ECON 147 Final EXAM

YOUR NAME:

Winter 20??,
3:00pm - 6:00pm

Instruction

- This is a closed book and closed note exam.
- Try to answer all questions and write all answers **within this exam sheets**. You will hand in this exam sheets. Please write legibly.
- Total points are 120, the bonus questions will give you extra points but your score will be the minimum of your actual score and 120.
- *Examination Rules from Department Policy will be strictly followed.*
- The following results may be useful:

$$
\begin{array}{ll}
\Pr(Z \leq 0.5) = 0.69, & \Pr(Z \leq -1.1) = 0.136, \\
\Pr(Z > -1.96) = 0.975, & \Pr(Z \leq 1.6449) = 0.95, \\
\Pr(Z \leq -1.64) = 0.05, & \Pr(Z \leq -0.70) = 0.242, \\
\Pr(Z \leq -0.35) = 0.363, & e^{-2.31} = 0.100, \\
e^{-0.148} = 0.862, & e^{-1.92} = 0.150, \\
e^{-3.21} = 0.041, & e^{-2.87} = 0.055,
\end{array}
$$

where Z is a standard normal random variable.

1. **(18 pts)** Let $\{Y_t\}$ be the first order autoregressive (AR(1)) process

$$\begin{aligned} Y_t - \mu &= \phi\left(Y_{t-1} - \mu\right) + \varepsilon_t, \;\; |\phi| < 1, \\ \varepsilon_t &\sim iid\left(0, \sigma_\varepsilon^2\right), \end{aligned}$$

and define $X_t = Y_t - \mu$.

1.1.a. [**4 pts**] Show that for any integers t and $j > 0$, we can write:

$$\begin{aligned} X_{t+j} &= \phi^{j+1} X_{t-1} + \phi^0 \varepsilon_{t+j} + \phi^1 \varepsilon_{t+j-1} + \cdots + \phi^j \varepsilon_t \\ &= \phi^{j+1} X_{t-1} + \sum_{s=0}^{j} \phi^s \varepsilon_{t+j-s}. \end{aligned}$$

From the expression above, find $\frac{\partial X_{t+j}}{\partial \varepsilon_t}$, i.e., the derivative of X_{t+j} with respect to ε_t.

Answer: By the definition of X_t and the definition of AR(1) process,

$$X_{t+j} = \phi X_{t+j-1} + \varepsilon_{t+j}. \;\; \textbf{(1 pt)}$$

Iterate the above equation, we get

$$\begin{aligned} X_{t+j} &= \phi^2 X_{t+j-2} + \phi\varepsilon_{t+j-1} + \varepsilon_{t+j} \\ &= \phi^3 X_{t+j-3} + \phi^2 \varepsilon_{t+j-2} + \phi\varepsilon_{t+j-1} + \varepsilon_{t+j} \\ &= \cdots \\ &= \phi^{j+1} X_{t+j-(j+1)} + \phi^j \varepsilon_{t+j-j} + \cdots \phi\varepsilon_{t+j-1} + \varepsilon_{t+j} \\ &= \phi^{j+1} X_{t-1} + \phi^0 \varepsilon_{t+j} + \phi^1 \varepsilon_{t+j-1} + \cdots + \phi^j \varepsilon_t. \;\; \textbf{(2 pt)} \end{aligned}$$

From the expression derived above,

$$\frac{\partial X_{t+j}}{\partial \varepsilon_t} = \phi^j. \;\; \textbf{(1 pt)}$$

1.1.b [**2 pts**] Let X_t be the time series of US GDP, and ε_{t-j} be some unexpected news (say, a sudden drop of the federal fund rate) from time $t - j$. What will be the economic interpretation of the $\frac{\partial X_{t+j}}{\partial \varepsilon_t}$ above? What happens to this $\frac{\partial X_{t+j}}{\partial \varepsilon_t}$ if j gets larger and larger ($j \to \infty$)?

Answer: From the expression

$$X_{t+j} = \phi^{j+1} X_{t-1} + \phi^0 \varepsilon_{t+j} + \phi^1 \varepsilon_{t+j-1} + \cdots + \phi^j \varepsilon_t$$

we see that $\frac{\partial X_{t+j}}{\partial \varepsilon_t}$ is the marginal effect of the unexpected news at time t on the US GDP at time $t+j$. **(1 pt)**
Since $|\phi| < 1$, we see that $\phi^j \to 0$ as $j \to \infty$ which means that the marginal effect of the unexpected news at time t on the US GDP at time $t+j$ vanishes as j becomes larger and larger. **(1 pt)**

1.2. **[9 pts]** Using the covariance stationarity of X_t, compute (a) $Var(X_t)$, (b) $Cov(X_t, X_{t-1})$ and (c) $Corr(X_t, X_{t-1})$. [**Hint:** using the equation $X_t = \phi X_{t-1} + \varepsilon_t$, we have

$$\begin{aligned} E[X_t] &= E[\phi X_{t-1} + \varepsilon_t], \\ Var(X_t) &= Var(\phi X_{t-1} + \varepsilon_t), \\ E[X_t X_{t-1}] &= E[(\phi X_{t-1} + \varepsilon_t) X_{t-1}] \end{aligned}$$

for any t.]

