CHAPTER 10.

FORECASTING THE LONG TERM: DETERMINISTIC AND STOCHASTIC TRENDS

SOLUTIONS
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Exercise 1
For a deterministic trend model, where the trend is a polynomial of order 3, i.e., $Y_t = \beta_0 + \beta_1 t + \beta_2 t^2 + \beta_3 t^3 + \varepsilon_t$, we choose different values of the parameters to generate an upward or downward trend or no trend at all:

- Model with upward trend: the easiest choice is when all parameters are positive as in

$$Y_t = 1.0 + 0.05t + 0.02t^2 + 0.01t^3 + \varepsilon_t, \quad \varepsilon_t \sim N(0, 20^2)$$

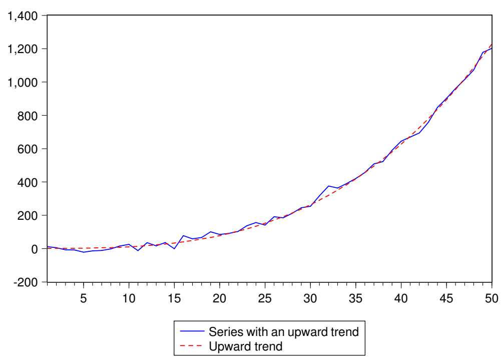
- Model with downward trend: all parameters, except the constant, are negative as in

$$Y_t = 1.0 - 0.05t - 0.02t^2 - 0.01t^3 + \varepsilon_t, \quad \varepsilon_t \sim N(0, 20^2)$$

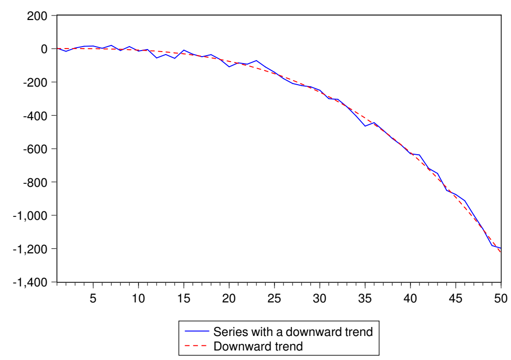
- Model with no trend: all parameters, except the constant, are zero, as in

$$Y_t = 1.0 + \varepsilon_t, \quad \varepsilon_t \sim N(0, 1)$$

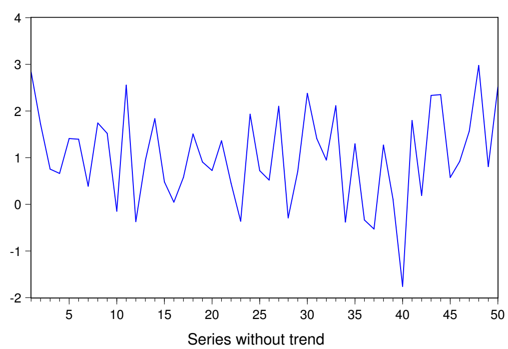
These models are plotted in Figure 1. Note that we could always mix negative and positive parameters to generate upward or downward or no trend at all. The direction of the trend will depend not only on the sign of the parameters but also on their magnitude.



(a) Series with upward trend



(b) Series with downward trend



(c) Series with no trend

Figure 1: Time series plots with polynomial trends of order 3

Exercise 2

We generate models with exponential and logistic-type trends as follows:

- Model with exponential trend:

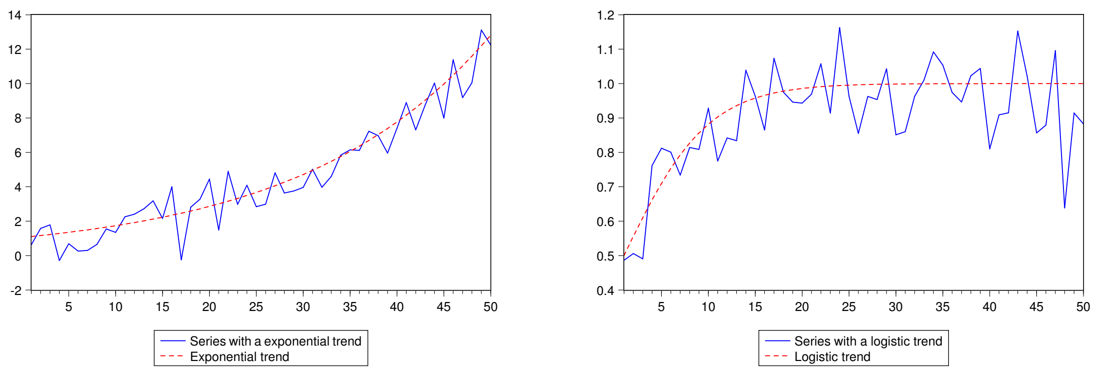
$$y_t = \exp(\beta_0 + \beta_1 t) + \varepsilon_t = \exp(0.1 + 0.05t) + \varepsilon_t, \quad \varepsilon_t \sim N(0,1)$$

.

- Model with logistic-type trend:

$$y_t = \frac{1}{(\beta_0 + \beta_1 x^t)} + \varepsilon_t = \frac{1}{(1 + x^t)} + \varepsilon_t = \frac{1}{(1 + 0.8^t)} + \varepsilon_t, \quad \varepsilon_t \sim N(0, 0.1^2)$$

.



(a) Series with upward exponential trend (b) Series with logistic-type downward trend

Figure 2: Time Series with exponential and logistic-type trends

The growth rate for the series with exponential and logistic-type trends will depend on time t. For the time series with exponential trend, the growth rate is

$$\frac{dY_t}{dt} = \beta_1 \exp(\beta_0 + \beta_1 t) = \beta_1 Y_t = 0.05 Y_t$$

which depends on Y_t. Compute $\frac{dY_t}{dt}$ at time $t = 3$. We have a growth rate of 0.0642, that is, $\frac{dY_t}{dt} = 0.05Y_t = 0.05\exp(0.1 + 0.05 \times 3) = 0.0642$. Compute $\frac{dY_t}{dt}$ at time $t = 4$. We have a growth rate of 0.0675, that is, $\frac{dY_t}{dt} = 0.05Y_t = 0.05\exp(0.1 + 0.05 \times 4) = 0.0675$.

For the time series with a logistic-type trend, the growth rate is

$$\frac{dY_t}{dt} = \frac{-\beta_1 x^t \ln x}{(\beta_0 + \beta_1 x^t)^2} = \frac{-(\ln 0.8)0.8^t}{(1 + 0.8^t)^2}$$

which depends on time t. Compute $\frac{dY_t}{dt}$ at time $t = 3$. We have a growth rate of 0.05, that is, $\frac{dY_t}{dt} = \frac{-(\ln 0.8)0.8^3}{(1+0.8^3)^2} = 0.05$. Compute $\frac{dY_t}{dt}$ at time $t = 4$. We have a growth rate of 0.046, that is, $\frac{dY_t}{dt} = \frac{-(\ln 0.8)0.8^4}{(1+0.8^4)^2} = 0.046$. Observe that, eventually when time is very large, the time series stabilizes around 1, and the growth rate becomes zero.

