CHAPTER 9.

FORECASTING PRACTICE II:

ASSESSMENT OF FORECASTS AND COMBINATION OF FORECASTS

SOLUTIONS

by
Wei Lin and Yingying Sun
(University of California, Riverside)

Note: In the textbook (Chapters 8 and 9), we work with the time series Conventional Mortgage Home Price Index (CMHPI). In February 2011, Freddie Mac discontinued the publication of CMHPI and replaced it with the Freddie Mac House Price Index (FMHPI). Thus, for Exercises 1 to 7, we need to work with this new data set FMHPI. Recall that in Chapter 8, Exercise 2, we also worked with FMHPI. Based on the autocorrelograms of the time series of San Diego home price growth (Chp.8-Exercise 2), we will entertain the following three models, AR(4), AR(6) and AR(7), to answer Exercises 1 to 7 instead of the original models proposed for the CMHPI-San Diego time series.

Exercise 1
The forecasting environment is described as follows:

- Forecast horizon: 1-step-ahead forecast ($h = 1$); thus we are interested in the forecast of the next quarter growth of the San Diego house price.
- Information set: From 1975.Q2 to 2008.Q2.
- Fixed sampling scheme with an estimation sample from 1975.Q2 to 2008.Q2 and a prediction sample from 2008.Q3 to 2012.Q2.
- We consider three time series models: AR(4), AR(6) and AR(7).

Within the above forecasting environment, we obtain optimal forecasts for all these models assuming an asymmetric Linex loss function.The estimation results are reported in Tables 1, 2, and 3. Based on the estimates, the optimal forecasts of y_{t+1} are constructed as follows. First, we obtain the conditional mean of y_{t+1}, i.e.

$$
\begin{aligned}
\widehat{\mu}_{t+1|t} &= \widehat{\mu} + \widehat{\phi}_1 y_t + \widehat{\phi}_2 y_{t-1} + \widehat{\phi}_3 y_{t-2} + \widehat{\phi}_4 y_{t-3} \quad \text{(for AR(4))} \\
\widehat{\mu}_{t+1|t} &= \widehat{\mu} + \widehat{\phi}_1 y_t + \widehat{\phi}_2 y_{t-1} + \widehat{\phi}_3 y_{t-2} + \widehat{\phi}_4 y_{t-3} + \widehat{\phi}_5 y_{t-4} + \widehat{\phi}_6 y_{t-5} \quad \text{(for AR(6))} \\
\widehat{\mu}_{t+1|t} &= \widehat{\mu} + \widehat{\phi}_1 y_t + \widehat{\phi}_2 y_{t-1} + \widehat{\phi}_3 y_{t-2} + \widehat{\phi}_4 y_{t-3} + + \widehat{\phi}_5 y_{t-4} + \widehat{\phi}_6 y_{t-5} + \widehat{\phi}_7 y_{t-6} \quad \text{(for AR(7))}.
\end{aligned}
$$

Secondly, we construct the optimal forecast $f^*_{t,1}$, under the assumption of conditional normality of Y_{t+1} and the linex loss function as follows

$$f^*_{t,1} = \mu_{t+1|t} + \frac{a}{2}\sigma^2_{t+1|t}$$

In the absence of any information about the asymmetry of the loss function, we will choose a value $a = -1$. This choice implies that negative forecast errors are more penalized than positive errors; in other words, the forecaster is not overly optimistic about the real estate market in San Diego and she prefers to take a conservative approach by lowering the forecast with respect to the average forecast $\mu_{t+1|t}$. Alternatively, we can also think of a forecaster who is interested on assessing risk for a portfolio of mortgages and she needs to provide adverse events, which will have low probability (a type of value-at-risk calculation) but nevertheless they need to be considered in order to protect the portfolio from eventual losses.

Dependent Variable: SDG				
Method: Least Squares				
Sample (adjusted): 1976Q2 2008Q2				
Included observations: 129 after adjustments				
Convergence achieved after 4 iterations				
Variable	Coefficient	Std. Error	t-Statistic	Prob.
C	0.784642	1.677701	0.467689	0.6408
AR(1)	0.305708	0.083185	3.675041	0.0004
AR(2)	0.010264	0.087776	0.116936	0.9071
AR(3)	0.12766	0.088474	1.442899	0.1516
AR(4)	0.393037	0.086789	4.528645	0.0000
R-squared	0.364819	Mean dependent var		1.714706
Adjusted R-squared	0.34433	S.D. dependent var		3.465103
S.E. of regression	2.805814	Akaike info criterion		4.939253
Sum squared resid	976.2015	Schwarz criterion		5.050099
Log likelihood	-313.582	Hannan-Quinn criter.		4.984292
F-statistic	17.80502	Durbin-Watson stat		1.978034
Prob(F-statistic)	0.000000			
Inverted AR Roots	0.94	.02+.78i	.02-.78i	-0.68

Table 1: Estimation results of AR(4) model

Dependent Variable: SDG				
Method: Least Squares				
Sample (adjusted): 1976Q4 2008Q2				
Included observations: 127 after adjustments				
Convergence achieved after 4 iterations				
Variable	Coefficient	Std. Error	t-Statistic	Prob.
C	0.962901	1.519018	0.633897	0.5274
AR(1)	0.31782	0.091038	3.491078	0.0007
AR(2)	0.027803	0.096331	0.288621	0.7734
AR(3)	0.129782	0.090092	1.440545	0.1523
AR(4)	0.400472	0.090589	4.420774	0.0000
AR(5)	-0.01688	0.097666	-0.17285	0.8631
AR(6)	-0.04126	0.095385	-0.43257	0.6661
R-squared	0.369488	Mean dependent var		1.705258
Adjusted R-squared	0.337962	S.D. dependent var		3.486451
S.E. of regression	2.836775	Akaike info criterion		4.976754
Sum squared resid	965.6754	Schwarz criterion		5.13352
Log likelihood	-309.024	Hannan-Quinn criter.		5.040446
F-statistic	11.72025	Durbin-Watson stat		1.991433
Prob(F-statistic)	0.000000			
Inverted AR Roots	0.92	0.32	.02+.81i	.02-.81i
	-0.32	-0.64		

