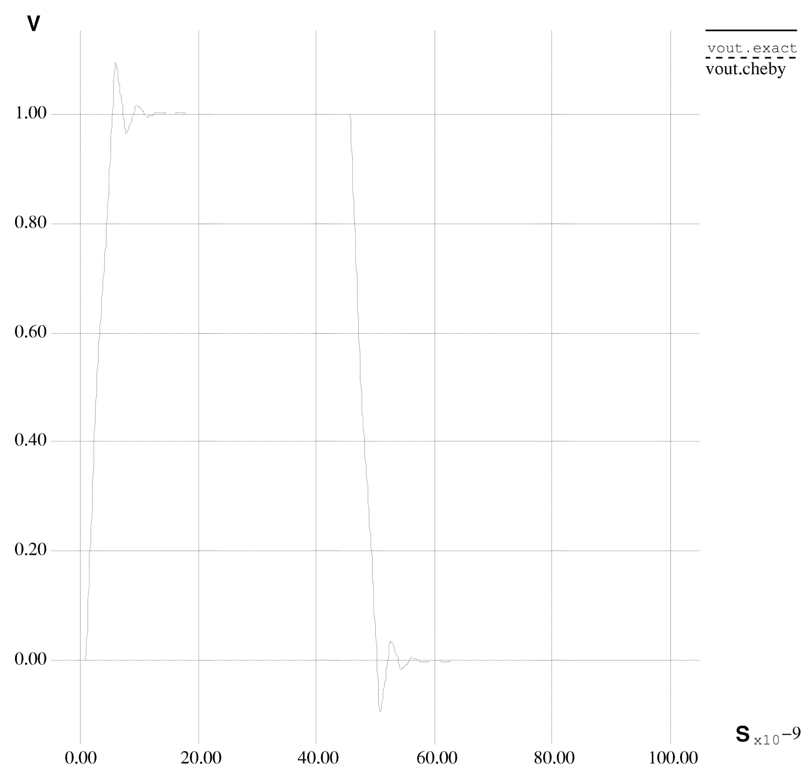


Fig. 3. Time-domain response of Example 2.

and the output equations

$$\overline{y}_d(t) = B^T \overline{x}_d(t) \tag{7.4}$$

and we denote $(M, N^T, C, B) = \text{dual}(M, N, B, C)$.

As mentioned in the previous sections, when model-order reduction is concerned, we often work with the impulse response of a system, and we start from the block form of the system. In this case, the block form of the state and output equations of the dual system are

$$M \frac{dx_d(t)}{dt} + N^T x_d(t) = C \tag{7.5}$$

and

$$y_d(t) = B^T x_d(t). \tag{7.6}$$

Note that when $M, N \in R^{n\times n}$, $B \in R^{n\times m}$ and $C \in R^{n\times p}$, the block matrices of the state and output vectors of the original system are $x(t) \in R^{n\times m}$ and $y(t) \in R^{p\times m}$, but those for the dual system are $x_d(t) \in R^{n\times p}$ and $y_d(t) \in R^{m\times p}$. In the case that $p \neq m$, the sizes of $x(t)$ and $x_d(t)$ are different, and when $p \ll m$ [e.g., in an multiple input single output (MISO) system], the size of $x_d(t)$ may be much smaller than that of $x(t)$. As the number of columns of the transformation matrix V is proportional to the number of input variables, in the case that $p \ll m$, the use of a dual system may be much more sparing than the use of the original system. This is one of the main reasons that we are interested in the dual system.

Next, we will give some theorems related to the use of a dual system in the model-order reduction.

Lemma 6: Let $H(s)$ and $H_d(s)$ be the transfer functions of the original and dual system, respectively, then

$$H_d(s) = H^T(s). \tag{7.7}$$

Proof:

$$\begin{aligned} H^T(s) &= \left[C^T(sM+N)^{-1}B\right]^T \\ &= B^T(sM+N^T)^{-1}C = H_d(s). \end{aligned}$$ □

Let $H(s) = L\{h(t)\}$ and $H_d(s) = L\{h_d(t)\}$, then from (7.7) we have

$$h_d(t) = h^T(t). \tag{7.8}$$

Lemma 7: Let $R(S) = (\hat{M}, \hat{N}, \hat{B}, \hat{C})$ be the reduced system w.r.t. the system $S = (M, N, B, C)$ and transformation matrix V. Then,

$$\text{dual}(R(S)) = R(\text{dual}(S)) \tag{7.9}$$

i.e., the operators "reduction" and "duality" are interchangeable.

Proof:

$$\begin{aligned} \text{dual}(R(S)) &= (\hat{M}, \hat{N}^T, \hat{C}, \hat{B}) \\ &= (V^TMV, V^TN^TV, V^TC, V^TB) \\ &= R(\text{dual}(S)). \end{aligned}$$ □

We first consider the moment-matching method and have the following theorem.

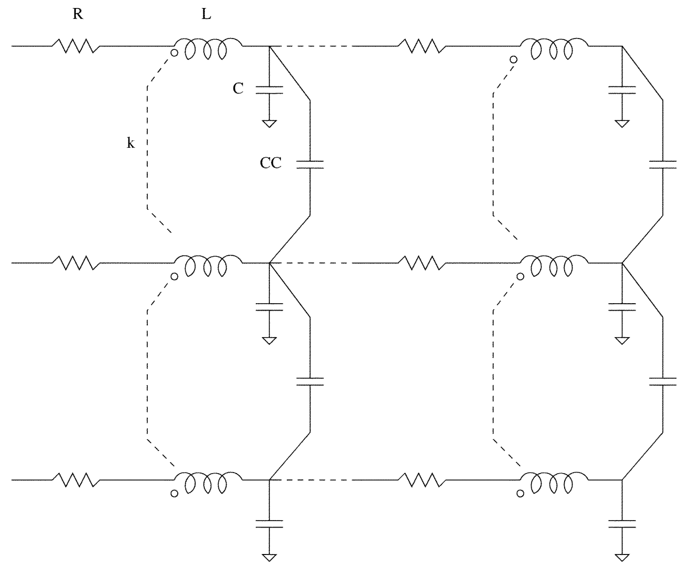


Fig. 4. Circuit of Example 3.

