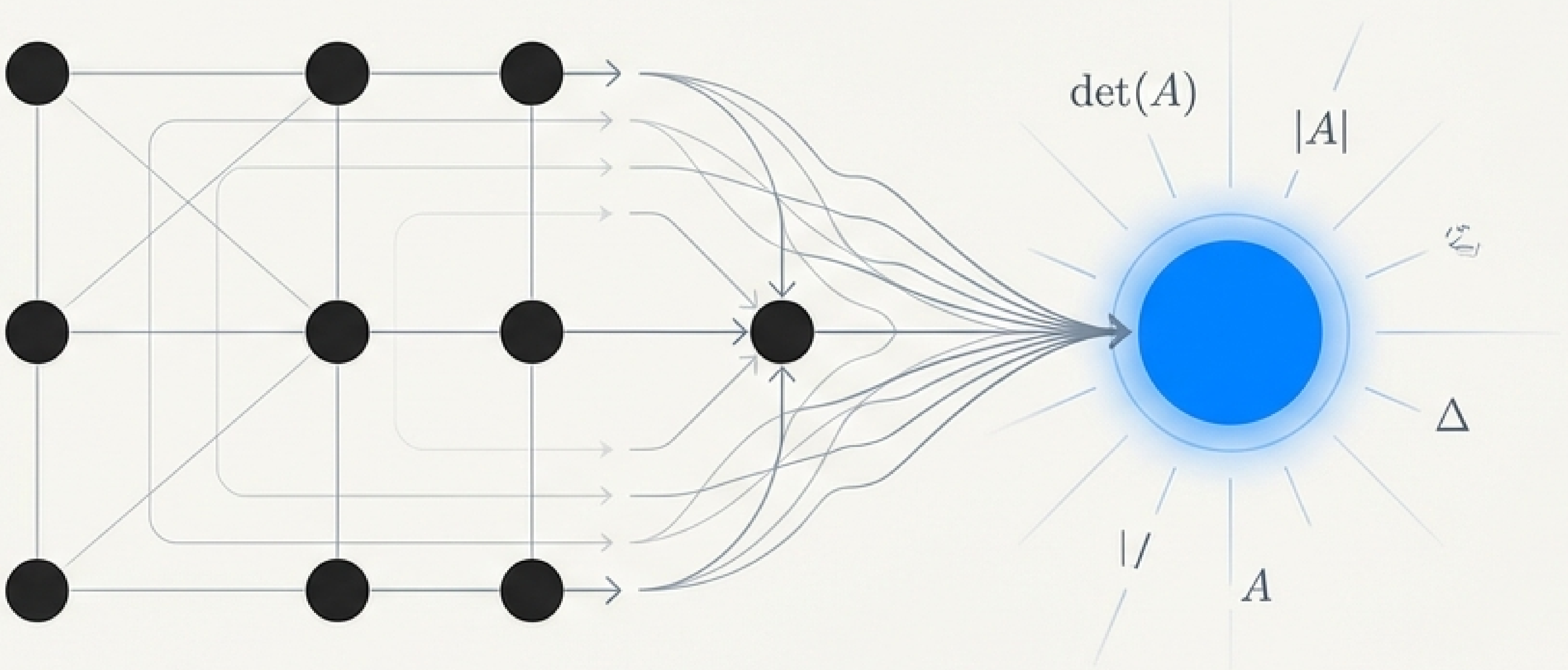
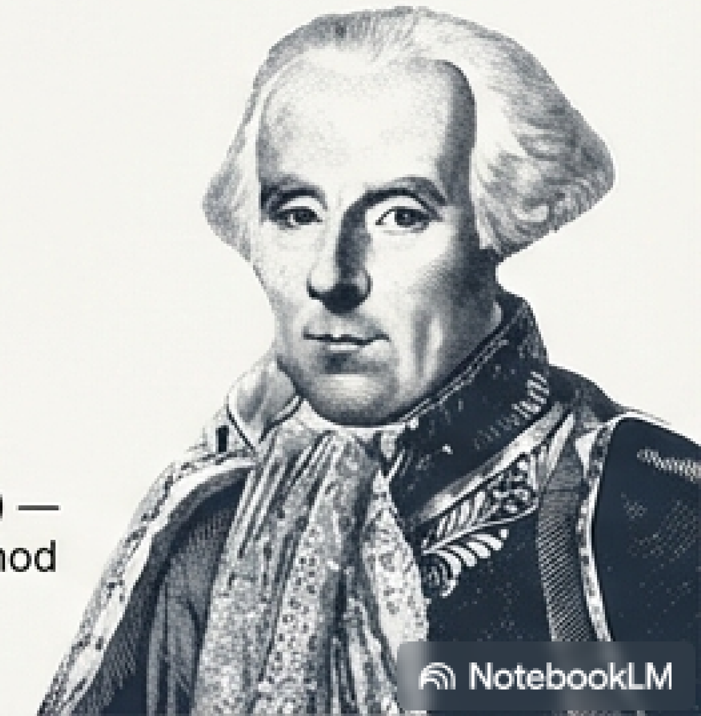
Determinants: The Decision Engine.

