Econ 147 (Winter 2018)
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Complicated Formula

1 Probability

1. **Skewness - Measure of symmetry**

$$\begin{aligned}\text{Skew}(X) &= E\left[\left(\frac{X-\mu_X}{\sigma_X}\right)^3\right] \\ &= \sum_{x\in S_X}\left(\frac{x-\mu_X}{\sigma_X}\right)^3 f(x) \text{ if } X \text{ is discrete} \\ &= \int_{-\infty}^{\infty}\left(\frac{x-\mu_X}{\sigma_X}\right)^3 f(x)dx \text{ if } X \text{ is continuous.}\end{aligned}$$

2. **Kurtosis - Measure of tail thickness**

$$\begin{aligned}\text{Kurt}(X) &= E\left[\left(\frac{X-\mu_X}{\sigma_X}\right)^4\right] \\ &= \sum_{x\in S_X}\left(\frac{x-\mu_X}{\sigma_X}\right)^4 f(x) \text{ if } X \text{ is discrete} \\ &= \int_{-\infty}^{\infty}\left(\frac{x-\mu_X}{\sigma_X}\right)^4 f(x)dx \text{ if } X \text{ is continuous}\end{aligned}$$

3. If X has a Student's t distribution with degrees of freedom parameter v, denoted $X \sim t_v$, then its pdf has the form

$$f(x) = \frac{\Gamma\left(\frac{v+1}{2}\right)}{\sqrt{v\pi}\Gamma\left(\frac{v}{2}\right)}\left(1+\frac{x^2}{v}\right)^{-\left(\frac{v+1}{2}\right)}, \quad -\infty < x < \infty,\ v > 0.$$

where $\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}dt$ denotes the gamma function. It can be shown that

$$\begin{aligned}E[X] &= 0,\ v > 1 \\ \text{var}(X) &= \frac{v}{v-2},\ v > 2, \\ \text{skew}(X) &= 0,\ v > 3, \\ \text{kurt}(X) &= \frac{6}{v-4} + 3,\ v > 4.\end{aligned}$$

4. Let X and Y be distributed bivariate normal. The joint pdf is given by

$$f(x,y) = \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} \times$$
$$e^{\left\{-\frac{1}{2(1-\rho^2)}\left[(\frac{x-\mu_X}{\sigma_X})^2 + (\frac{y-\mu_Y}{\sigma_Y})^2 - \frac{2\rho(x-\mu_X)(y-\mu_Y)}{\sigma_X\sigma_Y}\right]\right\}},$$

where $E[X] = \mu_X$, $E[Y] = \mu_Y$, $\mathrm{SD}(X) = \sigma_X$, $\mathrm{SD}(Y) = \sigma_Y$, and $\rho = \mathrm{cor}(X,Y)$.

5. Let $X_1,\ X_2, \cdots, X_N$ be rv's and let $a_1, a_2, \ldots, a_N$ be constants. Define

$$Z = a_1X_1 + a_2X_2 + \cdots + a_NX_N = \sum_{i=1}^{N} a_iX_i$$

Then

$$\begin{aligned}\sigma_Z^2 = \mathrm{Var}(Z) &= a_1^2\mathrm{Var}(X_1) + \cdots + a_N^2\mathrm{Var}(X_N) \\ &+ 2a_1a_2\mathrm{Cov}(X_1,\ X_2) + \cdots + 2a_1a_N\mathrm{Cov}(X_1,\ X_N) \\ &+ 2a_2a_3\mathrm{Cov}(X_2,\ X_3) + \cdots + 2a_2a_N\mathrm{Cov}(X_2,\ X_N) \\ &+ \cdots \\ &+ 2a_{N-1}a_N\mathrm{Cov}(X_{N-1}, X_N).\end{aligned}$$

Note: this formula will only be provided for $N \geq 3$.

2 Statistics

1. Suppose that $Y_t \sim iid\ N(\mu, \sigma^2)$. The standard deviation of the sample variance is

$$\mathrm{SE}(\hat{\sigma}^2) \approx \frac{\sqrt{2}\sigma^2}{\sqrt{T}}$$

where "$\approx$" denotes "approximately equal to" and approximation error $\longrightarrow 0$ as $T \longrightarrow \infty$.