Theorem 11: Let $M(s_0, k) = \{X^{(0)}(s_0), X^{(1)}(s_0), \ldots, X^k(s_0)\}$ be the set of moments of $X(s)$ at s_0 of the original system, and $M_d(s_0, l) = \{X_d^{(0)}(s_0), X_d^{(1)}(s_0), \ldots, X_d^l(s_0)\}$ be the set of moments of $X_d(s)$ at the same point s_0 of the dual system. Let $Y^{(i)}(s_0)$ and $\hat{Y}^{(i)}(s_0)$ be the ith-order moments of the output vectors of the original system and the reduced order system, respectively. If for the transformation matrix V

$$\{M(s_0, k), M_d(s_0, l)\} \in \text{colspan}(V). \tag{7.10}$$

Then

$$\hat{Y}^{(i)}(s_0) = Y^{(i)}(s_0), \qquad 0 \le i \le k + l + 1. \tag{7.11}$$

The theorem can be proved by using Lemma 1, Lemma 7, and (3.11), and the detailed proof is omitted.

This theorem means that if the transformation matrix V is formed by moment matching at the original system and the dual system, then the moment-matching order of the reduced system is accumulated. This provides a new way to overcome the numerical instability problem during the orthonormalization process in the formation of the matrix V. If we start from the Krylov subspace $\{X^{(0)}(s_0), X^{(1)}(S_0), \ldots, X^{(k)}(s_0)\}$ and find that for some k, the norm of matrix r (see Algorithm 1) after orthogonalization is much smaller than its original value, we may restart from the Krylov subspace of the dual system, then a higher order model may be obtained.

Now, we consider the model-order reduction algorithm based on the expansion on Chebyshev polynomials in the frequency domain.

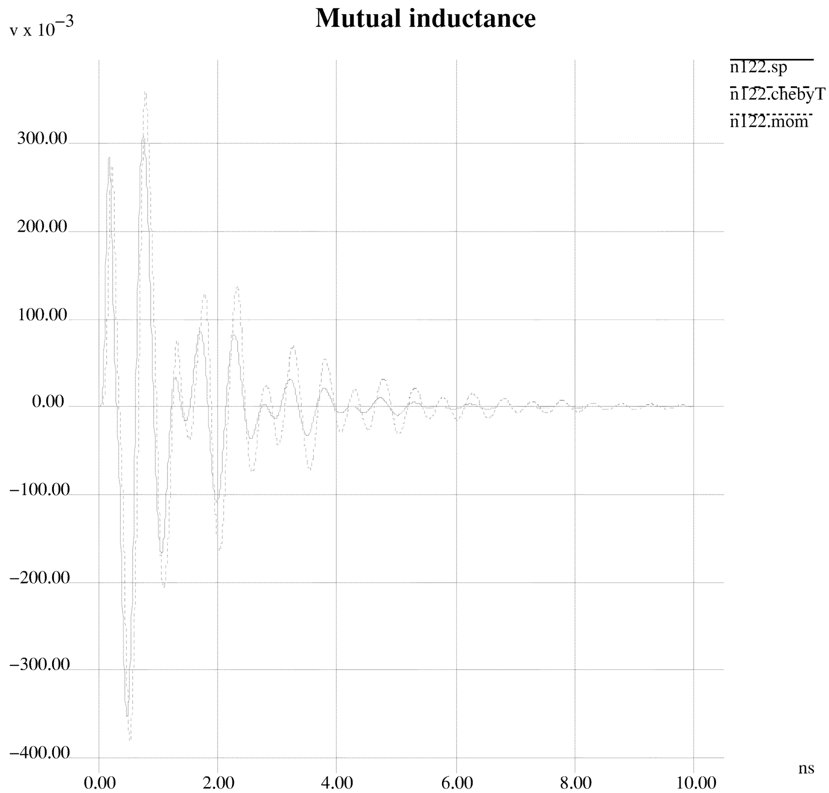


Fig. 5. Time-domain response of Example 3.

Lemma 8: Let C_{Yk} and D_{Yk} be the kth Chebyshev coefficient matrices of the output vectors of the original and dual system up to the order of K, respectively. Then, we have

$$C_{Yk} = D_{Yk}^T, \qquad 0 \le k \le K. \tag{7.12}$$

Proof:

$$H(s) = \sum_{k=0}^{K} C_{Yk} T_k(\varpi)$$

and from Lemma 7

$$H_d(s) = \sum_{k=0}^{K} D_{Yk} T_k(\varpi) = \sum_{k=0}^{K} C_{Yk}^T T_k(\varpi).$$

So, the lemma exists. □

Theorem 12: Let $\overline{X}_d(j\varpi) = \sum_{i=0}^{K} D_i T_i(\varpi)$. Suppose that

$$\{D_0, D_1, \ldots, D_K\} \in \text{colspan}(V) \tag{7.13}$$

and a reduced order system is generated by the congruence transform w.r.t. matrix V on the original system. Let C_{Yk} and $\hat{C}_{Yk}$ be the kth Chebyshev coefficient matrices of the output vectors of the original and reduced system, respectively. Then

$$\hat{C}_{Yk} = C_{Yk}, \qquad 0 \le k \le K. \tag{7.14}$$

Proof: Let $S = (M, N, B, C)$, $S_d = \text{dual}(S) = (M, N^T, C, B)$, $R(S) = (\hat{M}, \hat{N}, \hat{B}, \hat{C})$ be the reduced system of S, and $R(S_d)$ be the reduced system of S_d. Let C_{Yk}, D_{Yk}, $\hat{C}_{Yk}$ and $\hat{D}_{Yk}$ be the kth Chebyshev coefficient matrices of the output vectors of system S, S_d, $R(s)$ and $R(S_d)$, respectively. For $0 \le k \le K$, from Lemma 8, we have

$$C_{Yk} = D_{Yk}^T.$$

From Theorem 4 and the condition of Theorem 12, we have

$$\hat{D}_{Yk} = D_{Yk}.$$

From Lemmas 6 and 8, we have

$$\hat{C}_{Yk} = \hat{D}_{Yk}^T = D_{Yk}^T = C_{Yk}.$$ □

This theorem means that we can use the dual system to form the transformation matrix V with the preservation of the Chebyshev coefficient matrices of the output vectors of the dual system, and when the transformation w.r.t. matrix V is applied on the original system, the reduced model preserves the Chebyshev coefficient matrices of the output vectors of the original system. Note that the number of columns of the transformation matrix V is proportional to m (p) when the original (dual) system is used, and when $p < m$, it is more efficient to use the dual system.

Note that Theorem 12 is valid not only for the Chebyshev expansion-based frequency domain model-order reduction, but also for the Chebyshev expansion-based time domain model-order reduction. The proof in the case of time domain is similar to that in the frequency domain, and is not repeated.

For the model-order reduction based on the orthonormal basis functions in Hilbert and Hardy space, we have shown that such reduction with preservation of the expansion coefficient matrices can be transformed to an equivalent moment-matching problem. By using Theorem 11, it can be understood that the dual system can also be applied in this case.

