CHAPTER 15.

FINANCIAL APPLICATIONS OF TIME-VARYING VOLATILITY

SOLUTIONS

by

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Exercise 1

In Exercise 3 of Chapter 14, we worked with the updated daily time series of SP500 returns from January 3, 2000 to May 6, 2013 for a total of 3355 observations. We found that the best specification was a MA(1)-GARCH(2,1) model. Based on this model, we calculate the 1% VaR under the assumptions of conditional normality and Student-t.

Under normality, the conditional 1% VaR is calculated as

$$r_{t|t-1}^{VaR(1\%)} = \mu_{t|t-1} - 2.33\sigma_{t|t-1}$$

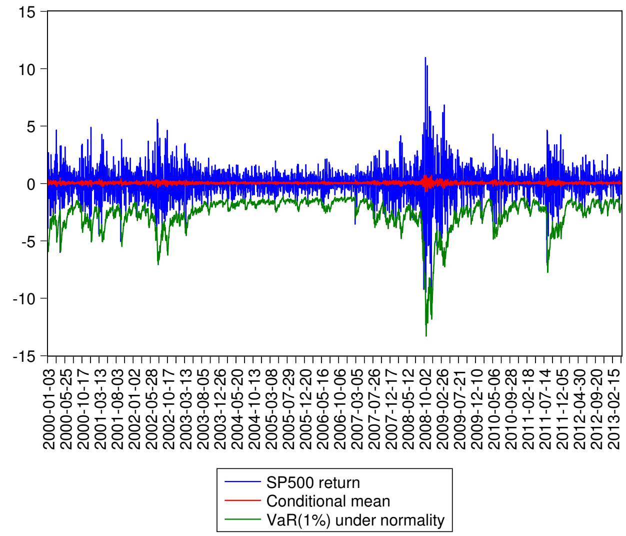
and the corresponding conditional expected shortfall as

$$r_{t|t-1}^{ES(1\%)} = \mu_{t|t-1} - 2.6426\sigma_{t|t-1}.$$

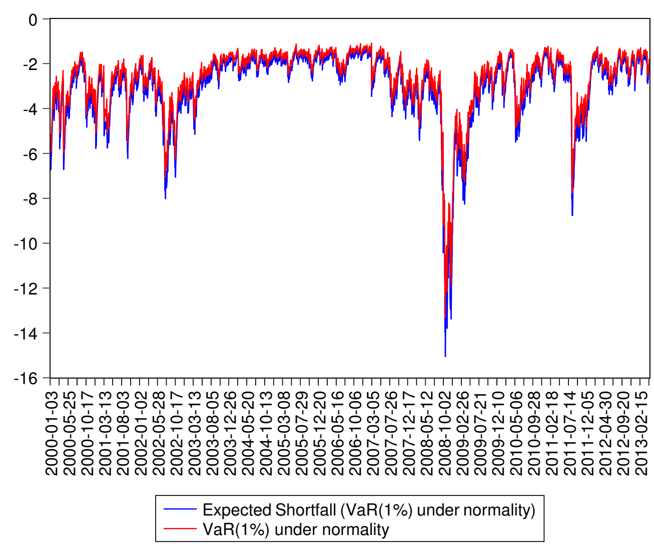
Under Student-t (8 d.f.), the conditional 1% VaR is calculated as

$$r_{t|t-1}^{VaR(1\%)} = \mu_{t|t-1} - 2.896\sqrt{(\nu-2)/\nu}\,\sigma_{t|t-1}$$

In Figure 1, we plot the conditional 1% VaR and the corresponding expected shortfall under the assumption of conditional normality, and in Figure 2, the 1% Var under the assumption of conditional Student-t with 8 degrees of freedom. We choose 8 because the kurtosis of the standardized residuals is small, about 4.2, and estimation of the model under Student-t delivers about this estimate for the degrees of freedom. As expected, the 1% VaR under normality is only slightly smaller (in magnitude) than that under Student-t (8 d.f.)



(a) 1% VaR under normality



(b) Expected Shortfall (1% VaR under normality)

