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Numerical Analysis

An 'empirical interpolation' method: application to efficient reduced-basis discretization of partial differential equations

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Abstract

We present an efficient reduced-basis discretization procedure for partial differential equations with *nonaffine* parameter dependence. The method replaces nonaffine coefficient functions with a collateral reduced-basis expansion which then permits an (effectively affine) offline–online computational decomposition. The essential components of the approach are (i) a good collateral reduced-basis approximation space, (ii) a stable and inexpensive interpolation procedure, and (iii) an effective a posteriori estimator to quantify the newly introduced errors. Theoretical and numerical results respectively anticipate and confirm the good behavior of the technique. ***To cite this article: M. Barrault et al., C. R. Acad. Sci. Paris, Ser. I 339 (2004).***

Résumé

Une méthode d'« interpolation empirique » : application à la discrétisation efficace par base réduite d'équations aux dérivées partielles. Nous présentons dans cette Note une méthode rapide de base réduite pour la résolution d'équations aux dérivées partielles ayant une dépendance non affine en ses paramètres. L'approche propose de remplacer le calcul des fonctionelles non affines par un développement en base réduite annexe qui conduit à une évaluation en ligne effectivement affine. Les points essentiels de cette approche sont (i) un bon système de base réduite annexe, (ii) une méthode stable et peu coûteuse d'interpolation dans cette base, et (iii) un estimateur a posteriori pertinent pour quantifier les nouvelles erreurs introduites. Des résultats théoriques et numériques viennent anticiper puis confirmer le bon comportement de cette technique. ***Pour citer cet article : M. Barrault et al., C. R. Acad. Sci. Paris, Ser. I 339 (2004).***

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Considérons une function $g(\cdot;\mu) \in L^\infty(\Omega)$ assez régulière. On propose tout d'abord une méthode constructive pour sélectionner une suite d'espaces emboités $W_M^g = \mathrm{Vect}\{\xi_m = g(\cdot;\mu_m^g),\ 1 \leqslant m \leqslant M\}$ avec $M \leqslant M_{\max}$ de dimension exactement M. On construit ensuite des ensembles emboités de points d'interpolation $T_M = \{t_1,\dots,t_M\}$, $1 \leqslant M \leqslant M_{\max}$, en posant tout d'abord $t_1 = \arg\operatorname{ess\,sup}_{x\in\Omega} |\xi_1(x)|$, $q_1 = \xi_1(x)/\xi_1(t_1)$. Puis, pour $M = 2,\dots,M_{\max}$, on résout le système linéaire $\sum_{j=1}^{M-1} \sigma_j^{M-1} q_j(t_i) = \xi_M(t_i)$, $1 \leqslant i \leqslant M-1$, et on pose $r_M(x) = \xi_M(x) - \sum_{j=1}^{M-1} \sigma_j^{M-1} q_j(x)$, on définit alors le point suivant d'interpolation $t_M = \arg\operatorname{ess\,sup}_{x\in\Omega} |r_M(x)|$ et on pose $q_M(x) = r_M(x)/r_M(t_M)$. On approche enfin $g(x;\mu)$ par $g_M(x;\mu) = \sum_{m=1}^M \beta_m(\mu) q_m(x)$, où $\sum_{j=1}^M \beta_j(\mu) q_j(t_i) = g(t_i;\mu)$, $1 \leqslant i \leqslant M$.