(a) **Answer:** First,

$$\begin{aligned} Var(X_t) &= Var(\phi X_{t-1} + \varepsilon_t), \\ &= \phi^2 Var(X_{t-1}) + Var(\varepsilon_t) + 2Cov(\phi X_{t-1}, \varepsilon_t) \\ &= \phi^2 Var(X_t) + \sigma_\varepsilon^2 \ \textbf{(2 pts)} \end{aligned}$$

where the first equality is by the second hint, the second equality is by the formula of variance of sum of random variables, the last equality is by the covariance stationarity and $\varepsilon_t \sim iid\ (0, \sigma_\varepsilon^2)$. From the above equation, we immediately get

$$Var(X_t) = \frac{\sigma_\varepsilon^2}{1 - \phi^2} \ \textbf{(1 pts).}$$

Second, from the first hint, the covariance stationarity and $\varepsilon_t \sim iid$ $(0, \sigma_\varepsilon^2)$,

$$E[X_t] = \phi E[X_{t-1}] + E[\varepsilon_t] = \phi E[X_t]$$

which implies that $E[X_t] = 0$. **(1 pts).**
Third,

$$\begin{aligned} E[X_t X_{t-1}] &= E[(\phi X_{t-1} + \varepsilon_t) X_{t-1}] \\ &= E[\phi X_{t-1}^2 + \varepsilon_t X_{t-1}] \\ &= E[\phi X_{t-1}^2] + E[\varepsilon_t X_{t-1}] \\ &= \phi Var\left(X_{t-1}\right) = \phi \frac{\sigma_\varepsilon^2}{1-\phi^2} \end{aligned}$$

where the first equality is by the third hint, the fourth equality is by the covariance stationarity, $E[X_t] = 0$ and $\varepsilon_t \sim iid\ (0, \sigma_\varepsilon^2)$, the variance of X_t derived above. **(2 pts)**
Since $E[X_t] = 0$ for any t,

$$Cov\left(X_t, X_{t-1}\right) = E[X_t X_{t-1}] = \phi \frac{\sigma_\varepsilon^2}{1-\phi^2}. \ \textbf{(1 pts)}$$

Finally,

$$Corr\left(X_t, X_{t-1}\right) = \frac{Cov\left(X_t, X_{t-1}\right)}{\sqrt{Var\left(X_t\right)}\sqrt{Var\left(X_{t-1}\right)}} = \frac{\frac{\phi\sigma_\varepsilon^2}{1-\phi^2}}{\frac{\sigma_\varepsilon^2}{1-\phi^2}} = \phi. \ \textbf{(2 pts)}$$

Note: **4 points** if no derivation and only the results.

1.3. **[3 pts]** Realizations from three different AR(1) processes (with $\phi = 0$, 0.5 and 0.99 respectively) and their (estimated) autocorrelation functions (ACF) are given in Figure 1 below. Which process do you think is generated from $\phi = 0$? Which one seems from $\phi = 0.99$? Briefly justify your answers.

(a) **Answer:** Note that true ACF of AR(1) is

$$\rho_j = \phi^{|j|}.$$

The process y2 appears to be the AR(1) with $\phi = 0$. Because the ACF of y2 are close to zero for all $j \neq 0$ which corresponds to the ACF of the AR(1) with $\phi = 0$. **(1.5 pts)**

The process y3 appears to be the AR(1) with $\phi = 0.99$. Because the ACF of y3 are large for many js and it goes to zero slowly for larger j, which corresponds to the ACF of the AR(1) with ϕ close to 1. **(1.5 pts)**