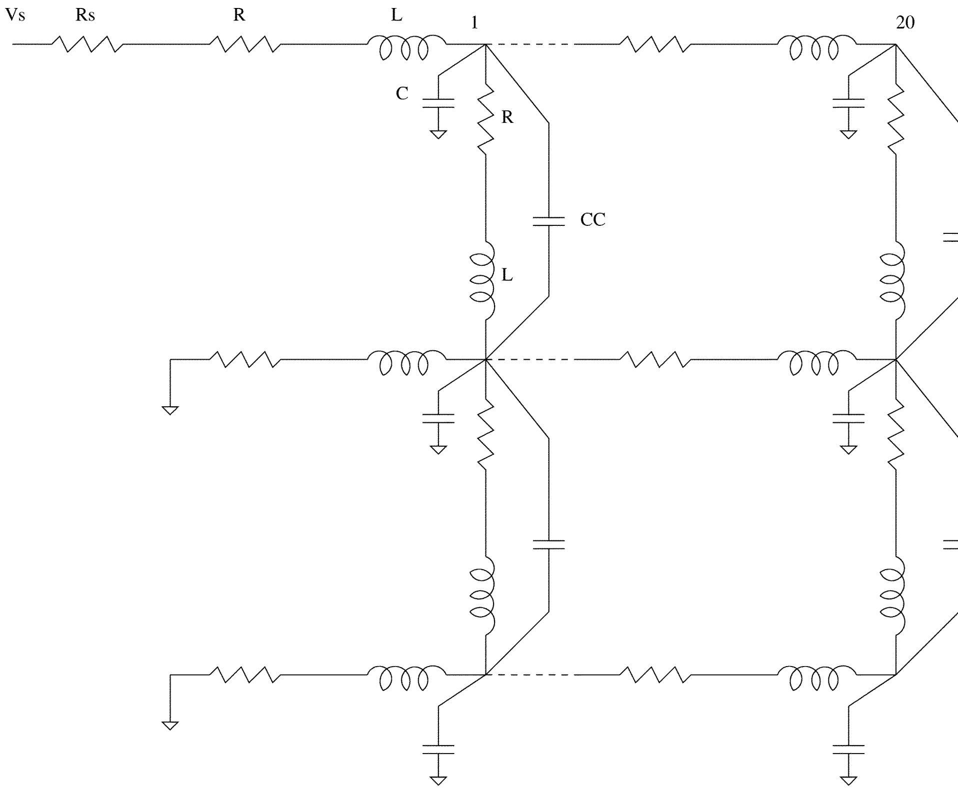


Fig. 6. Circuit of Example 4.

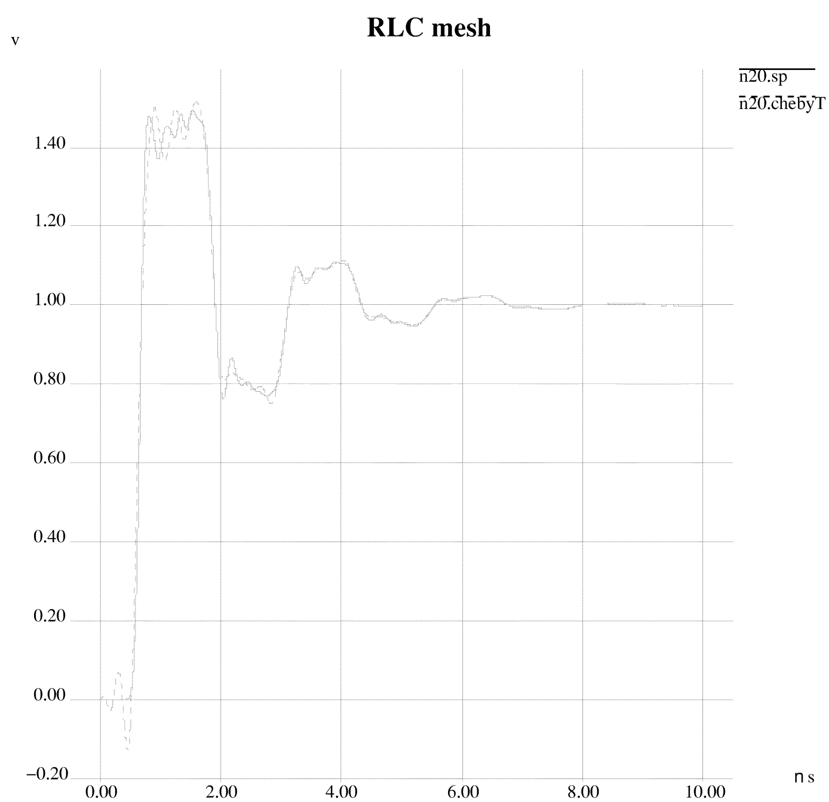


Fig. 7. Example 4-Chebyshev approximation-based model.

For the BT method, it is easy to show that the controllability and observability grammians of the original system are the observability and controllability grammians of the dual system, respectively, and nothing is gained by using the dual system.

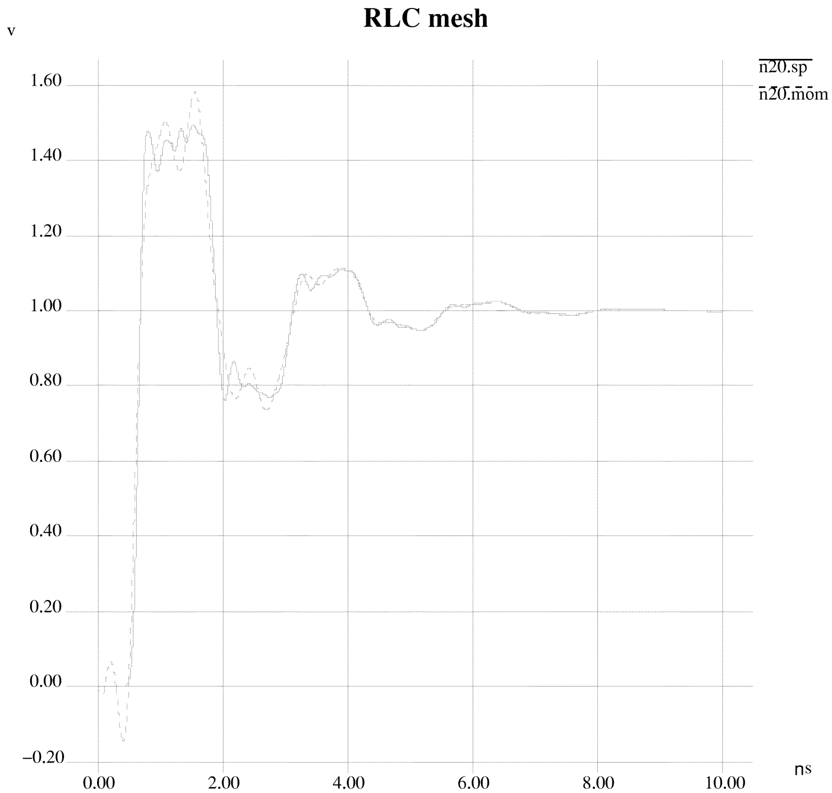


Fig. 8. Example 4—Moment-matching model.