Ce procédé d'interpolation peut être justifié. A priori tout d'abord, en introduisant la constante de type Lebesgue $\Lambda_M = \sup_{x\in\Omega} \sum_{m=1}^M |V_m^M(x)|$ où V_m^M est le seul élément de W_M^g tel que $V_m^M(t_i) = \delta_{im}$. On peut montrer que Λ_M est bornée par $2^M - 1$. L'erreur d'interpolation $\varepsilon_M(\mu) \equiv \|g(\cdot;\mu) - g_M(\cdot;\mu)\|_{L^\infty(\Omega)}$ vérifie alors $\varepsilon_M(\mu) \leqslant (1+\Lambda_M)\varepsilon_M^*(\mu)$ où $\varepsilon_M^*(\mu) \equiv \inf_{z\in W_M^g} \|g(\cdot;\mu) - z\|_{L^\infty(\Omega)}$, $\forall \mu \in \mathcal{D}$. Comme on le verra (et comme il est classique en approximation polynomiale) la borne sur la constante de Lebesgue bien que pessimiste est souvent compensée par la très rapide convergence de l'autre terme. On peut aussi proposer une approximation a posteriori en introduisant l'estimateur $\hat{\varepsilon}_M(\mu) \equiv |g(t_{M+1};\mu) - g_M(t_{M+1};\mu)|$, exact si $g(\cdot;\mu) \in W_{M+1}^g$ et asymptotique dans le cas contraire.

Le Tableau 1 synthétise les résultats numérique obtenus par la mise en oeuvre de cette interpolation. Il illustre le bon comportement de la méthode et des estimateurs sur le cas $g(x;\mu) \equiv \mathcal{V}((x_1,x_2);\ (\mu_1,\mu_2)) \equiv ((x_1-\mu_1)^2 + (x_2-\mu_2)^2)^{-1/2}$ pour $x \in \Omega \equiv]0,1[^2$ et $\mu \in \mathcal{D} \equiv [-1,-0.01]^2$. La convergence de la méthode est très rapide et la valeur moyenne $\bar{\eta}_M$ de $\hat{\varepsilon}_M(\mu)/\varepsilon_M(\mu)$ modérée.

Cette approche est ensuite couplée avec une méthode de discrétisation en base réduite pour l'approximation de la solution du problème : soit $\mu \in \mathcal{D}$, trouver $u(\mu) \in H_0^1(]0,1[^2)$ telle que $a(u,v;\mu) = f(v;\mu)$, $\forall v \in H_0^1(]0,1[^2$, où a est la forme introduite en (1), avec $g(x;\mu) \equiv \mathcal{V}(x;\mu)$; et $f(v;\mu) = \int_\Omega \mathcal{V}(x;\mu) v$.

La méthode en base réduite est alors : pour $\mu \in \mathcal{D}$, $u_{N,M}(\mu) \in W_N^u$ est la solution de $\int_\Omega \nabla u_{N,M}(\mu)\cdot\nabla v + \int_\Omega g_M(x;\mu) u_{N,M}(\mu) v = \int_\Omega \mathcal{V}(x;\mu) v$, $\forall v \in W_N^u$. L'espace en base réduite W_N^u est défini de façon classique par $W_N^u = \mathrm{Vect}\{\zeta_n \equiv u(\mu_n^u),\ 1 \leqslant n \leqslant N\}$ où $\{\mu_n^u\}_{n=1,\dots,N_{\max}}$ est un jeu de paramètres bien choisis et $g_M(x;\mu) = \sum_{m=1}^M \beta_m(\mu) q_m(x)$ est l'interpolé « empirique » introduit ci-dessus. Le Tableau 2 présente les résultats de cette multiple approximation (base réduite + interpolation empirique) ainsi que la pertinence de l'estimateur a posteriori qui peut être construit en combinant l'estimateur précédent sur l'interpolation empirique et les estimateurs classiques en base réduite proposés par exemple dans [6,8].

1. Introduction

We consider a parametrized evaluation problem: Given a $\mu \in \mathcal{D} \subset \mathbb{R}^P$, evaluate $s(\mu) = \ell(u(\mu))$, where $u \in X$ is the solution of a second-order coercive elliptic partial differential equation $a(u,v;\mu) = f(v)$, $\forall v \in X$. Here μ and $\mathcal{D}$ are the parameter and parameter domain, respectively; X is a Hilbert space with associated inner product $(w,v)_X$ and norm $\|w\|_X$; $\Omega \subset \mathbb{R}^2$ is our spatial domain, a point in which shall be denoted (x_1,x_2); ℓ and f are linear bounded functionals; and, for any $\mu \in \mathcal{D}$, $a(\cdot,\cdot;\mu) : X \times X \to \mathbb{R}$ is a coercive continuous bilinear form.