Exercise 3
A model with a cubic polynomial trend is specified as follows:

$$Y_t = 1.0 + 0.05t + 0.02t^2 + 0.01t^3 + \varepsilon_t, \quad \varepsilon_t \sim N(0, 20^2).$$

- Unconditional mean
$$E(Y_t) = 1.0 + 0.05t + 0.02t^2 + 0.01t^3 \equiv \mu_t$$
- Unconditional variance
$$\sigma_Y^2 = E(Y_t - \mu_t)^2 = E(\varepsilon_t)^2 = \sigma_\varepsilon^2 \equiv \gamma = 20^2$$
- Autocovariance of order k
$$\gamma_k \equiv E(Y_t - \mu_t)E(Y_{t-k} - \mu_{t-k}) = E(\varepsilon_t \varepsilon_{t-k}) = 0$$
- Autocorrelation for order k
$$\rho_k = \frac{\gamma_k}{\gamma_0} = \frac{0}{\sigma_\varepsilon^2} = 0$$

A model with an exponential trend is specified as follows:

$$y_t = \exp(\beta_0 + \beta_1 t) + \varepsilon_t = \exp(0.1 + 0.05t) + \varepsilon_t, \quad \varepsilon_t \sim N(0, 1).$$

- Unconditional mean
$$E(Y_t) = \exp(\beta_0 + \beta_1 t) = \exp(0.1 + 0.05t) \equiv \mu_t$$
- Unconditional variance
$$\sigma_Y^2 = E(Y_t - \mu_t)^2 = E(\varepsilon_t)^2 = \sigma_\varepsilon^2 \equiv \gamma = 1$$
- Autocovariance of order k
$$\gamma_k \equiv E(Y_t - \mu_t)E(Y_{t-k} - \mu_{t-k}) = E(\varepsilon_t \varepsilon_{t-k}) = 0$$
- Autocorrelation for order k
$$\rho_k = \frac{\gamma_k}{\gamma_0} = \frac{0}{\sigma_\varepsilon^2} = 0$$

Observe that the unconditional mean μ_t is time varying for both cubic and exponential trends, and that the unconditional variances, autocovariances and autocorrelation functions do no depend on time. A covariance stationary stochastic process requires that all random variables have the same mean, same variance and the covariances do not depend on time. Since the unconditional mean of the above trend processes is not constant, a stochastic process with cubic or exponential trend is not covariance stationary.

Exercise 4

A model with a linear trend is specified as follows: $Y_t = \beta_0 + \beta_1 t + \varepsilon_t, \quad \varepsilon_t \sim N(0, \sigma^2_\varepsilon)$. We compute the point forecast, the forecast error, its variance, and the density forecast of the process for $h = 2, 3$, and 4 under the assumption of a quadratic loss function.

OPTIMAL FORECAST WHEN $h = 2$

- Optimal point forecast

$$f_{t,2} = \mu_{t+2|t} = E(Y_{t+2}|I_t) = E(\beta_0 + \beta_1(t+2) + \varepsilon_{t+2}|I_t) = \beta_0 + \beta_1(t+2)$$

- Forecast error

$$e_{t,2} = Y_{t+2} - f_{t,2} = \varepsilon_{t+2}$$

- Variance of the forecast error

$$\sigma^2_{t+2|t} = E(Y_{t+2} - f_{t,2}|I_t)^2 = E(e^2_{t,2}) = E(\varepsilon^2_{t+2}) = \sigma^2_\varepsilon$$

- Density forecast

$$f(Y_{t+2}|I_t) \to N(\mu_{t+2|t}, \sigma^2_{t+2|t}) = N(\beta_0 + \beta_1(t+2), \sigma^2_\varepsilon)$$

OPTIMAL FORECAST WHEN $h = 3$

- Optimal point forecast

$$f_{t,3} = \mu_{t+3|t} = E(Y_{t+3}|I_t) = E(\beta_0 + \beta_1(t+3) + \varepsilon_{t+3}|I_t) = \beta_0 + \beta_1(t+3)$$

- Forecast error

$$e_{t,3} = Y_{t+3} - f_{t,3} = \varepsilon_{t+3}$$

- Variance of the forecast error

$$\sigma^2_{t+3|t} = E(Y_{t+3} - f_{t,3}|I_t)^2 = E(e^2_{t,3}) = E(\varepsilon^2_{t+3}) = \sigma^2_\varepsilon$$

- Density forecast

$$f(Y_{t+3}|I_t) \to N(\mu_{t+3|t}, \sigma^2_{t+3|t}) = N(\beta_0 + \beta_1(t+3), \sigma^2_\varepsilon)$$

OPTIMAL FORECAST WHEN $h = 4$

- Optimal point forecast

$$f_{t,4} = \mu_{t+4|t} = E(Y_{t+4}|I_t) = E(\beta_0 + \beta_1(t+4) + \varepsilon_{t+4}|I_t) = \beta_0 + \beta_1(t+4)$$

- Forecast error

$$e_{t,4} = Y_{t+4} - f_{t,4} = \varepsilon_{t+4}$$

- Variance of the forecast error

$$\sigma^2_{t+4|t} = E(Y_{t+4} - f_{t,4}|I_t)^2 = E(e^2_{t,4}) = E(\varepsilon^2_{t+4}) = \sigma^2_\varepsilon$$

- Density forecast

$$f(Y_{t+4}|I_t) \to N(\mu_{t+4|t}, \sigma^2_{t+4|t}) = N(\beta_0 + \beta_1(t+4), \sigma^2_\varepsilon)$$

Observe that the optimal forecast always moves along the linear trend, and since the error is the only source of uncertainty, the variance of the forecast error is always the same and, in this case, it is always constant. We could also write a process where the error follows some ARMA process, and in this case, the forecast error, its variance, and the density forecast will follow the rules that we have seen in Chapters 6 and 7.

Exercise 5
We download the updated data on outstanding mortgage debt from the website: `http://www.federalreserve.gov/datadownload/Choose.aspx?rel=Z1`

The updated series runs from 1992:Q1 to 2012:Q2. Comparing Figure 3 with Figure 10.5 in the textbook, we see that the upward trend has been broken in the updated series. We still could fit a polynomial trend. The estimation sample runs up to 2008Q2, and we reserve the rest of the observations to evaluate the forecast. The estimation results are shown in Table 1 (polynomial trend of order 4) and Table 2 (polynomial trend of order 5). We prefer the model with a polynomial trend of order 4. The contribution of t^5 is too small to be included in the model although it is statistically significant. Though we have fitted a polynomial of order 4 in Table 1, the estimates of the trend coefficients for t^2, t^3 and t^4 have opposite signs to those in Table 10.2 in the textbook, which is expected because the updated series has a break in the upward trend, in fact the trend moves downwards from 2006:Q4 to the most recent quarter.