Unlocking consistency, calculating areas, and solving linear systems.





P.S. Laplace (1749-1827) —
Pioneer of the expansion method

The Function $f : M \to K$

To every square matrix A, we associate a unique number (real or complex).

This function is denoted as $|A|$, $\det A$, or Δ.

Set of Square Matrices (M)

$$[A] = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Input: Square Matrix

Function f (Determinant)

Set of Numbers (K)

$$k$$

Output: Scalar Value (Real or Complex)

$f(A) = k$, where $A \in M$ and $k \in K$

CAUTION:
$|A|$ is read as ‘determinant of A’, not ‘modulus of A’. Only square matrices have determinants.

Calculating Value: Orders 1 & 2

Order 1

For $A = [a], \det A = a.$

Order 2

$$+\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}-$$

$$\det A = a_{11}a_{22} - a_{21}a_{12}$$

Example: $\begin{vmatrix} 2 & 4 \\ -1 & 2 \end{vmatrix} = 2(2) - 4(-1) = 4 + 4 = 8.$

Calculating Value: Order 3 (Expansion)

$(+a_{11}) \qquad (-a_{12})$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$$

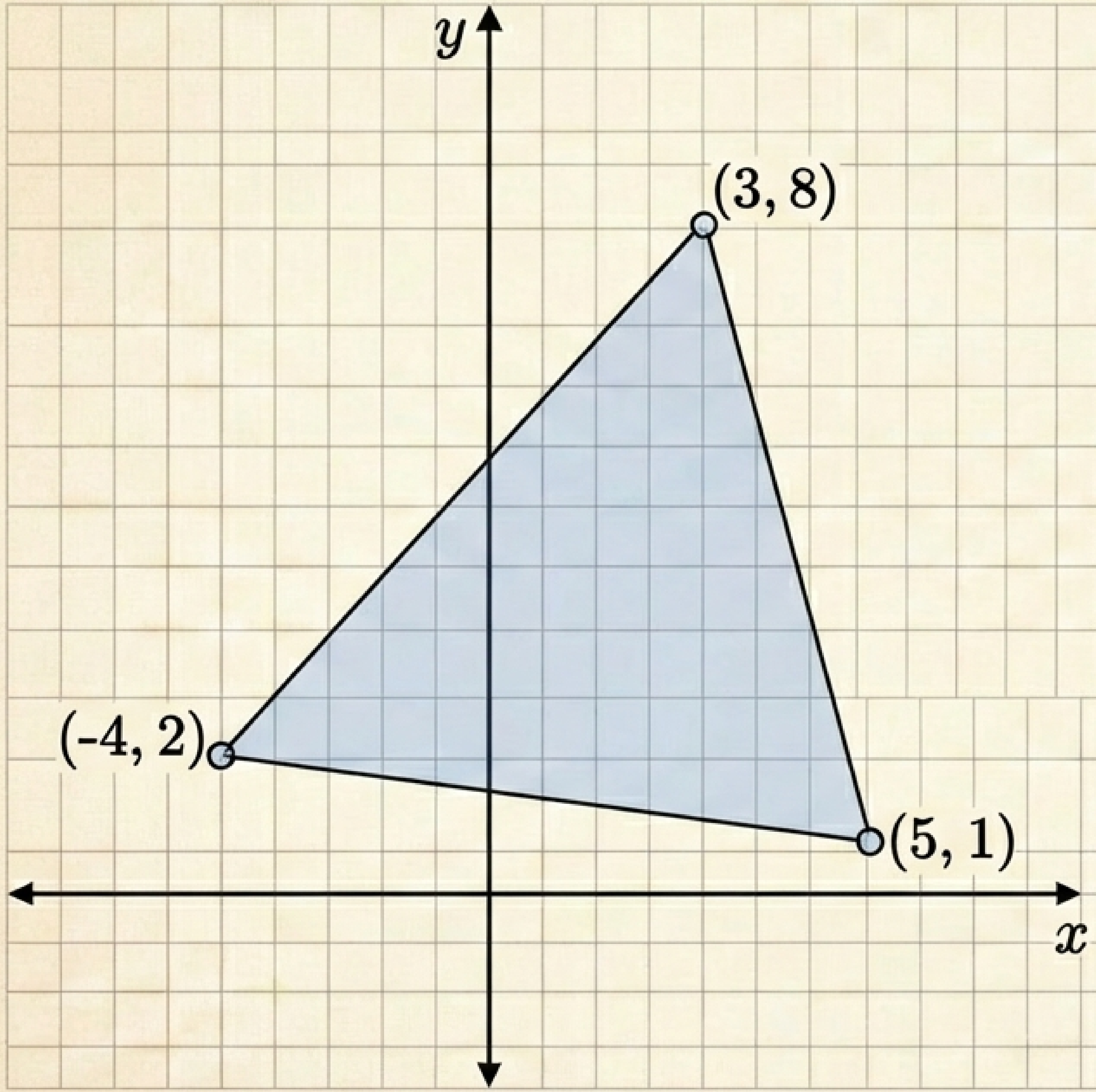
$$\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} \; (-a_{12})$$

$$\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \; (+a_{13})$$

Pro Tip: Expanding along any row or column yields the same value. Always choose the row/column with the most Zeros.

$$\Delta = (-1)^{1+1} a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + (-1)^{1+2} a_{12} \begin{vmatrix} a_{21} & \\ a_{31} & \end{vmatrix} + (-1)^{1+3} a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

Application: Area of a Triangle



$$\Delta = \frac{1}{2}\begin{vmatrix} 3 & 8 & 1 \\ -4 & 2 & 1 \\ 5 & 1 & 1 \end{vmatrix}$$

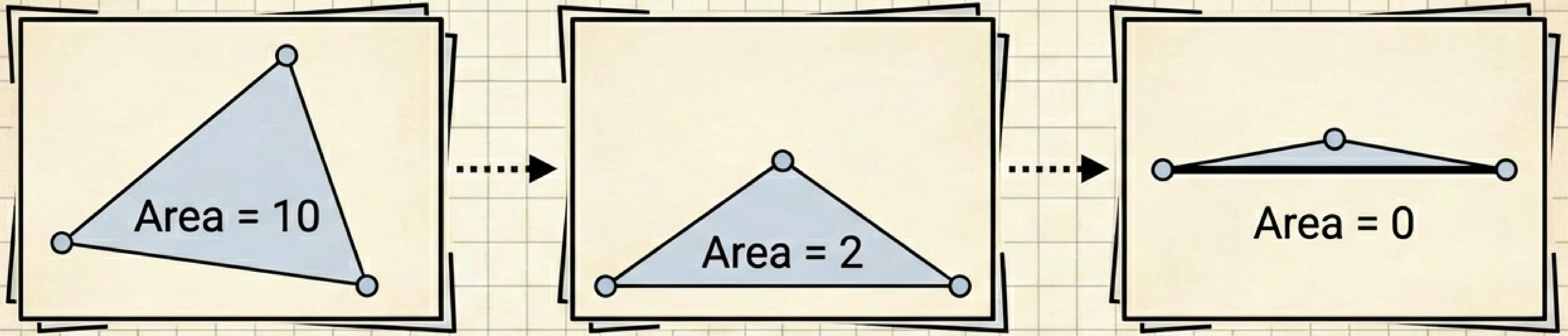
$$\Delta = \frac{1}{2}[3(2-1) - 8(-4-5) + 1(-4-10)]$$