In the reduced-basis approach [1–3,5,6] we first introduce nested parameter samples $S_N^u \equiv \{\mu_1^u,\dots,\mu_N^u\}$ and associated approximation spaces $W_N^u = \mathrm{span}\{\zeta_n \equiv u(\mu_n^u), 1 \leqslant n \leqslant N\}$ for $N = 1,\dots,N_{\max}$; in actual practice, of course, $u(\mu_n^u)$ is replaced with a 'truth approximation' on (say) a suitably fine piecewise-linear finite element subspace of typically large dimension $\mathcal{N}$. The reduced-basis approximation is then: Given $\mu \in \mathcal{D}$, evaluate $s_N(\mu) = \ell(u_N(\mu))$, where $u_N(\mu) \in W_N^u$ is the solution of $a(u_N(\mu),v;\mu) = f(v)$, $\forall v \in W_N^u$. In general, $u_N(\mu) \to u(\mu)$ very rapidly as N increases [2,4].

We now expand $u_N(\mu) = \sum_{j=1}^N u_{Nj}(\mu)\zeta_j$. The u_{Nj}, $1 \leqslant j \leqslant N$, will then satisfy $\sum_{j=1}^N a(\zeta_j,\zeta_i;\mu) u_{Nj} = f(\zeta_i)$, $1 \leqslant i \leqslant N$; we may subsequently evaluate $s_N(\mu) = \sum_{j=1}^N u_{Nj}(\mu)\ell(\zeta_j)$. If $a(w,v;\mu)$ is affine in μ,

$a(w,v;\mu)=\sum_{k=1}^{K}\Theta^k(\mu)a^k(w,v)$, then an extremely efficient offline–online computational strategy (relevant in the many-query and real-time contexts) may be developed. In the offline stage we form $a^k(\zeta_j,\zeta_i)$, $1\leqslant i,j\leqslant N_{\max}$, $1\leqslant k\leqslant K$; in the online stage we need only assemble and invert $a(\zeta_j,\zeta_i;\mu)=\sum_{k=1}^{K}\Theta^k(\mu)a^k(\zeta_j,\zeta_i)$, $1\leqslant i,j\leqslant N$. The online cost to evaluate $s_N(\mu)$ is thus KN^2+N^3+N — independent of $\mathcal{N}$; since $N\ll\mathcal{N}$, large computational economies can be realized.

Unfortunately, if a is not affine in the parameter, the online complexity is no longer independent of $\mathcal{N}$. For example, for *general* $g(x;\mu)$, the bilinear form

$$a(w,v;\mu)\equiv\int_{\Omega}\nabla w\cdot\nabla v+\int_{\Omega}g(x;\mu)wv \tag{1}$$

will not admit an efficient online–offline decomposition. In this paper we describe a technique that recovers online $\mathcal{N}$ independence even in the presence of non-affine parameter dependence. Our approach (applied to (1), say) is simple: we develop a 'collateral' reduced-basis expansion $g_M(x;\mu)$ for $g(x;\mu)$; we then replace $g(x;\mu)$ in (1) with the (necessarily) affine approximation $g_M(x;\mu)$. The essential ingredients are (i) a 'good' collateral reduced-basis approximation space, (ii) a stable and inexpensive interpolation procedure, and (iii) an effective a posteriori estimator to quantify the newly introduced error terms.

In Section 2 we develop our coefficient-function approximation method; in Section 3 we present a priori and a posteriori error analyses; and in Section 4 we incorporate our coefficient-function approximation into the reduced-basis method. In both Sections 3 and 4 we present numerical results relevant to our model problem (1). In a future paper we provide further details, extend the method to (highly) nonlinear problems, and develop more realistic elliptic and parabolic examples.