Figure 1: Daily SP500 Returns. 1% VaR and Corresponding Expected Shortfall

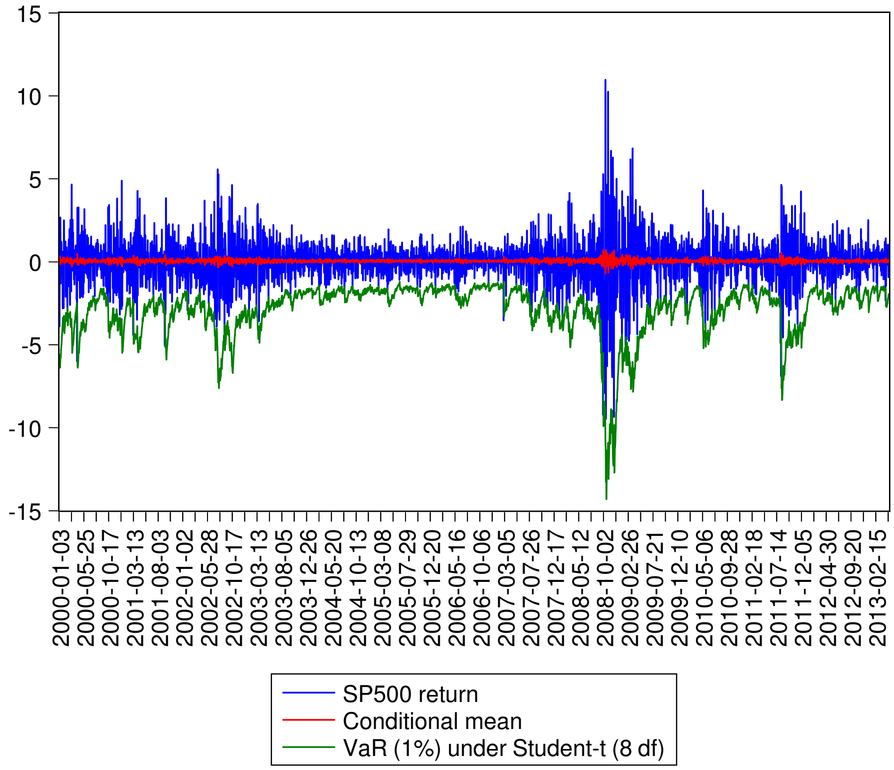


Figure 2: Daily SP500 Returns. 1% VaR under Student-t with 8 degrees of freedom

Exercise 2

We work with Amgen (AMGN) stock returns analyzed in Exercise 9 of Chapter 14. The best specification was a GARCH(1,1) model for the daily returns from January 2, 2003 to April 29, 2013. In this model, there are not dynamics in the conditional mean and the constant is statistically zero, thus $\mu_{t|t-1} = 0$. Then, under normality, the conditional 1% and 5% VaR are calculated as

$$r_{t|t-1}^{VaR(1\%)} = -2.33\sigma_{t|t-1}$$

$$r_{t|t-1}^{VaR(5\%)} = -1.645\sigma_{t|t-1}$$

We plot these conditional VaR series in Figure 3.

Under Student-t (5 df), the conditional 1% and 5% VaR are calculated as

$$r_{t|t-1}^{VaR(1\%)} = -3.365\sqrt{(\nu-2)/\nu}\sigma_{t|t-1}$$

$$r_{t|t-1}^{VaR(5\%)} = -2.015\sqrt{(\nu-2)/\nu}\sigma_{t|t-1}$$

Under Student-t (6 df), the conditional 1% and 5% VaR are calculated as

$$r_{t|t-1}^{VaR(1\%)} = -3.143\sqrt{(\nu-2)/\nu}\sigma_{t|t-1}$$

$$r_{t|t-1}^{VaR(5\%)} = -1.943\sqrt{(\nu-2)/\nu}\sigma_{t|t-1}$$

Under Student-t (7 df), the conditional 1% and 5% VaR are calculated as

$$r_{t|t-1}^{VaR(1\%)} = -2.998\sqrt{(\nu-2)/\nu}\sigma_{t|t-1}$$

$$r_{t|t-1}^{VaR(5\%)} = -1.895\sqrt{(\nu-2)/\nu}\sigma_{t|t-1}$$

We plot these conditional VaR series in Figure 4.