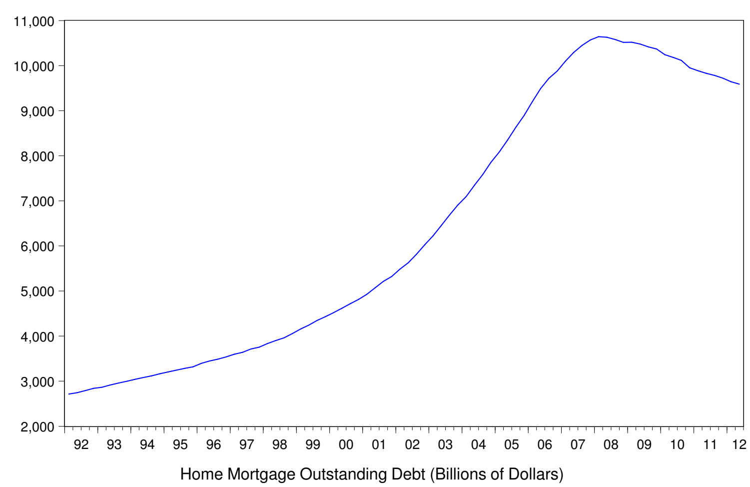


Figure 3: Time series plot of US outstanding mortgage debt

Dependent Variable: DEBT				
Method: Least Squares				
Sample: 1992Q1 2008Q2				
Included observations: 66				
Variable	Coefficient	Std. Error	t-Statistic	Prob.
C	2418.064	73.81297	32.75934	0.000000
TREND	129.6796	15.02073	8.633375	0.000000
TREND^2	-7.39367	0.902036	-8.19664	0.000000
TREND^3	0.210702	0.020157	10.4532	0.000000
TREND^4	-0.00149	0.000149	-10.0098	0.000000
R-squared	0.998263	Mean dependent var		5500.213
Adjusted R-squared	0.998149	S.D. dependent var		2538.57
S.E. of regression	109.2039	Akaike info criterion		12.29704
Sum squared resid	727454.9	Schwarz criterion		12.46293
Log likelihood	-400.803	Hannan-Quinn criter.		12.36259
F-statistic	8765.98	Durbin-Watson stat		0.168185
Prob(F-statistic)	0.000000			

Table 1: Estimation results of a polynomial trend of order 4

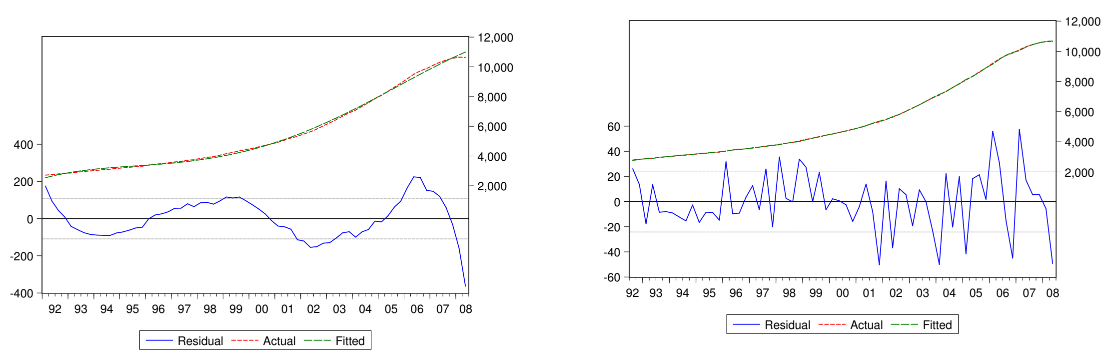
Dependent Variable: DEBT				
Method: Least Squares				
Sample: 1992Q1 2008Q2				
Included observations: 66				
Variable	Coefficient	Std. Error	t-Statistic	Prob.
C	2816.217	38.56921	73.01723	0.00000
TREND	-31.5975	11.32354	-2.79042	0.007
TREND^2	8.952828	1.027861	8.71015	0.00000
TREND^3	-0.43241	0.038567	-11.212	0.00000
TREND^4	0.009262	0.000633	14.64019	0.00000
TREND^5	-6.42E-05	3.76E-06	-17.0847	0.00000
R-squared	0.999704	Mean dependent var		5500.213
Adjusted R-squared	0.999679	S.D. dependent var		2538.57
S.E. of regression	45.46752	Akaike info criterion		10.55838
Sum squared resid	124037.7	Schwarz criterion		10.75744
Log likelihood	-342.427	Hannan-Quinn criter.		10.63704
F-statistic	40512.63	Durbin-Watson stat		0.313768
Prob(F-statistic)	0.000000			

Table 2: Estimation results of a polynomial trend of order 5

The estimation results in Table 1 assume that the error term is a white noise process. However, Figure 4a shows that the residuals are persistent. They remain above or below zero for several periods at a time, which is not what we expect from a white noise process. The autocorrelograms of the residuals are shown in Figure 5a, and clearly indicate that they are not white noise. The residual process is autoregressive. We estimate a model that combines a polynomial trend of order 4 with AR(2) terms. The estimation results are shown in Table 3. Comparing Table 1 with Table 3, we see that the extended model is preferred because the autoregressive parameters are significant; AIC, SIC and the residual variance are smaller than those in the model with only the polynomial trend. In addition, the residuals from the extended model in Figure 4b do not show any autocorrelation.

Dependent Variable: DEBT				
Method: Least Squares				
Sample (adjusted): 1992Q3 2008Q2				
Included observations: 64 after adjustments				
Convergence achieved after 137 iterations				
Variable	Coefficient	Std. Error	t-Statistic	Prob.
C	-31131	152931.6	-0.20356	0.8394
TREND	2501.025	7858.685	0.31825	0.7515
TREND^2	-78.2164	168.2437	-0.4649	0.6438
TREND^3	1.215156	1.743204	0.697082	0.4886
TREND^4	-0.00697	0.007203	-0.96735	0.3375
AR(1)	1.429181	0.152185	9.391044	0.0000
AR(2)	-0.46573	0.169946	-2.74046	0.0082
R-squared	0.999917	Mean dependent var		5586.766
Adjusted R-squared	0.999908	S.D. dependent var		2529.382
S.E. of regression	24.21259	Akaike info criterion		9.314541
Sum squared resid	33416.23	Schwarz criterion		9.550669
Log likelihood	-291.065	Hannan-Quinn criter.		9.407564
F-statistic	114577.5	Durbin-Watson stat		2.116146
Prob(F-statistic)	0.000000			

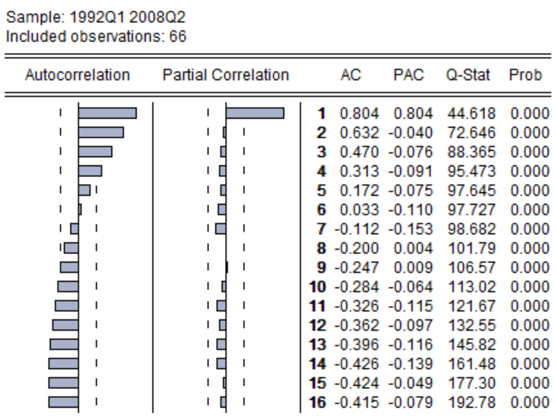
Table 3: Estimation results of trend and AR model