2. Coefficient-function approximation: empirical interpolation

We are given a function $g(\cdot;\mu)\in L^\infty(\Omega)$ of sufficient regularity. To begin, we choose μ_1^g, and define $S_1^g=\{\mu_1^g\}$, $\xi_1\equiv g(x;\mu_1^g)$, and $W_1^g=\operatorname{span}\{\xi_1\}$; we assume that $\xi_1\neq 0$. Then, for $M\geqslant 2$, we set $\mu_M^g=\arg\max_{\mu\in\Xi^g}\inf_{z\in W_{M-1}^g}\|g(\cdot;\mu)-z\|_{L^\infty(\Omega)}$, where Ξ^g is a suitably fine parameter sample over $\mathcal{D}$. We then set $S_M^g=S_{M-1}^g\cup\mu_M^g$, $\xi_M=g(x;\mu_M^g)$, and $W_M^g=\operatorname{span}\{\xi_m,\ 1\leqslant m\leqslant M\}$. Note that, thanks to our truth approximation, μ_M^g is the solution of a *standard linear program*.

We suppose that $M_{\max}$ is chosen such that the dimension of $\{g(\cdot;\mu)\,|\,\mu\in\mathcal{D}\}$ exceeds $M_{\max}$; we can then prove

Lemma 2.1. *For any $M\leqslant M_{\max}$, the space W_M^g is of dimension M.*

Proof. We first introduce some notation: $g_{M-1}^*(x;\mu)\equiv\arg\min_{z\in W_{M-1}^g}\|g(\cdot;\mu)-z\|_{L^\infty(\Omega)}$ and $\varepsilon_{M-1}^*(\mu)\equiv\|g(\cdot;\mu)-g_{M-1}^*(\cdot;\mu)\|_{L^\infty(\Omega)}$. It directly follows from our hypothesis on $M_{\max}$ that $\varepsilon_0\equiv\varepsilon_{M_{\max}}^*(\mu_{M_{\max}+1}^g)>0$; our 'arg max' construction then implies $\varepsilon_{M-1}^*(\mu_M^g)\geqslant\varepsilon_0$, $2\leqslant M\leqslant M_{\max}$. We now prove Lemma 1 by induction. Clearly, $\dim(W_1^g)=1$. Assume $\dim(W_{M-1}^g)=M-1$; then if $\dim(W_M^g)\neq M$, $g(\cdot;\mu_M^g)\in W_{M-1}^g$; however, the latter contradicts $\varepsilon_{M-1}^*(\mu_M^g)\geqslant\varepsilon_0>0$. □

We now construct nested sets of interpolation points $T_M=\{t_1,\dots,t_M\}$, $1\leqslant M\leqslant M_{\max}$. We first set $t_1=\arg\operatorname{ess\,sup}_{x\in\Omega}|\xi_1(x)|$, $q_1=\xi_1(x)/\xi_1(t_1)$, $B_{11}^1=1$. Then for $M=2,\dots,M_{\max}$, we solve the linear system $\sum_{j=1}^{M-1}\sigma_j^{M-1}q_j(t_i)=\xi_M(t_i)$, $1\leqslant i\leqslant M-1$, and set $r_M(x)=\xi_M(x)-\sum_{j=1}^{M-1}\sigma_j^{M-1}q_j(x)$, $t_M=\arg\operatorname{ess\,sup}_{x\in\Omega}|r_M(x)|$, $q_M(x)=r_M(x)/r_M(t_M)$, and $B_{ij}^M=q_j(t_i)$, $1\leqslant i,j\leqslant M$. It remains to demonstrate

Lemma 2.2. *The construction of the interpolation points is well-defined, and the functions $\{q_1,\dots,q_M\}$ form a basis for W_M^g.*