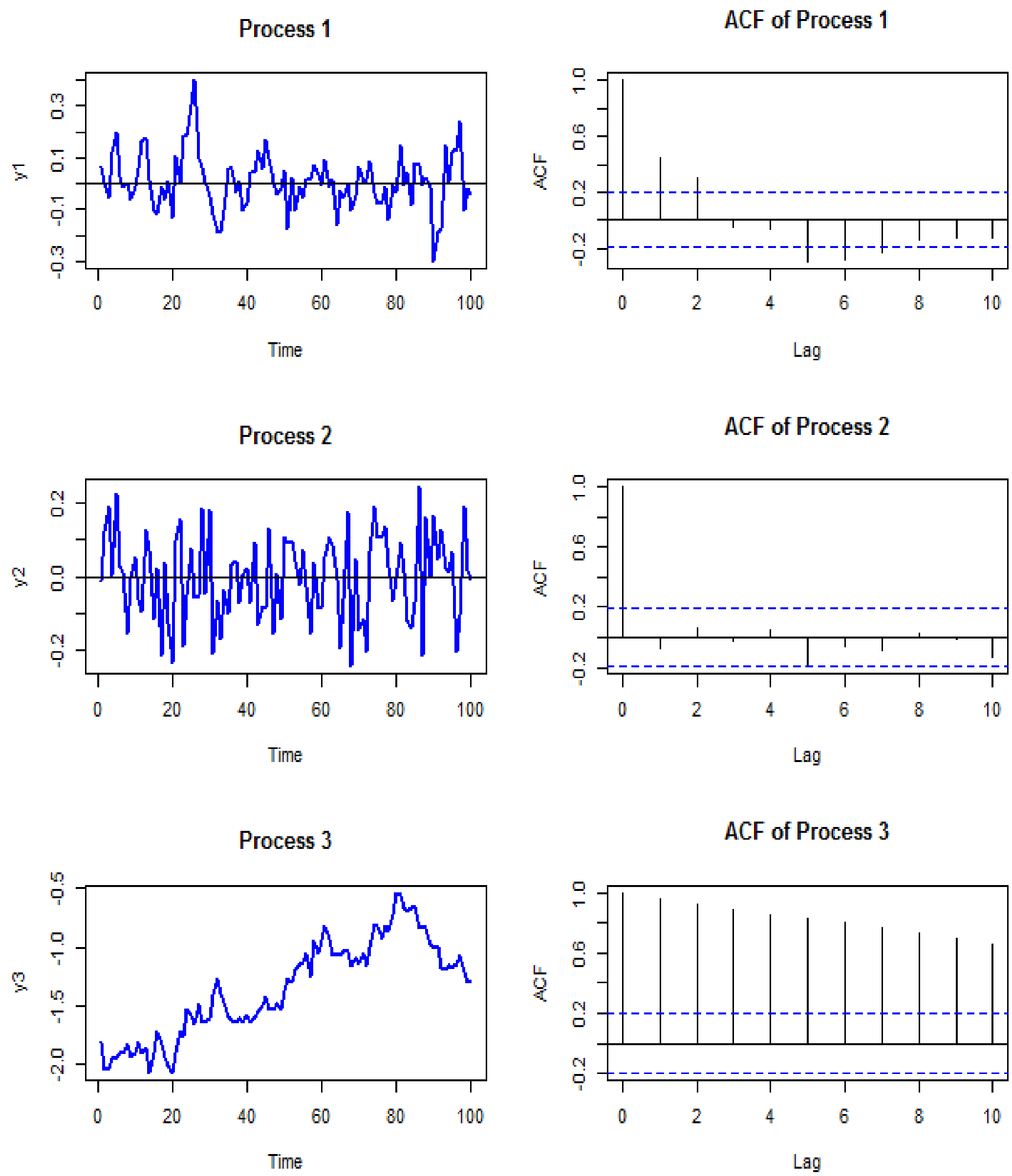


Figure 1: Realizations from three AR(1) processes

2. **(30 pts)** Motivated by the empirical stylized facts, the following ARCH(1) model has been introduced:

$$\begin{aligned} r_t &= \sigma_t e_t,\ e_t \sim iid\ N(0,1) \\ \sigma_t^2 &= \omega + \alpha_1 r_{t-1}^2. \end{aligned} \tag{1}$$

This model successfully generates the stylized facts on financial returns.

2.1. [**5 pts**] Figure 2 below presents the daily cc returns, the estimated ACF, the Box-plot and the QQ-plot based on apple shares. Summarize the empirical stylized facts from Figure 2.