$$\Delta = \frac{1}{2}[3 + 72 - 14] = 37.5$$

Key Rules:

1. Area is always absolute (positive).
2. For calculations given an area, use ± values.

Linearity: The Case of the Zero Area



If $\Delta = 0$, the points are **Collinear**.

A determinant of 0 implies a loss of dimension.
The shape collapses.

Deconstruction Part I: Minors (M_{ij})

The **Minor** of an element a_{ij} is the determinant obtained by deleting row i and column j.

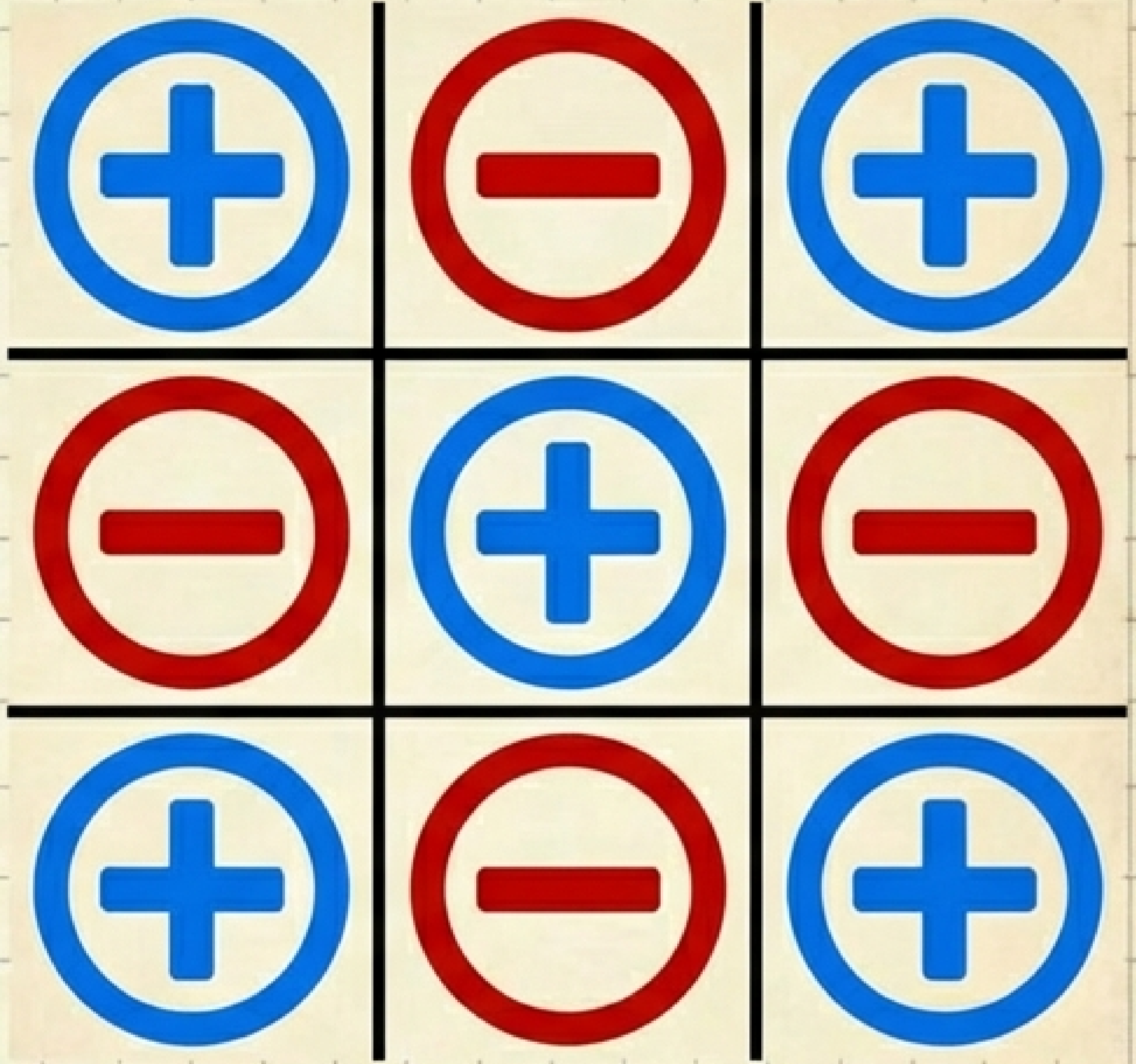
$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix}$$

