ECON 147 Final EXAM

YOUR NAME:

Winter 20??
3:00pm - 6:00pm

Instruction

- This is a closed book and closed note exam.
- Try to answer all questions and write all answers **within this exam sheets**. You will hand in the exam sheets. Please write legibly.
- Total points are 120, the bonus questions will give you extra points but the total point will still be no more than 120. Your score will be the minimum of your actual score and 120.
- *Examination Rules from Department Policy will be strictly followed.*
- The following results may be useful:

$$
\begin{array}{ll}
\Pr(Z \leq 0.5) = 0.69, & \Pr(Z \leq -1.1) = 0.136, \\
\Pr(Z > -1.96) = 0.975, & \Pr(Z \leq 1.6449) = 0.95, \\
\Pr(Z \leq -1.64) = 0.05, & \Pr(Z \leq -0.70) = 0.242, \\
\Pr(Z \leq -0.35) = 0.363, & e^{-2.31} = 0.100, \\
e^{-0.148} = 0.862, & e^{-1.92} = 0.150, \\
e^{-3.21} = 0.041, & e^{-2.87} = 0.055,
\end{array}
$$

  where Z is a standard normal random variable.

1. **(18 pts)** Let $\{Y_t\}$ be the first order autoregressive (AR(1)) process

$$\begin{aligned} Y_t - \mu &= \phi(Y_{t-1} - \mu) + \varepsilon_t, \quad |\phi| < 1, \\ \varepsilon_t &\sim iid\ \left(0, \sigma_\varepsilon^2\right), \end{aligned}$$

and define $X_t = Y_t - \mu$.

1.1.a. [**4 pts**] Show that for any integers t and $j > 0$, we can write:

$$\begin{aligned} X_{t+j} &= \phi^{j+1} X_{t-1} + \phi^0 \varepsilon_{t+j} + \phi^1 \varepsilon_{t+j-1} + \cdots + \phi^j \varepsilon_t \\ &= \phi^{j+1} X_{t-1} + \sum_{s=0}^{j} \phi^s \varepsilon_{t+j-s}. \end{aligned}$$

From the expression above, find $\frac{\partial X_{t+j}}{\partial \varepsilon_t}$, i.e., the derivative of X_{t+j} with respect to ε_t.

1.1.b [**2 pts**] Let X_t be the time series of US GDP, and ε_{t-j} be some unexpected news (say, a sudden drop of the federal fund rate) from time $t - j$. What will be the economic interpretation of the $\frac{\partial X_{t+j}}{\partial \varepsilon_t}$ above? What happens to this $\frac{\partial X_{t+j}}{\partial \varepsilon_t}$ if j gets larger and larger ($j \to \infty$)?

1.2. **[9 pts]** Using the covariance stationarity of X_t, compute (a) $Var\left(X_t\right)$, (b) $Cov\left(X_t, X_{t-1}\right)$ and (c) $Corr\left(X_t, X_{t-1}\right)$. [**Hint:** using the equation $X_t = \phi X_{t-1} + \varepsilon_t$, we have

$$
\begin{aligned}
E[X_t] &= E[\phi X_{t-1} + \varepsilon_t], \\
Var\left(X_t\right) &= Var\left(\phi X_{t-1} + \varepsilon_t\right), \\
E[X_t X_{t-1}] &= E[(\phi X_{t-1} + \varepsilon_t) X_{t-1}]
\end{aligned}
$$

for any t.]

1.3. **[3 pts]** Realizations from three different AR(1) processes (with $\phi = 0$, 0.5 and 0.99 respectively) and their (estimated) autocorrelation functions (ACF) are given in Figure 1 below. Which process do you think is generated from $\phi = 0$? Which one seems from $\phi = 0.99$? Briefly justify your answers.