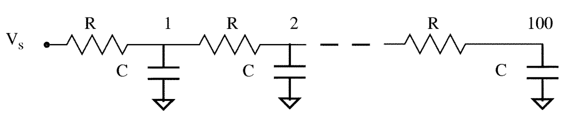


Fig. 9. An *RC* ladder.

C. Separatability in Model-order Reduction

In the previous sections, for coefficient matching algorithms (Sections III–V), we use the block form of the state equations to derive a number of theorems. By using the block form, all input variables are treated equally. For example, for a two input one output system, when moment-matching method is used, the two transfer functions keep the same order of moment matching at the same point.

To make the model-order reduction method more flexible to fit the different need for different transfer functions, we may treat the input variables separately. Now we use a two input system as an example to explain the idea.

Let $B = [b_1, b_2]$, $x(t) = [x_1(t), x_2(t)]$ and $y(t) = [y_1(t), y_2(t)]$ such that for $i = 1, 2$

$$M\,\frac{dx_i(t)}{dt} + Nx_i(t) = b_i\delta(t) \tag{7.15}$$

and

$$y_i(t) = C^T x_i(t) \tag{7.16}$$

and in the frequency domain

$$(sM + N)X_i(s) = b_i \tag{7.17}$$

and

$$Y_i(s) = C^T X_i(s). \tag{7.18}$$

Now, suppose that it is needed that for the reduced order system, $\hat{Y}_1^{(i_1)}(s_1) = Y_1^{(i_1)}(s_1)$ for $i_1 = 0, 1, \ldots, k_1$ and $\hat{Y}_2^{(i_2)}(s_2) = Y_2^{(i_2)}(s_2)$ for $i_2 = 0, 1, \ldots, k_2$. Then, we can satisfy the requirement by letting the transformation matrix V be such that

$$\left\{X_1^{(0)}(s_1), \ldots, X_1^{(k_1)}(s_1), X_2^{(0)}(s_2), \ldots, X_2^{(k_2)}(s_2)\right\} \in \text{colspan}(V). \tag{7.19}$$

Note that if $C \in R^{n\times p}$ and $p > 1$, then in each $Y_i(s)$, there are p components. If (7.19) is satisfied, for the p components, their matching points and matching orders are the same. If it is needed that for different components there are different matching points and matching orders, we may form the dual system of the system described by (7.17) and (7.18) and separate the input variables of the dual system again. In the general case, when $B \in R^{n\times m}$ and $C \in R^{n\times p}$, there are mp transfer functions, and we can treat them individually with different need for model-order reduction by using the separated systems and their dual systems. Note that the above technique can be used for any coefficient matching method.

D. Additivity

One important property of projection-based coefficient matching model-order reduction algorithm is the “additivity,” which means that if there are several sets of coefficient matching required, then the formation of the transformation matrix V can be implemented to meet the coefficient matching

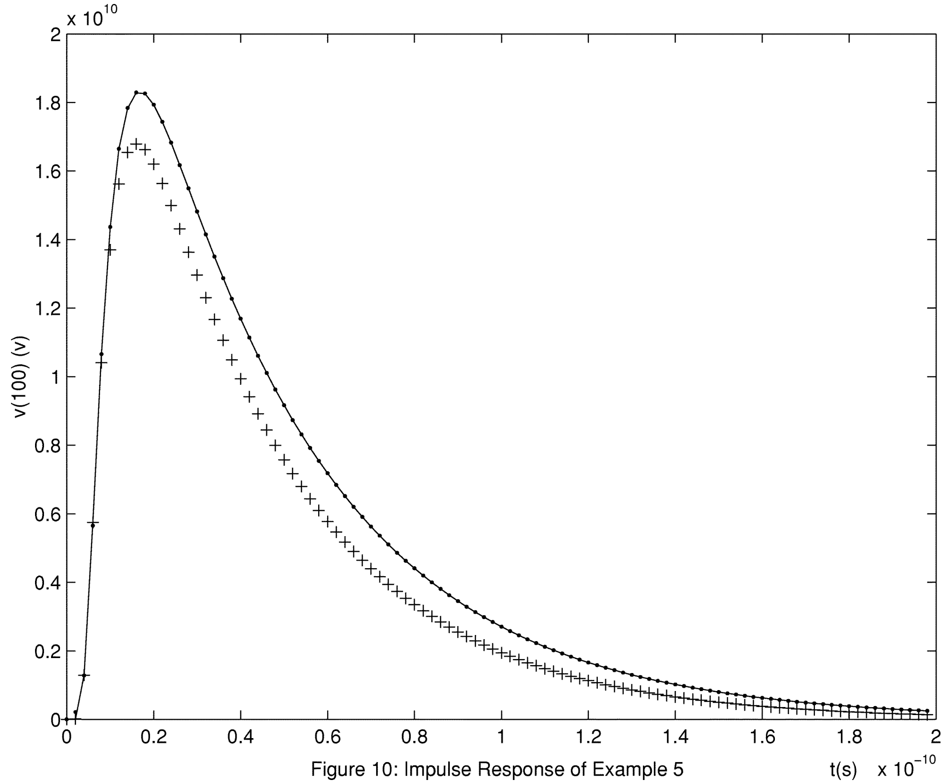


Fig. 10. Impulse response of Example 5.

one set after another, and finally the matching of all sets can be realized by using one transformation matrix.

For example, when coefficient matching based on the orthonormal basis functions in Hilbert space is used, because of the error due to truncation, the initial point of the impulse response of the reduced model may not be the same as that of the original model. As the matching of the initial point of the impulse response is important, after formation of the first part of V by coefficient matching of the expansion on the orthonormal functions, a moment matching at ∞ frequency can be added to form the second part of V. Then, the reduced order model will keep the coefficient matching based on the orthonormal basis functions and the series expansion on $1/s$ simultaneously.