Proof. We proceed by induction. Clearly, $W_1^g = \text{span}\{q_1\}$. Assume $W_{M-1}^g = \text{span}\{q_1, \dots, q_{M-1}\}$; if (i) B^{M-1} is invertible, and (ii) $|r_M(t_M))| > 0$, then our construction may proceed and we may form $W_M^g = \text{span}\{q_1, \dots, q_M\}$. To prove (i), we need only note that B^{M-1} is lower triangular with unity diagonal; to prove (ii), we observe that $|r_M(t_M)| \geqslant \varepsilon_{M-1}^*(\mu_M^g) \geqslant \varepsilon_0 > 0$. □

Lemma 2.3. *For any M-tuple $(\alpha_i)_{i=1,\dots,M}$ of real numbers, there exists a unique element $w \in W_M^g$ such that $\forall i$, $1 \leqslant i \leqslant M$, $w(t_i) = \alpha_i$.*

Proof. It is a straightforward consequence of the invertibility of B^M. □

Finally, our coefficient function approximation is the interpolant of g over T_M as defined from Lemma 2.3: $g_M(x;\mu) = \sum_{m=1}^M \beta_m(\mu) q_m(x)$, where $\sum_{j=1}^M B_{ij}^M \beta_j(\mu) = g(t_i;\mu)$, $1 \leqslant i \leqslant M$. We define $\varepsilon_M(\mu) \equiv \|g(\cdot;\mu) - g_M(\cdot;\mu)\|_{L^\infty(\Omega)}$.

3. Error analyses for the empirical interpolation procedure

3.1. A priori framework

We define a 'Lebesgue constant' [7] $\Lambda_M = \sup_{x\in\Omega} \sum_{m=1}^M |V_m^M(x)|$, where V_m^M is the only element in W_M^g such that $V_m^M(t_n) = \delta_{mn}$ (the V_m^M are the characteristic functions as defined from Lemma 2.3). Note that Λ_M depends on W_M^g and T_M, but not on μ nor on our choice of basis for W_M^g. Observe also that $\sum_{j=1}^M B_{ji}^M V_j^M(x) = q_i(x)$, $1 \leqslant i \leqslant M$. We can then prove

Lemma 3.1. *The interpolation error $\varepsilon_M(\mu)$ satisfies $\varepsilon_M(\mu) \leqslant \varepsilon_M^*(\mu)(1 + \Lambda_M)$, $\forall \mu \in \mathcal{D}$.*

Proof. We define $e_M^*(x;\mu) = g(x;\mu) - g_M^*(x;\mu)$ and $g_M(x;\mu) - g_M^*(x;\mu) = \sum_{m=1}^M \delta_m^M(\mu) q_m(x)$. We then readily derive that $e_M^*(t_i;\mu) = \sum_{m=1}^M \delta_m^M(\mu) q_m(t_i) = \sum_{m=1}^M B_{im}^M \delta_m^M(\mu)$, $1 \leqslant i \leqslant M$. It thus follows that $|\varepsilon_M(\mu) - \varepsilon_M^*(\mu)| \leqslant \|\sum_{m=1}^M \delta_m^M(\mu) q_m(x)\|_{L^\infty(\Omega)} = \|\sum_{k=1}^M \sum_{m=1}^M B_{km}^M \delta_m^M(\mu) V_k^M(x)\|_{L^\infty(\Omega)} = \|\sum_{i=1}^M e_M^*(t_i;\mu) \times V_i^M(x)\|_{L^\infty(\Omega)} \leqslant \varepsilon_M^*(\mu)\Lambda_M$, since $|e_M^*(t_i;\mu)| \leqslant \varepsilon_M^*(\mu)$, $1 \leqslant i \leqslant M$. □

We can further show

Proposition 3.2. *The Lebesgue constant Λ_M satisfies $\Lambda_M \leqslant 2^M - 1$.*

