Assignment 5

(MA6.102) Probability and Random Processes, Monsoon 2023

Date: 13 October 2023, Due on 19 October 2023 (Thursday).

Instructions

- Discussions with other students are not discouraged. However, all write-ups must be done individually with your own solutions.
- Any plagiarism when caught will be heavily penalized.
- Be clear and precise in your writing. Also, clearly state the assumptions made (if any) that are not specified in the question.

Problem 1 (5 marks)**.** Let X be a random variable with PDF

$$f_X(x) = \begin{cases} \frac{x}{4}, & \text{if } 1 \le x \le 3, \\ 0, & \text{otherwise,} \end{cases} \tag{1}$$

and let A be the event $\{X \ge 2\}$. Find $P(A)$, $f_{X|A}(x)$, $\mathbb{E}[X|A]$, and $\mathbb{E}[X]$.

Problem 2 (5 Marks)**.** Consider two jointly continuous random variables X and Y with joint PDF

$$f_{XY}(x,y) = \begin{cases} 4x^2, & 0 < y < x < 1, \\ 0, & \text{otherwise.} \end{cases} \tag{2}$$

Compute f_Y and $f_{X|Y}$.

Problem 3 (5 Marks)**.** Consider a random variable X with the following two-sided exponential PDF

$$f_X(x) = \begin{cases} p\lambda \mathsf{e}^{-\lambda x}, & \text{if } x \ge 0, \\ (1-p)\lambda \mathsf{e}^{\lambda x}, & \text{if } x < 0, \end{cases}$$

where λ and p are scalars with $\lambda > 0$ and $p \in [0,1]$. Using the total expectation theorem, find the mean and the variance of X.

Problem 4. Let $X_1, X_2, \ldots, X_n, \ldots$ be a sequence of independent and identically distributed continuous random variables with the common PDF $f_X(x)$. Note that $P(X = \alpha) = 0$, for all $\alpha \in \mathbb{R}$, and that $P(X_i = X_j) = 0$, for all $i \neq j$. For $n \ge 2$, define X_n as a *record-to-date* of the sequence if $X_n > X_i$, for all $i < n$.
(a) [2 Marks] Find the probability that X_2 is a record-to-date.
(b) [2 Marks] Find the probability that X_n is a record-to-date.
(c) [4 Marks] Let N_1 be the index of the first record-to-date in the sequence. Find $P(N_1 > n)$, for each $n \ge 2$.

Problem 5. Consider a discrete random variable Y with CDF

$$F_Y(k) = 1 - \frac{2}{(k+1)(k+2)}, \text{ for integer values } k \geq 0.$$

(a) [2 Marks] Compute $\mathbb{E}[Y]$.
(b) [3 Marks] Let X be another integer-valued random variable with the conditional PMF given by

$$P_{X|Y}(x|y) = \frac{1}{y}, \text{ for } x \in \{1, 2, \ldots, y\}.$$

Find $\mathbb{E}[X]$.

Problem 6 (10 Marks)**.** Prove that two random variables X and Y (either both continuous or both discrete) are independent if and only if $F_{XY}(x, y) = F_X(x)F_Y(y)$, for all x, y.

Problem 7. A family has three children, A, B, and C, of height X_1, X_2, and X_3, respectively. If X_1, X_2, and X_3 are independent and identically distributed continuous random variables, evaluate the following probabilities:
(a) [2 Marks] $P(A$ is the tallest child$)$.
(b) [2 Marks] $P(A$ is taller than $B \mid A$ is taller than $C)$.
(c) [2 Marks] $P(A$ is taller than $B \mid B$ is taller than $C)$.