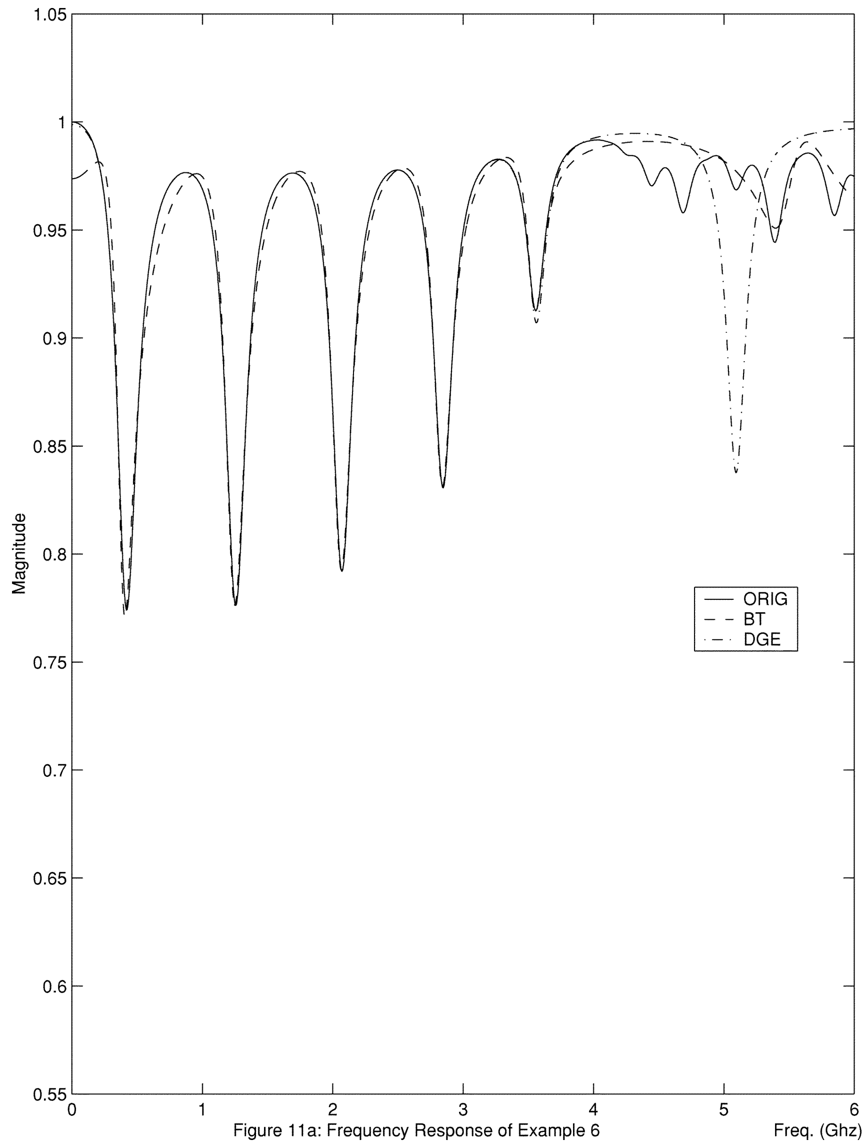
The reason that the property of additivity for coefficient matching algorithms exists is that in all the theorems related to the coefficient matching, the condition is that some coefficient set is covered by the column span of matrix V. If one coefficient set has been covered by column span of $V_1 \in V$, no matter how many new columns are added to V, such condition is always satisfied. Also, if this coefficient set is covered by column span $V_2 \in V$, no matter how many columns in V before V_2 is added, it is covered by $\text{colspan}(V)$. Therefore, a different coefficient set may be covered by the column span of a different part of V, and in the model-order reduction the matching of all coefficient sets is obtained.

VIII. EXAMPLES

We provide six examples here to show the result of various model-order reduction algorithms. The first four examples compare the time-domain response of the reduced model with those of the SPICE simulation, where the source voltage is a pulse. Among them, Examples 1 and 2 use model-order reduction based on the Chebyshev approximation in frequency domain, and Examples 3 and 4 use model-order reduction based on the Chebyshev approximation in time domain. In Example 5, the impulse responses of the reduced model based on the generalized orthonormal basis functions are compared with the exact solution, and in Example 6, the frequency domain response of the reduced model based on the grammian approach is compared with the exact solution.

Example 1: The circuit consists of a single transmission line with parameters $R = 0.01$ Ω/cm, $L = 2.5$ nH/cm $C = 1$ pF/cm length $d = 1$ cm, a load resistor and a source resistor of 50 Ω, which match the characteristic impedance of the line at high frequencies. A three point moment-matching method was used to obtain a nearly exact time domain response, and the order of the reduced model is 10 [5]. When Chebyshev approximation in frequency domain is used, only the zeroth-order coefficient matrix C_0 is used to form the transformation matrix, as the norm of C_k with $k > 0$ is much smaller than the norm of C_0, which results in a model with order of 2. The time-domain response of the reduced model by Chebyshev expansion is shown in Fig. 1. Compared with SPICE simulation, the waveform is nearly exact.

Example 2: This is an example borrowed from [30]. The circuit is shown in Fig. 2, which has two coupled line systems, each of which consists of three coupled lines. The time-domain responses of V_{out} are shown in Fig. 3. The solid line represents the exact solution where the coupled lines are represented by their exact characteristic model. The dashed line corresponds the result from the model by Chebyshev expansion in frequency domain. These two curves are indistinguishable. The reduced



(a)