Proof. We need only note that (i) B^M is lower triangular with unity diagonal — $q_m(t_m) = 1$, $1 \leqslant m \leqslant M$, and (ii) all entries of B^M are of modulus no greater than unity — $\|q_m\|_{L^\infty(\Omega)} \leqslant 1$, $1 \leqslant m \leqslant M$. Hence $|V_m^M(x)| \leqslant |q_m(x)| + \sum_{i=m+1}^M |V_i^M(x)| \leqslant 1 + \sum_{i=m+1}^M |V_i^M(x)|$. It follows, since $|V_M^M(x)| \leqslant 1$, that $|V_{M+1-m}^M(x)| \leqslant 2^{m-1}$, $1 \leqslant m \leqslant M$, and thus $\sum_{m=1}^M |V_m^M(x)| \leqslant 2^M - 1$. □

Proposition 3.2 is very pessimistic and of little practical value (though $\varepsilon_M^*(\mu)$ does often converge sufficiently rapidly that $\varepsilon_M^*(\mu) 2^M \to 0$ as $M \to \infty$); this is not surprising given analogous results in the theory of polynomial interpolation [7]. However, Proposition 3.2 does provide some notion of stability.

3.2. A posteriori estimators

Given an approximation $g_M(x;\mu)$ for $M \leqslant M_{\max} - 1$, we define $\mathcal{E}_M(x;\mu) \equiv \hat{\varepsilon}_M(\mu) q_{M+1}(x)$, where $\hat{\varepsilon}_M(\mu) \equiv |g(t_{M+1};\mu) - g_M(t_{M+1};\mu)|$. We can then prove

Table 1

$\varepsilon^*_{M,\max}$, $\bar{\rho}_M$, Λ_M, and $\bar{\eta}_M$ as a function of M

M	$\varepsilon^*_{M,\max}$	$\bar{\rho}_M$	Λ_M	$\bar{\eta}_M$
4	2.65E–01	0.64	1.79	1.79
8	4.20E–02	0.65	2.07	2.01
12	8.66E–03	0.54	3.14	2.23
16	1.45E–03	0.85	2.09	2.62
20	1.85E–04	0.46	3.57	2.10

Proposition 3.3. *If* $g(\cdot;\mu) \in W^g_{M+1}$, *then* (i) $g(x;\mu) - g_M(x;\mu) = \pm\mathcal{E}_M(x;\mu)$ (*either* $\mathcal{E}_M(x;\mu)$ *or* $-\mathcal{E}_M(x;\mu)$), *and* (ii) $\|g(\cdot;\mu) - g_M(\cdot;\mu)\|_{L^\infty(\Omega)} \leqslant \hat{\varepsilon}_M(\mu)$.

Proof. Since by assumption $g(\cdot;\mu) \in W^g_{M+1}$, $g(x;\mu) - g_M(x;\mu) = \sum_{m=1}^{M+1} \kappa_m q_m(x)$. We may thus consider the linear system $\sum_{m=1}^{M+1} \kappa_m q_m(t_i) = g(t_i;\mu) - g_M(t_i;\mu)$, $1 \leqslant i \leqslant M+1$. However, $g(t_i;\mu) - g_M(t_i;\mu) = 0$, $1 \leqslant i \leqslant M$; thus, since the matrix $q_m(t_i)$ is lower triangular, $\kappa_m = 0$, $1 \leqslant m \leqslant M$, and since $q_{M+1}(t_{M+1}) = 1$, $\kappa_{M+1} = g(t_{M+1};\mu) - g_M(t_{M+1};\mu)$; this concludes the proof of (i). The proof of (ii) then directly follows from $\|q_{M+1}\|_{L^\infty(\Omega)} = 1$. $\square$

Of course, in general $g(\cdot;\mu) \notin W^g_{M+1}$, and hence our estimator $\hat{\varepsilon}_M(\mu)$ is not quite a rigorous upper bound; however, if $\varepsilon_M(\mu) \to 0$ very fast, we expect that the effectivity, $\eta_M(\mu) \equiv \hat{\varepsilon}_M(\mu)/\varepsilon_M(\mu)$, shall be close to unity. Furthermore, the estimator is very inexpensive – *one additional evaluation* of $g(\cdot;\mu)$.