Table 2: Estimation results of AR(6) model

Dependent Variable: SDG				
Method: Least Squares				
Sample (adjusted): 1977Q1 2008Q2				
Included observations: 126 after adjustments				
Convergence achieved after 4 iterations				
Variable	Coefficient	Std. Error	t-Statistic	Prob.
C	0.380962	2.322899	0.164003	0.87
AR(1)	0.320282	0.090715	3.530633	0.0006
AR(2)	0.034152	0.096083	0.355447	0.7229
AR(3)	0.064547	0.096642	0.667899	0.5055
AR(4)	0.380661	0.090678	4.197961	0.0001
AR(5)	-0.01631	0.097074	-0.16797	0.8669
AR(6)	-0.08479	0.097605	-0.86872	0.3868
AR(7)	0.16636	0.094876	1.753455	0.0821
R-squared	0.383976	Mean dependent var		1.681023
Adjusted R-squared	0.347432	S.D. dependent var		3.489613
S.E. of regression	2.818967	Akaike info criterion		4.972005
Sum squared resid	937.6958	Schwarz criterion		5.152086
Log likelihood	-305.236	Hannan-Quinn criter.		5.045166
F-statistic	10.50729	Durbin-Watson stat		1.949439
Prob(F-statistic)	0.000000			
Inverted AR Roots	0.96	.43+.48i	.43-.48i	-.04+.86i
	-.04-.86i	-.70-.27i	-.70+.27i	

Table 3: Estimation results of AR(7) model

In Table 4, we report the optimal one-step-ahead forecasts, forecast errors and density forecasts under the asymmetric Linex loss function with $a = -1$. Observe that the forecasts are very conservative (well below the conditional mean) and they fall in the lower tail of the density forecasts. It is advisable to compute the probability associated with these forecasts. The forecast errors are all positive, which is expected, as the forecaster is avoiding negative errors. Note that the variance of the density forecast is not constant because it takes into consideration the uncertainty induced by the estimation of the parameters of the model. If we do not take parameter uncertainty into consideration, the variance would be constant over the prediction sample and equal to the square of the standard error of the regression, i.e. $\hat{\sigma}_\varepsilon = 2.805$ for AR(4); $\hat{\sigma}_\varepsilon = 2.836$ for AR(6); and $\hat{\sigma}_\varepsilon = 2.818$ for AR(7). Remember that the uncertainty of the one-step-ahead forecast is always equal to the variance of the innovation of the model. These numbers are not very different from those reported in Table 4, which are slightly larger because parameter uncertainty is included in the uncertainty of the forecasts.

Date	Actual	Forecast AR(4)	Forecast error AR(4)	Density forecast AR(4)	Forecast AR(6)	Forecast error AR(6)	Density forecast AR(6)	Forecast AR(7)	Forecast error AR(7)	Density forecast AR(7)
2008Q3	-5.041	-8.230	3.189	$N(-3.897, 2.944^2)$	-8.386	3.345	$N(-4.071, 2.996^2)$	-8.332	3.291	$N(-3.999, 2.978^2)$
2008Q4	-6.620	-9.586	2.967	$N(-5.252, 2.944^2)$	-9.717	3.097	$N(-5.263, 2.988^2)$	-9.718	3.098	$N(-4.671, 2.989^2)$
2009Q1	-3.595	-9.380	5.784	$N(-5.111, 2.922^2)$	-9.585	5.990	$N(-4.960, 2.991^2)$	-9.536	5.940	$N(-4.753, 2.975^2)$
2009Q2	3.991	-7.222	11.213	$N(-3.031, 2.895^2)$	-7.495	11.486	$N(-2.756, 2.988^2)$	-7.458	11.449	$N(-3.043, 2.976^2)$
2009Q3	3.516	-5.714	9.230	$N(-1.515, 2.898^2)$	-5.914	9.430	$N(-1.190, 2.966^2)$	-5.931	9.448	$N(-1.731, 2.972^2)$
2009Q4	1.272	-6.011	7.284	$N(-1.817, 2.896^2)$	-6.154	7.427	$N(-1.487, 2.945^2)$	-6.165	7.438	$N(-2.219, 2.949^2)$
2010Q1	0.926	-4.489	5.415	$N(-0.350, 2.877^2)$	-4.690	5.616	$N(0.075, 2.946^2)$	-4.615	5.541	$N(-0.567, 2.921^2)$
2010Q2	0.262	-1.622	1.884	$N(2.442, 2.851^2)$	-1.892	2.153	$N(2.894, 2.944^2)$	-1.834	2.095	$N(1.919, 2.924^2)$
2010Q3	-1.794	-2.282	0.488	$N(1.762, 2.844^2)$	-2.557	0.763	$N(1.939, 2.939^2)$	-2.526	0.731	$N(0.726, 2.928^2)$
2010Q4	-1.913	-3.824	1.912	$N(0.201, 2.837^2)$	-4.021	2.108	$N(0.018, 2.906^2)$	-4.046	2.133	$N(-0.964, 2.914^2)$
2011Q1	-3.081	-4.071	0.989	$N(-0.077, 2.826^2)$	-4.217	1.135	$N(-0.244, 2.877^2)$	-4.245	1.164	$N(0.092, 2.887^2)$
2011Q2	0.687	-4.942	5.629	$N(-0.960, 2.822^2)$	-5.071	5.758	$N(-1.053, 2.868^2)$	-5.055	5.742	$N(-0.555, 2.862^2)$
2011Q3	-2.400	-4.636	2.236	$N(-0.643, 2.826^2)$	-4.756	2.356	$N(-0.701, 2.868^2)$	-4.726	2.325	$N(-0.511, 2.858^20)$
2011Q4	-0.955	-5.740	4.785	$N(-1.744, 2.827^2)$	-5.863	4.908	$N(-1.715, 2.870^2)$	-5.805	4.850	$N(-1.460, 2.850^2)$
2012Q1	1.327	-5.325	6.652	$N(-1.312, 2.833^2)$	-5.427	6.753	$N(-1.233, 2.869^2)$	-5.368	6.695	$N(-1.238, 2.848^2)$
2012Q2	1.328	-3.533	4.862	$N(0.488, 2.836^2)$	-3.623	4.952	$N(0.665, 2.867^2)$	-3.569	4.898	$N(0.464, 2.848^2)$