(a) **Answer:** From the first panel of the graph, we see that: i) the daily cc return is variating around its mean which is close to zero; ii) the volatility of the daily cc return is dependent or clustering. (**2 pts**) From the ACF, we see that the daily cc return is (almost) an uncorrelated process since its autocorrrelation coefficient function is close to zero. (**1 pts**) The box plot shows that the cc returns have many outlies. (**1 pts**) The QQ plot in the last panel shows that it is not normally distributed and has heavier tails than the normal distribution. (**1 pts**)

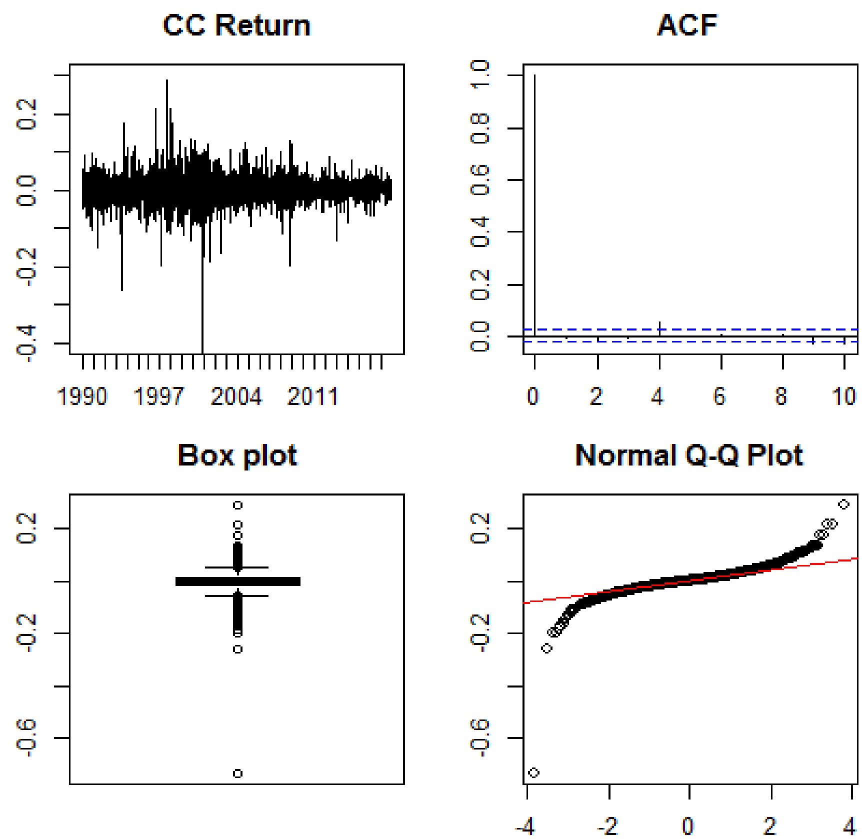


Figure 2: Daily CC Return of Apple

2.2. [**3 pts**] Show that the mds property $E[r_t|\mathcal{F}_{t-1}] = 0$ holds.

Answer: Since $\sigma_t^2 = \omega + \alpha_1 r_{t-1}^2$ is known at time $t-1$,

$$\begin{aligned} E[r_t|\mathcal{F}_{t-1}] &= E[\sigma_t e_t|\mathcal{F}_{t-1}] \quad \text{by the definition of } r_t \ (\mathbf{1\ pts}) \\ &= \sigma_t E[e_t|\mathcal{F}_{t-1}] \quad \text{by } \sigma_t^2 \text{ is known at } t-1 \ (\mathbf{1\ pts}) \\ &= 0 \quad \text{by } e_t \sim iid\ N(0,1) \ (\mathbf{1\ pts}) \end{aligned}$$

This shows that $\{r_t\}_t$ is an m.d.s. and hence mean zero and uncorrelated process.

2.3. [**3 pts**] In the ARCH(1) model above, we can write

$$r_t^2 = \omega + \alpha_1 r_{t-1}^2 + v_t$$

where $v_t = r_t^2 - \sigma_t^2$. The empirically estimated parameter value for α_1 is typically close to 1, say, 0.99. Discuss which stylized fact of financial return data is well captured by this result.

Answer: The squared returns are the proxies of the variances of the cc returns. The expression

$$r_t^2 = \omega + \alpha_1 r_{t-1}^2 + v_t$$

shows that r_t^2 is an AR(1) process. (**1 pts**) When α_1 is close to 1, the AR(1) process is close to be a random walk which is a persistent process. (**1 pts**) Therefore, when α_1 is close to 1, the r_t^2 is a strongly dependent process which captures the volatility clustering show up in the first panel of Figure 2. (**1 pts**)

2.4. **[8 pts]** Show that

$$\frac{E\left[r_t^4\right]}{(E\left[r_t^2\right])^2} \geq 3,$$

and discuss why this result explains financial return distribution better than *iid normal* model.

