Arithmetic, Geometric & Harmonic Progressions

Complete Course Material for Class 11 & 12 Students

Table of Contents

1. Introduction to Sequences and Series {#introduction}

What is a Sequence?

A **sequence** is an ordered collection of numbers arranged according to a definite pattern or rule.

Examples:

- 2, 4, 6, 8, 10, ... (even numbers)
- 1, 1, 2, 3, 5, 8, ... (Fibonacci sequence)
- 1, 4, 9, 16, 25, ... (perfect squares)

What is a Series?

A **series** is the sum of the terms of a sequence.

Example: If sequence is 2, 4, 6, 8, 10
Then series is 2 + 4 + 6 + 8 + 10 = 30

Types of Progressions

1. **Arithmetic Progression (AP)** - Constant difference between consecutive terms
2. **Geometric Progression (GP)** - Constant ratio between consecutive terms
3. **Harmonic Progression (HP)** - Reciprocals form an arithmetic progression

2. Arithmetic Progression (AP) {#arithmetic-progression}

Definition

An **Arithmetic Progression** is a sequence where the difference between consecutive terms is constant.

This constant difference is called the **common difference (d)**.

General Form

If **a** is the first term and **d** is the common difference:
a, a+d, a+2d, a+3d, a+4d, ...

Key Formulas

nth Term Formula

$a_n = a + (n-1)d$

Where:

- a_n = nth term
- a = first term
- d = common difference
- n = position of term

Sum of n Terms

$S_n = n/2 [2a + (n-1)d]$

Alternative form: $S_n = n/2$ [first term + last term]

Common Difference

$d = a_{n+1} - a_n$

Properties of AP

1. If a constant is added/subtracted to each term, the resulting sequence is also an AP
2. If each term is multiplied/divided by a non-zero constant, the resulting sequence is also an AP
3. Three numbers a, b, c are in AP if and only if **2b = a + c**

Examples

Example 1: Find the 15th term of AP: 3, 7, 11, 15, ...

- First term (a) = 3
- Common difference (d) = 7 - 3 = 4
- 15th term = a + (n-1)d = 3 + (15-1)×4 = 3 + 56 = 59

Example 2: Find the sum of first 20 terms of AP: 5, 8, 11, 14, ...

- a = 5, d = 3, n = 20
- S_{20} = 20/2 [2×5 + (20-1)×3] = 10[10 + 57] = 670

3. Geometric Progression (GP) {#geometric-progression}

Definition

A **Geometric Progression** is a sequence where each term after the first is obtained by multiplying the previous term by a constant called the **common ratio (r)**.

General Form

If **a** is the first term and **r** is the common ratio:
$a, ar, ar^2, ar^3, ar^4, \ldots$

Key Formulas

nth Term Formula

$a_n = ar^{(n-1)}$

Where:

- a_n = nth term
- a = first term
- r = common ratio
- n = position of term

Sum of n Terms (Finite GP)

When $r \neq 1$: **$S_n = a(r^n - 1)/(r - 1)$**

When r = 1: **$S_n = na$**

Sum to Infinity (Infinite GP)

When $|r| < 1$: **$S_\infty = a/(1 - r)$**

When $|r| \geq 1$: Series diverges (no finite sum)

Common Ratio

$r = a_{n+1}/a_n$

Properties of GP

1. Three numbers a, b, c are in GP if and only if **$b^2 = ac$**
2. If all terms are positive and in GP, their logarithms form an AP
3. If each term is raised to the same power, the resulting sequence is also a GP

Examples

Example 1: Find the 8th term of GP: 2, 6, 18, 54, ...

- First term (a) = 2
- Common ratio (r) = 6/2 = 3
- 8th term = ar^(n-1) = 2 × 3^(8-1) = 2 × 3^7 = 2 × 2187 = 4374

Example 2: Find the sum of infinite GP: 1, 1/2, 1/4, 1/8, ...

- a = 1, r = 1/2
- Since $|r| = 1/2 < 1$, sum exists
- S_∞ = a/(1-r) = 1/(1-1/2) = 1/(1/2) = 2

4. Harmonic Progression (HP) {#harmonic-progression}

Definition

A sequence is called a **Harmonic Progression** if the reciprocals of its terms form an Arithmetic Progression.