$$M_{23} = \begin{vmatrix} 1 & 2 \\ 7 & 8 \end{vmatrix} = 8 - 14 = -6$$

Deconstruction Part II: Cofactors (A_{ij})

Cofactor = Signed Minor

$$A_{ij} = (-1)^{i+j} M_{ij}$$



The sign depends on the position.
If $i + j$ is even, the sign is positive.
If odd, negative.

The Expansion Theorem

The value of a determinant is the sum of the products of elements of any row with their corresponding Cofactors.

$$\Delta = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}$$

Note: Elements of a row multiplied by Cofactors of a different row sum to zero.

$$a_{11}A_{21} + a_{12}A_{22} + a_{13}A_{23} = 0$$

The Transformation: Adjoint (adj A)

The Transpose of the Cofactor Matrix.

Cofactor Matrix $[A_{ij}]$

$$\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$$

Transpose
(Swap Rows & Cols)

Adjoint A

$$\begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$$

$$A(adj\ A) = |A|\ I$$

The Checkpoint: Singular vs. Non-Singular

Calculate $|A|$

$= 0$

SINGULAR

No Inverse exists.

$\neq 0$

NON-SINGULAR

Invertible.

A square matrix A is invertible if and only if A is non-singular.

The Inverse Matrix (A^{-1})

$$A^{-1} = \frac{1}{|A|}(\text{adj}\, A)$$

The Reason for Non-Singularity

Since we divide by the determinant, it cannot be zero. Division by zero is undefined.

Derived from $A(\text{adj}\, A) = |A|\, I \Rightarrow A\left(\frac{1}{|A|}\text{adj}\, A\right) = I$.

The Problem: Systems of Linear Equations

$$a_1x + b_1y + c_1z = d_1$$
$$a_2x + b_2y + c_2z = d_2$$
$$a_3x + b_3y + c_3z = d_3$$

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

Coefficient Matrix A

Variables X

Constants B

The Solution: Matrix Method

Given $|A| \neq 0$ (Unique Solution).

