ECON 147 MIDTERM EXAM

YOUR NAME:

Spring 2018, May 7th
12:30pm - 1:45pm

Instruction

- This is a closed book and closed note exam. All necessary information will be provided, so no cheat sheet is needed.
- Try to answer all questions and write all answers **within this exam sheets**. You will hand in this exam sheets. Please write legibly.
- *Examination Rules from Department Policy will be strictly followed.*

1. **(30 pts)** Consider the following joint distribution of X and Y:

Bivariate pmf

		Y = 0	Y = 1	
X	0	1/4	1/8	3/8
	1	1/8	1/8	2/8
	2	1/8	1/4	3/8
		1/2	1/2	

1.1. Find the marginal probability mass functions of X and Y. Using these distributions, compute $E[X]$, $Var(X)$, $E[Y]$ and $Var(Y)$. **[8 pts]**

$$
\begin{aligned}
E[X] &= 0 \times \frac{3}{8} + 1 \times \frac{2}{8} + 2 \times \frac{3}{8} = 1; \ \textbf{(2pts)} \\
E[X^2] &= 0^2 \times \frac{3}{8} + 1^2 \times \frac{2}{8} + 2^2 \times \frac{3}{8} = \frac{7}{4}; \\
Var(X) &= E[X^2] - (E[X])^2 = \frac{3}{4}; \ \textbf{(2pts)} \\
E[Y] &= 0 \times \frac{1}{2} + 1 \times \frac{1}{2} = \frac{1}{2}; \ \textbf{(2pts)} \\
E[Y^2] &= 0^2 \times \frac{1}{2} + 1^2 \times \frac{1}{2} = \frac{1}{2}; \\
Var(Y) &= E[Y^2] - (E[Y])^2 = \frac{1}{4}. \textbf{(2pts)}
\end{aligned}
$$

1.2. Compute the conditional pmf $f(x|Y=0)$ and $f(x|Y=1)$, and also write down the marginal pmf $f(x)$. **[5 pts]**

$$
\begin{aligned}
f(x|Y &= 0) = \begin{cases} 1/2, & x = 0 \\ 1/4, & x = 1 \\ 1/4, & x = 2 \end{cases} ; \textbf{(2 pts)} \\
f(x|Y &= 1) = \begin{cases} 1/4, & x = 0 \\ 1/4, & x = 1 \\ 1/2, & x = 2 \end{cases} ; \textbf{(2 pts)} \\
f(x) &= \begin{cases} 3/8, & x = 0 \\ 2/8, & x = 1 \\ 3/8, & x = 2 \end{cases} . \textbf{(1 pts)}
\end{aligned}
$$

1.3. Are X and Y independent? Fully justify your answer. **[5 pts]**

They are not independent since:
$f(x|Y=0) \neq f(x)$ and $f(x|Y=1) \neq f(x)$.

1.4. Compute $E[X|Y=0]$, $Var[X|Y=0]$ and $Cov(X,Y)$. **[6 pts]**

$$\begin{aligned}
E[X|Y &= 0] = \sum_{x \in S_x} x f(x|Y=0) \\
&= 0 \times \frac{1}{2} + 1 \times \frac{1}{4} + 2 \times \frac{1}{4} = \frac{3}{4}; \qquad \textbf{(2 pts)} \\
E[X^2|Y &= 0] = \sum_{x \in S_x} x^2 f(x|Y=0) \\
&= 0^2 \times \frac{1}{2} + 1^2 \times \frac{1}{4} + 2^2 \times \frac{1}{4} = \frac{5}{4}; \\
Var[X|Y &= 0] = E[X^2|Y=0] - (E[X|Y=0])^2 = \frac{5}{4} - \frac{9}{16} = \frac{11}{16}; \ \textbf{(2 pts)} \\
E[XY] &= \sum_{x \in S_x} \sum_{y \in S_y} xy f(x,y) = \frac{5}{8}; \\
Cov(X,Y) &= E[XY] - E[X]E[Y] = \frac{5}{8} - 1 \times \frac{1}{2} = \frac{1}{8}. \ \textbf{(2 pts)}
\end{aligned}$$