3.3. Numerical results

We consider $g(x;\mu) \equiv \mathcal{V}((x_1,x_2);(\mu_1,\mu_2)) \equiv ((x_1-\mu_1)^2 + (x_2-\mu_2)^2)^{-1/2}$ for $x \in \Omega \equiv]0,1[^2$ and $\mu \in \mathcal{D} \equiv [-1,-0.01]^2$; we choose for Ξ^g a random sample of 225 parameter points; and we take $\mu^g_1 = (-0.01,-0.01)$. We then construct S^g_M, W^g_M, T_M, and B^M, $1 \leqslant M \leqslant M_{\max}$, following the procedure of Section 2. We introduce a random parameter test sample Ξ^g_{Test} of size $Q_{\text{Test}} = 121$, and define $\varepsilon^*_{M,\max} = \max_{\mu\in\Xi^g_{\text{Test}}} \varepsilon^*_M(\mu)$, $\bar{\rho}_M = Q^{-1}_{\text{Test}} \sum_{\mu\in\Xi^g_{\text{Test}}} (\varepsilon_M(\mu)/(\varepsilon^*_M(\mu)(1+\Lambda_M)))$, $\bar{\eta}_M = Q^{-1}_{\text{Test}} \sum_{\mu\in\Xi^g_{\text{Test}}} \eta_M(\mu)$. We present in Table 1 $\varepsilon^*_{M,\max}$, $\bar{\rho}_M$, Λ_M, and $\bar{\eta}_M$ as a function of M ($M_{\max} = 20$). We observe that $\varepsilon^*_{M,\max}$ converges rapidly with M; that the Lebesgue constant provides a reasonably sharp measure of the interpolation-induced error; that the Lebesgue constant grows very slowly — $\varepsilon_M(\mu)$ is *only slightly larger that the min max result* $\varepsilon^*_M(\mu)$; and that the error estimator effectivity is reasonably close to unity. (Note also that B^M is quite well-conditioned given our choice of basis.)

4. Reduced-basis approximation

We consider the following model problem: Given $\mu \in \mathcal{D} \equiv [-1,-0.01]^2$, find $u(\mu) \in X$ such that $a(u,v;\mu) = f(v;\mu)$, $\forall v \in X$. Here $\Omega =]0,1[^2$; $X = H^1_0(\Omega)$; $(w,v)_X = \int_\Omega \nabla w \cdot \nabla v$; a is the bilinear form (1) for $g(x;\mu) \equiv \mathcal{V}(x;\mu)$; and $f(v;\mu) = \int_\Omega \mathcal{V}(x;\mu)v$. The solution develops a boundary layer in the vicinity of $x = (0,0)$ for μ near the 'corner' $(-0.01,-0.01)$. For our output, we consider $s(\mu) = \ell(u(\mu))$ for $\ell(v) = \int_\Omega v$.

Our reduced-basis approximation is thus: Given $\mu \in \mathcal{D}$, evaluate $s_{N,M}(\mu) = \ell(u_{N,M}(\mu))$, where $u_{N,M}(\mu) \in W^u_N$ is the solution of $\int_\Omega \nabla u_{N,M}(\mu) \cdot \nabla v + \int_\Omega g_M(x;\mu) u_{N,M}(\mu) v = \int_\Omega g_M(x;\mu) v$, $\forall v \in W^u_N$. Here W^u_N is defined in Section 1, and $g_M(x;\mu) = \sum_{m=1}^M \beta_m(\mu) q_m(x)$ is our coefficient-function approximation defined in Section 2 and analyzed in Section 3. Our discrete equations for $u_{N,Mj}$ ($u_{N,M}(\mu) = \sum_{j=1}^N u_{N,Mj}(\mu)\zeta_j$) are therefore $\sum_{j=1}^N (\int_\Omega \nabla\zeta_j \cdot \nabla\zeta_i + \sum_{m=1}^M \int_\Omega \beta_m(\mu) q_m(x)\zeta_j\zeta_i) u_{N,Mj} = \sum_{m=1}^M \int_\Omega \beta_m(\mu) q_m(x)\zeta_i$, $1 \leqslant i \leqslant N$. It is now a simple matter to develop an offline–online computational procedure: the online complexity is $\mathrm{O}(N^2 M) + O(N^3)$