$$\mathbf{AX = B}$$

Pre-multiply by Inverse: $\mathbf{A^{-1}(AX) = A^{-1}B}$

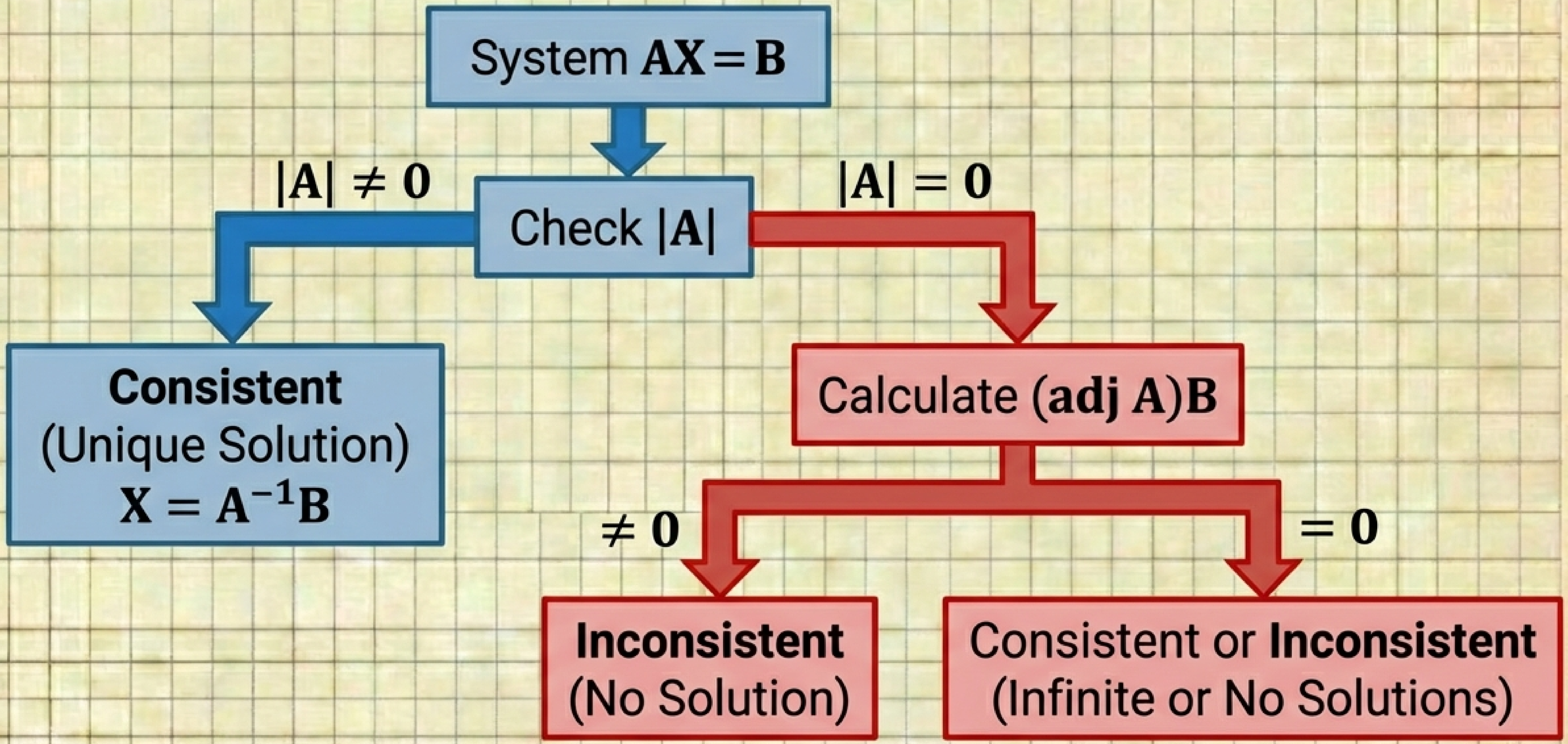
Associativity: $\mathbf{(A^{-1}A)X = A^{-1}B}$

Identity Property: $\mathbf{IX = A^{-1}B}$

Result: $\boxed{\mathbf{X = A^{-1}B}}$

Find Inverse A^{-1} $\xrightarrow{\times B}$ Multiply by Constants → Solution **X**

Consistency Summary



Methods pioneered by Seki Kowa (1683), Laplace (1772), and Cauchy (1812).