1.5. Now suppose X and Y represent outcomes of two lotteries, and you would like to invest your \$100 to these lotteries. Assume you are extremely risk averse, hence only care about the risk (preferably low variance) of your investment. How do you want to distribute your \$100, i.e., how do you choose your $\alpha \in [0, 1]$ for $\alpha X + (1 - \alpha)Y$? **[6 pts]**

$$\begin{aligned} Var(\alpha X + (1-\alpha)Y) &= \alpha^2\sigma_x^2 + (1-\alpha)^2\sigma_y^2 + 2\alpha(1-\alpha)\sigma_{x,y} \\ &= \frac{3}{4}\alpha^2 + \frac{1}{4}(1-\alpha)^2 + 2\alpha(1-\alpha)\frac{1}{8} \end{aligned}$$

which implies that

$$\begin{aligned} \frac{\partial Var(\alpha X + (1-\alpha)Y)}{\partial \alpha} &= \frac{3}{2}\alpha - \frac{1}{2}(1-\alpha) + \frac{1}{4}(1-2\alpha) \\ &= 2\alpha - \frac{1}{2} - \frac{1}{2}\alpha + \frac{1}{4} \\ &= \frac{3\alpha}{2} - \frac{1}{4}. \end{aligned}$$

Therefore, the portfolio with smallest variance will be $\alpha = \frac{1}{6}$. That is, put 100/6 on X and 500/6 on Y.

2. **(25 pts)** Consider a 3-year period and that there are 3 mutual funds. The performance of each mutual fund relative to market is random in the sense that each fund has a 50-50 chance of outperforming the market in any year and that performance is independent from year to year, and across each fund. Let $t = 1, ..., 3$ (year) and $i = 1, ..., 3$ (mutual funds), and

$$Y_{it} = \begin{cases} 1, & \text{if fund } i \text{ outperforms the market in year } t \\ 0, & \text{otherwise.} \end{cases}$$

2.1. What is the probability that at least one fund outperforms the market in all three years? **[8 pts]**

Answer: The probability that fund i outperforms the market in all three years is $(0.5)^3 = \frac{1}{8}$. The probability that fund i fails to outperform the market in all three years is $1-\frac{1}{8} = 7/8$. Therefore, none of the funds outperform the market in all three years is $(7/8)^3$ and the probability that at least one fund outperforms the market in all three years is

$$1 - (7/8)^3 = 0.33.$$

2.2. Find the $E[Y_{it}]$ and $Var(Y_{it})$. **[3 pts]**

Answer:

$$\begin{aligned} E[Y_{it}] &= 1 * \frac{1}{2} + 0 * \frac{1}{2} = \frac{1}{2}; \\ Var(Y_{it}) &= E\left[Y_{it}^2\right] - (E[Y_{it}])^2 = \frac{1}{2} - \frac{1}{4} = \frac{1}{4} \end{aligned}$$

2.3. Now assume that $P(Y_{it} = 1) = p_i$ is unknown. Given a random sample $\{Y_{i1}, Y_{i2}, Y_{i3}\}$, we may want to use $\hat{p}_i = \frac{1}{3}\sum_{t=1}^{3} Y_{it}$ as an estimator for p_i. Show that $\hat{p}_i$ is an unbiased estimator for p_i.**[4 pts]**

Answer: By the linearity of the expectation operator:

$$E[\hat{p}_i] = \frac{1}{3}\sum_{t=1}^{3} E[Y_{it}] = \frac{1}{3}\sum_{t=1}^{3} p_i = p_i.$$

2.4. Suppose we are interested in estimating the odds ratio $\gamma = \frac{p_i}{1-p_i}$, then a natural estimator is $\hat{\gamma} = \frac{\hat{p}_i}{1-\hat{p}_i}$. Is $\hat{\gamma}$ an unbiased estimator for γ? Explain why or why not. **[6 pts]**