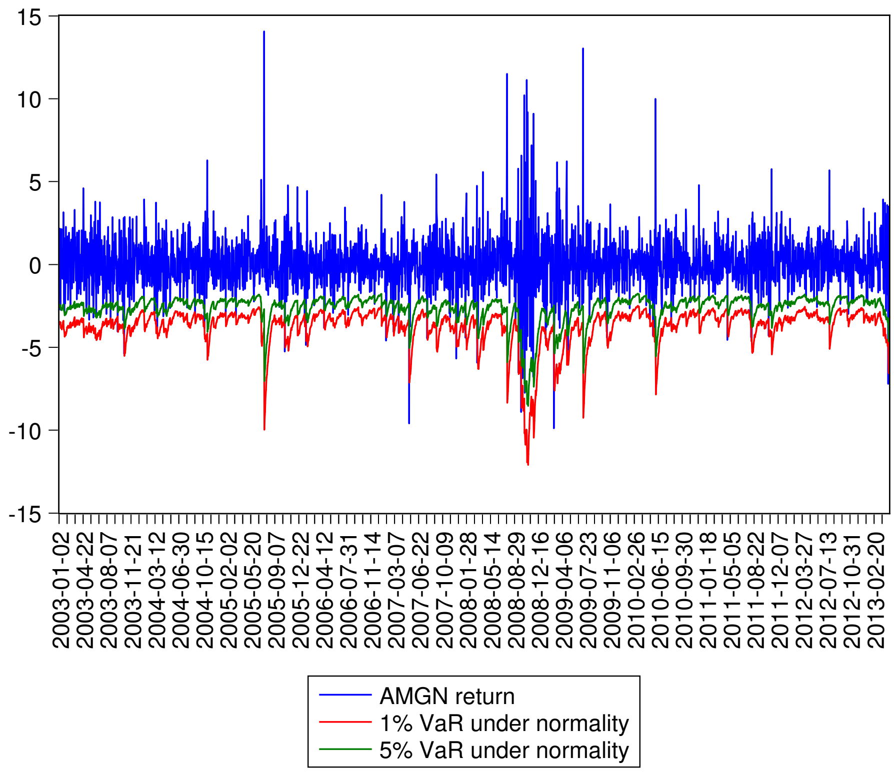
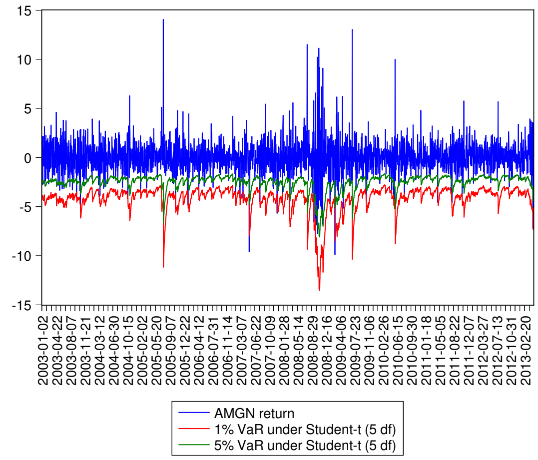
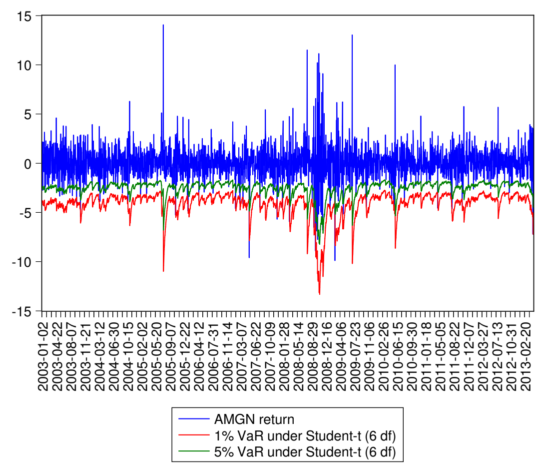


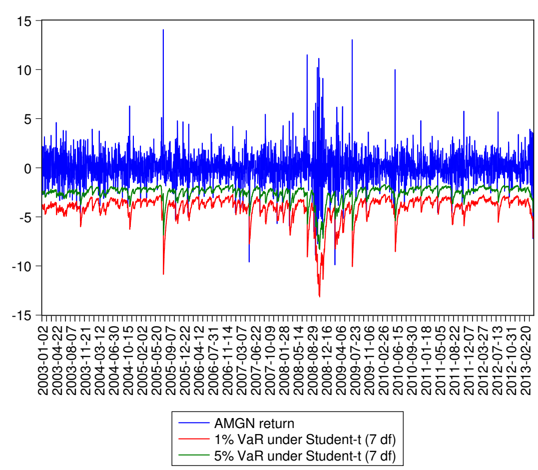
Figure 3: AMGN Returns. 1% and 5 % VaR under Normality



(a) 1% and 5% VaR under Student-t (5 df)



(b) 1% and 5% VaR under Student-t (6 df)



(c) 1% and 5% VaR under Student-t (7 df)

Figure 4: AMGN Returns. VaR under Student-t

Exercise 3

For the AMGN returns, the conditional expected shortfalls associated with the 1% and 5% VaR under normality are calculated as follows

$$r_{t|t-1}^{ES(1\%)} = -2.6426\sigma_{t|t-1}$$

$$r_{t|t-1}^{ES(5\%)} = -2.0622\sigma_{t|t-1}$$

We plot both expected shortfall series in Figure 5.