Answer: First

$$\begin{aligned} & E\left[r_t^4|\mathcal{F}_{t-1}\right] \\ = \; & E\left[\sigma_t^4 e_t^4|\mathcal{F}_{t-1}\right] \quad \text{by the definition of } v_t \\ = \; & \sigma_t^4 E\left[e_t^4|\mathcal{F}_{t-1}\right] \quad \text{by } \sigma_t^4 \text{ is known at } t-1 \\ = \; & \sigma_t^4 E\left[e_t^4\right] \quad \text{by } e_t \sim iid\ N(0,1) \\ = \; & 3\sigma_t^4 \quad \text{by } E\left[e_t^4\right] = 3 \ (\mathbf{3\ pts}) \end{aligned}$$

We will use the Jensen's inequality: $E[X^2] \geq (E[X])^2$ for any random variable X. Therefore

$$\begin{aligned} & E[r_t^4] \\ = \; & 3E[\sigma_t^4] \quad \text{by } E\left[r_t^4|\mathcal{F}_{t-1}\right] = 3\sigma_t^4 \\ = \; & 3E[(E\left[r_t^2|\mathcal{F}_{t-1}\right])^2] \quad \text{by } \sigma_t^2 = E\left[r_t^2|\mathcal{F}_{t-1}\right] \\ \geq \; & 3(E[E\left[r_t^2|\mathcal{F}_{t-1}\right]])^2 \quad \text{by the Jensen's inequality with } X = E\left[r_t^2|\mathcal{F}_{t-1}\right] \\ = \; & 3(E[r_t^2])^2 \quad \text{by the property of the conditional expectation } (\mathbf{3\ pts}) \end{aligned}$$

which shows that

$$\frac{E[r_t^4]}{(E[r_t^2])^2} \geq 3$$

and hene the ARCH(1) model can generate heavier tail than the normal distribution. (**2 pts**).

2.5. [**2 pts**] Suppose that $\omega = 0.1$ and $\alpha_1 = 0.8$. At time t, we observe that $r_t = 0.1$. What is the best forecast of r_{t+1} based on ARCH(1) given the information we have at time t?

Answer: The best preduction will be $E[r_t|\mathcal{F}_{t-1}]$ which will be zero by mds. (**2 pts**)

2.6. [**4 pts**] Suppose that $\omega = 0$ and $\alpha_1 = 0.81$. At time t, we observe that $r_t = 0.1$. What is the 0.95 confidence interval of r_{t+1} given the information we have at time t?

Answer: We know that the 0.95 confidence interval of r_{t+1} is

$$CI = [-\sigma_{t+1} \cdot q^Z_{0.975},\ \sigma_{t+1} q^Z_{0.975}].\ (\mathbf{1\ pts})$$

Since $\sigma^2_{t+1} = \omega + \alpha_1 r_t^2 = 0.81 * (0.1)^2$ which implies that $\sigma_{t+1} = 0.09$. (**2 pts**) Therefore

$$CI = [-0.09 \cdot 1.96,\ 0.09 \cdot 1.96] = [-0.176,\ 0.176].\ (\mathbf{1\ pts})$$

2.7. [**5 pts**] Suppose that $\omega = 0$ and $\alpha_1 = 0.81$. At time t, we observe that $r_t = 0.1$. If we invest $W_0 = 100$ in Apple at time t, what is the value at risk ($\alpha = 0.05$) of our investment at time $t + 1$ given the information we have at time t?

Answer: Since $r_{t+1} \sim N(0, \sigma^2_{t+1})$ given the information at time t and $\sigma^2_{t+1} = 0.81 * (0.1)^2$, we have

$$r_{t+1} \sim N(0, 0.81 \cdot (0.1)^2).\ (\mathbf{2\ pts})$$

Therefore,

$$q^{r_{t+1}}_{0.05} = 0 + \sigma_{t+1} q^Z_{0.05} = -0.09 * 1.6449 = -0.148.\ (\mathbf{1\ pts})$$

Therefore,

$$VaR_{0.05} = W_0 * (e^{q^{r_{t+1}}_{0.05}} - 1) = 100(0.8624 - 1) = -13.76.\ (\mathbf{2\ pts})$$

3. **(42 pts)** Consider the constant expected return model for the two Northwest stocks (MSFT and SBUX)

$$R_{it} = \mu_i + \epsilon_{it} \quad t = 1, \cdots, T;$$
$$i = 1, 2 \text{ (MSFT and SBUX, respectively)},$$
$$\epsilon_{it} \sim \text{iid } N(0, \sigma_i^2),\ \text{cov}(\epsilon_{it},\ \epsilon_{jt}) = \sigma_{ij},\ \text{cor}(\epsilon_{it},\ \epsilon_{jt}) = \rho_{ij}$$

3.1. [**3 pts**] Let $\mathbf{x}$ denote the vector of portfolio shares. Transfer the model into vector and matrix forms, i.e., define the following vector and matrices.