2. The Jarque-Bera statistic

$$\mathrm{JB} = \frac{T}{6}\left(\widehat{\mathrm{skew}}^2 + \frac{(\widehat{\mathrm{kurt}} - 3)^2}{4}\right),$$

where

$$\begin{aligned}\widehat{\mathrm{skew}}^2 &= \frac{1}{T-1}\sum_{t=1}^{T}(r_t - \overline{r})^3/s_r^3, \\ \widehat{\mathrm{kurt}} &= \frac{1}{T-1}\sum_{t=1}^{T}(r_t - \overline{r})^4/s_r^4,\end{aligned}$$

and s_r^2 denote the sample variance of data $\{r_t\}_{t=1}^T$. If $H_0 : r_t \sim$ iid $N(\mu, \sigma^2)$ is true then

$$\mathrm{JB} \sim \chi^2(2)$$

where $\chi^2(2)$ denotes a chi-square distribution with 2 degrees of freedom (d.f.).

3 Time Series Econometrics

1. Suppose we have data $\{Y_t\}_{t=1}^T$. The sample ACF: $\hat{\rho}_j$ as a function of j where

$$\widehat{\rho}_j = \frac{\widehat{\gamma}_j}{\widehat{\gamma}_0} \text{ for } |j| < T,$$

where

$$\widehat{\gamma}_j = (T-j)^{-1} \sum_{t=1}^{T-j} (Y_t - \overline{Y})(Y_{t+j} - \overline{Y}),\ \overline{Y} = T^{-1} \sum_{t=1}^{T} Y_t.$$

2. For a given positive integer m, the Ljung-Box statistic is

$$Q_m = (T+2) \sum_{j=1}^{m} (1 - j/T) \widehat{\rho}_j^2,$$

where Q_m is approximated a Chi-square random variable with degree of freedom m.

3. $\{Y_t\}_t$ is called an ARMA (1,1) process if

$$Y_t - \mu = \phi(Y_{t-1} - \mu) + \varepsilon_t + \theta\varepsilon_{t-1}, \varepsilon_t \sim mds(0, \sigma_\varepsilon^2),\ |\phi| < 1.$$

The mean, variance, autocovariance function and the autocorrelation function of $\{Y_t\}_t$ are

$$\begin{aligned}
E[Y_t] &= \mu, \\
\operatorname{var}(Y_t) = \sigma_Y^2 &= \frac{1 + \theta^2 + 2\phi\theta}{1 - \phi^2} \sigma_\varepsilon^2, \\
\gamma_j = \operatorname{cov}(Y_t, Y_{t-j}) &= \begin{cases} \frac{(1+\phi\theta)(\phi+\theta)}{1-\phi^2}, & |j| = 1 \\ \phi\gamma_{j-1}, & j \geq 2 \\ \gamma_{-j}, & j \leq -2 \end{cases}, \\
\rho_j = \operatorname{corr}(Y_t, Y_{t-j}) &= \begin{cases} \frac{(1+\phi\theta)(\phi+\theta)}{1+\theta^2+2\phi\theta}, & |j| = 1 \\ \phi\rho_{j-1}, & j \geq 2 \\ \rho_{-j}, & j \leq -2 \end{cases},
\end{aligned}$$

respectively.