Table 4: Forecast summary for AR(4), AR(6) and AR(7) under Linex loss functions (fixed scheme)

Exercise 2

The forecasting environment is the same as in Exercise 1. We construct the optimal forecast under both the symmetric MSE and asymmetric Linex loss functions. The forecasting results under Linex loss function are reported in Table 4. We report the results for the optimal forecasts under symmetric loss function in Table 5 . We plot the forecasts under both loss functions in Figures 1 and 2. The forecasts under Linex are clearly biased and the mean of the forecast errors is positive as the forecaster avoids negative errors. As we mention in the previous exercise, the forecasts under Linex are very conservative and they lie in the tails of the density forecast. On the contrary, the forecasts under MSE loss are unbiased and the mean of the forecast errors is around zero. Overall, the forecasts from the three models under both loss functions are very similar reflecting the fact that the differences among the statistical models are somehow small.

Date	Actual	Forecast AR(4)	Forecast error AR(4)	Forecast AR(6)	Forecast error AR(6)	Forecast AR(7)	Forecast error AR(7)
2008Q3	-5.041	-3.897	-1.144	-4.071	-0.970	-3.999	-1.042
2008Q4	-6.620	-5.252	-1.368	-5.263	-1.357	-4.671	-1.949
2009Q1	-3.595	-5.111	1.515	-4.960	1.365	-4.753	1.158
2009Q2	3.991	-3.031	7.022	-2.756	6.747	-3.043	7.034
2009Q3	3.516	-1.515	5.031	-1.190	4.706	-1.731	5.247
2009Q4	1.272	-1.817	3.089	-1.487	2.759	-2.219	3.491
2010Q1	0.926	-0.350	1.277	0.075	0.851	-0.567	1.494
2010Q2	0.262	2.442	-2.180	2.894	-2.632	1.919	-1.657
2010Q3	-1.794	1.762	-3.556	1.939	-3.733	0.726	-2.520
2010Q4	-1.913	0.201	-2.113	0.018	-1.931	-0.964	-0.949
2011Q1	-3.081	-0.077	-3.004	-0.244	-2.837	0.092	-3.173
2011Q2	0.687	-0.960	1.646	-1.053	1.740	-0.555	1.242
2011Q3	-2.400	-0.643	-1.758	-0.701	-1.699	-0.511	-1.889
2011Q4	-0.955	-1.744	0.789	-1.715	0.760	-1.460	0.505
2012Q1	1.327	-1.312	2.639	-1.233	2.560	-1.238	2.565
2012Q2	1.328	0.488	0.841	0.665	0.663	0.464	0.864

Table 5: Forecast summary for AR(4), AR(6) and AR(7) (MSE loss function and fixed scheme)

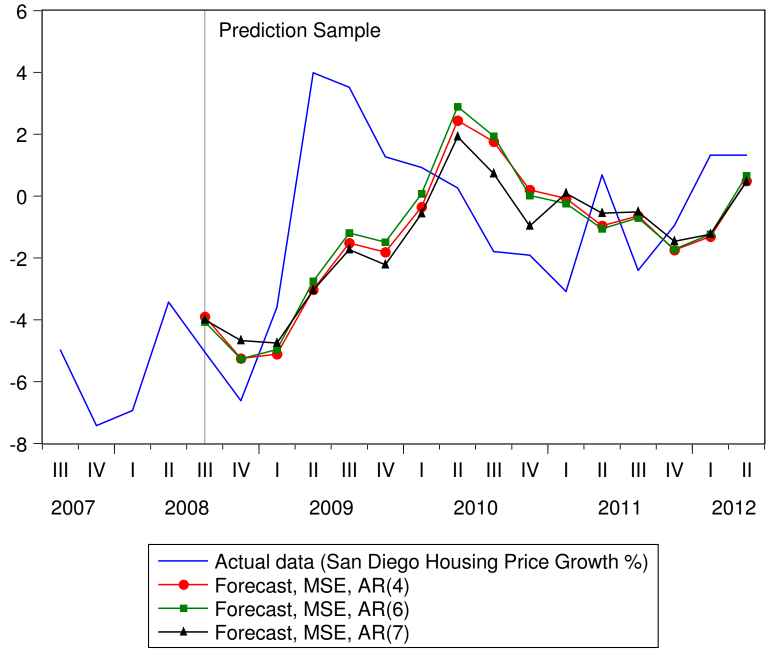

Figure 1: Forecasts from AR(4), AR(6), and AR(7) models under MSE loss function

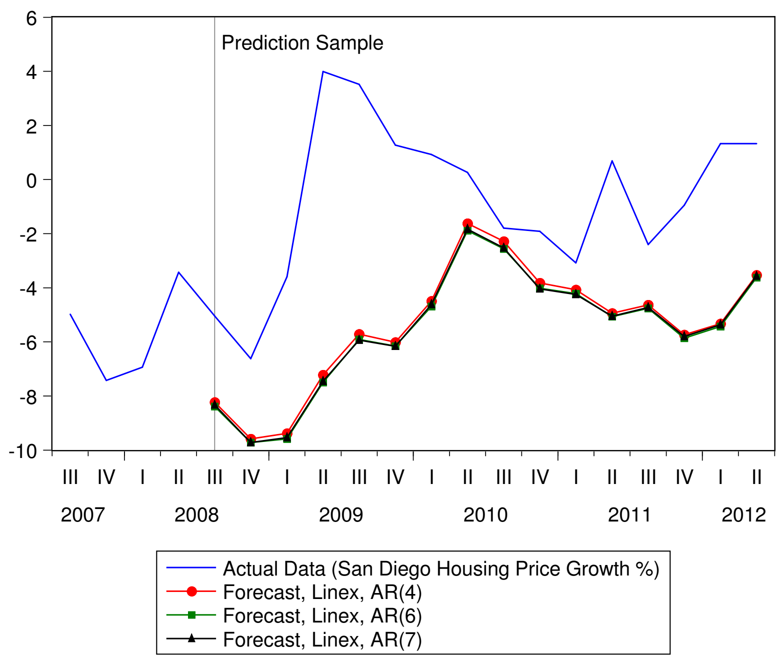

Figure 2: Forecasts from AR(4), AR(6), and AR(7) models under Linex loss function