(a) Residuals from model with only trend

(b) Residuals from model with trend and AR terms

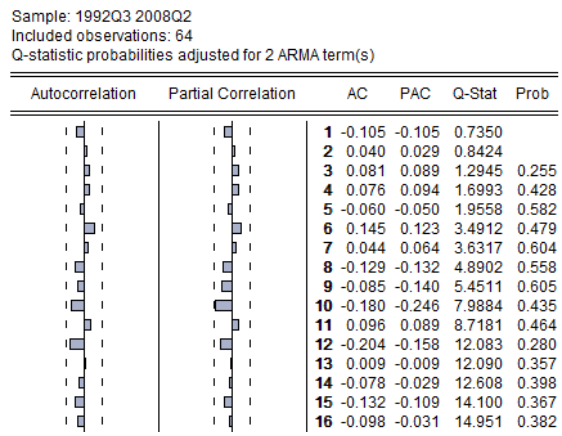
Figure 4: Residuals from both models



Sample: 1992Q1 2008Q2
Included observations: 66

	AC	PAC	Q-Stat	Prob
1	0.804	0.804	44.618	0.000
2	0.632	-0.040	72.646	0.000
3	0.470	-0.076	88.365	0.000
4	0.313	-0.091	95.473	0.000
5	0.172	-0.075	97.645	0.000
6	0.033	-0.110	97.727	0.000
7	-0.112	-0.153	98.682	0.000
8	-0.200	0.004	101.79	0.000
9	-0.247	0.009	106.57	0.000
10	-0.284	-0.064	113.02	0.000
11	-0.326	-0.115	121.67	0.000
12	-0.362	-0.097	132.55	0.000
13	-0.396	-0.116	145.82	0.000
14	-0.426	-0.139	161.48	0.000
15	-0.424	-0.049	177.30	0.000
16	-0.415	-0.079	192.78	0.000

(a) Correlograms of residuals from model with trend



Sample: 1992Q3 2008Q2
Included observations: 64
Q-statistic probabilities adjusted for 2 ARMA term(s)

	AC	PAC	Q-Stat	Prob
1	-0.105	-0.105	0.7350	
2	0.040	0.029	0.8424	
3	0.081	0.089	1.2945	0.255
4	0.076	0.094	1.6993	0.428
5	-0.060	-0.050	1.9558	0.582
6	0.145	0.123	3.4912	0.479
7	0.044	0.064	3.6317	0.604
8	-0.129	-0.132	4.8902	0.558
9	-0.085	-0.140	5.4511	0.605
10	-0.180	-0.246	7.9884	0.435
11	0.096	0.089	8.7181	0.464
12	-0.204	-0.158	12.083	0.280
13	0.009	-0.009	12.090	0.357
14	-0.078	-0.029	12.608	0.398
15	-0.132	-0.109	14.100	0.367
16	-0.098	-0.031	14.951	0.382

(b) Correlograms of residuals from model with trend and AR terms

Figure 5: Correlograms of residuals

With the extended model (polynomial trend of order 4 and AR(2) terms), we construct a short-term (one-quarter-ahead) and a longer-term (three-quarters-ahead) forecasts. The forecast sample runs from 2008:Q3 to 2012:Q2. We implement a fixed scheme, so that the model is estimated once (Table 3) and we update the information set as needed.

The short-term (one-quarter-ahead) forecast is constructed as follows:

$$f_{t,1} = c + \beta_1(t+1) + \beta_2(t+1)^2 + \beta_3(t+1)^3 + \beta_4(t+1)^4 + \phi_1 Y_t + \phi_2 Y_{t-1}.$$

In order to construct the long-term $f_{t,3}$ (three-quarter-ahead) forecast, we need $f_{t,1}$, already constructed above, and $f_{t,2}$. Thus,

$$\begin{aligned} f_{t,2} &= c + \beta_1(t+2) + \beta_2(t+2)^2 + \beta_3(t+2)^3 + \beta_4(t+2)^4 + \phi_1 f_{t,1} + \phi_2 Y_t \\ f_{t,3} &= c + \beta_1(t+3) + \beta_2(t+3)^2 + \beta_3(t+3)^3 + \beta_4(t+3)^4 + \phi_1 f_{t,2} + \phi_2 f_{t,1} \end{aligned}$$

In Figure 6, we plot both forecasts and the realized series. Obviously, the forecasts are rather poor. The estimated trend is too strong and forces the forecast towards a steep downward path, in particular at longer horizons than one-quarter-ahead. Observe that the forecast errors (short and long term) are all positive, thus a MPE test will reject this model easily. It is fair to say that the model is not adequate for long term forecasting. We could also implement a rolling or a recursive scheme in which the model is re-estimated every time that new information becomes available. On doing this, the estimates of the trend coefficients will be more adaptive to the progression of the trend. This is left as an exercise for the student.

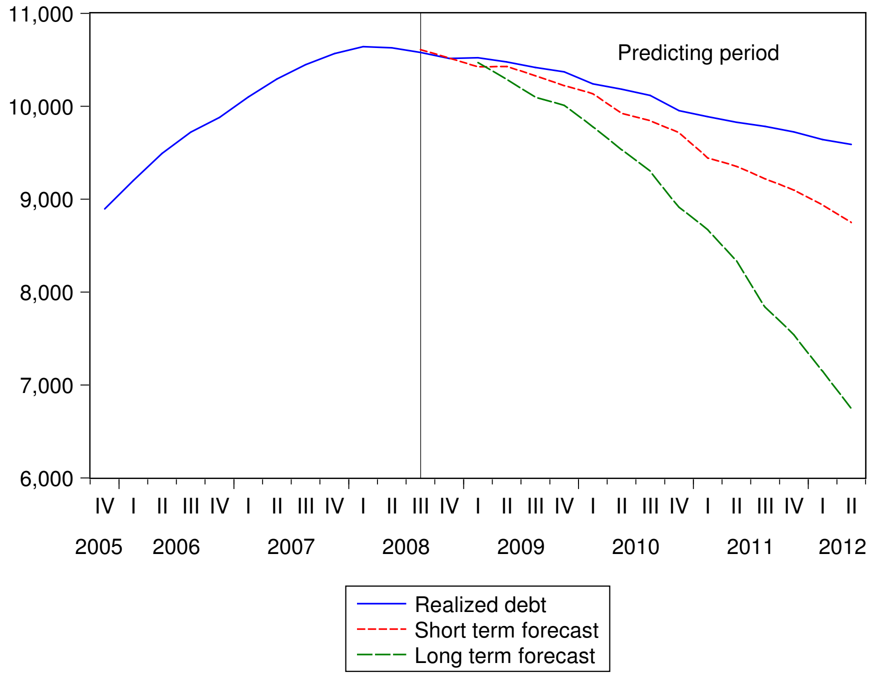


Figure 6: Short term and long term forecasts of US outstanding mortgage debt

Exercise 6
We simulate a random walk with drift (series 1) and without drift (series 2) for a sample of 10,000 observations. The two data generating processes are as follows,

$$\begin{aligned} \text{Series 1}: &\quad Y_t = Y_{t-1} + \varepsilon_t \\ \text{Series 2}: &\quad Y_t = 0.02 + Y_{t-1} + \varepsilon_t, \end{aligned}$$

where ε_t is a white noise process $N(0,1)$.

