Assignment 6

(MA6.102) Probability and Random Processes, Monsoon 2023

Date: 31 October 2023, Due on 6 November 2023 (Monday).

INSTRUCTIONS

- Discussions with other students are not discouraged. However, all write-ups must be done individually with your own solutions.
- Any plagiarism when caught will be heavily penalized.
- Be clear and precise in your writing. Also, clearly state the assumptions made (if any) that are not specified in the question.

Problem 1 (5 Marks)**.** Let X be a continuous random variable with PDF f_X, and Y be a function of X defined as

$$Y \triangleq \begin{cases} X, & \text{if } X \geq 0, \\ X^2, & \text{if } X^2 \leq 0. \end{cases}$$

Compute the PDF of Y in terms of f_X.

Problem 2 (5 Marks)**.** Let X and Y be independent random variables that are uniformly distributed on the interval $[0, 1]$. Find the PDF of the random variable $Z = XY$.

Problem 3. Suppose X and Y are jointly continuous random variables with joint PDF f_{XY}. Let $Z = \min\{X, Y\}$ and $W = \max\{X, Y\}$.

(a) (5 Marks) Find the joint PDF f_{ZW} in terms of f_{XY}.

(b) (5 Marks) If X and Y are independent and uniformly distributed on $[0, 1]$, compute f_{ZW}.

Problem 4 (5 marks)**.** Two points are chosen randomly and independently from the interval $[a, b]$ according to a uniform distribution. Find the expected distance between the two points.

Problem 5 (5 Marks)**.** Let X and Y be two random variables with the associated MGFs $M_X(s)$ and $M_Y(s)$, respectively. Let Z and W be random variables with the respective MGFs given by

$$M_Z(s) = M_X(s)^5,$$
$$M_W(s) = M_X(s)^2 M_Y(s)^3.$$

Find $\mathbb{E}[Z]$, var(Z), $\mathbb{E}[W]$, and var(W) in terms of $\mathbb{E}[X]$, var(X), $\mathbb{E}[Y]$, and var(Y).

Problem 6. (a) (5 Marks) Find the probability that the quadratic polynomial $Ax^2 + Bx + 1$, where the coefficients A and B are determined by drawing independent and identically distributed (i.i.d.) $U[0, 1]$ random variables, has at least one real root.

(b) (5 Marks) Find the probability that the quadratic polynomial $ax^2 + bx + c$ has a complex root, where the coefficients a, b, c are determined by drawing i.i.d. $U[0, 1]$ random variables.

Problem 7. (a) (3 Marks) Let a random variable $X \sim N(0,1)$ and $Y = X^2$. Find the pdf of Y.
(b) (3 Marks) Let X, Y be i.i.d. $\sim U(0,1)$. Find the distribution of $T = X + Y$.
(c) (4 Marks) Let X and Y be i.i.d. Expo(λ), and T = $\log \frac{X}{Y}$. Find the CDF and PDF of T.