Exercise 3
We work within the same forecasting environment as in Exercise 1 and perform the optimality forecast tests under symmetric and asymmetric loss functions. Tables 4 and 5 contain information on the optimal forecasts and forecast errors under MSE loss function and asymmetric Linex loss function. The results of the MPE test and the information efficiency test are displayed in Table 6. For the MPE test, we fail to reject the null hypothesis of a zero mean prediction error for all the three models under the MSE loss function at the 5% significance level. We also fail to reject the null hypothesis of a positive mean prediction error under Linex ($a = -1$) for all the three models. The test of informational efficiency is also passed by all models for both loss function as we fail to reject the null hypothesis of no correlation between the forecast error and the forecast at the 5% significant level. At this stage, we cannot discard any of the three models.[1]

	$H_0 : \alpha = 0$ $H_1 : \alpha \neq 0$ (RMSE)	$H_0 : \alpha \geq 0$ $H_1 : \alpha < 0$ (Linex, $a = -1$)
AR(4)	$t = 0.737$	$t = 6.220$
AR(6)	$t = 0.609$	$t = 6.444$
AR(7)	$t = 0.907$	$t = 6.413$

Table 6: MPE test

	$H_0 : \alpha_1 = 0$ $H_1 : \alpha_1 \neq 0$ (RMSE)	$H_0 : \alpha_1 = 0$ $H_1 : \alpha_1 \neq 0$ (Linex, $a = -1$)
AR(4)	$t_1 = -1.506$	$t_1 = -1.633$
AR(6)	$t_1 = -1.420$	$t_1 = -1.629$
AR(7)	$t_1 = -1.143$	$t_1 = -1.625$

Table 7: Information efficiency test

Exercise 4
Table 8 summarizes the sample average loss for each model for the prediction period from 2008Q3 to 2012Q2. The preferred model is in bold type. We observe that, for each loss function, the preferred model is different. Under RMSE, AR(6) is preferred; under MAE, AR(7) is preferred; and under MAPE and Linex, AR(4) is preferred. This disagreement across functions stresses the importance of choosing *a priori* the loss functions.

	RMSE	MAE	MAPE(%)	Linex ($a = -1$)
AR(4)	2.918	2.436	**146.253**	**3.752**
AR(6)	**2.815**	2.332	171.375	3.907
AR(7)	2.855	**2.299**	167.597	3.880

Table 8: Descriptive evaluation of the average loss

[1] Unlike the results of the informational efficiency test in Table 9.4 in the textbook, the t-statistics are not identical for symmetric and asymmetric loss functions. This is because the variance of the forecasts errors varies from one period to the next due to the inclusion of parameter uncertainty.

Exercise 5

From Table 8, we choose AR(6) as the preferred model because it delivers the lowest average loss under the MSE loss function (mean square error, 2.815). We also consider an alternative moving average smoothed forecast (SF) such as

$$f_{t,1} = (y_t + y_{t-1} + y_{t-2} + y_{t-3} + y_{t-4} + y_{t-5})/6.$$

which does not require estimation. In Table 9, we report the optimal forecasts and forecast errors from the AR(6) model and from the smoothed forecast. We compare both forecasts by implementing the test of unconditional predictive ability. The null hypothesis is that the two models have equal predictive ability so that they deliver the same expected loss:

$$H_0 : E(L(e_{t,1}^{(AR)})) = E(L(e_{t,1}^{(SF)})) \Rightarrow E(L(e_{t,1}^{(AR)}) - L(e_{t,1}^{(SF)})) = E(\Delta L_{t,1}) = 0,$$

To perform this test, first we calculate the sample losses (square of forecast errors under MSE function) for both models, which are reported in Table 9. Secondly, we regress the differences of the sample losses of the two models on a constant,

$$\Delta L_{t+j,1} = \beta_0 + \varepsilon_{t+j}, \quad (j = 0, 1, 2, \cdots, T - t - 1)$$

where ε_{t+j} is a zero-mean regression error. The regression output is in Table 10. The hypothesis of interest is:

$$H_0 : \beta_0 = 0$$

$$H_1 : \beta_0 < 0$$

where the alternative hypothesis considers that the average loss associated with the SF forecast is larger than that associated with the AR(6) model, so that the question is whether this difference is statistically different from zero. We implement a t-ratio to test the hypothesis, i.e.

$$t = \frac{\widehat{\beta}_0}{\widehat{\sigma}_{\widehat{\beta}}} \xrightarrow{A} N(0, 1).$$

