Assignment 4

(MA6.102) Probability and Random Processes, Monsoon 2023

Date: 27 September 2023, Due on 7 October 2023 (Saturday).

INSTRUCTIONS

- Discussions with other students are not discouraged. However, all write-ups must be done individually with your own solutions.
- Any plagiarism when caught will be heavily penalized.
- Be clear and precise in your writing. Also, clearly state the assumptions made (if any) that are not specified in the question.

Problem 1 (5 Marks). Consider a random variable X with the following two-sided exponential PDF

$$f_X(x) = \begin{cases} p\lambda e^{-\lambda x}, & \text{if } x \geq 0, \\ (1-p)\lambda e^{\lambda x}, & \text{if } x < 0, \end{cases}$$

where λ and p are scalars with $\lambda > 0$ and $p \in [0, 1]$. Find the mean and the variance of X.

Problem 2. Let X be a Gaussian random variable with PDF

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}},$$

where $\mu \in \mathbb{R}$ and $\sigma \in (0, \infty)$. Show the following.
(a) (2 Marks) $\int_{-\infty}^{\infty} f_X(x) \, dx = 1$.
(b) (1 Mark) $\mathbb{E}[X] = \mu$.
(c) (2 Marks) $\text{var}(X) = \sigma^2$.

Problem 3 (5 Marks). A stick of length 1 is split at a point U that is uniformly distributed over $[0, 1]$. Determine the expected length of the substick that contains the point p, $0 \leq p \leq 1$. Also, find the value of p that maximizes this expected length.

Problem 4 (5 Marks). For a non-negative continuous random variable X, show that

$$\mathbb{E}[X^n] = \int_0^{\infty} nx^{n-1} P(X > x) \, dx.$$

Problem 5. Let X be a continuous random variable with CDF F_X. Suppose F_X is strictly increasing over $\{x : 0 < F_X(x) < 1\}$. Show that
(a) (2.5 Marks) the random variable $Y \triangleq F_X(X)$ is uniformly distributed on $[0, 1]$,
(b) (2.5 Marks) the random variable $Z \triangleq -\log F_X(X)$ is exponentially distributed.

Problem 6 (5 Marks). Let X be an exponentially distributed random variable with parameter λ. Let $Y = \lfloor X \rfloor$, which is the integer part of X, and let $R = X - \lfloor X \rfloor$. Find the PMF of Y and the PDF of R.