In Figures 7a and 7b, we plot the two series and their unconditional means. We observe that each process crosses its unconditional mean only a few times in 10,000 periods. The crossing takes place about a couple of thousands periods, which is a quite long time.

For the random walk without drift (series 1), i.e.,

$$Y_t = Y_{t-1} + \varepsilon_t = \varepsilon_t + \varepsilon_{t-1} + \cdots + \varepsilon_1,$$

the unconditional mean is

$$\mu \equiv E(Y_t) = E(\varepsilon_t + \varepsilon_{t-1} + \cdots + \varepsilon_1) = 0,$$

the unconditional variance is

$$\sigma^2_{Y_t} = E(Y_t - \mu)^2 = E(\varepsilon_t + \varepsilon_{t-1} + \cdots + \varepsilon_1)^2 = t\sigma^2_\varepsilon = t,$$

the autocovariance of order k is

$$\begin{aligned}\gamma_{t,t-k} &\equiv E(Y_t-\mu)(Y_{t-k}-\mu)\\ &= E(\varepsilon_t+\varepsilon_{t-1}+\cdots+\varepsilon_1)(\varepsilon_{t-k}+\cdots+\varepsilon_1)\\ &= (t-k)\sigma_\varepsilon^2 = t-k,\end{aligned}$$

and the autocorrelation functions of order k is

$$\rho_{t,t-k} = \frac{\gamma_{t,t-k}}{\sigma_{Y_t}\sigma_{Y_{t-k}}} = \sqrt{\frac{t-k}{t}}.$$

Similarly, for the random walk with drift (series 2), i.e.,

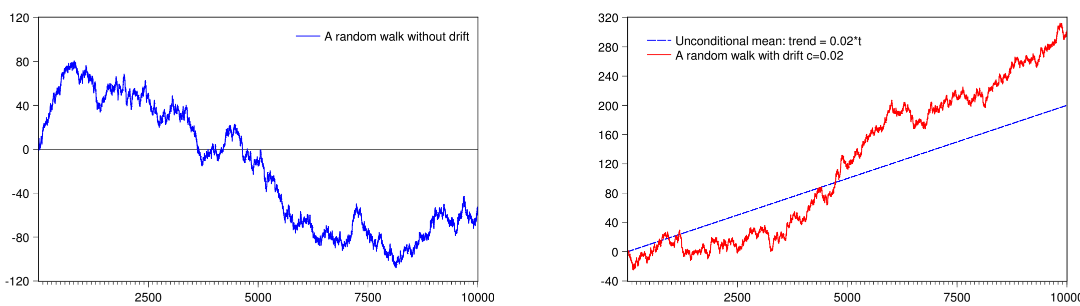
$$Y_t = c + Y_{t-1} + \varepsilon_t = ct + \varepsilon_t + \varepsilon_{t-1} + \cdots + \varepsilon_1.$$

the unconditional mean is

$$\mu_t \equiv E(Y_t) = ct.$$

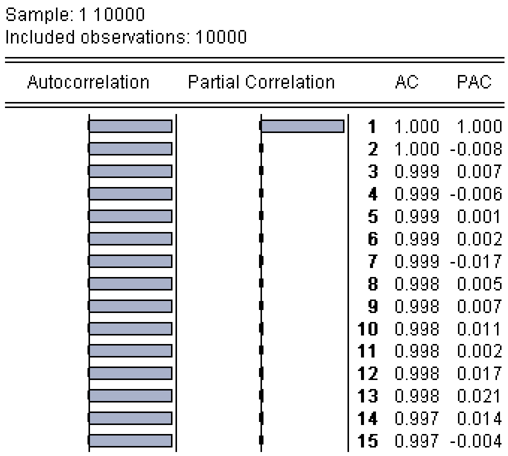
The unconditional variance, autocovariance of order k, and autocorrelation function of order k are identical to those of the random walk without drift.

In Figures 8a and 8b, we report the sample autocorrelation functions of both processes, which are identical. As expected, the profile of the ACF and PACF shows strong persistence and the presence of an autoregressive process with a unit root.

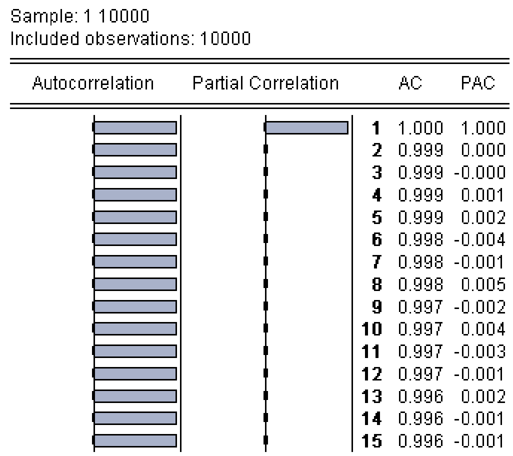


(a) Without drift (b) With drift ($c = 0.02$)

Figure 7: Time series plot of random walks



(a) Random walk without drift



(b) Random walk with drift ($c = 0.02$)

Figure 8: AC and PAC functions

Exercise 7

In Tables 4 and 5, we present the results of the Augmented Dickey-Fuller unit root test on series 1 and series 2 in Exercise 6. Because we have simulated the data, the choice of case I, II, or III (Table 10.5 in the textbook) is trivial: we run case I for series 1 and case III for series 2.[1] If the DGPs were not known, the clear upward trend in series 2 would suggest the choice of case III; and the reverting behavior of series 1 would suggest that either case I or II are plausible choices. The test results for both series indicate that we cannot reject the null hypothesis of a unit root. For series 1, the ADF is -0.531, which clearly falls in the acceptance region of the test according to the MacKinnon critical values, e.g. at the 5% level the critical value is -1.94. For series 2, the ADF is -2.668 and the 5% critical value is -3.41.

ADF Test statistic	-0.531967	1% Critical Value*		-2.56519
		5% Critical Value		-1.94086
		10% Critical Value		-1.61668
*MacKinnon critical values for rejection of hypothesis of a unit root.				
Augmented Dickey-Fuller Test Equation				
Dependent Variable: D(Y1)				
Method: Least Squares				
Sample (adjusted): 2 10000				
Included observations: 9999 after adjustments				
Variable	Coefficient	Std. Error	t-Statistic	Prob.
Y1(-1)	-8.94E-05	0.000168	-0.531967	0.5948
R-squared	-0.000008	Mean dependent var		-0.00603
Adjusted R-squared	-0.000008	S.D. dependent var		0.996111
S.E. of regression	0.996116	Akaike info criterion		2.830193
Sum squared resid	9920.479	Schwarz criterion		2.830914
Log likelihood	-14148.55	Hannan-Quinn criter.		2.830437
Durbin-Watson stat	1.966598			

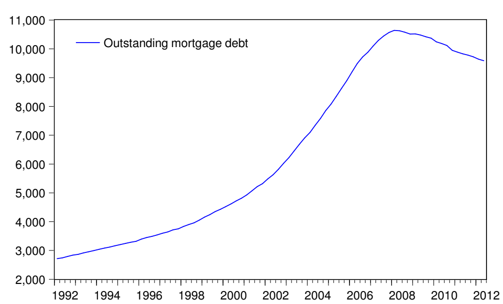
Table 4: ADF test on series 1 (Exercise 6)

[1]From Table 10.5 in the textbook: in case II, only intercept is included in the equation of the ADF test; in case III, both intercept and linear time trend are included; in case I, no constant and no trend are included.

