

Fig. 1. Scattering of plane wave by parabolic cylinder.

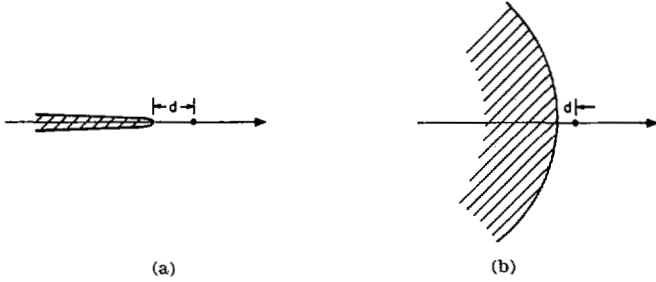


Fig. 2. Two cases for discussing field behavior near edge. (a) $\xi_0^2 \ll d$. (b) $\xi_0^2 \gg d$.

(Fig. 2(a)), while the case B arises when d is small (Fig. 2(b)). The behavior of $|E_z|$ for these two cases is

$$|E_z| \sim \begin{cases} d^{1/2}, & \text{for case A (Fig. 2(a))} \\ d, & \text{for case B (Fig. 2(b)).} \end{cases} \quad (5)$$

Two comments on the results in (5) and (6) are now in order.

1) In case A, the magnetic field becomes very large as the observation point approaches the edge. As far as the field near the edge is concerned, it is not different from a half-plane with a sharp edge, and its behavior is predictable by the Meixner's edge condition.

2) In case B, the field decreases linearly with distance d . This is caused by the fact that a smooth cylinder was assumed at the beginning of this example purely for mathematical convenience. In practice, when the length can be measured as small as d in Fig. 2(b), a smooth cylinder is by no means easier to make mechanically than a sharp edge. In other words, for sufficiently small ξ_0^2 the results in (6) have little physical significance since the roughness of a surface in reality, however small, will eventually invalidate its usefulness.

In conclusion, the so-called "new condition" is based on an observation that "the edge of all physical apertures must be smooth, not sharp in the exact geometric sense." In reality, the smoothness or sharpness can only be measured according to some scale. When the scale is small enough, no object is smooth. On the other hand, when the scale of measuring distance is larger than the thickness of a smooth object, the field behavior near the tip of the object is not different from that of a sharp half-plane.

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Kirchhoff Theory: Scalar, Vector, or Dyadic?

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Abstract—The superior nature of the Franz formula as compared to that of Stratton and Chu is pointed out. A connection between the two distinct formulas is shown. The compactness of the dyadic version of Franz' formula is emphasized.

I. INTRODUCTION

This communication is prompted by a recent note of Wait [1] commenting on the pitfall of the scalar electromagnetic formulation of Kirchhoff's theory. Wait further implied that the general vector formulation to use is the Stratton-Chu formula [2]. The purpose of this communication is to point out the superior nature of Franz' formula [3] and to show a simple connection between these two distinct formulas. Finally, we wish to emphasize the compactness and convenience of adopting the dyadic version of the Franz formulation in presenting Kirchhoff's theory and Huygens' principle.

The so-called Stratton-Chu formula actually is identical to the Larmor-Tedone formula as described by Baker and Copson [4]. However, presumably because of the simplicity with which Stratton and Chu derived their formula, the formula bears the names of these two authors. In any event, the formula has enjoyed its popularity in formulating diffraction problems and in antenna theory, particularly in works published in this country. Recent books such as the ones by Bowman *et al.* [5] and by Collin and Zucker [6], still treat it as one of the basic formulas in electromagnetic theory. An alternate formulation due to Franz [3], which appears to be less restrictive and more elegant, has not received the popularity it deserves, since its inception. Franz' formula has been quoted by some notable authors such as Sommerfeld [7] and Jones [8]. The latter, however, did not give the reference. A derivation of the Franz formula is also found in the book by Mentzer [9]. The popularity of the Stratton-Chu formula is illustrated by the following incident. In 1960 this author submitted a communication on Huygens' principle [10] to this TRANSACTIONS for consideration of publication. A reviewer commented that the formulation does not agree with the Stratton-Chu formula, hence expressed his dismay. This incident is indicative of the influence of Stratton and Chu's work in the minds of some people.

