Assignment 7

(MA6.102) Probability and Random Processes, Monsoon 2023

Date: 14 November 2023, Due on 21 November 2023 (Tuesday).

INSTRUCTIONS

- Discussions with other students are not discouraged. However, all write-ups must be done individually with your own solutions.
- Any plagiarism when caught will be heavily penalized.
- Be clear and precise in your writing. Also, clearly state the assumptions made (if any) that are not specified in the question.

Problem 1 (5 Marks)**.** Minimize $\mathbb{E}[X]$ over all non-negative discrete random variables X such that $P(X \geq a) = \alpha$, for some $a > 0$ and $0 < \alpha < 1$. Prove Markov's inequality using the solution to this optimization problem. Also, construct a PMF P_X that achieves equality in Markov's inequality.

Problem 2 (5 Marks)**.** Let X be a random variable with mean μ and variance σ^2. Then prove that, for $c > 0$,

$$P(X - \mu \geq c) \leq \frac{\sigma^2}{\sigma^2 + c^2}.$$

Problem 3 (5 Marks)**.** For any two random variables X and Y, Cauchy-Schwarz inequality states that

$$|\mathbb{E}[XY]| \leq \sqrt{\mathbb{E}[X^2]\mathbb{E}[Y^2]}$$

with equality if and only if $X = \alpha Y$, for some constant $\alpha \in \mathbb{R}$. Prove this and use it to show that $|\rho(X,Y)| \leq 1$, where $\rho(X,Y)$ is the correlation coefficient of X and Y given by

$$\rho(X,Y) = \frac{\mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]}{\sqrt{\text{var}(X)\text{var}(Y)}}.$$

Problem 4 (5 Marks)**.** In order to estimate f, the true fraction of smokers in a large population, Alvin selects n people at random. His estimator M_n is obtained by dividing S_n, the number of smokers in his sample, by n, i.e., $M_n = \frac{S_n}{n}$. Alvin chooses the sample size n to be the smallest possible number for which the Chebyshev's inequality yields a guarantee that

$$P(|M_n - f| \geq \epsilon) \leq \delta,$$

where ϵ and δ are some prespecified tolerances. Determine how the value of n recommended by the Chebyshev's inequality changes in the following cases.

(a) The value of ϵ is reduced to $\frac{2}{3}$ of its original value.

(b) The probability δ is reduced to $\frac{3}{5}$ of its original value.

Problem 5. Let $X_1, X_2, \ldots, X_n$ be independent and identically distributed random variables with common mean $\mu = 2$ and variance $\sigma^2 = 9$, and let $Y_n = \frac{X_n}{2^n}$, for $n \in \mathbb{N}$. Also, define $T_n = \sum_{i=1}^{n} Y_i$ and $A_n = \frac{T_n}{n}$.

(a) [2 Marks] Evaluate the mean and variance of Y_n, T_n, and A_n.

(b) [1 Mark] Does Y_n converge in probability? If so, to what value?

(c) [1 Mark] Does T_n converge in probability? If so, to what value?

(c) [1 Mark] Does A_n converge in probability? If so, to what value?

Problem 6. Let $X_1, X_2, \dots, X_n$ be independent and identically distributed random variables with mean μ. For $n \in \mathbb{N}$, let

$$Y_n = \frac{1}{3}X_n + \frac{2}{3}X_{n+1}.$$

(a) [1 Mark] Are the random variables $Y_1, Y_2, \dots$ independent?

(b) [1 Mark] Are they identically distributed?

(c) [3 Marks] Does $M_n = \frac{\sum_{i=1}^n Y_i}{n}$ converge to μ in probability?

Problem 7 (5 Marks)**.** A machine processes parts, one at a time. The processing times of different parts are different random variables, uniformly distributed in $[1, 5]$. Using the central limit theorem, find an approximate value of the probability that the number of parts processed within 160 time units is at least 81.

Problem 8 (5 Marks)**.** A sequence of random variables $X_1, X_2, \dots$ converges almost surely (or with probability 1) to a random variable X if

$$P(\{\omega : \lim_{n\to\infty} X_n(\omega) = X(\omega)\}) = 1.$$

Consider the sample space $\Omega = [0, 1]$ with uniform probability law, i.e., $P([a, b]) = b - a$, for all $0 \le a \le b \le 1$. Define a sequence of random variables $(X_n)_{n\in\mathbb{N}}$ by $X_n(\omega) = \frac{n}{n+1}\omega + (1-\omega)^n$. Also, define another random variable X on this sample space as $X(\omega) = \omega$, for all $\omega \in \Omega$. Show that X_n converges to X almost surely (or with probability 1).

Problem 9. To cross a single lane of moving traffic, we require at least a duration d. Successive car interarrival times are independently and identically distributed with PDf $f_T(t)$. If an interval between successive cars is longer than d, we say that the interval represents a single opportunity to cross the lane. Assume that car lengths are small relative to intercar spacing and that our experiment begins the instant after the zeroth car goes by. Determine, in as simple form as possible, expressions for the probability that:

(a) [1 Mark] We can cross for the first time just before the nth car goes by.

(b) [2 Marks] We shall have had exactly m opportunities by the instant the nth car goes by.

(c) [2 Marks] The occurrence of the mth opportunity is immediately followed by the arrival of the nth car.

Problem 10 (5 Marks)**.** Consider a Poisson process of rate λ. Let N be the number of arrivals in $(0, t]$, and let M be the number of arrivals in $[0, t + s]$. Find the joint PMF of N and M, and $\mathbb{E}[NM]$.

Problem 11. (a) [3 Marks] Consider a random process $X(t) = A\cos\omega t + B\sin\omega t$, where A and B are random variables, and ω is a constant. Find necessary and sufficient conditions on the (joint) moments of A and B for $X(t)$ to be a wide-sense stationary (WSS) process.

(b) [2 Marks] Consider a WSS process $X(t)$ with autocorrelation $R_X(\tau) = \mathsf{e}^{-a|\tau|}$, where $a > 0$, for all $\tau \in \mathbb{R}$. Find the power spectral density of $X(t)$.