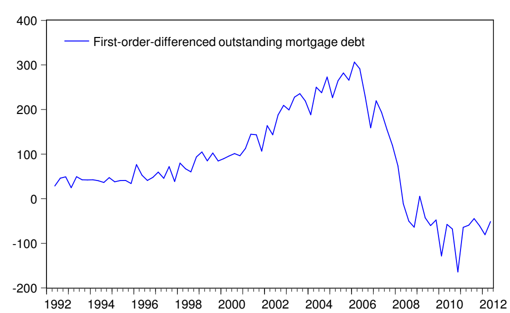
ADF test statistic	-2.66839	1% Critical Value*		-3.95895
		5% Critical Value		-3.41025
		10% Critical Value		-3.12687
*MacKinnon critical values for rejection of hypothesis of a unit root.				
Augmented Dickey-Fuller Test Equation				
Dependent Variable: D(Y2)				
Method: Least Squares				
Sample (adjusted): 2 10000				
Included observations: 9999 after adjustments				
Variable	Coefficient	Std. Error	t-Statistic	Prob.
Y2(-1)	-0.00115	0.000429	-2.668388	0.0076
C	-0.04654	0.029525	-1.576334	0.115
@TREND(1)	4.27E-05	1.50E-05	2.845375	0.0044
R-squared	0.000828	Mean dependent var		0.030123
Adjusted R-squared	0.000628	S.D. dependent var		0.99967
S.E. of regression	0.999356	Akaike info criterion		2.836889
Sum squared resid	9983.137	Schwarz criterion		2.839053
Log likelihood	-14180	Hannan-Quinn criter.		2.837622
F-statistic	4.142517	Durbin-Watson stat		1.990172
Prob(F-statistic)	0.01591			

Table 5: ADF test on series 2 (Exercise 6)

For the updated data of outstanding mortgage debt in Exercise 5, we perform ADF tests on the level and its first order differences. In Figures 9a and 9b, we plot the two series. We choose case II and include a constant in the equation of the ADF test because the upward trend in the original series is broken and there is not an obvious trend in the series in first-differences. The ADF test results are reported in Tables 6 and 7. In both instances, we fail to reject the null of unit root because the value of the test falls into the acceptance region according to MacKinnon critical values. In this case, we need to take another difference and run again the ADF test. The second-order-differenced series is stationary (Table is not reported), and we conclude that the mortgage debt series has two unit roots, and is *integrated of order 2*.



(a) Level



(b) First order differences

Figure 9: Time series plot of outstanding mortgage debt

ADF test statistic	-2.210038	1% Critical Value*		-3.51554
		5% Critical Value		-2.89862
		10% Critical Value		-2.58661
*MacKinnon critical values for rejection of hypothesis of a unit root.				
Augmented Dickey-Fuller Test Equation				
Dependent Variable: D(DEBT)				
Method: Least Squares				
Sample (adjusted): 1992Q4 2012Q2				
Included observations: 79 after adjustments				
Variable	Coefficient	Std. Error	t-Statistic	Prob.
DEBT(-1)	-0.003002	0.001358	-2.210038	0.0302
D(DEBT(-1))	0.691393	0.111311	6.211339	0.0000
D(DEBT(-2))	0.278532	0.112814	2.468948	0.0158
C	20.37554	10.08188	2.021006	0.0468
R-squared	0.90492	Mean dependent var		86.05608
Adjusted R-squared	0.901117	S.D. dependent var		108.5471
S.E. of regression	34.13339	Akaike info criterion		9.947736
Sum squared resid	87381.65	Schwarz criterion		10.06771
Log likelihood	-388.9356	Hannan-Quinn criter.		9.9958
F-statistic	237.9363	Durbin-Watson stat		2.063459
Prob(F-statistic)	0.000000			

Table 6: ADF test on the original series of outstanding mortgage debt (Exercise 5)

ADF test statistic	-1.254873	1% Critical Value*		-3.51443
		5% Critical Value		-2.89815
		10% Critical Value		-2.58635
*MacKinnon critical values for rejection of hypothesis of a unit root.				
Augmented Dickey-Fuller Test Equation				
Dependent Variable: D(DEBT,2)				
Method: Least Squares				
Sample (adjusted): 1992Q3 2012Q2				
Included observations: 80 after adjustments				
Variable	Coefficient	Std. Error	t-Statistic	Prob.
D(DEBT(-1))	-0.046843	0.037329	-1.25487	0.2133
C	3.055053	5.11893	0.596815	0.5524
R-squared	0.019789	Mean dependent var		-0.9995
Adjusted R-squared	0.007222	S.D. dependent var		35.64093
S.E. of regression	35.512	Akaike info criterion		10.0023
Sum squared resid	98365.96	Schwarz criterion		10.06185
Log likelihood	-398.092	Hannan-Quinn criter.		10.02618
F-statistic	1.574705	Durbin-Watson stat		2.414404
Prob(F-statistic)	0.213272			

Table 7: ADF test on the first-order-differenced series of outstanding mortgage debt

When there is a break in the trend of the series, a model based on a deterministic trend will produce very unreliable predictions because the shape of the trend is fixed by whatever functional form we choose to estimate (linear, polynomial, exponential, etc.). When the shape of the trend is stochastic, the model is more robust to trend changes, and the forecast is more adaptive as new information comes. The modeling of the second-order-differenced series, its corresponding forecast, and the comparison with the forecasts in Exercise 5 are left as an exercise.

Exercise 8
We download and update the index of total hours worked in Spain from the OECD website.

The updated series runs from 1970 to 2011. The red dashed line in Figure 10 represents the Spanish series.

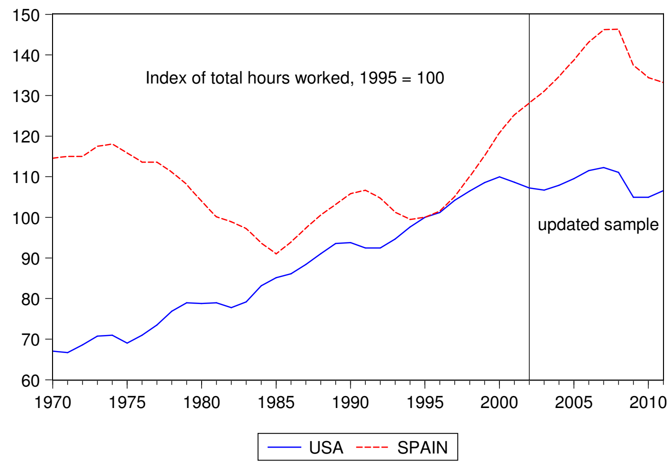


Figure 10: Index of total hours worked in United States and in Spain

Since we do not observe an obvious trend in the data, the testing for unit root should start with case II including an intercept in the equation of the test. The result of the ADF unit root test is presented in Table 8. Because the test statistic -0.326626 falls in the acceptance region of the distribution of the test ($-0.326626 > -2.61287$), we fail to reject the null hypothesis of unit root. Thus, we will model the first-difference of the series. We propose ARIMA(1,1,0) model as follows,

$$\Delta Y_t = \phi \Delta Y_{t-1} + \varepsilon_t$$

where the number of lags is chosen by minimizing SIC. In Table 9, we present the estimation results. The specification of the model with the updated data is the same as the model in Section 10.2 in the textbook, and the estimation results very similar.