Answer: 0.5 point each.

$$\mathbf{R}_t = \begin{pmatrix} R_{1t} \\ R_{2t} \end{pmatrix},\ \boldsymbol{\mu} = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix},\ \boldsymbol{\epsilon}_t = \begin{pmatrix} \epsilon_{1t} \\ \epsilon_{2t} \end{pmatrix}$$
$$\boldsymbol{\Sigma} = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{pmatrix}, \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix},\ \mathbf{1} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

3.2. [**6 pts**] Write down the optimization problem and give the Lagrangian used to determine the global minimum variance portfolio. Let $\mathbf{m}$ denote the vector of portfolio weights in the global minimum variance portfolio. Show that the solution is

$$\mathbf{m} = \frac{\boldsymbol{\Sigma}^{-1}\mathbf{1}}{\mathbf{1}'\boldsymbol{\Sigma}^{-1}\mathbf{1}}.$$

Answer: The optimization problem is:

$$\min_{\mathbf{m}=(m_1,m_2)} \left(\mathbf{m}'\boldsymbol{\Sigma}\mathbf{m} + \boldsymbol{\lambda}(\mathbf{m}'\mathbf{1} - \mathbf{1})\right). \ (\mathbf{2\ pts})$$

The first order condition is

$$2\boldsymbol{\Sigma}\mathbf{m} + \boldsymbol{\lambda}\mathbf{1} = \mathbf{0} \quad , \tag{2}$$
$$\mathbf{m}'\mathbf{1} - \mathbf{1} = \mathbf{0}. \quad (\mathbf{2\ pts}) \tag{3}$$

Multiplying $\boldsymbol{\Sigma}^{-1}2^{-1}$ in the both sides of (2), we get

$$\mathbf{m} + \frac{\boldsymbol{\lambda}}{2}\boldsymbol{\Sigma}^{-1}\mathbf{1} = \mathbf{0}. \tag{4}$$

Multiplying $\mathbf{1}'$ in the both sides of (4), we get

$$\mathbf{1}'\mathbf{m}+\frac{\boldsymbol{\lambda}}{2}\mathbf{1}'\boldsymbol{\Sigma}^{-1}\mathbf{1}=\mathbf{0}. \tag{5}$$

Since $\mathbf{1}'\mathbf{m}$ =1, from (5) we can solve

$$\boldsymbol{\lambda}=-\frac{2}{\mathbf{1}'\boldsymbol{\Sigma}^{-1}\mathbf{1}}.(\mathbf{1\ pts}) \tag{6}$$

Plugging (6) in (4), we get

$$\mathbf{m}-\frac{\boldsymbol{\Sigma}^{-1}\mathbf{1}}{\mathbf{1}'\boldsymbol{\Sigma}^{-1}\mathbf{1}}=\mathbf{0}$$

which implies that

$$\mathbf{m}=\frac{\boldsymbol{\Sigma}^{-1}\mathbf{1}}{\mathbf{1}'\boldsymbol{\Sigma}^{-1}\mathbf{1}}.(\mathbf{1\ pts})$$

3.3. [**4 pts**] Write down the optimization problem used to determine the tangency portfolio when the risk free rate is given by r_f. Let $\mathbf{t}$ denote the vector of portfolio weights in the tangency portfolio. What does the following ratio represent,

$$\frac{\mathbf{t}'\boldsymbol{\mu}-r_f}{(\mathbf{t}'\sum\mathbf{t})^{1/2}}$$

in financial economics?

Answer: The optimization problem is

$$\max_{\mathbf{x}=(x_1,x_2,x_3,x_4)}\frac{\mathbf{x}'\boldsymbol{\mu}-r_f}{(\mathbf{x}'\sum\mathbf{x})^{1/2}}.\ (\mathbf{2\ pts})$$

The maximizer of the above optimization problem is the portfolio weights of the tangency portfolio $\mathbf{t}$. The ratio

$$\frac{\mathbf{t}'\boldsymbol{\mu}-r_f}{(\mathbf{t}'\sum\mathbf{t})^{1/2}}$$

is the Sharpe ratio of portfolio $\mathbf{t}$ which measures the excess expected return of portfolio $\mathbf{t}$ per unit risk. (**2 pts**)

3.4. [**5 pts**] It can be shown (no need to show) that

$$\mathbf{t} = \frac{\boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu} - r_f \cdot \mathbf{1})}{\mathbf{1}'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu} - r_f \cdot \mathbf{1})}.$$

Explain the economic concept of $(\boldsymbol{\mu} - r_f \cdot \mathbf{1})$. Discuss the main difference between

$$\mathbf{m} = \frac{\boldsymbol{\Sigma}^{-1}\mathbf{1}}{\mathbf{1}'\boldsymbol{\Sigma}^{-1}\mathbf{1}}$$

and $\mathbf{t}$.