Date	Actual	Forecast AR(6)	Forecast Error AR(6)	Quadratic loss AR(6)	Forecast Smoothed	Forecast Error Smoothed	Quadratic loss Smoothed	Quadratic loss differential (AR(6)-SM)
2008Q3	-5.041	-4.071	-0.970	0.940	-3.887	-1.154	1.331	-0.392
2008Q4	-6.620	-5.263	-1.357	1.841	-4.830	-1.790	3.203	-1.362
2009Q1	-3.595	-4.960	1.365	1.862	-5.738	2.143	4.592	-2.730
2009Q2	3.991	-2.756	6.747	45.518	-5.508	9.498	90.221	-44.703
2009Q3	3.516	-1.190	4.706	22.146	-3.605	7.121	50.711	-28.565
2009Q4	1.272	-1.487	2.759	7.614	-1.863	3.136	9.832	-2.218
2010Q1	0.926	0.075	0.851	0.724	-1.079	2.006	4.023	-3.299
2010Q2	0.262	2.894	-2.632	6.926	-0.085	0.347	0.120	6.806
2010Q3	-1.794	1.939	-3.733	13.935	1.062	-2.856	8.158	5.777
2010Q4	-1.913	0.018	-1.931	3.728	1.362	-3.275	10.725	-6.996
2011Q1	-3.081	-0.244	-2.837	8.051	0.378	-3.460	11.969	-3.918
2011Q2	0.687	-1.053	1.740	3.028	-0.721	1.408	1.983	1.045
2011Q3	-2.400	-0.701	-1.699	2.887	-0.819	-1.582	2.501	0.386
2011Q4	-0.955	-1.715	0.760	0.578	-1.373	0.419	0.175	0.402
2012Q1	1.327	-1.233	2.560	6.554	-1.576	2.903	8.428	-1.873
2012Q2	1.328	0.665	0.663	0.440	-1.056	2.384	5.685	-5.245
Average			0.437	7.923		1.078	13.354	-5.766
Square root				2.815			3.654	

Table 9: Optimal forecast from AR(6) and smoothed forecast under MSE loss function, fixed scheme

Dependent Variable: DIFFERENCE				
Method: Least Squares				
Sample: 2008Q3 2012Q2				
Included observations: 16				
Newey-West HAC Standard Errors & Covariance (lag truncation=2)				
Variable	Coefficient	Std. Error	t-Statistic	Prob.
C	-5.43035	3.93578	-1.37974	0.1879
R-squared	0.00000	Mean dependent var		-5.43035
Adjusted R-squared	0.00000	S.D. dependent var		13.02352
S.E. of regression	13.02352	Akaike info criterion		8.031852
Sum squared resid	2544.18	Schwarz criterion		8.080139
Log likelihood	-63.2548	Durbin-Watson stat		1.193998

Table 10: Unconditional predictability

From Table 10, the t-statistic for the intercept β_0 is -1.379 with a corresponding p-value of 0.083, so that we fail to reject the null of equal predictive ability and conclude that both models are, on average, equivalent. Note that we have used consistent standard errors in the regression output to guard against some serial correlation in the loss difference. Since the p-value is not that far from the 5% significance level, it is possible that using a more efficient estimation procedure, we would find statistical evidence in favor of the AR(6) model.

Exercise 6

We implement the three combination schemes with the forecasts derived from the AR(4), AR(6) and AR(7) models (under symmetric loss MSE, see Exercise 2). Table 11 reports the weights and MSEs of the three combined forecasts.

The equal-weighted forecast assigns 1/3 to each of the three models and delivers an average loss of 8.125. The MSE-inversely weighted forecast assigns very similar weights as the equal-weighted scheme with slightly higher weight (34.5%) to the AR(6) forecast, and generates an average loss of 8.117. In both cases, we do not have any diversification gains since their MSEs are slightly higher than that of the AR(6) forecast. The OLS weighted forecast tells a different story. To obtain the weights, we regress the realized values on the three individual forecasts as in,

$$y_{t+1} = \omega_0 + \omega_1 f_{t,1}^{(1)} + \omega_2 f_{t,1}^{(2)} + \omega_3 f_{t,1}^{(3)} + \varepsilon_{t+1}.$$

When we run the OLS regression, we observe that the MSE of the combined forecast is reduced to 4.566, which is much smaller than the lowest MSE (7.923) among the three individual forecasts. Therefore, the OLS-weighted forecast offers diversification gains.

	Equal Weight	MSE-inverse Weight	OLS Weight
AR(4) (MSE=8.515)	1/3	0.3205637	-8.135459
AR(6) (MSE=7.923)	1/3	0.344516	7.869928
AR(7) (MSE=8.150)	1/3	0.3349203	0.680511
MSE of Combined Forecast	8.125463	8.117675	**4.566127**

Table 11: Combination of forecasts – Weights and MSEs

Exercise 7

Based on the results in Exercises 1 to 6, we choose the AR(6) as the overall best model since it gives the smallest sample average loss. For this model, we compute the 1-step-ahead forecast and corresponding forecast errors under the three schemes (fixed, recursive and rolling). The forecast environment is described as follow,

- Forecast horizon: 1-step-ahead ($h = 1$).
- Initial information set: From 1975.Q1 to 2008.Q2.
- Fixed scheme: The estimation sample is fixed to the initial information set.
- Recursive scheme: The estimation sample is updated one observation at the time; for each update, the model is estimated again in order to calculate the new one-step-ahead forecast.
- Rolling scheme: The estimation sample is a rolling sample of 50 observations (i.e., observations corresponding to the most recent twelve and a half years). For each rolling sample, the model is estimated and the one-step-ahead forecast is calculated.

In Figure 3, we plot the forecasts, and in Table 12, we report the forecasts and their MSEs under the three schemes. The rolling scheme forecast has the smallest average loss. Observe that the rolling forecast is also the most volatile of the three because it is based on a smaller sample that that of the fixed and recursive schemes. These two forecasts (fixed and recursive) are very similar indicating that the estimates of the parameters of the model are quite stable.