ADF test statistic	-0.326626	1% Critical Value*		-3.6329
		5% Critical Value		-2.9484
		10% Critical Value		-2.61287
*MacKinnon critical values for rejection of hypothesis of a unit root.				
Augmented Dickey-Fuller Test Equation				
Dependent Variable: D(SPAIN)				
Method: Least Squares				
Sample (adjusted): 1973 2007				
Included observations: 35 after adjustments				
Variable	Coefficient	Std. Error	t-Statistic	Prob.
SPAIN(-1)	-0.009151	0.028018	-0.326626	0.7461
D(SPAIN(-1))	1.097932	0.170252	6.448858	0.0000
D(SPAIN(-2))	-0.303046	0.190366	-1.591913	0.1216
C	1.227649	3.054577	0.401905	0.6905
R-squared	0.717928	Mean dependent var		0.891619
Adjusted R-squared	0.69063	S.D. dependent var		3.051927
S.E. of regression	1.697513	Akaike info criterion		4.003416
Sum squared resid	89.32802	Schwarz criterion		4.18117
Log likelihood	-66.05977	Hannan-Quinn criter.		4.064776
F-statistic	26.30029	Durbin-Watson stat		1.764809
Prob(F-statistic)	0.000000			

Table 8: ADF test for the index of total hours worked in Spain

Dependent Variable: D(SPAIN)				
Method: Least Squares				
Sample (adjusted): 1972 2011				
Included observations: 40 after adjustments				
Convergence achieved after 2 iterations				
Variable	Coefficient	Std. Error	t-Statistic	Prob.
AR(1)	0.728966	0.109989	6.6276	0.0000
R-squared	0.520583	Mean dependent var		0.45423
Adjusted R-squared	0.520583	S.D. dependent var		3.304765
S.E. of regression	2.288217	Akaike info criterion		4.518105
Sum squared resid	204.2015	Schwarz criterion		4.560327
Log likelihood	-89.3621	Hannan-Quinn criter.		4.533371
Durbin-Watson stat	1.690472			
Inverted AR Roots	0.73			

Table 9: Estimation results of ARIMA(1,1,0) model for the Spanish time series

Based on the ARIMA(1,1,0) model, we construct the 1, 2, 3, and 4-year-ahead forecasts (short- and long-term) using the *first and second recursive relation* (**R1** and **R2**) proposed in the textbook. The forecast results are listed below.

$t = 2011$	$\Delta Y_{2011} = -1.25$		$Y_{2011} = 133.15$	
Forecasting Horizon	$f_{2011,s} = \phi^s \Delta Y_{2011}$	$\sigma_{t+s\|2011}$	$f^*_{2011,s} = f_{2011,s} + f^*_{2011,s-1}$	$\sigma^*_{t+s\|2011}$
$s = 1,\ 2012$	-0.91	2.29	132.24	2.29
$s = 2,\ 2013$	-0.66	2.84	131.58	4.58
$s = 3,\ 2014$	-0.48	3.08	131.09	6.91
$s = 4,\ 2015$	-0.35	3.21	130.74	9.19

In Figure 11, we present the lower and upper bounds of a 95% confidence interval for the Spanish index under the assumption that ε_t is normally distributed. Note that the uncertainty of the forecasts based on the ARIMA(1,1,0) increases with the forecast horizon, as it is expected in a process with a unit root.

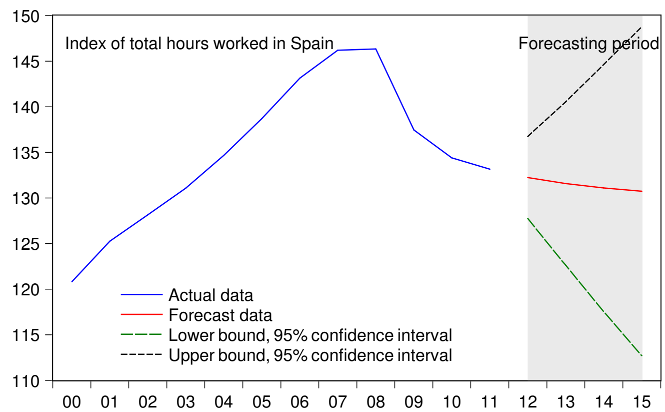


Figure 11: Forecast of index of total hours worked in Spain with 95% confidence bands

Exercise 9

Since the data have had some minor adjustments by the OECD, we re-estimate the deterministic linear trend model in Table 10.11 of the textbook, using the same sample range from 1970 to 2002. In Table 10, we present the estimation results based on the corrected data, which are similar to those presented in the textbook.

Dependent Variable: USA				
Method: Least Squares				
Sample (adjusted): 1972 2002				
Included observations: 31 after adjustments				
Convergence achieved after 3 iterations				
Variable	Coefficient	Std. Error	t-Statistic	Prob.
C	64.31205	0.882787	72.85113	0.0000
@TREND(1970)	1.413009	0.046965	30.08667	0.0000
AR(1)	1.125004	0.159922	7.034723	0.0000
AR(2)	-0.643565	0.153131	-4.202698	0.0003
R-squared	0.99278	Mean dependent var		88.34399
Adjusted R-squared	0.991977	S.D. dependent var		13.2695
S.E. of regression	1.188542	Akaike info criterion		3.303245
Sum squared resid	38.14105	Schwarz criterion		3.488276
Log likelihood	-47.2003	Hannan-Quinn criter.		3.363561
F-statistic	1237.465	Durbin-Watson stat		1.65612
Prob(F-statistic)	0.000000			
Inverted AR Roots	.56+.57i	.56-.57i		

Table 10: Estimation results of the deterministic linear trend model

Based on the results of Table 10, we construct the 1,2,3, and 4-step-ahead forecasts of the index of total hours worked in the United States. The forecasts are listed below:

Forecasting horizon	Actual	Forecast	Forecast uncertainty ($s.e.$)	Forecast error
$s = 1,\ 2003$	106.68	108.01	1.36	-1.33
$s = 2,\ 2004$	107.89	110.53	2.04	-2.65
$s = 3,\ 2005$	109.53	113.61	2.19	-4.08
$s = 4,\ 2006$	111.50	116.17	2.17	-4.67

In Figure 12, we plot the lower and upper bounds of a 95% confidence interval under the assumption that ε_t is normally distributed. The forecast seems too be a bit optimistic as it is always above the realized data, but nevertheless the actual data is contained most of the time within the 95% confidence bands. Note that, unlike the stochastic trend model, the width of the confidence interval does not explode and, therefore, forecast uncertainty remains the same in the long run.

