Assignment 2

(MA6.102) Probability and Random Processes, Monsoon 2023

Date: 21 August 2023, Due on 28 August 2023 (Monday).

INSTRUCTIONS

- Discussions with other students are not discouraged. However, all write-ups must be done individually with your own solutions.
- Any plagiarism when caught will be heavily penalized.
- Be clear and precise in your writing. Also, clearly state the assumptions made (if any) that are not specified in the question.

Problem 1. Consider three events A, B, and C.

(a) (3 Marks) If $P(A) = 0.5$, $P(B) = 0.46$, and $P(A \cup B) = 2P(A \cap B)$, find the value of $P(A \cup B)$.

(b) (4 Marks) Show that

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(A \cap C) + P(A \cap B \cap C).$$

(c) (3 Marks) If $P(A) = P(B) = P(C) = 0.5$, $P(A \cup B) = 0.55$, $P(A \cup C) = 0.7$, $P(B \cap C) = 0.3$, and $P(A \cap B \cap C) = 2P(A \cap B \cap C^{\mathsf{c}})$, find the values of $P(A \cap B \cap C)$ and $P(A \cup B \cup C)$.

Problem 2. Consider a family that has two children. Assume that every birth results in a boy with probability $\frac{1}{2}$, independent of other births and also that the parents in the family had decided to have exactly two children. Consider the following three different scenarios.

(a) (3 Marks) Given that the first child is a girl, what is the probability that both children are girls?

(b) (3 Marks) Given that at least of the children is a girl, what is the probability that both children are girls?

(c) (4 Marks) Assume that if a child is a girl, her name will be Lilly with probability $\frac{1}{2}$ independently from other child's name. If the child is a boy, his name will not be Lilly. Given that the family has at least one daughter named Lilly, what is the probability that both children are girls?

Problem 3 (5 Marks)**.** Alice and Bob want to choose between the opera and the movies by tossing a fair coin. Unfortunately, the only available coin is biased (though the bias is not known exactly). Assume that the probability of getting heads for the biased coin is not equal to 0 or 1. How can they use the biased coin to make a decision so that either option (the opera or the movies) is equally likely to be chosen?

Problem 4 (10 Marks)**.** For an arbitrary sequence of infinite number of events $A_1, A_2, \ldots$, prove that

$$\text{(i)}(\cup_{i=1}^{\infty} A_i)^{\mathsf{c}} = \cap_{i=1}^{\infty} A_i^{\mathsf{c}},$$
$$\text{(ii)}P(\cap_{i=1}^{\infty} A_i) = \lim_{n\to\infty} P(\cap_{i=1}^{n} A_i).$$

Problem 5 (5 Marks)**.** A hiker starts by taking one of n available trails denoted $1, 2, \ldots, n$. An hour into the hike, trail i subdivides into $i + 1$ subtrails, only one of which leads to the hiker's destination. The hiker has no map and makes uniformly random choices of trail and subtrail. What is the probability of reaching the destination?