4. Suppose we have data $\{(X_t, Y_t\}_{t=1}^T$. Sample covariance is defined as

$$\hat{\sigma}_{xy} = \frac{1}{T-1} \sum_{t=1}^{T} (X_t - \overline{X})(Y_t - \overline{Y})$$

where

$$\overline{X} = T^{-1} \sum_{t=1}^{T} X_t \text{ and } \overline{Y} = T^{-1} \sum_{t=1}^{T} Y_t.$$

Sample correlation is defined as

$$\hat{\rho}_{xy} = \frac{\hat{\sigma}_{xy}}{\hat{\sigma}_x \hat{\sigma}_y}.$$

4 Conditional Volatility Models

1. The final exam will **NOT** test anything on maximum likelihood estimation of ARCH or GARCH.

2. From the ARCH(1) model:

$$\begin{aligned} r_t &= \sigma_t e_t,\ e_t \sim iid\ N(0,1) \\ \sigma_t^2 &= \omega + \alpha_1 r_{t-1}^2 \end{aligned}$$

   the conditional distribution of $r_t|\mathcal{F}_{t-1} \sim N(0, \sigma_t^2)$ is given by

$$\begin{aligned} f\left(r_t|\mathcal{F}_{t-1};\theta\right) &= \frac{1}{\sqrt{2\pi\sigma_t^2}} \exp\left[-\frac{r_t^2}{2\sigma_t^2}\right] \\ &= \frac{1}{\sqrt{2\pi\left(\omega + \alpha_1 r_{t-1}^2\right)}} \exp\left[-\frac{r_t^2}{2\left(\omega + \alpha_1 r_{t-1}^2\right)}\right] \end{aligned}$$

   where $\theta = (\omega,\, \alpha_1)$ denotes the parameter to be estimated.

3. From the GARCH(1) model:

$$\begin{aligned} r_t &= \sigma_t e_t,\ e_t \sim iid\ N(0,1) \\ \sigma_t^2 &= \omega + \alpha_1 r_{t-1}^2 + \beta_1 \sigma_{t-1}^2 \end{aligned}$$

   the conditional distribution of $r_t|\mathcal{F}_{t-1} \sim N(0, \sigma_t^2)$ is given by

$$\begin{aligned} f\left(r_t|\mathcal{F}_{t-1};\theta\right) &= \frac{1}{\sqrt{2\pi\sigma_t^2}} \exp\left[-\frac{r_t^2}{2\sigma_t^2}\right] \\ &= \frac{\exp\left[-\frac{r_t^2}{2\left(\omega+\alpha_1 r_{t-1}^2+\beta_1\sigma_{t-1}^2\right)}\right]}{\sqrt{2\pi\left(\omega + \alpha_1 r_{t-1}^2 + \beta_1 \sigma_{t-1}^2\right)}} \end{aligned}$$

   where $\theta = (\omega,\, \alpha_1,\, \beta_1)$ denotes the parameter to be estimated.

5 Portfolio Theory with Two Risky Assets

1. Suppose we have two risky assets A and B with simple returns R_A and R_B respectively. Then the global minimum variance portfolio $(x_A^{\min}, x_B^{\min})$ is

$$x_A^{\min} = \frac{\sigma_B^2 - \sigma_{AB}}{\sigma_A^2 + \sigma_B^2 - 2\sigma_{AB}},\ x_B^{\min} = 1 - x_A^{\min}$$

   where σ_A^2, σ_B^2 and σ_{AB} denote the variance of R_A, the variance of R_B and the covariance of R_A and R_B respectively.

2. Suppose we have two risky assets A and B with simple returns R_A and R_B respectively. Then the tangency portfolio $(x_A^{\tan}, x_B^{\tan})$ is

$$\begin{aligned} x_A^{\tan} &= \frac{(\mu_A - r_f)\sigma_B^2 - (\mu_B - r_f)\sigma_{AB}}{(\mu_A - r_f)\sigma_B^2 + (\mu_B - r_f)\sigma_A^2 - (\mu_A - r_f + \mu_B - r_f)\sigma_{AB}}, \\ x_B^{\tan} &= 1 - x_A^{\tan}, \end{aligned}$$

where μ_A, μ_B and r_f denote the mean of R_A, the mean of R_B and the risk free return respectively.