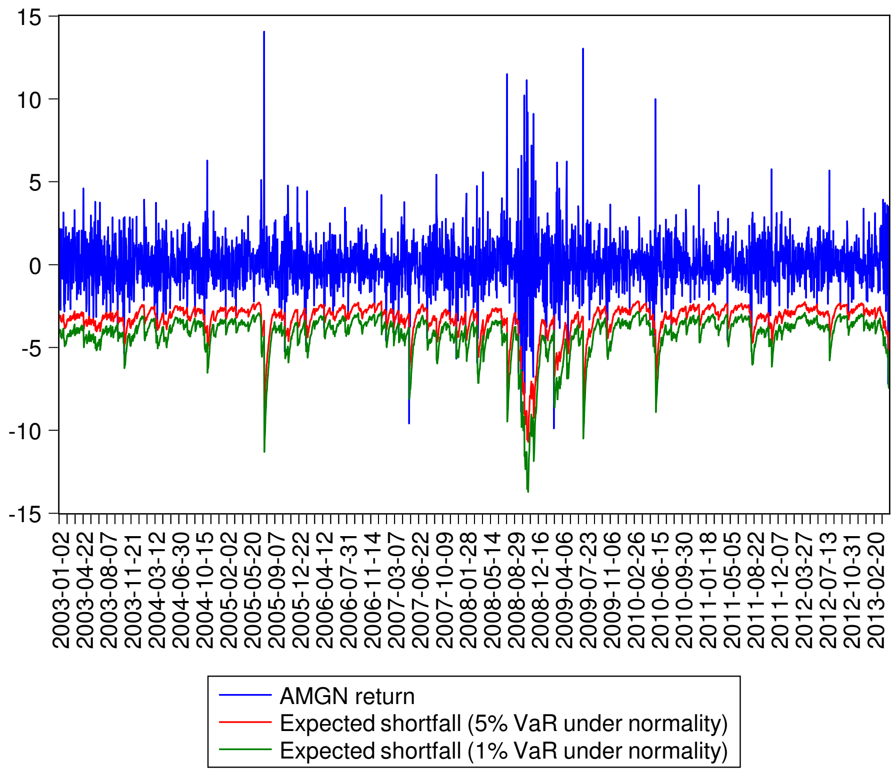


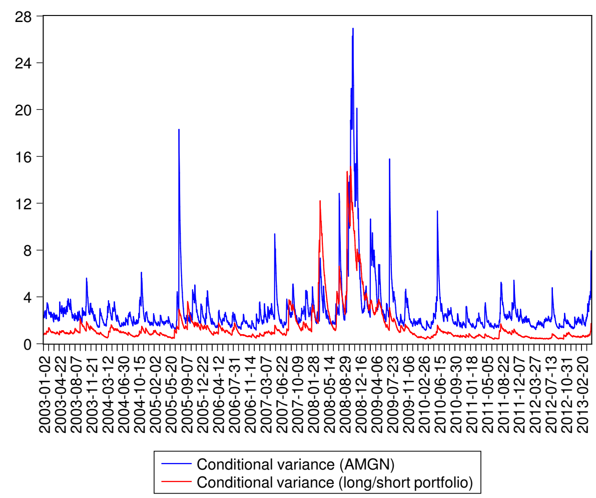
Figure 5: AMGN Returns. Expected Shortfall (1% and 5% VaR under normality)

Exercise 4

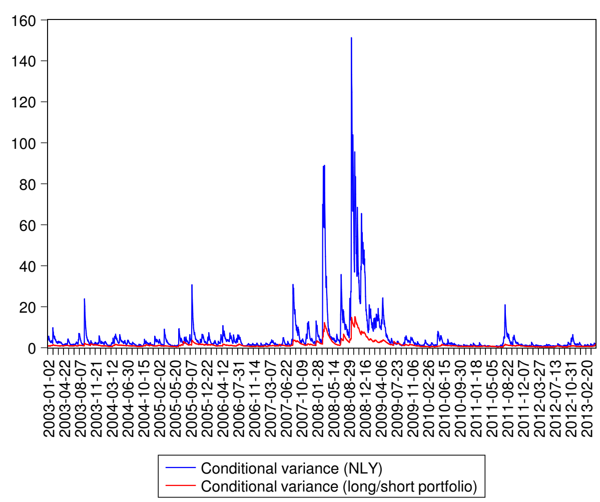
We form an equal weighted portfolio by buying the AMGN stock (pharmaceutical industry) and shorting the NLY (Annaly Capital Management) stock (mortgage industry) from January 2, 2003 to April 29, 2013. By including long/short positions, the portfolio is hedged and it should be less risky (less volatile) than any of the individual stocks. Though these two stocks belong to two very different industries, they show a small contemporaneous correlation (0.2) in their daily returns. This is due to their common (non-diversifiable) exposure to macro/market forces. AMGN returns have been already analyzed in Exercise 9 of Chapter 14. NLY returns are also heteroscedastic and the best specification is a GARCH(1,1) model. The portfolio returns are calculated as

$$r_t^P = 0.5 \times ret_t^{AMGN} - 0.5 \times ret_t^{NLY}$$

We find that the best model for the portfolio returns is a GARCH(1,1) with no dynamics in the conditional mean and a constant that is statistically zero. From this model, we compute the conditional variances of the portfolio, which are substantially lower than those of any of the two stocks. This is the result of diversification and hedging. See Figure 6 for the comparison of the conditional variances.



