Q1] X and Y are continuous R.V → $f_{X,Y}$

$Z = \min(X,Y)$ $W = \max(X,Y)$

$$f_{ZW}(z,w) = \frac{f_{XY}(x_1,y_1)}{|\delta_{XY}(x_1,y_1)|} + \frac{f_{XY}(x_2,y_2)}{|\delta_{XY}(x_2,y_2)|}$$

$x_1 = g_1(z,w)$ $y_1 = g_2(z,w)$

$(x_1,y_1$ & $x_2,y_2)$: are roots of function g_1 and g_2

These roots are

$x_1 y_1 \Rightarrow (z,w)$

$x_2 y_2 \Rightarrow (w,z)$

$$f_{ZW}(z,w) = \frac{f_{XY}(z,w)}{|\delta_{XY}(z,w)|} + \frac{f_{XY}(w,z)}{|\delta_{XY}(w,z)|}$$

Jacobian Matrix

$$\partial_{XY}(x,y) = \begin{vmatrix} \frac{\partial x}{\partial z} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial z} & \frac{\partial y}{\partial w} \end{vmatrix}$$

$$\partial_{XY}(z,w) = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$$

$$\partial_{XY}(w,z) : \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1$$

(a) $f_{Z,W}(z,w) = f_{XY}(z,w) + f_{XY}(w,z)$

(b) Computing f_{ZW} for independent uniform X and .

$f_{XY}(x,y) = 1$ for $0 \le x \le 1$ & $0 \le y \le 1$

$f_{ZW}(z,w) = f_{XY}(z,w) + f_{XY}(w,z)$

$$= \begin{cases} 2 & 0 \le z \le w \le 1 \\ 0 & \text{otherwise} \end{cases}$$

Q2]

Moment generating func of Normal Distribution $X \sim N(\mu, \sigma^2)$

$E[X] = \mu$ $\quad V(X) = \sigma^2$

$$N(x; \mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}(x-\mu)^2/\sigma^2}$$

$$M_x(t) = E[e^{xt}] = \int e^{xt} \frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{-1}{2}(x-\mu)^2/\sigma^2} dx$$

$$z = \frac{x-\mu}{\sigma}$$

$$x = z\sigma + \mu$$

$$M_x(t) = e^{\mu t} \int e^{z\sigma t} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}z^2} \left|\frac{dx}{dz}\right| dz$$

$$\frac{dx}{dz} = \sigma$$

$$M_x(t) = e^{\mu t} \int e^{z\sigma t} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz$$

$$= e^{\mu t} e^{\frac{1}{2}\sigma^2 t^2}$$

Q3]

Given

$$\lim_{n\to\infty} P(|X_n - X| > \varepsilon) = 0 \quad \cancel{for}\ \forall \varepsilon > 0$$

then we need to say if its equivalent to

$$\lim_{n\to\infty} P(|X_n - X| \geq \varepsilon) = 0$$

Case 1

we know

$$P(|X_n - X| > \varepsilon) < P(|X_n - X| \geq \varepsilon)$$

So if

$$\lim_{n\to\infty} P(|X_n - X| \geq \varepsilon) = 0$$

then $\lim_{n\to\infty} P(|X_n - X| > \varepsilon) = 0$

Now to prove the other way

its given

$\lim_{n\to\infty} P(|X_n - X| > \varepsilon) = 0 \ \forall\ \varepsilon > 0$

~~Let there be some~~ $\cancel{\varepsilon_0 > 0}$

~~such that~~ fix $\varepsilon' > 0$

$P(|X_n - X| \geq \varepsilon_0)$

if $\varepsilon_0 > \varepsilon'$

$P(|X_n - X| \geq \varepsilon_0) \leq P(|X_n - X| > \varepsilon')$

$\hookrightarrow \to 0$

as $n \to \infty$

$\therefore$ $P(|X_n - X| \geq \varepsilon_0)$ will also be 0 as $n \to \infty$.

$\therefore$ for every $\varepsilon_0 > 0$ there will be some ε', $\varepsilon_0 > \varepsilon'$

and then

$\lim_{n\to\infty} P(|X_n - X| \geq \varepsilon_0) \to 0$ for $\forall\ \varepsilon_0 > 0$

Question 4

From chebyschev's inequality, we get that:

(Since mean of a Mean RV corresponds to mean of variable itself, and σ_n corresponds to variance of mean rv.)

$$P(|M_n - f| \leq k\sigma_n) \leq \frac{1}{k^2}$$

So fitting out equation to this, we see that ϵ corresponds to $k\sigma_n$ while δ corresponds to $\frac{1}{k^2}$.