General Form

If 1/a, 1/(a+d), 1/(a+2d), ... is an AP, then:
a, a/(a+d), a/(a+2d), ... is an HP

Key Formulas

nth Term Formula

If reciprocals form AP with first term 1/A and common difference D:
$a_n = 1/[A + (n-1)D]$

Harmonic Mean

For three numbers a, b, c in HP:
$b = 2ac/(a + c)$

Properties of HP

1. No general formula exists for sum of HP
2. Three numbers a, b, c are in HP if $2/b = 1/a + 1/c$
3. HP problems are usually solved by converting to AP

Examples

Example 1: Check if 1/2, 1/4, 1/6, 1/8 are in HP

- Reciprocals: 2, 4, 6, 8
- Check if these form AP: 4-2 = 2, 6-4 = 2, 8-6 = 2
- Since common difference is constant, original sequence is in HP

Example 2: Find the harmonic mean of 4 and 12

- HM = 2×4×12/(4+12) = 96/16 = 6

5. Relationship between AP, GP, HP {#relationships}

AM, GM, HM Inequality

For positive numbers a and b:
AM ≥ GM ≥ HM

Where:

- **Arithmetic Mean (AM)** = $(a + b)/2$
- **Geometric Mean (GM)** = $\sqrt{ab}$
- **Harmonic Mean (HM)** = $2ab/(a + b)$

Key Relationship

$AM \times HM = GM^2$

Proof of AM ≥ GM ≥ HM

Proof of AM ≥ GM:
$(\sqrt{a} - \sqrt{b})^2 \geq 0$
$a + b - 2\sqrt{(ab)} \geq 0$
$(a + b)/2 \geq \sqrt{(ab)}$
Therefore, AM ≥ GM

Proof of GM ≥ HM:
$GM^2 - HM \times AM = ab - 2ab/(a+b) \times (a+b)/2 = ab - ab = 0$
Since $AM \times HM = GM^2$, and AM ≥ GM, we get GM ≥ HM

Examples

Example: For numbers 9 and 16

- $AM = (9 + 16)/2 = 12.5$
- $GM = \sqrt{(9 \times 16)} = \sqrt{144} = 12$
- $HM = 2 \times 9 \times 16/(9+16) = 288/25 = 11.52$

Verification: 12.5 ≥ 12 ≥ 11.52 ✓

6. Advanced Topics {#advanced-topics}

Arithmetic-Geometric Progression (AGP)

A sequence where each term is the product of corresponding terms of an AP and GP.

Example: If AP is 1, 3, 5, 7, ... and GP is 2, 4, 8, 16, ...
Then AGP is 2, 12, 40, 112, ...

Method of Differences

Used to find sum of series like $1^2 + 2^2 + 3^2 + ... + n^2$

Special Series Formulas

1. **Sum of first n natural numbers:** $\Sigma n = n(n+1)/2$
2. **Sum of squares:** $\Sigma n^2 = n(n+1)(2n+1)/6$
3. **Sum of cubes:** $\Sigma n^3 = [n(n+1)/2]^2$

Telescoping Series

Series where consecutive terms cancel out.

Example: $1/(1\times2) + 1/(2\times3) + 1/(3\times4) + \ldots + 1/[n(n+1)]$
$= (1 - 1/2) + (1/2 - 1/3) + \ldots + (1/n - 1/(n+1))$
$= 1 - 1/(n+1) = n/(n+1)$

7. Problem Solving Strategies {#problem-solving}

General Approach

1. **Identify the type** of progression (AP, GP, or HP)
2. **Find the first term** and common difference/ratio
3. **Apply appropriate formulas**
4. **Verify the answer**

Common JEE Techniques

For AP Problems:

- Use arithmetic mean property for three terms in AP
- Apply sum formula when finding sum of terms
- Use the fact that if $S_n = An^2 + Bn$, then sequence is AP

For GP Problems:

- Check if $|r| < 1$ for infinite series
- Use geometric mean property for three terms in GP
- Apply AM-GM inequality when needed

For HP Problems:

- Always convert to AP first
- Use harmonic mean formula for three terms
- Remember that HP has no direct sum formula

Word Problem Tips

1. **Set up equations** based on given conditions
2. **Use substitution** to simplify complex expressions
3. **Check boundary conditions** (like $r < 1$ for infinite GP)
4. **Verify answers** by substituting back

8. Practice Questions {#practice-questions}

Basic Level (Class 11)

Question 1: Find the 20th term of the AP: 3, 8, 13, 18, ...
Solution: $a = 3$, $d = 5$, $n = 20$
$a_{20} = 3 + (20-1)\times5 = 3 + 95 = 98$

Question 2: Find the sum of first 15 terms of GP: 2, 6, 18, ...
Solution: $a = 2$, $r = 3$, $n = 15$
$S_{15} = 2(3^{15} - 1)/(3 - 1) = (3^{15} - 1) = 14{,}348{,}906$

Question 3: Check if 1/3, 1/5, 1/7 are in HP.
Solution: Reciprocals are 3, 5, 7
$5 - 3 = 2$, $7 - 5 = 2$
Common difference = 2, so they form AP
Therefore, original terms are in HP.