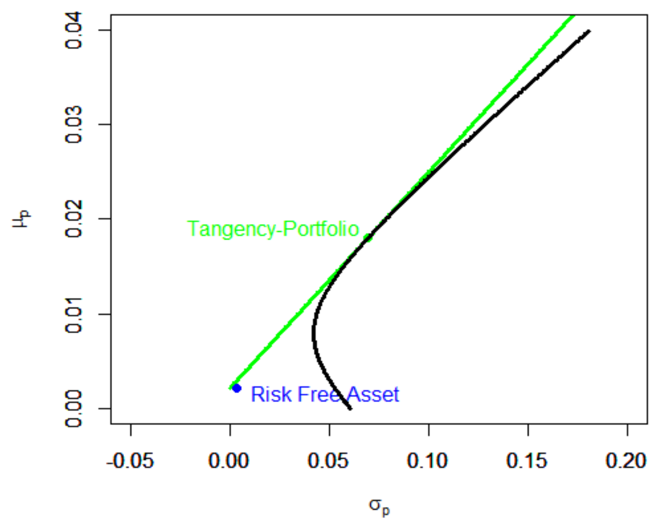
Answer: By definition

$$(\boldsymbol{\mu} - r_f \cdot \mathbf{1}) = \begin{pmatrix} \mu_1 - r_f \\ \mu_2 - r_f \end{pmatrix}$$

which is the vector of excess expected returns of MSFT and SBUX, respectively. (**1 pt**) $\mathbf{m}$ in 3.3 is the global minimum variance portfolio when there is no risk free asset. When there is no risky asset, $\mathbf{m}$ will be a efficient portfolio. However when there is a risk free asset, the global minimum variance portfolio will be dominated by the efficient portfolio constructed by the risk free asset and the tangency portfolio. (**2 pt**) $\mathbf{t}$ represents the tangency portfolio which has the highest Sharpe ratio. From the efficient portfolio theory, we know that $\mathbf{t}$ is an efficient portfolio, but $\mathbf{m}$ is not if there exists risk free asset. (**2 pts**)

3.5. [**4 pts**] Denote the simple return of the tangency portfolio $R_{\text{tan}} = \mathbf{t}'R$. State the Mutual Fund Separation Theorem and draw a portfolio frontier (with $i = 1, 2$, hence two risky assets) and tangency portfolio line. Based on the Theorem, discuss the resulting weights determination (x_f for r_f and x_{tan} for R_{tan}) according to an investor's risk preference.

Answer: The Mutual Fund Separation Theorem states that the efficient portfolios are combinations of two portfolios: the risky free asset and the tangency portfolio. Therefore, All investors hold assets MSFT and SBUX according to their proportions in the tangency portfolio regardless of their risk preferences. (**1 pt**)



(**1 pt**)

The green solid line in the above picture is the efficient portfolio frontier which represents the expected mean and standard deviation of the simple returns of all efficient portfolios. The black solid line is the efficient portfolio frontier when there is no risk free asset. Since there is risk free asset, the consumer will choose portfolio according to the green solid line. The risk averse consumers tends to choose portfolios whose expected return and risk are closer to the risk free asset. Hence they will choose high x_f and low $x_{\tan}$. The less risk averse consumers will choose portfolios whose expected return and risk are far away from the risk free asset. Hence they will choose low x_f and high $x_{\tan}$. (**2 pts**)

3.6. [**6 pts**] In the rest questions (3.6) - (3.10), we assume that

$$r_f = 0.01,\ \boldsymbol{\mu} = \begin{pmatrix} 0.03 \\ 0.05 \end{pmatrix} \text{ and } \boldsymbol{\Sigma} = \begin{pmatrix} 0.01 & -0.008 \\ -0.008 & 0.04 \end{pmatrix}.$$

Consider the portfolio $(x_f, x_1, x_2) = (0, 0.4, 0.6)$. Find the mean and standard deviation of the simple return of this portfolio.