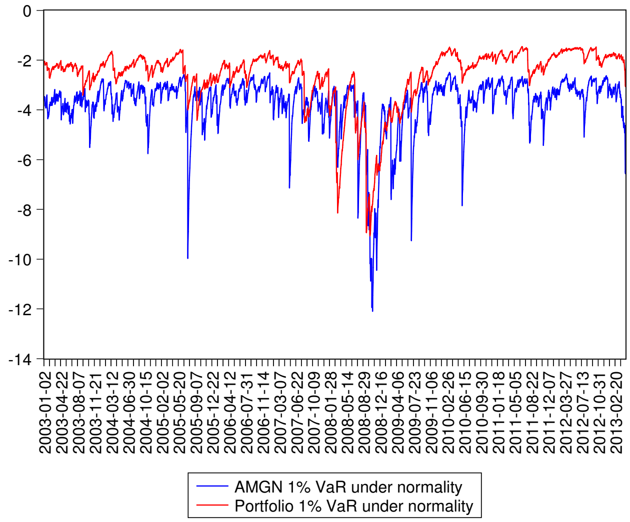
(a) Conditional variances (AMGN and Portfolio)



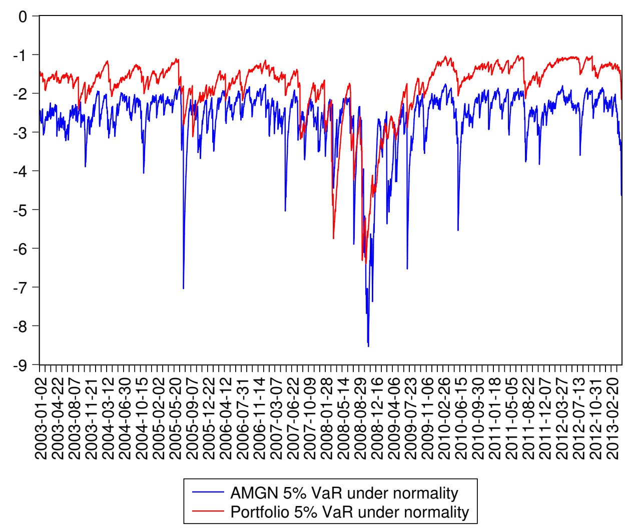
(b) Conditional variances (NLY and Portfolio)

Figure 6: Comparison of Conditional Variances

Consequently, the 1% and 5% conditional VaR values are smaller (in magnitude) than the corresponding VaRs of any of the two stocks. In Figure 7 we plot the 1% and 5% VaRs of the portfolio and compare with those of AMGN.



(a) 1% VaR under normality (AMGN and Portfolio)



(b) 5% VaR under normality (AMGN and Portfolio)

Figure 7: Comparison of AMGN and Portfolio VaRs

Exercise 5

For the portfolio returns in Exercise 4, the conditional expected shortfalls associated with the 1% and 5% VaR under normality are calculated as follows

$$r^{P(1\%)}_{t|t-1} = -2.6426\sigma_{t|t-1}$$

$$r_{t|t-1}^{P(5\%)} = -2.0622\sigma_{t|t-1}$$

which will be smaller (in magnitude) than those for any individual stock in the portfolio. We plot both expected shortfall series in Figure 8.

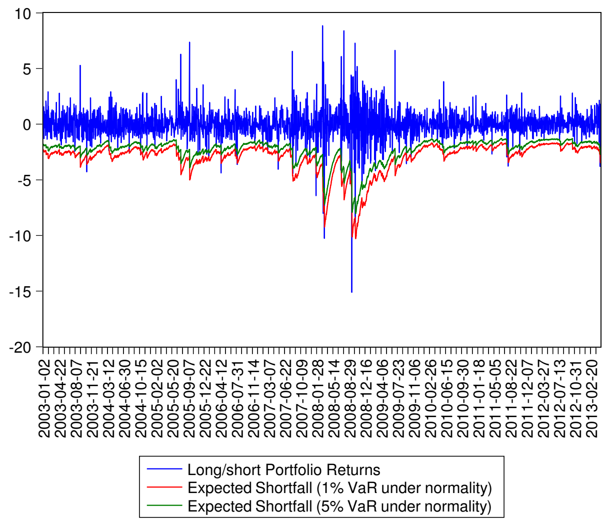


Figure 8: Expected Shortfall for Long/Short Portfolio

Exercise 6

We update the time series of daily returns for AAPL and FCX with the new sample running from January 2, 2008 to May 31, 2013 for a total of 1363 observations. Over this period, the average daily return is 0.063% for AAPL and -0.027% for FCX, which means that the portfolio will be long in AAPL and short in FCX. For each stock, the best specification is a GARCH(1,1) and there is no dynamics in the conditional mean. Thus, the conditional mean is equal to the unconditional mean. For each stock, we compute the conditional variances and apply the formulas of optimal weights given in the textbook with the objective of obtaining an average daily portfolio return of 0.15%. Observe that this objective is very ambitious and, given the unconditional average performance of each stock, we will need to load on AAPL substantially. The time series of the weights are plotted in Figure 9, and their descriptive statistics are shown in Table 1. The average weight is 2.19 for AAPL and -0.45 for FCX, which means that for 2.19 dollars that we buy of AAPL, we will need to sell 0.45 dollars of FCX to obtain an average daily return of 0.15%.