Answer: Biased estimator.

$$\hat{\gamma} = \frac{\hat{p}_i}{1-\hat{p}_i} = \frac{1}{1-\hat{p}_i} - 1.$$

Since $(1-x)^{-1}$ is a nonlinear function of x,

$$E\left[\frac{1}{1-\hat{p}_i}\right] \neq \frac{1}{1-E[\hat{p}_i]}$$

where $E[\hat{p}_i] = p_i$ as we show in 2.3. Therefore,

$$E\left[\hat{\gamma}\right] = E\left[\frac{1}{1-\hat{p}_i}\right] - 1 \neq \frac{1}{1-E[\hat{p}_i]} - 1 = p$$

2.5. Now assume that $t = 1, ..., 100$ (years) which is large enough (sample period is $T = 100$), and that $\hat{p}_i = 0.5$. Construct an 95% asymptotic confidence interval for p_i. (You can use the fact $SE\left(\hat{p}_i\right) = \sqrt{\frac{\hat{p}_i(1-\hat{p}_i)}{100}}$ and $q^Z_{0.025} = -2$). **[4 pts]**

Answer: Using the formula

$$\hat{p}_i \pm q^Z_{0.975} \cdot SE\left(\hat{p}_i\right)$$

we get

$$\begin{aligned} 0.5 \pm 2\sqrt{\frac{0.5\,(1-0.5)}{100}} &= 0.5 \pm 0.1; \\ \text{or } 0.5 \pm 1.96\sqrt{\frac{0.5\,(1-0.5)}{100}} &= 0.5 \pm 0.98. \end{aligned}$$

3. **(30 pts)** Let Y_1, Y_2, Y_3, Y_4 and Y_5 be iid (μ, σ^2).

3.1. Let $W_1 = \frac{1}{8}Y_1 + \frac{1}{8}Y_2 + \frac{1}{2}Y_3 + \frac{1}{8}Y_4 + \frac{1}{8}Y_5$. Find the mean and variance of W_1? **[4 pts]**

Answer: By definition

$$\begin{aligned} E[W_1] &= \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{2} + \frac{1}{8} + \frac{1}{8}\right)\mu = \mu, \ (\textbf{2 pts}) \\ Var(W_1) &= \left(\frac{1}{64} + \frac{1}{64} + \frac{1}{4} + \frac{1}{64} + \frac{1}{64}\right)\sigma^2 = \frac{5\sigma^2}{16} \ (\textbf{2 pts}) \end{aligned}$$

3.2. Let $W_2 = \frac{1}{4}Y_1 + \frac{1}{4}Y_2 + \frac{1}{5}Y_3 + \frac{1}{4}Y_4 + \frac{1}{4}Y_5$. Find the mean and variance of W_2? **[4 pts]**

Answer: By definition

$$\begin{aligned} E[W_2] &= \left(\frac{1}{4} + \frac{1}{4} + \frac{1}{5} + \frac{1}{4} + \frac{1}{4}\right)\mu = \frac{6}{5}\mu, \ (\textbf{2 pts}) \\ Var(W_2) &= \left(\frac{1}{16} + \frac{1}{16} + \frac{1}{25} + \frac{1}{16} + \frac{1}{16}\right)\sigma^2 = \frac{29\sigma^2}{100} \ (\textbf{2 pts}) \end{aligned}$$

3.3. Which estimator of μ (W_1 or W_2) do you prefer? Fully justify your answer. **[13 pts]**

Answer: By definition

$$\begin{aligned} MSE(W_1) &= \frac{5}{16}\sigma^2 \ (\textbf{3 pts}) \\ MSE(W_2) &= \frac{\mu^2}{25} + \frac{29}{100}\sigma^2 \ (\textbf{3 pts}) \end{aligned}$$

Therefore

$$\begin{aligned} MSE(W_1) - MSE(W_2) &= \frac{125 - 116}{400}\sigma^2 - \frac{\mu^2}{25} \\ &= \frac{9}{400}\sigma^2 - \frac{\mu^2}{25} \ (\textbf{3 pts}) \end{aligned}$$