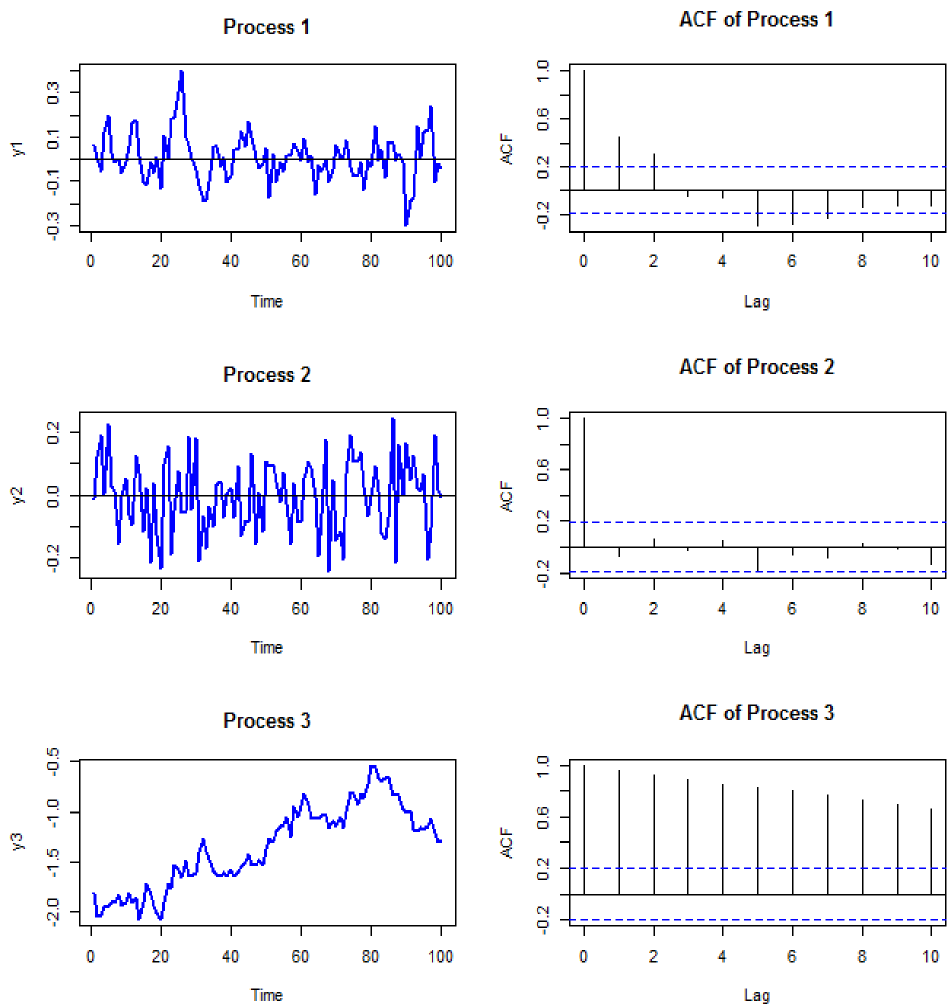

Figure 1: Realizations from three AR(1) processes

2. **(30 pts)** Motivated by the empirical stylized facts, the following ARCH(1) model has been introduced:

$$\begin{aligned} r_t &= \sigma_t e_t, \ e_t \sim iid\ N(0,1) \\ \sigma_t^2 &= \omega + \alpha_1 r_{t-1}^2. \end{aligned} \tag{1}$$

This model successfully generates the stylized facts on financial returns.

2.1. [**5 pts**] Figure 2 below presents the daily cc returns, the estimated ACF, the Box-plot and the QQ-plot based on apple shares. Summarize the empirical stylized facts from Figure 2.