To compute the conditional betas of both stocks, we need to find their correlation with the market returns, which is represented by the SP500 returns and follows a MA(1)-GARCH(2,1) specification. The contemporaneous correlation between the market and AAPL is 0.65 and between the market and FCX is 0.72. Thus, the conditional beta is computed as

$$\beta_{i,t} = \rho_{i,m} \frac{\sigma_{i,t|t-1}}{\sigma_{m,t|t-1}}$$

We plot the conditional betas in Figure 10. In this period of time, FCX is riskier than AAPL. The average beta of AAPL is 1.17 but that of FCX is 1.94.

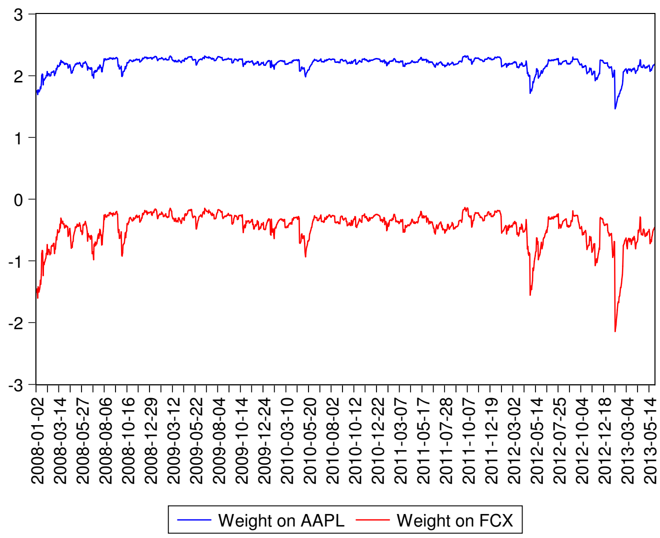


Figure 9: Optimal weights on AAPL and FCX for $\mu_p = 0.15\%$

Sample: 1 1363		
	W (AAPL)	W (FCX)
Mean	2.18898	-0.447936
Median	2.219808	-0.376004
Maximum	2.322062	-0.13741
Minimum	1.462476	-2.143112
Std. Dev.	0.110256	0.257265
Skewness	-2.543455	-2.543455
Kurtosis	12.12232	12.12232

Table 1: Descriptive Statistics of Optimal Weights

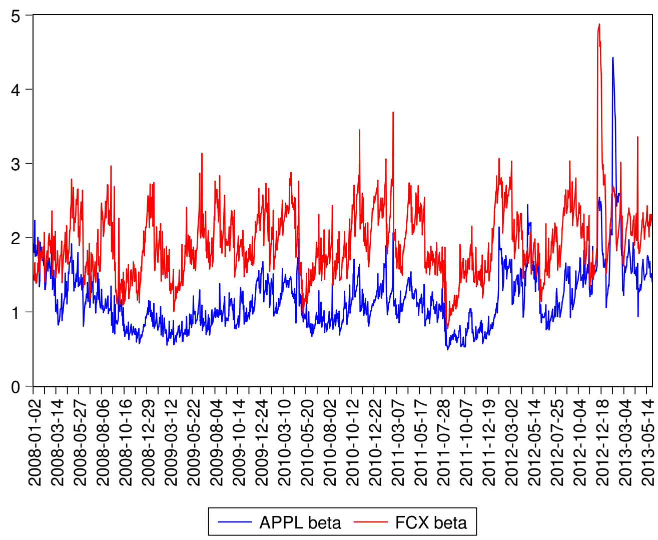


Figure 10: Conditional beta of AAPL and FCX

Exercise 7

We keep on working with AAPL (technology sector) and NLY (financial sector) stocks and we add Exxon Mobil Corporation XOM (oil sector) to form a diversified portfolio. We work with daily returns from January 2, 2008 to May 31, 2013 for a total of 1363 observations. We would like to allocate capital optimally so that we minimize risk and we obtain an average daily return $\mu_p = 0.15\%$. The optimization problem is to choose weights w_1, w_2, and w_3 such that

$$\min \sigma_P^2 = w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + w_3^2\sigma_3^2$$
$$s.t. \;\; \mu_p = w_1\mu_1 + w_2\mu_2 + w_3\mu_3$$