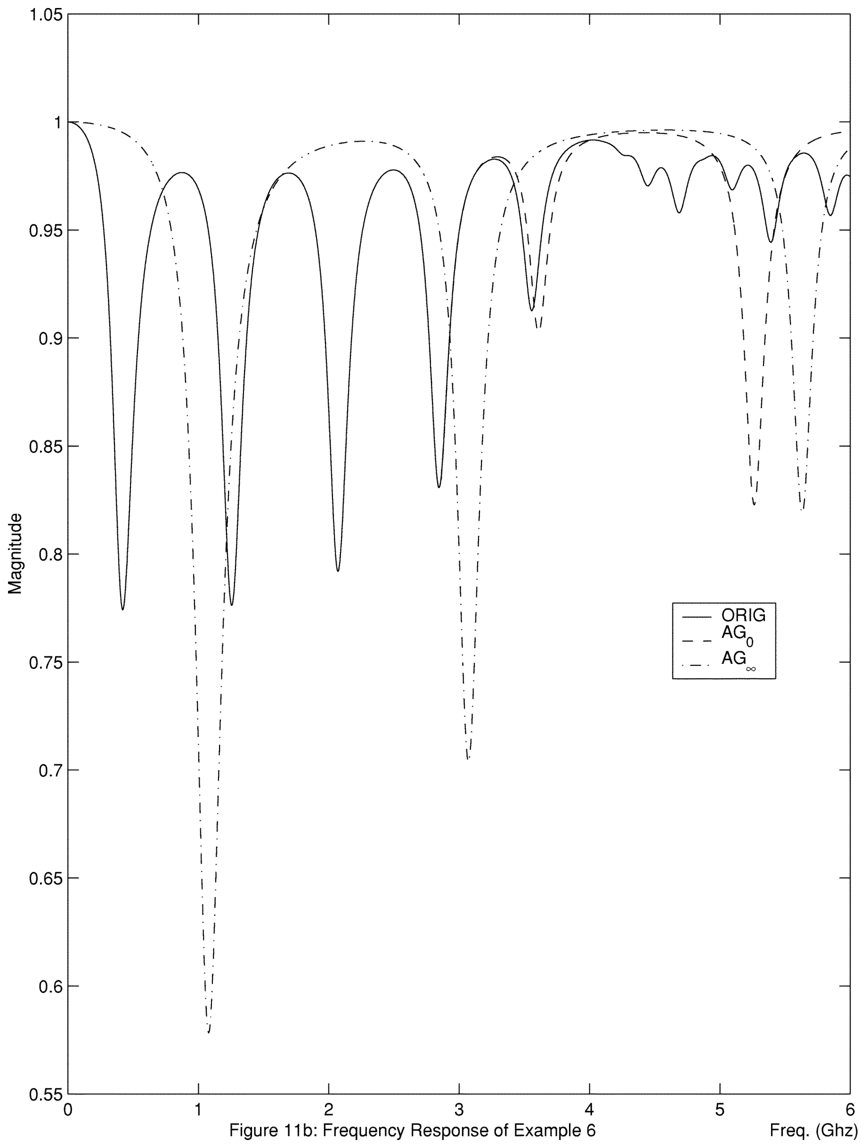
Fig. 11. (a) Frequency response of Example 6.

model order for each transmission line system is 5. In [30], a single moment-matching method is used with an order up to 40 to model the line system, and in [5], a multipoint moment-matching method is used with an order of 13. It can be seen that multipoint moment-matching method works better than the single moment-matching method, and the Chebyshev expansion method works the best among the three methods.

Example 3: The circuit is shown in Fig. 4, which consists of three *RLC* lines with capacitive and inductive coupling between adjacent lines. Each line is modeled by 200 *RLC* sections. The circuit parameters are: $R = 1\ \Omega$, $L = 1$ nh, the ground capacitance C and coupling capacitance CC are 1 pF, and the inductance coupling coefficient $k = 0.2$. The voltage supply is connected to the middle line, and we test the output of the upper victim line. The time-domain responses are shown in Fig. 5, where the solid, dashed, and dotted line corresponds to the result of SPICE simulation, the model formed by Chebyshev expansion in the time domain, and the moment-matching model at $s = 0$, respectively. The order of the two reduced model is 5. It can be seen that the result from the Chebyshev model is indistinguishable with that from the SPICE simulation, but the result from the moment-matching model is far from them.

Example 4: The circuit is an RLC mesh shown in Fig. 6. The circuit parameters are: $R_s = R = 1\ \Omega$, $L = 1$ nh, $C = 1$ pF and $CC = 0.5$ pF. We test the voltage at node 20, and compare the results from Chebyshev expansion in time domain with order 7 (Fig. 7, dashed line) and moment-matching model at $s = 0$ with the same order (Fig. 8, dashed line) with the SPICE simulation result (Figs. 7 and 8, solid line). It can be seen that the Chebyshev model works better than the moment-matching model.

Example 5: This is an *RC* ladder with 100 sections. The circuit is shown in Fig. 9, where $C = 0.01$ pF and $R = 1\ \Omega$. The impulse responses at node 100 are shown in Fig. 10, where the solid line corresponds to the exact solution, the dotted line to the solution of the reduced model based on the series expansion of orthonormal basis functions with multiple parameters, and the crossing line to that of the reduced model based on the



(b)

Fig. 11. *(Continued)* (b) Frequency response of Example 6.

series expansion of Laguerre functions. The order of the two reduced models is 12. It can be seen that the dotted line matches the solid line perfectly, and there is a big discrepancy between the crossing line and the solid line. This shows that for the same order, the reduced model based on the orthonormal functions with multiple parameters works better than that with only one parameter.

Example 6: We use the same circuit shown in Example 4 to test the BT method and its modifications in the frequency domain. The reduced order is 16. The frequency response of $F(s) = V_{20}(s)/V_s(s)$ $(s = j\omega)$ is shown in Fig. 11(a) and (b), where the exact solution, the results from BT method, from the dominant grammian eigenspace (DGE) method proposed by [37], and from the approximate grammian approach with scheme 1 and 2 are denoted by "ORIG," "BT," "DGE," "AG_∞" and "AG_0," respectively. It can be seen from the simulation result that while the BT method can essentially capture the peak values of the Bode plot over a wide frequency range with minimal average errors, it cannot capture the zero frequency response in the general case. The DGE method works better than BT method at the frequencies lower than about 3.5 Ghz but worse than BT at higher frequencies. The AG_0 behaves similar to DGE, but AG_∞ does not work well.

IX. Conclusion

In this paper, we summarized our experience in model-order reduction, and provided new algorithms and improvements on existing algorithms for model-order reduction.

Among the model-order reduction algorithms mentioned in this paper, the moment-matching method is the basic one, as some of other algorithms, e.g., the frequency-domain Chebyshev approximation-based method, the orthonormal basis function-based method, and the approximate grammian-based method can be implemented via the moment-matching method or moment computation. The moment-matching method with only one matching point generally needs high order to well approximate a frequency response in a wide frequency range or a time-domain response with widely spread time constants, and the problem for its efficient use is how to select the matching points scattered in a wide range. The frequency-domain Chebyshev expansion-based method provides one way to determine the matching points. The advantage of this method is that it has a very efficient way to determine the order of approximation, and it may result in a very low-order approximation (e.g., Examples 1 and 2) because of the exponential convergence property of the Chebyshev expansion. However, the collocation points of Chebyshev expansion are located more densely at high frequencies than at low frequencies, which is not good for interconnects with low-pass characteristics. The time-domain Chebyshev expansion-based method works better than the moment-matching method with a single matching point. Its disadvantage is the computation complexity, which is higher than the frequency domain Chebyshev expansion-based method. The orthonormal basis function-based model-order reduction has the advantage that the coefficients of the basis functions can be selected to fit for the characteristics of the system of interest, and it can be easily transformed to an equivalent moment-matching method and can be implemented very efficiently. We have successfully used it to model frequency dependent RL parameters of transmission lines at high frequencies. We have also done some work on using the orthonormal basis functions with complex parameters for model reduction of RLC interconnects [8], and continued the study in this respect. The BT method has a good estimation for the approximation error. However, it can hardly compete with the above mentioned algorithms because of its high computation complexity and no guarantee of the passivity for *RLC* interconnects. We provide some improvements on the two folds, and we think more work is needed in order to make it widely accepted by circuit simulators.

