Assignment 3

(MA6.102) Probability and Random Processes, Monsoon 2023

Date: 6 September 2023, Due on 16 September 2023 (Saturday).

INSTRUCTIONS

- Discussions with other students are not discouraged. However, all write-ups must be done individually with your own solutions.
- Any plagiarism when caught will be heavily penalized.
- Be clear and precise in your writing. Also, clearly state the assumptions made (if any) that are not specified in the question.

Problem 1. Alvin shops for probability books for K hours, where K is a random variable that is equally likely to be $1, 2, 3,$ or 4. The number of books N that he buys is random and depends on how long he shops according to the conditional PMF $P_{N|K}(n|k) = \frac{1}{k}$, for $n = 1, 2, \ldots, k$.

(a) (2 Marks) Find the joint PMF of K and N.

(b) (2 Marks) Find the marginal PMF of N.

(c) (1 Mark) Find the conditional PMF of K given that $N = 2$.

(d) (5 Marks) Find the conditional mean and conditional variance of K, given that he bought at least 2 but no more than 3 books.

(e) (5 Marks) The cost of each book is a random variable with mean Rs. 30. What is the expected value of his total expenditure?

Problem 2 (5 Marks)**.** Let N be a non-negative integer valued random random variable. Show that

$$\mathbb{E}[N] = \sum_{i=1}^{\infty} P(N \geq i).$$

Problem 3 (10 Marks)**.** A permutation on the numbers in $[1 : n]$ can be represented as a function $\pi : [1 : n] \to [1 : n]$, where $\pi(i)$ is the position of i in the ordering given by the permutation. A fixed point of a permutation $\pi : [1 : n] \to [1 : n]$ is a value x for which $\pi(x) = x$. Let X be number of fixed points of a permutation chosen uniformly at random from all permutations. Find $\mathbb{E}[X]$ and $\text{Var}(X)$.

Problem 4 (5 Marks)**.** Two chess players plan to play a best-of-7 match (the match will end as soon as either team has won 4 games). Each game ends in a win for one team and a loss for the other team. Assume that each team is equally likely to win each game, and that the games played are independent. Find the mean and variance of the number of games played.

Problem 5 (5 Marks)**.** A non-zero integer random Variable X has pmf $p_X(x) = 2^{-(|x|+1)}$. Show that this is a valid pmf. Also, calculate $E[X]$.

Problem 6 (5 Marks)**.** In this problem, we want you to show that the geometric random variable is memoryless. Let $X \sim$ Geometric(p).

Show that,

$$P(X > m + l | X > m) = P(X > l), \forall m, l \in 1, 2, 3, ...$$

We can interpret this in the following way: Remember that a geometric random variable can be obtained by tossing a coin repeatedly until observing the first heads. If we toss the coin several times, and do not observe a heads, from now on it is like we start all over again. In other words, the failed coin tosses do not impact the distribution of waiting time from this point forward. The reason for this is that the coin tosses are independent.

Problem 7 (5 Marks)**.** The number of students entering the library is a Poisson Random Variable. On average, 20 students arrive per hour. Let X be the number of students coming to the library from 9:30AM to 10:30AM. What is $P(15 \leq X \leq 20)$?

Problem 8. For three random variables X, Y, and Z, show that

(a) (2 Marks) $\mathbb{E}[X|X] = X$.

(b) (4 Marks) $\mathbb{E}[Xg(Y)|Y] = g(Y)\mathbb{E}[X|Y]$. From this, deduce that $\mathbb{E}[g(Y)|Y] = g(Y)$.

(c) (4 Marks) $\mathbb{E}[\mathbb{E}[X|Y, Z]|Y] = \mathbb{E}[X|Y]$.