Observe that we are assuming that the stocks in the portfolio are uncorrelated, otherwise the variance of the portfolio needs to be augmented with covariance terms. To find the solution to this problem, we follow the optimization procedure explained in the appendix of Chapter 15. The optimal weights are

$$w_i = \mu_p \frac{\mu_i/\sigma_i^2}{\mu_1^2/\sigma_1^2 + \mu_2^2/\sigma_2^2 + \mu_3^2/\sigma_3^2}$$

We proceed to find the best time series model for each of the stock returns. For AAPL and XOM, we find a GARCH(1,1) with no dynamics in the conditional mean, and for NLY a MA(1)-GARCH(1,1). Based on these models, we compute the conditional means and conditional variances and calculate the time-varying version of the optimal weights[1]. We plot the weights in Figure 11. Observe that the weights on AAPL and XOM are all positive over time, so the portfolio is long in these two

[1] There is a small correlation among the three stocks that we are not considering in the computation of the weights.

stocks; on the contrary, the weights on NLY are positive in some days and negative in others (this is the effect of the MA(1) term in the conditional mean), which means that NLY can be long or short in the portfolio. In Table 2, we provide the descriptive statistics of the optimal weights over the sample period. On average, AAPL has the largest weight in the portfolio and XOM the least. NLY weights are the most volatile and XOM weights the least.

Sample: 1 1363			
	W3_AAPL	W3_NLY	W3_XOM
Mean	0.525266	0.21638	0.153923
Median	0.183332	0.370683	0.046592
Maximum	2.357967	3.470142	2.090844
Minimum	0.002034	-4.97781	0.000175
Std. Dev.	0.672478	1.127633	0.25212
Skewness	1.435306	-0.41117	3.228305
Kurtosis	3.766908	3.417555	17.08342

Table 2: Descriptive Statistics of Optimal Weights

Exercise 8

We compute the conditional betas of AAPL, NLY, and XOM. We need the contemporaneous correlation between the market and each of the three stocks. As in Exercise 6, the market is represented by the SP500 returns. For AAPL, the correlation is 0.65, for NLY 0.59, and for XOM 0.82. The conditional beta is computed as

$$\beta_{i,t} = \rho_{i,m} \frac{\sigma_{i,t|t-1}}{\sigma_{m,t|t-1}}$$

We plot the conditional betas in Figure 12. In this period of time, AAPL is riskier than NLY and XOM. The average beta of AAPL is 1.17, of NLY 0.84, and of XOM 0.96. The most volatile betas are those of AAPL and NLY with standard deviations of 0.44 and 0.42 respectively. The standard deviation of the betas of XOM is only 0.18.