6 Portfolio Theory with Matrix

1. Let $\mathbf{A}$ be an $n \times n$ symmetric matrix, and let $\mathbf{x}$ and $\mathbf{y}$ be an $n \times 1$ vectors. Then

$$\frac{\partial}{\partial \mathbf{x}}(\mathbf{x}'\mathbf{y}) = \begin{pmatrix} \frac{\partial}{\partial x_1}\mathbf{x}'\mathbf{y} \\ \vdots \\ \frac{\partial}{\partial x_n}\mathbf{x}'\mathbf{y} \end{pmatrix} = \mathbf{y},$$

$$\frac{\partial}{\partial \mathbf{x}}(\mathbf{x}'\mathbf{A}\mathbf{x}) = \begin{pmatrix} \frac{\partial}{\partial x_1}\mathbf{x}'\mathbf{A}\mathbf{x} \\ \vdots \\ \frac{\partial}{\partial x_n}\mathbf{x}'\mathbf{A}\mathbf{x} \end{pmatrix} = 2\mathbf{A}\mathbf{x}.$$

2. Suppose we have three risky assets A, B and C with simple returns R_A, R_B and R_C respectively. Let $\mathbf{R} = (R_A, R_B, R_C)'$, $\boldsymbol{\mu} = E[\mathbf{R}]$ and $\Sigma = Var(\mathbf{R})$. These notations will be used in the formula below.

3. The global minimum variance portfolio $\mathbf{x} = (x_A^{\tan}, x_B^{\tan}, x_C^{\tan})'$ is the solution of the following linear equations:

$$\begin{pmatrix} 2\Sigma & \mathbf{1}_{3\times 1} \\ \mathbf{1}_{1\times 3} & 0 \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ \lambda \end{pmatrix} = \begin{pmatrix} \mathbf{0}_{3\times 1} \\ 1 \end{pmatrix}.$$

Solving the above equations, we get

$$\begin{pmatrix} \mathbf{x} \\ \lambda \end{pmatrix} = \begin{pmatrix} 2\Sigma & \mathbf{1}_{3\times 1} \\ \mathbf{1}_{1\times 3} & 0 \end{pmatrix}^{-1} \begin{pmatrix} \mathbf{0}_{3\times 1} \\ 1 \end{pmatrix}.$$

The first three elements of $\begin{pmatrix} \mathbf{x} \\ \lambda \end{pmatrix}$ are the portfolio weights $\mathbf{x} = (x_A^{\tan}, x_B^{\tan}, x_C^{\tan})'$ for the global minimum variance portfolio.

4. The efficient portfolio $\mathbf{x} = (x_A, x_B, x_C)'$ with target expected return $\mu_{p,0}$ is the solution of the following linear equations:

$$\begin{pmatrix} 2\Sigma & \boldsymbol{\mu} & \mathbf{1}_{3\times 1} \\ \boldsymbol{\mu}' & 0 & 0 \\ \mathbf{1}_{1\times 3} & 0 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} \mathbf{0}_{3\times 1} \\ \mu_{p,0} \\ 1 \end{pmatrix}.$$

Solving the above equations, we get

$$\begin{pmatrix} \mathbf{x} \\ \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} 2\Sigma & \boldsymbol{\mu} & \mathbf{1}_{3\times 1} \\ \boldsymbol{\mu}' & 0 & 0 \\ \mathbf{1}_{1\times 3} & 0 & 0 \end{pmatrix}^{-1} \begin{pmatrix} \mathbf{0}_{3\times 1} \\ \mu_{p,0} \\ 1 \end{pmatrix}.$$

The first three elements of $\begin{pmatrix} \mathbf{x} \\ \lambda_1 \\ \lambda_2 \end{pmatrix}$ are the portfolio weights $\mathbf{x} = (x_A, x_B, x_C)'$ for the efficient portfolio with target expected return $\mu_{p,0}$.

5. The tangency portfolio $\mathbf{t} = (x_A^{\tan}, x_B^{\tan}, x_C^{\tan})'$ is

$$\mathbf{t} = \frac{\Sigma^{-1}(\boldsymbol{\mu} - r_f \cdot \mathbf{1}_{3\times 1})}{\mathbf{1}_{1\times 3}\Sigma^{-1}(\boldsymbol{\mu} - r_f \cdot \mathbf{1}_{3\times 1})}$$

where r_f denotes the risk free return.