There are some problems commonly met in practice. One is the numerical instability problem with high-order approximation. We provide the dual system theory which is helpful to deal with the problem and may also lead to some lower order approximation with the same order of accuracy. Another one is that by using a single algorithm some requirements may not be met exactly, e.g., exact matching of the starting point of an impulse response and/or exact matching of the frequency response at the dc point. By using the additivity property of the projection-based parameter matching methods, these requirements can be met with very easily. Our experience shows that by using projection methods of one kind in a wide frequency (time) range in addition to some parameter matching at special points of interest may be the best practical way to achieve a low-order model with high accuracy.

ACKNOWLEDGMENT

The authors wish to thank the reviewers for their helpful comments to this paper.

REFERENCES

[1] L. T. Pillage and R. A. Rohrer, "Asymptotic waveform evaluation for timing analysis," *IEEE Trans. Computer-Aided Design*, vol. 9, pp. 352–366, Apr. 1990.

[2] P. Feldmann and R. W. Freund, "Efficient linear circuit analysis by *Padé via Lanczos* process," *IEEE Trans. Computer-Aided Design*, vol. 14, pp. 639–649, May 1995.

[3] K. J. Kerns, I. L. Wemple, and A. T. Yang, "Stable and efficient reduction of substrate model networks using congruence transforms," in *Proc. ICCAD'95*, Nov. 1995, pp. 207–214.

[4] A. Odabasioglu, M. Celik, and L. T. Pillegi, "PRIMA: Passive reduced-order interconnect macromodeling algorithm," in *Proc. ICCAD'97*, Nov. 1997, pp. 58–65.

[5] Q. Yu, J. M. Wang, and E. S. Kuh, "Passive multipoint moment-matching model order reduction on multiport distributed interconnect networks," *IEEE Trans. Circuits Syst. I*, vol. 46, pp. 140–160, Jan. 1999.

[6] J. M. Wang, E. S. Kuh, and Q. Yu, "The Chebyshev expansion based passive model for distributed interconnect networks," in *Proc. ICCAD'99*, Nov. 1999, pp. 370–375.

[7] Q. Yu, J. M. Wang, and E. S. Kuh, "Passive model order reduction based on Chebyshev expansion of impulse response of interconnect networks," in *Proc. DAC'00*, June 2000, pp. 520–525.

[8] Q. Yu and E. S. Kuh, "Passive time-domain model order reduction via orthonormal basis functions," in *Proc. ECCTD'01*, Sept. 2001.

[9] K. Gallivan, E. Grimme, and P. Van Dooren, "Multi-point *Padé* approximants of large-scale systems via a two-sided rational Krylov algorithm," Univ. of Illinois, Urbana, IL 61 801, Tech. Rep., 1994.

[10] B. C. Moore, "Principal component analysis in linear system: Controllability, observability, and model reduction," *IEEE Trans. Automat. Contr.*, vol. 26, pp. 17–32, Feb. 1981.

[11] M. Green, "Balanced stochastic realizations," *Linear Algebra Appl.*, vol. 98, pp. 211–247, 1988.

[12] K. Glover, "All optimal Hankel-norm approximations of linear multivariable systems and their L^{∞} error bounds," *Int. J. Control*, vol. 39, no. 6, pp. 1115–1193, 1984.

[13] X. Chen and J. T. Wen, "Positive realness preserving model reduction with H_{∞} norm bounds," *IEEE Trans. Circuits Syst. I*, vol. 42, pp. 23–29, Jan. 1995.

[14] M. G. Safanov and R. Y. Chiang, "A Schur method for balanced truncation model reduction," *IEEE Trans. Automat. Contr.*, vol. 34, pp. 729–733, July 1989.

[15] Y. Saad, "Numerical solution of large Lyapunov equations," in *Proc. Int. Symp. Signal Processing, Scattering and Operator Theory and Numerical Methods. MTINS'89*, vol. 3, 1990, pp. 856–869.

[16] P. K. Gunupudi and M. S. Nakhla, "Model-reduction of nonlinear circuits using Krylov-space techniques," in *Proc. 36th DAC*, June 1999, pp. 13–16.

[17] C.-K. Cheng, J. Lillis, S. Lin, and N. Chang, *Interconnect Analysis and Synthesis* New York, 1999, ch. 5.

[18] C. Canuto, M. Y. Hussaini, A. Quarteroni, and T. A. Zang, *Spectral Methods in Fluid Dynamics*. Berlin, Germany: Springer-Verlag, 1987.

[19] W. H. Kautz, "Transient synthesis in the time domain," *IEEE Trans. Circuit Theory*, vol. 1, pp. 29–39, Sept. 1954.

[20] D. C. Ross, "Orthonormal exponentials," *IEEE Trans. Commun. Electron.*, pp. 173–176, 1964.

[21] W. H. Huggins, "Signal theory," *IEEE Trans. Circuit Theory*, vol. 3, pp. 210–216, Dec. 1956.

[22] H. Akcay and B. Ninness, "Orthonormal basis functions for modeling continuous-time systems," *Signal Processing*, vol. 77, pp. 261–274, 1999.

[23] P. S. C. Heuberger, P. M. J. Van Den Holf, and O. H. Bosgra, "A generalized orthonormal basis for linear dynamic systems," *IEEE Trans. Automat. Contr.*, vol. 40, pp. 451–465, Mar. 1995.

[24] P. M. J. Van Den Hof, P. S. Heuberger, and J. Bokor, "System identification with generalized orthonormal basis functions," *Automatica*, vol. 31, no. 12, pp. 1821–1834, 1995.

[25] B. Wahberg, "System identification using Laguerre models," *IEEE Trans. Automat. Contr.*, vol. 36, pp. 551–562, May 1991.