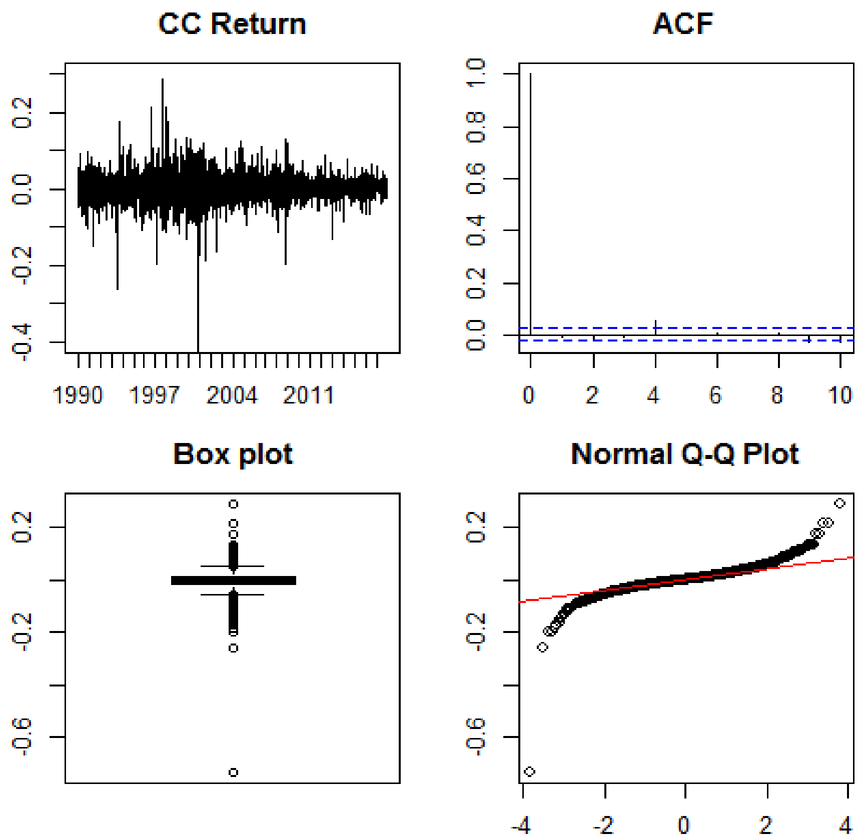

Figure 2: Daily CC Return of Apple

2.2. **[3 pts]** Show that the mds property $E\left[r_t|\mathcal{F}_{t-1}\right] = 0$ holds.

2.3. **[3 pts]** In the ARCH(1) model above, we can write

$$r_t^2 = \omega + \alpha_1 r_{t-1}^2 + v_t$$

where $v_t = r_t^2 - \sigma_t^2$. The empirically estimated parameter value for α_1 is typically close to 1, say, 0.99. Discuss which stylized fact of financial return data is well captured by this result.

2.4. [**8 pts**] Show that

$$\frac{E\left[r_t^4\right]}{(E\left[r_t^2\right])^2} \geq 3,$$

and discuss why this result explains financial return distribution better than *iid normal* model.

2.5. [**2 pts**] Suppose that $\omega = 0.1$ and $\alpha_1 = 0.8$. At time t, we observe that $r_t = 0.1$. What is the best forecast of r_{t+1} based on ARCH(1) given the information we have at time t?

2.6. [**4 pts**] Suppose that $\omega = 0$ and $\alpha_1 = 0.81$. At time t, we observe that $r_t = 0.1$. What is the 0.95 confidence interval of r_{t+1} given the information we have at time t?

2.7. [**5 pts**] Suppose that $\omega = 0$ and $\alpha_1 = 0.81$. At time t, we observe that $r_t = 0.1$. If we invest $W_0 = 100$ in Apple at time t, what is the value at risk ($\alpha = 0.05$) of our investment at time $t + 1$ given the information we have at time t?

3. **(42 pts)** Consider the constant expected return model for the two Northwest stocks (MSFT and SBUX)

$$R_{it} = \mu_i + \epsilon_{it} \quad t = 1, \cdots, T;$$
$$i = 1, 2 \text{ (MSFT and SBUX, respectively)},$$
$$\epsilon_{it} \sim \text{iid } N(0, \sigma_i^2),\ \text{cov}(\epsilon_{it},\ \epsilon_{jt}) = \sigma_{ij},\ \text{cor}(\epsilon_{it},\ \epsilon_{jt}) = \rho_{ij}$$

3.1. [**3 pts**] Let $\mathbf{x}$ denote the vector of portfolio shares. Transfer the model into vector and matrix forms, i.e., define the following vector and matrices.

$$\mathbf{R}_t = \begin{pmatrix} \\ \\ \end{pmatrix}, \qquad \boldsymbol{\mu} = \begin{pmatrix} \\ \\ \end{pmatrix}, \qquad \boldsymbol{\epsilon}_t = \begin{pmatrix} \\ \\ \end{pmatrix}$$

$$\boldsymbol{\Sigma} = \begin{pmatrix} \quad & \quad \\ \quad & \quad \end{pmatrix}, \qquad \mathbf{x} = \begin{pmatrix} \\ \\ \end{pmatrix}, \qquad \mathbf{1} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

3.2. [**6 pts**] Write down the optimization problem and give the Lagrangian used to determine the global minimum variance portfolio. Let $\mathbf{m}$ denote the vector of portfolio weights in the global minimum variance portfolio. Show that the solution is

$$\mathbf{m} = \frac{\boldsymbol{\Sigma}^{-1}\mathbf{1}}{\mathbf{1}'\boldsymbol{\Sigma}^{-1}\mathbf{1}}.$$

3.3. [**4 pts**] Write down the optimization problem used to determine the tangency portfolio when the risk free rate is given by r_f. Let $\mathbf{t}$ denote the vector of portfolio weights in the tangency portfolio. What does the following ratio represent,

$$\frac{\mathbf{t}'\boldsymbol{\mu} - r_f}{(\mathbf{t}' \sum \mathbf{t})^{1/2}}$$

in financial economics?