If $\frac{9}{400}\sigma^2 > \frac{\mu^2}{25}$, we prefer W_2. If $\frac{9}{400}\sigma^2 < \frac{\mu^2}{25}$, we prefer W_1. If $\frac{9}{400}\sigma^2 = \frac{\mu^2}{25}$, there is no difference between these two estimators. (**4 pts**)

3.4. An estimator W_a is called a linear estimator of μ if it takes the following form

$$W = a_1 Y_1 + a_2 Y_2 + a_3 Y_3 + a_4 Y_4 + a_5 Y_5.$$

Among all possible unbiased linear estimators of μ based on Y_1, Y_2, Y_3, Y_4 and Y_5, does there exist a best (unbiased) estimator which has smallest mean square error? Fully justify your answer. [Hint: the inequality $\left(\sum_{i=1}^{5} a_i\right)^2 \leq 5 \sum_{i=1}^{5} a_i^2$ will be useful.] **[9 pts]**

Answer: for any linear unbiased estimator,

$$\begin{aligned} Var\left(W\right) &= \left(\sum_{i=1}^{5} a_i^2\right) \sigma^2 \ (\mathbf{1\ pt}) \\ E[W] &= \mu \ (\mathbf{1\ pt}) \\ MSE(W) &= \left(\sum_{i=1}^{5} a_i^2\right) \sigma^2 \ (\mathbf{1\ pt}\text{, or 3 pts if directly write MSE of W}). \end{aligned}$$

While for the sampe average

$$\begin{aligned} Var\left(\bar{Y}\right) &= \frac{\sigma^2}{n} \ (\mathbf{1\ pt}) \\ E[\bar{Y}] &= \mu \ (\mathbf{1\ pt}) \\ MSE(\bar{Y}) &= \frac{\sigma^2}{n} \ (\mathbf{1\ pt}\text{, or 3 pts if directly write MSE of } \bar{Y}). \end{aligned}$$

Since $\frac{\sigma^2}{5} \leq \left(\sum_{i=1}^{5} a_i^2\right) \sigma^2$ by the inequality in hint,

$$Var\left(\bar{Y}\right) \leq Var\left(W_a\right)$$

so $\bar{Y}$ is the *best* in MSE sense (**2 pts**).

4. **(15 pts, 5pts each)** Indicate whether the following statements are true or false (circle one). Briefly discuss why it is so.

4.1. If r_t is *continuously compounded (cc)* monthly return at month t for $t = 1, 2, \ldots, 12$, then the annualized cc return is $r_A = 12r_t$.

True False

Why?

Answer: False. The annualized cc return is $r_A = r_1 + r_2 + \cdots + r_{12}$.

4.2. Let $r_{GS,t}$ and $r_{AIG,t}$ be *cc* 1-month returns for Goldman Sachs Group (GS) and American International Group (AIG). If we construct a portfolio using the share $\alpha \in [0, 1]$ for GS, the portfolio cc return is $r_{p,t} = \alpha r_{GS,t} + (1 - \alpha)\, r_{AIG,t}$.

True False

Why?

Answer: False. Because cc return is not additive across portfolio shares.

4.3. In 5.2., if 5% quantile of the portfolio *simple return* is given as $q_{0.05}^{R_p} = -0.5$, then 5% monthly Value-at-Risk for the \$10,000 investment in this portfolio is $\$10,000 \times (-0.5) = -\$5,000$.

Answer: True. Because by the definition of VaR.

5. Bonus Questions **(5 pts, 1 puts each)**. Briefly explain what are the following R commands:

5.1. colnames

Answer: Show the column names of data

5.2. tail

Answer: Show the last 6 observations of the data

5.3. rmvnorm

Answer: generate multivariate normal random variable

5.4. cumprod

Answer: calculate the commulative products

5.5. qnorm

Answer: calculate the quantile of normal random variable