Table 2
$\varepsilon^u_{N,M,\max}$ and $\bar{\eta}^u_{N,M}$ as a function of N (for $M = N$)

N	4	8	12	16	20
$\varepsilon^u_{N,M,\max}$	9.70E–02	5.53E–03	1.76E–03	4.53E–04	2.71E–05
$\bar{\eta}^u_{N,M}$	2.02	3.46	3.11	3.14	5.28

to respectively assemble and solve the requisite stiffness system and then $\mathrm{O}(N)$ to evaluate $s_{N,M}(\mu)$; the essential point is that the online complexity is independent of $\mathcal{N}$.

It is readily demonstrated that the error $e_{N,M}(\mu) = u(\mu) - u_{N,M}(\mu)$ satisfies $\int_\Omega \nabla e_{N,M}(\mu)\cdot\nabla v + \int_\Omega g(x;\mu)\times e_{N,M}(\mu)v = R_{N,M}(v;\mu) + \int_\Omega (g(x;\mu) - g_M(x;\mu))v - \int_\Omega (g(x;\mu) - g_M(x;\mu))u_{N,M}(\mu)v,\ \ \forall v \in X$, where $R_{N,M}(v;\mu) \equiv \int_\Omega g_M(x;\mu)v - \int_\Omega \nabla u_{N,M}(\mu)\cdot\nabla v - \int_\Omega g_M(x;\mu)u_{N,M}(\mu)v$. It follows that, if we suppose $g(x;\mu) \in W^g_{M+1}$, then $\|e_{N,M}(\mu)\|_X \leqslant \Delta_{N,M}(\mu)$, where $\Delta_{N,M}(\mu) \equiv \hat{\varepsilon}_M(\mu)\sup_{v\in X}\frac{\int_\Omega q_{M+1}(x)(1-u_{N,M}(\mu))v}{\|v\|_X} + \sup_{v\in X}\frac{R_{N,M}(v;\mu)}{\|v\|_X}$. (Note an associated error bound on $s(\mu) - s_{N,M}(\mu)$ can be readily developed from standard duality considerations [6].) It is now possible [6] to develop an offline–online computational procedure for $\Delta_{N,M}(\mu)$: the online complexity to evaluate the requisite dual norms is $\mathrm{O}(N^2M^2)$ – independent of $\mathcal{N}$. (We may invoke these inexpensive error estimators to develop good samples S^u_N: given S^u_{N-1}, we choose μ^u_N to be the arg max over (a fine sample in) $\mathcal{D}$ of $\Delta_{N,M_{\max}}(\mu)$ [8].)

We now introduce a random parameter test sample Ξ^u_{Test} of size $Q^u_{\mathrm{Test}} = 289$, and define $\varepsilon^u_{N,M,\max} = \max_{\mu\in\Xi^u_{\mathrm{Test}}}\|e_{N,M}(\mu)\|_X$ and $\bar{\eta}^u_{N,M} = (Q^u_{\mathrm{Test}})^{-1}\sum_{\mu\in\Xi^u_{\mathrm{Test}}}(\Delta_{N,M}(\mu)/\|e_{N,M}(\mu)\|_X)$. We present in Table 2 $\varepsilon^u_{N,M,\max}$ and $\bar{\eta}^u_{N,M}$ as a function of N for the particular choice $M = N$. We observe that the error decreases very rapidly, and that our error bound is quite sharp. Indeed, the results are largely indistinguishable from the standard Galerkin projection. However, the latter suffers from $\mathrm{O}(\mathcal{N})$ online complexity, and is thus much more expensive than the coefficient-function approximation/empirical interpolation approach developed in this paper.

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