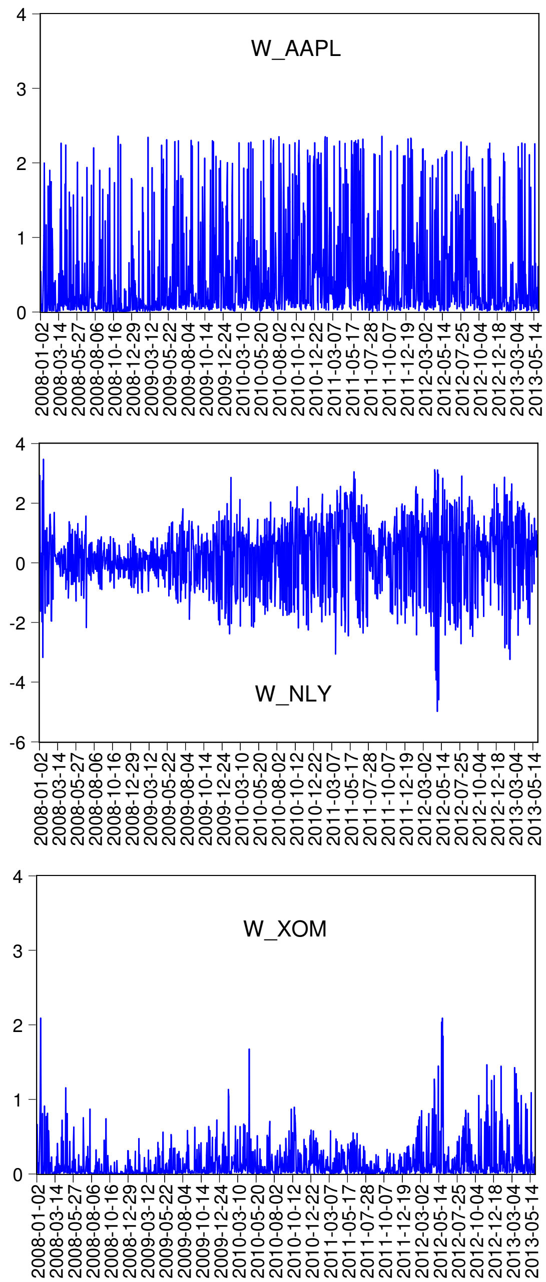


Figure 11: Optimal weights on AAPL, NLY, and XOM for $\mu_p = 0.15\%$

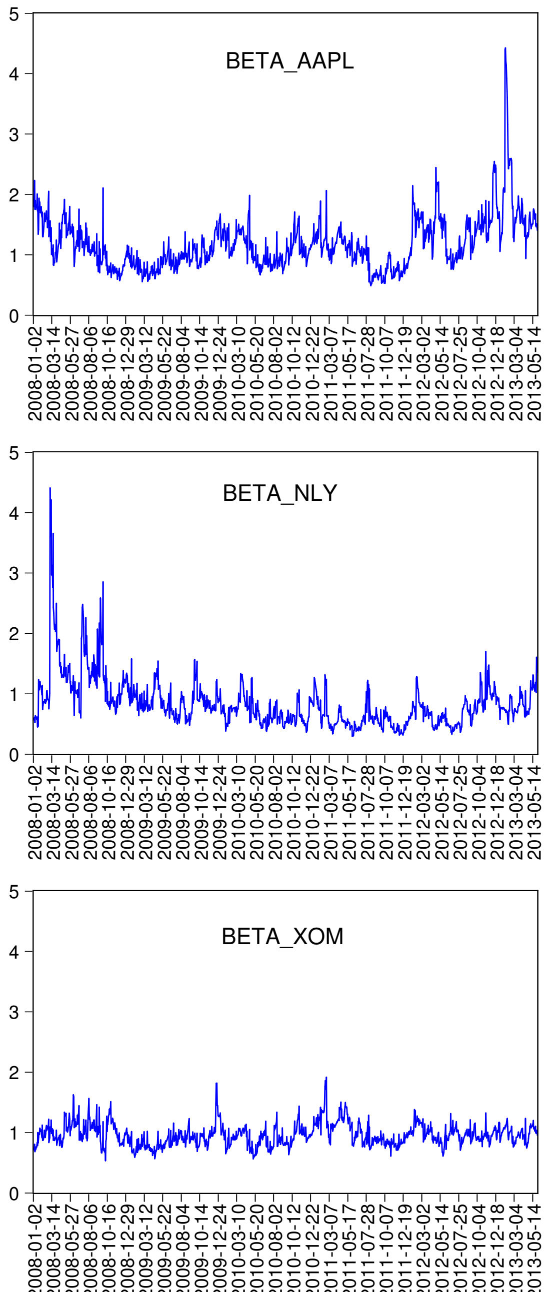


Figure 12: Conditional betas of AAPL, NLY, and XOM

Exercises 9 and 10

On June 7 (Friday), 2013, the SP500 index is $S_t = 1643.38$. On this day, we would like to know the price of call options (SPX) written on the index that will be expiring in one week from today, on June 14 (Friday), 2013.

We download weekly prices (Friday to Friday) of the SP500 index from the week of January 3, 2005 to the week of June 3, 2013 for a total of 440 observations. We compute the weekly returns and search for the best dynamic specification of returns. A GARCH(1,1) with just a constant in the conditional mean suffices to model the dynamics. For this period, the unconditional variance of the returns is $\sigma^2 = 2.7021^2$, and the one-week-ahead forecast of the conditional variance is $\sigma^2_{t+1|t} = 2.6217$, which is obtained from the fitted GARCH(1,1) model. Observe that there is a large difference between the conditional and unconditional variances, so that the differences in the option prices will also be large. We choose three call options: at-the-money $K = 1640$, in-the-money $K = 1630$, and out-of-the-money $K = 1650$; all will expire in one week, thus $T = 1$. We also need to choose the risk-free rate, which is practically zero these days, $r_f = 0.02\%$. With these choices, we implement the Black-Scholes (BS) formula to price the call options. The results are presented in Table 3. In general, the three options are overpriced by the BS formula. The call prices based on the unconditional variances are excessively large, in particular for the at-the-money and out-of-the-money options. Note that the period from 2005 to 2013 contains the high volatility episode of the 2008 financial crisis, which weighs heavily on the value of the unconditional variance. On the contrary, the conditional volatility gives more weight to the recent history than to the far away history and 2012-2013 has been a much calmer period than 2008, thus the lower value of the conditional variance. The call prices based on the conditional variance are very much on target with the actual ask-price.

K	$\sigma^2 = 2.7021^2$	$\sigma^2_{t+1\|t} = 2.6217$	Actual prices on June 7, 2013
1630 in-the-money	$C = 25.14$	$C = 18.59$	$C_{bid} = 15.30, C_{ask} = 20.30$ $C_{ave} = 17.80$
1640 at-the-money	$C = 19.44$	$C = 12.38$	$C_{bid} = 10.00, C_{ask} = 11.50$ $C_{ave} = 10.75$
1650 out-of-the-money	$C = 14.64$	$C = 7.66$	$C_{bid} = 4.20, C_{ask} = 7.90$ $C_{ave} = 6.05$

Table 3: Black-Scholes Call Option Pricing