[26] B. Wahberg and P. M. Mäkilä, "On approximation of stable linear dynamical systems using Laguerre and Kautz functions," *Automatica*, vol. 32, no. 5, pp. 693–708, 1996.

[27] L. Knockaert and D. D. Zutter, "Passive reduced order multiport modeling: The Padé–Laguerre, Krylov–Arnoldi–SVD connection," *AEÜ Int. J. Electron. Commun.*, vol. 53, no. 5, pp. 254–260, 1999.

[28] A. Ruhe, "Rational Krylov sequence methods for eigenvalue computation," *Linear Algebra and Its Applications*, vol. 58, pp. 391–405, 1984.

[29] G. H. Golub and C. F. Van Loan, *Matrix Computations*, 3rd ed. Baltimore, MD: Johns Hopkins Univ. Press, 1996, ch. 9.
[30] M. Celik and A. C. Cangellaris, "Simulation of dispersive multiconductor transmission lines by *Padé* approximation via the Lanczos process," *IEEE Trans. on Microwave Theory Tech.*, vol. 44, pp. 2525–2535, Dec. 1996.
[31] K. Glover, "All optimal Hankel-norm approximations of linear multivariable systems and their L^{∞}-error bounds," *Int. J. Control*, vol. 39, no. 6, pp. 1115–1193, 1984.
[32] M. G. Safonov and R. Y. Chiang, "A Shur method for balanced-truncation model reduction," *IEEE Trans. Automat. Contr.*, vol. 34, pp. 729–733, July 1989.
[33] C. S. Kenney and A. J. Laub, "The matrix sign function," *IEEE Trans. Automat. Contr.*, vol. 40, pp. 1330–1348, Aug. 1995.
[34] X. Chen and J. T. Wen, "Positive realness preserving model reductions with H_{∞} error bounds," *IEEE Trans. Circuits Syst. I*, vol. 42, pp. 23–29, Jan. 1995.
[35] P. C. Opdenacker and E. A. Jonckheere, "A contraction mapping preserving balanced reduction scheme and its infinity norm bounds," *IEEE Trans. Circuits Syst.*, vol. 35, pp. 184–189, Feb. 1988.
[36] R. E. Skelton, *Dynamic System Control: Linear Systems Analysis and Synthesis*. New York: Wiley, 1988.
[37] J. Li and J. White, "Efficient model reduction of interconnect via approximate system grammians," in *Proc. ICCAD'99*, Nov. 1999, pp. 380–383.
[38] V. Sreeram and P. Agathoklis, "On the computation of the gram matrix in time domain and its application," *IEEE Trans. Automat. Contr.*, vol. 38, pp. 1516–1520, Oct. 1993.

Janet Meiling Wang received the B.S. degree from Nanjing University of Science of Technology (former East China Institute of Technology), Nanjing, China, in 1991, the M.E. degree from the Chinese Academy of Sciences, Beijing, China, in 1994, and the Master's and Ph.D. degrees in electrical engineering and computer sciences, from the University of California at Berkeley, in 1997 and 2000, respectively.

She was with Intel, Santa Clara, CA, and Cadence Santa Clara, CA, from 2000 to 2002. She has recently joined the Electrical Computing Engineering Department at University of Arizona, Tucson, as a Faculty Member.

Chia-Chi Chu (S'92–M'95) was born in Taipei, Taiwan, R.O.C.,on September 4, 1965. He received the B.S. and M.S. degrees in electrical engineering from National Taiwan University, Taipei, Taiwan, R.O.C., and the Ph.D. degree in electrical engineering from Cornell University, Ithaca, NY, in 1996.

From 1995 to 1996, he was a member of the technical staff at Avant! Corp., Fremont, CA. Since 1996, he has been a Faculty Member of Electrical Engineering, Chang Gung University, Tao-Yuan, Taiwan, R.O.C., where he is currently an Associate Professor. He was a Visiting Scholar at the University of California at Berkeley in 1999. His current research interests include interconnect analysis techniques for deep submicron ICs, and applications of nonlinear circuit theory.

Dr. Chu was the recipient of the Young Author Award of the IEEE 1997 Control of Oscillations and Chaos Conference (COC'97).

Qingjian Yu was born in Shanghai, China on March 27, 1939. He received the graduate degree from East China Institute of Technology (ECIT), Nanjing, China, in 1961.

He was a Teaching Assistant at ECIT from September 1961 to July 1977, a Lecturer, from August 1977 to March 1981, and an Associate Professor from April 1981 to October 1986. He has been a Full Professor at ECIT, now called Nanjing University of Science and Technology from November 1986 to March 1999 until he retired. Between 1982–1984, he was a Visiting Scholar at Columbia University, New York, NY. In 1984, he was a Visiting Scholar at MIT, Cambridge, MA. From November 1992 to October 1993, he was invited as a Visiting Fellow to the Chinese University of Hong Kong, Hong Kong. He was employed as a Visiting Scholar at University of California at Berkeley in 1994, 1997, and from June 1999 to June 2002. He is currently working at Celestry Design Technologies, San Jose, CA.

Prof. Yu is a member of the CAD Committee of the Electronics Society of China, and a member of the CAD Committee of the Electrical Engineering Society of China.

Ernest S. Kuh (S'49–M'57–F'65–LF'94) received the B.S. degree from the University of Michigan, Ann Arbor, in 1949, the M.S. degree from the Massachusetts Institute of Technology, Cambridge, in 1950, the Ph.D. degree from Stanford University, Stanford, CA, in 1952, the Doctor of Engineering, Honoris Causa, from Hong Kong University of Science and Technology, Hong Kong, in 1997, and the Doctor of Engineering degree from the National Chiao Tung University, Hsin Chu, Taiwan, R.O.C., in 1999.

From 1952 to 1956, he was a member of the Technical Staff at Bell Telephone Laboratories in Murray Hill, NJ. He is the William S. Floyd, Jr. Professor Emeritus in Engineering and a Professor, the Graduate School of the Department of Electrical Engineering and Computer Sciences, University of California, Berkeley, which he joined in 1956. From 1968 to 1972, he served as Chair of the Department, and from 1973 to 1980, as Dean of the College of Engineering.

Prof. Kuh is a member of the National Academy of Engineering, the Academia Sinica, and a foreign member of the Chinese Academy of Sciences. He is a Fellow of AAAS. He has received numerous awards and honors, including the ASEE Lamme Medal, the IEEE Centennial Medal, the IEEE Education Medal, the IEEE Circuits and Systems Society Award, the IEEE Millennium Medal, the 1996 C&C Prize, and the 1998 EDAC Phil Kaufman Award.