		Fixed Scheme		Recursive Scheme		Rolling Scheme	
Date	Actual	Forecast	Error	Forecast	Error	Forecast	Error
2008Q3	-5.040843	-4.071338	-0.96951	-4.071338	-0.96951	-4.615846	-0.425
2008Q4	-6.619759	-5.263014	-1.35675	-5.345226	-1.27453	-7.200343	0.580584
2009Q1	-3.595453	-4.960078	1.364625	-5.127212	1.531758	-6.285526	2.690073
2009Q2	3.990836	-2.755878	6.746714	-2.774296	6.765132	-2.262995	6.25383
2009Q3	3.516095	-1.189895	4.70599	-0.650641	4.166735	3.730279	-0.21418
2009Q4	1.2723	-1.487068	2.759368	-0.684256	1.956556	0.840871	0.431429
2010Q1	0.926239	0.075421	0.850817	0.991697	-0.06546	2.868764	-1.94253
2010Q2	0.261861	2.893633	-2.63177	3.623253	-3.36139	4.652546	-4.39068
2010Q3	-1.794278	1.938683	-3.73296	2.032769	-3.82705	1.312726	-3.107
2010Q4	-1.91266	0.018234	-1.93089	-0.190009	-1.72265	-1.765315	-0.14735
2011Q1	-3.081349	-0.243903	-2.83745	-0.319583	-2.76177	-0.929182	-2.15217
2011Q2	0.686884	-1.053118	1.740002	-1.131055	1.817939	-2.738614	3.425498
2011Q3	-2.400453	-0.701288	-1.69917	-0.397184	-2.00327	1.022496	-3.42295
2011Q4	-0.954674	-1.714698	0.760024	-1.473201	0.518527	-2.844225	1.889551
2012Q1	1.326932	-1.23318	2.560112	-0.900841	2.227773	-0.694764	2.021696
2012Q2	1.328486	0.665163	0.663323	1.183056	0.14543	2.054169	-0.72568
MSE		7.923289		7.561428		**7.255625**	

Table 12: Optimal forecasts and forecast errors under the three schemes

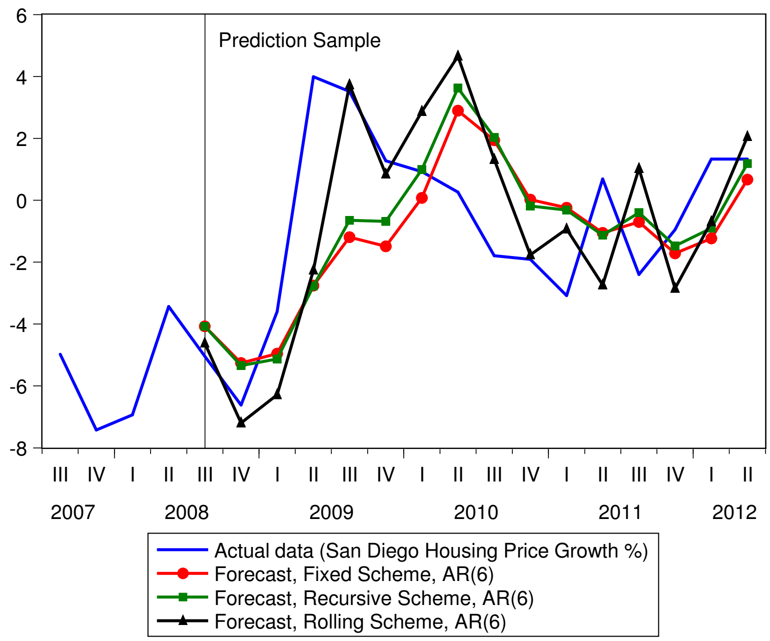

Figure 3: Forecasts from fixed, recursive and rolling schemes

Note: In Exercise 5 of Chapter 8, for U.S. GDP quarterly growth, we have already selected the best ARMA models as AR(3) and ARMA(2,2). Exercises 8 to 10 are based on that data set and these two ARMA models.

Exercise 8

Before implementing the forecast optimality tests (MPE and Informational Efficiency Tests), let us describe the forecasting environment:

- Forecast horizon: 1-step-ahead forecast ($h = 1$); thus we are interested in the forecast of U.S. GDP growth for the next quarter.
- Information set: From 1947.Q1 to 2008.Q2.
- Fixed sampling scheme with an estimation sample from 1947.Q1 to 2008.Q2 and prediction sample from 2008.Q3 to 2012.Q2.
- We consider two time series models: AR(3) and ARMA(2,2).
- We obtain optimal forecasts for both models under the symmetric MSE loss function.

In Table 13 and Table 14 we report the estimation results of these two models. Based on the estimates, we construct the optimal forecasts for y_{t+1} as $f_{t,1}$,

$$\begin{aligned}
\widehat{f}_{t,1} &= \widehat{\mu} + \widehat{\phi}_1 y_t + \widehat{\phi}_2 y_{t-1} + \widehat{\phi}_3 y_{t-2} \quad \text{(for AR(3))} \\
\tilde{f}_{t,1} &= \tilde{\mu} + \tilde{\phi}_1 y_t + \tilde{\phi}_2 y_{t-1} + \tilde{\theta}_1 \tilde{\varepsilon}_{t-1} + \tilde{\theta}_2 \tilde{\varepsilon}_{t-2} \quad \text{(for ARMA(2,2))},
\end{aligned}$$

Table 15 contains information on the optimal forecast and forecast error under MSE loss function. In Figure 4, we plot the optimal forecasts with actual data. Now, we proceed to assess the optimality of the forecasts by implementing the MPE test and the the informational efficiency test. The results are displayed in Table 16. With MPE test, we reject the null hypothesis of the mean prediction error being zero for both models at the 5% significance level. This is not surprising given the results of Table 15, where we observe the dominance of negative forecast errors. Thus, the mean prediction error is significantly negative. The forecasts from both models are rather optimistic; the models are not able to predict the sharp drop in GDP, a consequence of the 2008 financial meltdown. However, once we move from the crisis of 2008-2009, the predictions improve and they follow the actual data more closely. The test of informational efficiency is passed by both models. We fail to reject the null hypothesis of no correlation between the forecast error and the forecast at the customary 5% significance level.