3.4. [**5 pts**] It can be shown (no need to show) that

$$\mathbf{t} = \frac{\boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu} - r_f \cdot \mathbf{1})}{\mathbf{1}'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu} - r_f \cdot \mathbf{1})}.$$

Explain the economic concept of $(\boldsymbol{\mu} - r_f \cdot \mathbf{1})$. Discuss the main difference between

$$\mathbf{m} = \frac{\boldsymbol{\Sigma}^{-1}\mathbf{1}}{\mathbf{1}'\boldsymbol{\Sigma}^{-1}\mathbf{1}}$$

and $\mathbf{t}$.

3.5. [**4 pts**] Denote the simple return of the tangency portfolio $R_{\tan} = \mathbf{t}'R$. State the Mutual Fund Separation Theorem and draw a portfolio frontier (with $i = 1, 2$, hence two risky assets) and tangency portfolio line. Based on the Theorem, discuss the resulting weights determination (x_f for r_f and $x_{\tan}$ for $R_{\tan}$) according to an investor's risk preference.

3.6. [**6 pts**] In the rest questions (3.6) - (3.10), we assume that

$$r_f = 0.01,\ \boldsymbol{\mu} = \begin{pmatrix} 0.03 \\ 0.05 \end{pmatrix} \text{ and } \boldsymbol{\Sigma} = \begin{pmatrix} 0.01 & -0.008 \\ -0.008 & 0.04 \end{pmatrix}.$$

Consider the portfolio $(x_f, x_1, x_2) = (0, 0.4, 0.6)$. Find the mean and standard deviation of the simple return of this portfolio.

3.7. [**6 pts**] The tangency portfolio $(x_1^{\tan}, x_2^{\tan})$ is determined by

$$x_1^{\tan} = \frac{(\mu_1 - r_f)\sigma_2^2 - (\mu_2 - r_f)\sigma_{12}}{(\mu_1 - r_f)\sigma_2^2 + (\mu_2 - r_f)\sigma_1^2 - (\mu_1 + \mu_2 - 2r_f)\sigma_{12}}$$

and $x_2^{\tan} = 1 - x_1^{\tan}$. Find the tangency portfolio.

3.8. [**4 pts**] Find the efficient portfolio which has the same expected return as MSFT.

3.9. [**4 pts**] Find the efficient portfolio which has the same risk (same standard deviation of simple return) as MSFT.

4. Indicate whether the following statements are true or false (circle one). Briefly discuss why it is so. **(30 pts, 6 puts each)**

4.1. If $\varepsilon_t \sim mds\ (0, \sigma_\varepsilon^2)$, then $\varepsilon_t \sim iid\ (0, \sigma_\varepsilon^2)$.

True False

Why?

4.2. If $\varepsilon_t \sim mds\ (0, \sigma_\varepsilon^2)$, then $\varepsilon_t \sim WN\ (0, \sigma_\varepsilon^2)$.

True False

Why?

4.3. If $\{\varepsilon_t\}_t$ is strictly stationary, then it is also covariance stationary.

True False

Why?

4.4. Let Y_1 and Y_2 be iid random variables with mean μ and variance σ^2, and let $\hat{\mu}_1 = Y_1$ and $\hat{\mu}_2 = \frac{Y_1+Y_2}{2}$ are two different point estimators for μ. From the MSE criteria, we prefer to use $\hat{\mu}_1$ rather than $\hat{\mu}_2$.

True False

Why?

4.5. MA(1) process $Y_t = \mu + \varepsilon_t + \theta\varepsilon_{t-1}$, $\varepsilon_t \sim mds(0, \sigma_\varepsilon^2)$ is *not covariance stationary* when $|\theta| = 1$.

True False

Why?

5. Bonus Questions **(5 pts, 1 puts each)**. Briefly explain what are the following R commands:

5.1. abline

5.2. tail

5.3. rmvnorm

5.4. solve(A)

5.5. qnorm