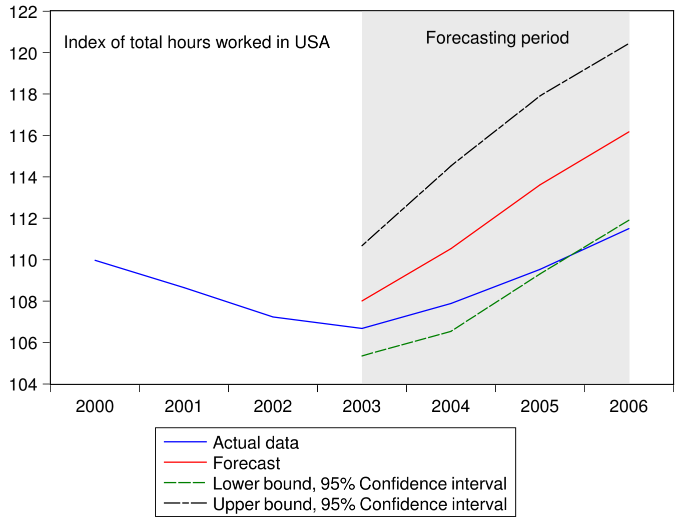


Figure 12: Forecast of index of total hours worked in USA (2003 – 2006)

Exercise 10
We download and update the index of total hours worked in USA from the OECD website.

The updated series runs from 1970 to 2011. The blue solid line in Figure 10 represents the USA series. Since we observe an obvious upward trend in the data, the testing for unit root should start with case III with both intercept and a linear trend in the equation of the test. The result of the ADF unit root test is shown in Table 11. Because the test statistic -2.14785 falls in the acceptance region of the distribution of the test ($-2.14785 > -3.19461$), we fail to reject the null hypothesis of unit root. Recall that in the textbook we mentioned that the rejection of the unit root was borderline. Now with the updated data, the unit root is more plausible. Consequently, we proceed to take first differences in the original series and to model the resulting time series.

We propose an ARIMA(2,1,0) model as follows,

$$\Delta Y_t = c + \phi_1 \Delta Y_{t-1} + \phi_2 \Delta Y_{t-2} + \varepsilon_t$$

where the number of lags is chosen by minimizing SIC. In Table 10, we report the estimation results of the model.

ADF test statistic	-2.14785	1% Critical Value*		-4.205
		5% Critical Value		-3.52661
		10% Critical Value		-3.19461
*MacKinnon critical values for rejection of hypothesis of a unit root.				
Augmented Dickey-Fuller Test Equation				
Dependent Variable: D(USA)				
Method: Least Squares				
Sample (adjusted): 1972 2011				
Included observations: 40 after adjustments				
Variable	Coefficient	Std. Error	t-Statistic	Prob.
USA(-1)	-0.19496	0.090768	-2.147846	0.0385
D(USA(-1))	0.57193	0.170596	3.352541	0.0019
C	13.60708	5.791855	2.349347	0.0244
@TREND(1970)	0.221171	0.118181	1.871456	0.0694
R-squared	0.294418	Mean dependent var		0.996714
Adjusted R-squared	0.23562	S.D. dependent var		1.884539
S.E. of regression	1.647631	Akaike info criterion		3.931193
Sum squared resid	97.72874	Schwarz criterion		4.100081
Log likelihood	-74.6239	Hannan-Quinn criter.		3.992258
F-statistic	5.007242	Durbin-Watson stat		1.728785
Prob(F-statistic)	0.005279			

Table 11: ADF test for the index of total hours worked in the United States

Dependent Variable: D(USA)				
Method: Least Squares				
Sample (adjusted): 1973 2011				
Included observations: 39 after adjustments				
Convergence achieved after 3 iterations				
Variable	Coefficient	Std. Error	t-Statistic	Prob.
C	0.96895	0.33565	2.886791	0.0065
AR(1)	0.589922	0.153997	3.830738	0.0005
AR(2)	-0.370758	0.153581	-2.414092	0.021
R-squared	0.299863	Mean dependent var		0.974527
Adjusted R-squared	0.260966	S.D. dependent var		1.903875
S.E. of regression	1.636705	Akaike info criterion		3.897051
Sum squared resid	96.43696	Schwarz criterion		4.025017
Log likelihood	-72.9925	Hannan-Quinn criter.		3.942964
F-statistic	7.709252	Durbin-Watson stat		2.08249
Prob(F-statistic)	0.001634			
Inverted AR Roots	.29+.53i	.29-.53i		

Table 12: Estimation results of ARIMA(2,1,0) for the U.S. time series of hours worked

Based on the ARIMA(2,1,0) model, we construct the 1, 2, 3, and 4-year-ahead forecasts (short- and long-term) using the first and second recursion relation (**R1** and **R2**) in the textbook. The forecast results are listed below:

$t = 2011$	$\Delta Y_{2011} = 1.64$		$Y_{2011} = 106.57$	
Forecasting Horizon	$f_{2011,s} = c + \phi_1 f_{2011,s-1} + \phi_2 f_{2011,s-2}$	$\sigma_{t+s\|2011}$	$f^*_{2011,s} = f_{2011,s} + f^*_{2011,s-1}$	$\sigma^*_{t+s\|2011}$
$s = 1,\ 2012$	1.73	1.67	108.30	1.67
$s = 2,\ 2013$	1.17	1.93	109.46	3.13
$s = 3,\ 2014$	0.80	1.93	110.27	4.07
$s = 4,\ 2015$	0.80	1.97	111.07	4.64

In Figure 13, we present the lower and upper bounds of a 95% confidence interval for the U.S. index under the assumption that ε_t is normally distributed. Note that the uncertainty of the forecasts based on the ARIMA(2,1,0) increases with the forecast horizon as a result of the unit root in the data.

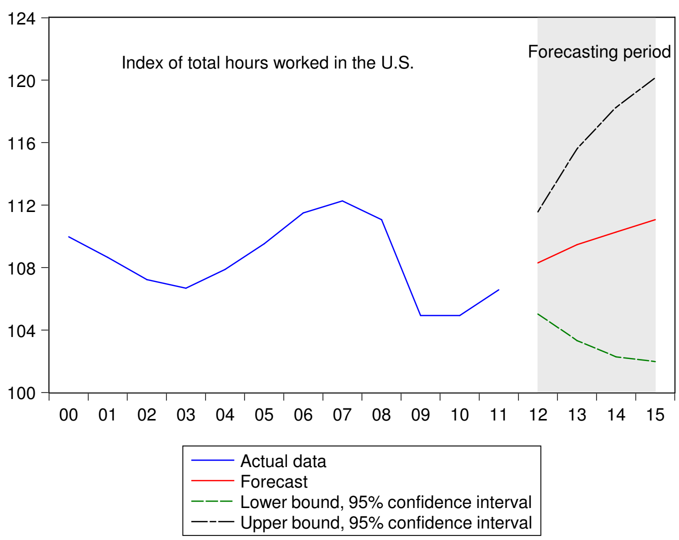


Figure 13: Forecast of Index of Total Hours Worked in USA with 95% Confidence Bands