In the original work of Franz, the formula was derived by means of the method of dyadic Green's function. The same method, presented in a different manner, was later used by Levine and Schwinger [11]. To show the connection between the Stratton-Chu formula and the Franz formula, we reproduce here the relevant formulas as applied to some effective source field located on a closed surface.

Stratton-Chu Formula:

$$\vec{E}_s(\vec{R}) = \iint [\omega\mu(\hat{n} \times \vec{H})g + (\hat{n} \times \vec{E}) \times \nabla'g + (\hat{n} \cdot \vec{E})\nabla'g] dS' \quad (1)$$

$$\vec{H}_s(\vec{R}) = \iint [-i\omega\epsilon(\hat{n} \times \vec{E})g + (\hat{n} \times \vec{H}) \times \nabla'g + (\hat{n} \cdot \vec{H})\nabla'g] dS' \quad (2)$$

where

$$g = \frac{\exp(ik|\vec{R} - \vec{R}'|)}{4\pi|\vec{R} - \vec{R}'|}$$

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\vec{R} = position vector designating point of observation

\vec{R}' = position vector designating source field

$$\nabla' = \left(\hat{x} \frac{\partial}{\partial x'} + \hat{y} \frac{\partial}{\partial y'} + \hat{z} \frac{\partial}{\partial z'} \right).$$

Franz Formula:

$$\vec{E}(\vec{R}) = \nabla \times \iint (\hat{n} \times \vec{E}) g dS' + \frac{i}{\omega\epsilon} \nabla \times \nabla \times \iint (\hat{n} \times \vec{H}) g dS' \quad (3)$$

$$\vec{H}(\vec{R}) = \nabla \times \iint (\hat{n} \times \vec{H}) g dS' - \frac{i}{\omega\mu} \nabla \times \nabla \times \iint (\hat{n} \times \vec{E}) g dS'. \quad (4)$$

It seems obvious that the Franz formula is conceptually simpler since it requires only the tangential components of the field on the closed surface, while the Stratton-Chu formula requires the normal components as well. Most important of all, when the field has an edge singularity on the surface of integration the Larmor-Tedone formula or the Stratton-Chu formula must be modified as shown by Kottler [2], [4] in order to make the resultant field Maxwellian. Such a modification, however, is not needed in Franz' formula since the field due to the edge current is already embedded in that formulation. In spite of the restricted nature of the Larmor-Tedone or Stratton-Chu formula one can use it to derive Franz' formula by a simple manipulation. Of course, this is not the way that Franz originally derived the formula. For example, by taking the curl of (2) we obtain

$$\begin{aligned} \nabla \times \vec{H}_s \\ = \nabla \times \iint [-i\omega\epsilon(\hat{n} \times \vec{E})g + (\hat{n} \times \vec{H}) \times \nabla'g + (\hat{n} \cdot \vec{H})\nabla'g] dS'. \end{aligned} \quad (5)$$

Now

$$\begin{aligned} \nabla \times \nabla \times \iint (\hat{n} \times \vec{H}) g dS' &= \nabla \times \iint (\hat{n} \times \vec{H}) \times \nabla'g dS' \\ \nabla \times \iint (\hat{n} \cdot \vec{H}) \nabla'g dS' &= 0. \end{aligned}$$

Thus in regions where \vec{J} the current density vanishes we have

$$\begin{aligned} \vec{E}(\vec{R}) &= \frac{i}{\omega\epsilon} \nabla \times \vec{H}_s(\vec{R}) = \nabla \times \iint (\hat{n} \times \vec{E}) g dS' \\ &\quad + \frac{i}{\omega\epsilon} \nabla \times \nabla \times \iint (\hat{n} \times \vec{H}) \times \nabla'g dS' \quad (6) \end{aligned}$$

which is identical to (3). The derivation of (4) from (1) follows the same procedure. In regions where \vec{J} is nonvanishing, the volume integral in the Stratton-Chu formula should be retained. The proof is more tedious but straightforward. It is interesting and perhaps unexpected that a restricted formula can be changed to a superior formula by such a simple manipulation.

Finally, it may not be superfluous to stress that as far as the exhibition of Kirchhoff's theory and the presentation of Huygens' principle are concerned, the most compact formulation appears to be the one based on the dyadic Green's function pertaining to the vector wave equation for \vec{E} and \vec{H} or by forming a combined field vector as discussed by the author [10].

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