Dependent Variable: GGDP				
Method: Least Squares				
Sample (adjusted): 1948Q1 2008Q2				
Included observations: 242 after adjustments				
Convergence achieved after 3 iterations				
Variable	Coefficient	Std. Error	t-Statistic	Prob.
C	0.833884	0.086909	9.594954	0.0000
AR(1)	0.318897	0.064123	4.973245	0.0000
AR(2)	0.12347	0.066981	1.843351	0.0665
AR(3)	-0.1274	0.064159	-1.98566	0.0482
R-squared	0.134012	Mean dependent var		0.836575
Adjusted R-squared	0.123096	S.D. dependent var		0.988958
S.E. of regression	0.926091	Akaike info criterion		2.700702
Sum squared resid	204.1194	Schwarz criterion		2.758371
Log likelihood	-322.785	Hannan-Quinn criter.		2.723933
F-statistic	12.27685	Durbin-Watson stat		2.027551
Prob(F-statistic)	0.000000			
Inverted AR Roots	.40-.32i	.40+.32i	-0.48	

Table 13: Estimation results of AR(3) model

Dependent Variable: GGDP				
Method: Least Squares				
Sample (adjusted): 1947Q4 2008Q2				
Included observations: 243 after adjustments				
Convergence achieved after 59 iterations				
MA Backcast: 1947Q2 1947Q3				
Variable	Coefficient	Std. Error	t-Statistic	Prob.
C	0.828082	0.068338	12.11737	0.0000
AR(1)	1.437971	0.061739	23.29129	0.0000
AR(2)	-0.86597	0.054737	-15.8205	0.0000
MA(1)	-1.25871	0.078817	-15.97	0.0000
MA(2)	0.758661	0.074766	10.14708	0.0000
R-squared	0.16137	Mean dependent var		0.839375
Adjusted R-squared	0.147276	S.D. dependent var		0.987877
S.E. of regression	0.912236	Akaike info criterion		2.674526
Sum squared resid	198.0577	Schwarz criterion		2.7464
Log likelihood	-319.955	Hannan-Quinn criter.		2.703476
F-statistic	11.44906	Durbin-Watson stat		1.807412
Prob(F-statistic)	0.000000			
Inverted AR Roots	.72-.59i	.72+.59i		
Inverted MA Roots	.63-.60i	.63+.60i		

Table 14: Estimation results of ARMA(2,2) model

Date	Actual	Forecast AR(3)	Forecast Error AR(3)	Forecast ARMA(2,2)	Forecast Error ARMA(2,2)
2008Q3	-0.92859	0.567471	-1.49606	0.604465	-1.533055
2008Q4	-2.30077	0.372376	-2.67315	0.420422	-2.721192
2009Q1	-1.33892	-0.31909	-1.01983	0.112237	-1.451157
2009Q2	-0.07867	-0.02152	-0.05715	0.183618	-0.262288
2009Q3	0.35981	0.673944	-0.31413	0.62997	-0.27016
2009Q4	0.99163	0.84684	0.14479	1.081007	-0.089377
2010Q1	0.57873	0.941912	-0.36318	1.37631	-0.79758
2010Q2	0.55609	0.832389	-0.2763	1.264009	-0.707919
2010Q3	0.64441	0.693696	-0.04929	0.93887	-0.29446
2010Q4	0.59297	0.771668	-0.1787	0.633074	-0.040104
2011Q1	0.01973	0.769053	-0.74932	0.476136	-0.456406
2011Q2	0.61363	0.568645	0.044985	0.413354	0.200276
2011Q3	0.31814	0.693814	-0.37567	0.621368	-0.303228
2011Q4	1.00775	0.745941	0.261809	0.814128	0.193622
2012Q1	0.48657	0.85371	-0.36714	1.054273	-0.567703
2012Q2	0.38204	0.810298	-0.42826	1.042878	-0.660838

Table 15: Optimal one-step-ahead forecasts and forecast errors (MSE loss function, fixed scheme)

Tests		AR(3)	ARMA(2,2)
Mean Prediction Error Test	t-ratios	-2.705993	-3.257680
	(p-values)	(0.0163)	(0.0053)
Informational Efficiency Test	t-ratios	1.290146	0.751543
	(p-values)	(0.2179)	(0.4648)

Table 16: Optimality of the forecast under MSE loss function

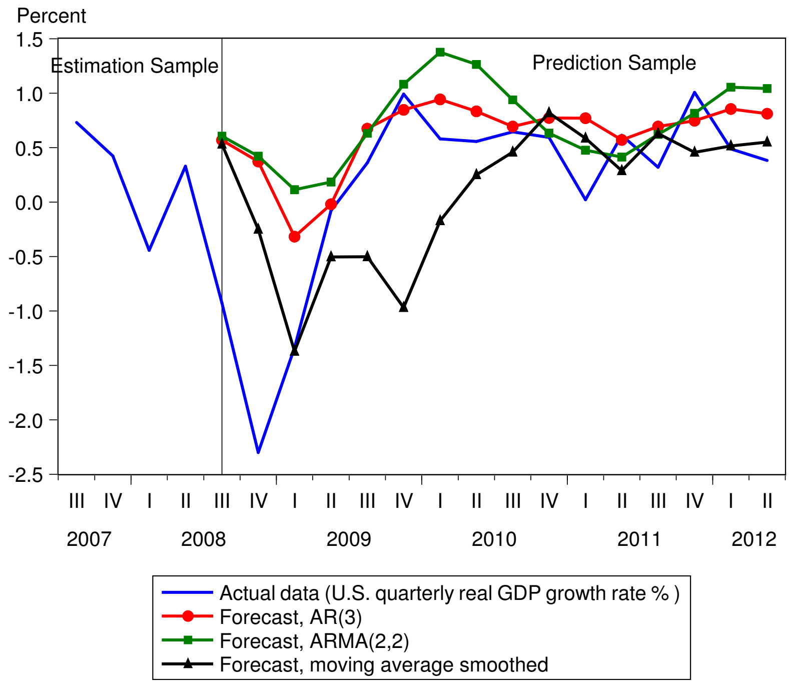

Figure 4: One-step-ahead forecasts from AR(3), ARMA(2,2) and smoothed average models