From here, it's pretty simple, considering the fact that for a mean random variable, our $\sigma = (\frac{f(1-f)}{n})^{1/2}$

ie:

$$\epsilon = k(\frac{f(1-f)}{n})^{1/2}, \delta = \frac{1}{k^2}$$

a. So here to keep δ same, while reducing ϵ to two-thirds it's value, just have $n \leftarrow \frac{9n}{4}$.

b. So for making δ , 3/5 times its original value, While for doing that for δ, while keeping ϵ the same, make $k \leftarrow k\sqrt{\frac{5}{3}})$, and to keep ϵ constant, make $\leftarrow \frac{5n}{3}$ times its own value

1 Question 5

Since $X_n \xrightarrow{d} c$, we conclude that for any $\epsilon > 0$, we have

$$\lim_{n\to\infty} F_{X_n}(c-\epsilon) = 0,$$

(Convergence in distribution implies $P(X \leq x) = P(c \leq x)$.
(Keep in mind that c is a constant value.)
So if x is less than c, it simply would not be possible for c to be lesser than a quantity less than c!

$$\lim_{n\to\infty} F_{X_n}\left(c + \frac{\epsilon}{2}\right) = 1.$$

We can write for any $\epsilon > 0$,

$$\lim_{n\to\infty} P(|X_n - c| \geq \epsilon) = \lim_{n\to\infty} [P(X_n \leq c-\epsilon) + P(X_n \geq c+\epsilon)]$$

$$= \lim_{n\to\infty} P(X_n \leq c-\epsilon) + \lim_{n\to\infty} P(X_n \geq c+\epsilon)$$

$$= \lim_{n\to\infty} F_{X_n}(c-\epsilon) + \lim_{n\to\infty} P(X_n \geq c+\epsilon)$$

$$= 0 + \lim_{n\to\infty} P(X_n \geq c+\epsilon) \quad (\text{since } \lim_{n\to\infty} F_{X_n}(c-\epsilon) = 0)$$

$$\leq \lim_{n\to\infty} P\left(X_n > c + \frac{\epsilon}{2}\right)$$

$$= 1 - \lim_{n\to\infty} F_{X_n}\left(c + \frac{\epsilon}{2}\right)$$

$$= 0 \quad (\text{since } \lim_{n\to\infty} F_{X_n}\left(c + \frac{\epsilon}{2}\right) = 1).$$

Since $\lim_{n\to\infty} P(|X_n - c| \geq \epsilon) \geq 0$, we conclude that

$$\lim_{n\to\infty} P(|X_n - c| \geq \epsilon) = 0, \quad \text{for all } \epsilon > 0,$$

which means $X_n \xrightarrow{p} c$.

2 Question 6

For $n \in N$, define the following events:

$$A_n = \{|X_n - X| < \frac{\epsilon}{2}\}, \quad B_n = \{|Y_n - Y| < \frac{\epsilon}{2}\}.$$

Given that $X_n \xrightarrow{P} X$ and $Y_n \xrightarrow{P} Y$, we have for any $\epsilon > 0$:

$$\lim_{n\to\infty} P(A_n) = 1, \quad \lim_{n\to\infty} P(B_n) = 1.$$

Now, we can express the probability of their intersection as:

$$P(A_n \cap B_n) = P(A_n) + P(B_n) - P(A_n \cup B_n) \geq P(A_n) + P(B_n) - 1.$$

Hence,

$$\lim_{n\to\infty} P(A_n \cap B_n) = 1.$$

Definition of convergence in probability is that $P(\ |X_n - X| > \epsilon) = 0$, so simply taking the complement of that set, gives us the above expression, while also applying the fact that $\epsilon/2 > 0$ covers the same space as $\epsilon > 0$ given that $\epsilon > 0$.

Next, let us define the events C_n and D_n as follows:

$$C_n = \{|X_n - X| + |Y_n - Y| < \epsilon\}, \quad D_n = \{|X_n + Y_n - (X + Y)| < \epsilon\}.$$

It is clear that $(A_n \cap B_n) \subset C_n$, which implies $P(A_n \cap B_n) \leq P(C_n)$. Additionally, by applying the triangle inequality, we get:

$$|(X_n - X) + (Y_n - Y)| \leq |X_n - X| + |Y_n - Y|.$$

Thus, $C_n \subset D_n$, leading to:

$$P(C_n) \leq P(D_n).$$

Consequently, we have:

$$P(A_n \cap B_n) \leq P(C_n) \leq P(D_n).$$

Since $\lim_{n\to\infty} P(A_n \cap B_n) = 1$, it follows that $\lim_{n\to\infty} P(D_n) = 1$.
By definition, this shows that $X_n + Y_n \xrightarrow{P} X + Y$.