Intermediate Level (Class 12)

Question 4: Three numbers are in GP. Their sum is 21 and product is 216. Find the numbers.
Solution: Let numbers be a/r, a, ar
Sum: $a/r + a + ar = 21 \rightarrow a(1/r + 1 + r) = 21$
Product: $(a/r) \times a \times ar = a^3 = 216 \rightarrow a = 6$
Substituting: $6(1/r + 1 + r) = 21 \rightarrow 1/r + 1 + r = 3.5$
Solving: $r^2 - 2.5r + 1 = 0 \rightarrow r = 2$ or $r = 0.5$
Numbers are 3, 6, 12 or 12, 6, 3

Question 5: If AM and HM of two numbers are 10 and 8 respectively, find GM.
Solution: Using $AM \times HM = GM^2$
$GM^2 = 10 \times 8 = 80$
$GM = \sqrt{80} = 4\sqrt{5}$

Advanced Level (JEE)

Question 6: If a, b, c are in AP and a^2, b^2, c^2 are in GP, prove that a, b, c are in GP or $a = b = c$.
Solution:
Since a, b, c are in AP: $2b = a + c$
Since a^2, b^2, c^2 are in GP: $b^4 = a^2c^2$
From first condition: $c = 2b - a$
Substituting: $b^4 = a^2(2b - a)^2$
Expanding and simplifying: $(b^2 - a^2)(b^2 - ac) = 0$
This gives $b^2 = a^2$ or $b^2 = ac$
If $b^2 = a^2$, then $b = \pm a$. If $b = a$, then $c = a$ (all equal)
If $b^2 = ac$ and $2b = a + c$, then a, b, c are in GP.

Question 7: Sum the series: $1^2 + 2^2 + 3^2 + ... + n^2$
Solution: Using the formula for sum of squares:
$\Sigma n^2 = n(n+1)(2n+1)/6$

Question 8: Find the sum to infinity: 1 + 1/2 + 1/4 + 1/8 + ...
Solution: This is a GP with a = 1, r = 1/2
Since $|r| < 1$, sum exists
$S_\infty = 1/(1 - 1/2) = 2$

JEE Advanced Level

Question 9: Let $a_1, a_2, a_3, ...$ be an AP with $a_1 = 7$ and common difference 8. Let $T_1, T_2, T_3, ...$ be such that $T_1 = 3$ and $T_{n+1} - T_n = a_n$ for $n \geq 1$. Find T_{10}.

Solution:
$a_n = 7 + (n-1)\times8 = 8n - 1$
$T_2 - T_1 = a_1 = 7$, so $T_2 = 3 + 7 = 10$
$T_3 - T_2 = a_2 = 15$, so $T_3 = 10 + 15 = 25$
Generally: $T_{n+1} = T_1 + \Sigma^k{}_{=1}{}^n\, a_k = 3 + \Sigma^k{}_{=1}{}^n\, (8k - 1)$
$= 3 + 8\times n(n+1)/2 - n = 3 + 4n(n+1) - n = 3 + 4n^2 + 3n$
$T_{10} = 3 + 4\times100 + 30 = 433$

Question 10: If $\log_2 x, \log_2 y, \log_2 z$ are in AP, then x, y, z are in:
(a) AP (b) GP (c) HP (d) None

Solution:
If $\log_2 x, \log_2 y, \log_2 z$ are in AP, then:
$2\log_2 y = \log_2 x + \log_2 z$
$\log_2 y^2 = \log_2(xz)$
$y^2 = xz$
This is the condition for x, y, z to be in GP.
Answer: (b) GP

Formula Sheet

Arithmetic Progression

- nth term: $a_n = a + (n-1)d$
- Sum: $S_n = n/2[2a + (n-1)d]$
- Three terms in AP: $2b = a + c$

Geometric Progression

- nth term: $a_n = ar^{(n-1)}$
- Sum (finite): $S_n = a(r^n-1)/(r-1)$
- Sum (infinite): $S_\infty = a/(1-r)$ when $|r| < 1$
- Three terms in GP: $b^2 = ac$

Harmonic Progression

- Three terms in HP: $2/b = 1/a + 1/c$
- Convert to AP by taking reciprocals

AM-GM-HM Relations

- $AM = (a+b)/2$, $GM = \sqrt{(ab)}$, $HM = 2ab/(a+b)$
- $AM \geq GM \geq HM$
- $AM \times HM = GM^2$

This comprehensive course material covers all essential concepts of Arithmetic, Geometric, and Harmonic Progressions suitable for Class 11 and 12 students, including JEE preparation. Practice regularly and focus on understanding the underlying patterns and relationships.