Exercise 9

Under the MSE loss function, the sample average loss of the AR(3) model is 0.861715, and of ARMA(2,2) model is 0.947801. Therefore, we choose AR(3) as our preferred model because it delivers a smaller loss. We consider an alternative model, a moving average smoothed forecast (SF), which does not require any estimation, such as

$$f_{t,1} = (y_t + y_{t-3})/2.$$

Table 17 contains information on the optimal one-step-ahead forecasts and forecast errors from the AR(3) and from the SF models (which is plotted as black line in Figure 4). Now, we proceed to compare the predictability of the two models by implementing the test of unconditional predictive ability. The null hypothesis is that the two models have equal predictive ability, i.e.,

$$H_0 : E(L(e_{t,1}^{(AR)})) = E(L(e_{t,1}^{(SF)})) \Rightarrow E(L(e_{t,1}^{(AR)}) - L(e_{t,1}^{(SF)})) = E(\Delta L_{t,1}) = 0,$$

which is written in terms of the unconditional expectation of the loss difference. To perform this test, we first calculate the sample losses (square of forecast errors under MSE loss function) for both models. Then, we regress the differences of the sample losses of the two models on a constant,

$$\Delta L_{t+j,1} = \beta_0 + \varepsilon_{t+j}, \quad (j = 0, 1, 2, \cdots, T - t - 1)$$

where $\Delta L_{t+j,1} = e_{t+j,1}^{2(AR)} - e_{t+j,1}^{2(SF)}$ We test the following one-sided hypothesis :

$$H_0 : \beta_0 = 0$$

$$H_1 : \beta_0 < 0$$

for which we implement the t-ratio statistic

$$t = \frac{\widehat{\beta}_0}{\widehat{\sigma}_{\widehat{\beta}}} \xrightarrow{A} N(0,1).$$

We choose the negative side under the alternative hypothesis because the loss associated with the SF forecast is larger than the loss associated with the AR(3) forecast. Our goal is to assess whether these differences are significantly different from zero.

After running the OLS regression, we find that the t-statistic for the intercept β_0 is -0.158421 with a corresponding p-value of 0.437. Thus, we fail to reject the null of equal predictive ability. Hence, the AR(3) forecast is, on average, statistically equivalent to the moving average smoothed forecast.

Date	Actual	Forecast AR(3)	Forecast Error AR(3)	Forecast Smoothed	Forecast Error Smoothed
2008Q3	-0.92859	0.567471	-1.49606	0.52982	-1.45841
2008Q4	-2.30077	0.372376	-2.67315	-0.252535	-2.048235
2009Q1	-1.33892	-0.319091	-1.01983	-1.372505	0.033585
2009Q2	-0.07867	-0.021519	-0.05715	-0.504765	0.426095
2009Q3	0.35981	0.673944	-0.31413	-0.50363	0.86344
2009Q4	0.99163	0.84684	0.14479	-0.97048	1.96211
2010Q1	0.57873	0.941912	-0.36318	-0.173645	0.752375
2010Q2	0.55609	0.832389	-0.2763	0.25003	0.30606
2010Q3	0.64441	0.693696	-0.04929	0.45795	0.18646
2010Q4	0.59297	0.771668	-0.1787	0.81802	-0.22505
2011Q1	0.01973	0.769053	-0.74932	0.58585	-0.56612
2011Q2	0.61363	0.568645	0.044985	0.28791	0.32572
2011Q3	0.31814	0.693814	-0.37567	0.62902	-0.31088
2011Q4	1.00775	0.745941	0.261809	0.455555	0.552195
2012Q1	0.48657	0.85371	-0.36714	0.51374	-0.02717
2012Q2	0.38204	0.810298	-0.42826	0.5501	-0.16806

Table 17: Optimal one-step-ahead forecasts and forecast errors (MSE loss function, fixed scheme)

Exercise 10

We implement the four combination schemes with the forecasts derived from the ARMA(2,2) and AR(3) models (under symmetric loss MSE, Table 15). In Table 18, we report the weights and MSEs of the four combined forecasts.

The equal-weighted forecast assigns 0.5 weight to each of the two models and delivers an average loss of 0.805486. The MSE-inversely weighted forecast assigns similar weights as the equal-weighted scheme with slightly higher weight (54.7%) to the AR(3) forecast, and generates an average loss of 0.798227. The optimal weighted forecast, using the optimal weight formula, assigns even larger weights (79.3%) to the AR(3) forecast, and produces an average loss of 0.764923. In the three cases above, we do not have any diversification gains since their MSEs are slightly higher than that of the AR(3) forecast. Note that the optimal weighted forecast should offer diversification gains in combining *unbiased forecasts*. However, this is not the case in this exercise because the ARMA(2,2) and AR(3) forecasts have a statistically significant negative bias (see MPE tests in Table 16). The OLS weighted forecast shows a different story. To obtain the weights, we regress the realized values on the two individual forecasts, i.e.,

$$y_{t+1} = \omega_0 + \omega_1 f_{t,1}^{(1)} + \omega_2 f_{t,1}^{(2)} + \varepsilon_{t+1}$$

with the intercept ω_0 absorbing the bias of the forecasts. After running the OLS regression, we observe that the $\widehat{\omega}_0$ is -0.999332 and statistically significant different from zero, and that the MSE of the combined forecast is reduced to 0.438516, which offers substantial diversification gains. The OLS weights assigned to the AR(3) forecasts about three times more weight than to the ARMA(2,2) forecasts.

	Equal Weight	MSE-inverse Weight	Optimal Weight	OLS Weight
ARMA(2,2) (MSE=0.898326)	0.5	0.453	0.207	0.407
AR(3) (MSE=0.742554)	0.5	0.547	0.793	1.341
MSE of Combined Forecast	0.805486	0.798227	0.764923	**0.438516**

Table 18: Combination of forecasts: weights and MSEs

The student is encouraged to perform a similar exercise including the smoothed average forecast.