Answer: The simple return of any portfolio (x_f, x_1, x_2) is

$$R_p = x_f r_f + x_1 R_1 + x_2 R_2.$$

we have

$$\begin{aligned}\mu_p &= E[R_p] = x_f r_f + x_1\mu_1 + x_2\mu_2 \\ &= 0.4 * 0.03 + 0.6 * 0.05 = 0.032. \ \textbf{(3 pts)}\end{aligned}$$

Moreover,

$$\begin{aligned}\sigma_p^2 &= x_1^2\sigma_1^2 + x_2^2\sigma_2^2 + 2x_1x_2\sigma_{12} \\ &= 0.16 * 0.01 + 0.36 * 0.04 - 2 * 0.24 * 0.008 \\ &= 0.01216. \ \textbf{(3 pts)}\end{aligned}$$

3.7. [**6 pts**] The tangency portfolio $(x_1^{\tan}, x_2^{\tan})$ is determined by

$$x_1^{\tan} = \frac{(\mu_1 - r_f)\sigma_2^2 - (\mu_2 - r_f)\sigma_{12}}{(\mu_1 - r_f)\sigma_2^2 + (\mu_2 - r_f)\sigma_1^2 - (\mu_1 + \mu_2 - 2r_f)\sigma_{12}}$$

and $x_2^{\tan} = 1 - x_1^{\tan}$. Find the tangency portfolio.

Answer: Calculation show that

$$x_1^{\tan} = \frac{2}{3} \text{ and } x_2^{\tan} = \frac{1}{3} \ \textbf{(3 pts)}$$

3.8. [**4 pts**] Find the efficient portfolio which has the same expected return as MSFT.

Answer: We will use the following equation to find the efficient portfolio:

$$\mu_p^e = r_f + x_t(\mu_p^t - r_f) \ \textbf{(1 pt)}$$

where r_f is the risk free return and μ_p^t is the expected return of the tangency portfolio. Note that

$$\mu_p^t = x_1^{\tan}\mu_1 + x_2^{\tan}\mu_2 = \frac{2}{3}0.03 + \frac{1}{3}0.05 = \frac{0.11}{3}. \ \textbf{(1 pt)}$$

Therefore,

$$x_t = \frac{\mu_p^e - r_f}{\mu_p^t - r_f} = \frac{0.03 - 0.01}{\frac{0.11}{3} - 0.01} = \frac{3}{4}$$

and hence $x_f = 1 - x_t = \frac{1}{4}$. **(2 pts)**

3.9. [**4 pts**] Find the efficient portfolio which has the same risk (same standard deviation of simple return) as MSFT.

Answer: We will use the following equation to find the efficient portfolio:

$$\sigma_p^e = x_t \sigma_p^t \textbf{ (1 pt)}$$

where σ_p^t is the risk of the tangency portfolio. Note that

$$\begin{aligned}(\sigma_p^t)^2 &= (x_1^{\tan})^2\sigma_1^2 + (x_2^{\tan})^2\sigma_2^2 + 2x_1^{\tan}x_2^{\tan}\sigma_{12} \\ &= \frac{4}{9}0.01 + \frac{1}{9}0.04 - \frac{4}{9} * 0.008 = \frac{0.048}{9}. \textbf{ (1 pt)}\end{aligned}$$

Therefore,

$$x_t = \frac{\sigma_p^e}{\sigma_p^t} = \frac{0.3}{\sqrt{0.048}} = 1.37$$

and hence $x_f = 1 - x_t = -0.37$. **(2 pts)**

4. Indicate whether the following statements are true or false (circle one). Briefly discuss why it is so. **(30 pts, 6 puts each)**

4.1. If $\varepsilon_t \sim mds\ (0, \sigma_\varepsilon^2)$, then $\varepsilon_t \sim iid\ (0, \sigma_\varepsilon^2)$.

True False

Why?

Answer: False **(2 pts)**. For example, let $\varepsilon_t = e_t e_{t-1}$, where $e_t \sim iid$ $N(0, \sigma^2)$. Then

$$E[\varepsilon_t|\mathcal{F}_{t-1}] = E[e_t e_{t-1}|\mathcal{F}_{t-1}] = e_{t-1} E[e_t|\mathcal{F}_{t-1}] = e_{t-1} \cdot 0 = 0$$

and $E[\varepsilon_t^2] = E\left[e_t^2 e_{t-1}^2\right] = E[e_t^2]\, E\left[e_{t-1}^2\right] = 1$, which implies that $\varepsilon_t \sim$ $iid\ (0, 1)$. However, ε_t is not independent because

$$\begin{aligned} & Cov(\varepsilon_t^2, \varepsilon_{t-1}^2) \\ = \ & E[\varepsilon_t^2 \varepsilon_{t-1}^2] - E[\varepsilon_t^2] E[\varepsilon_{t-1}^2], \quad \text{by the definition of covariance} \\ = \ & E[e_t^2 e_{t-1}^4 e_{t-2}^2] - E[e_t^2 e_{t-1}^2] E[e_{t-1}^2 e_{t-2}^2], \quad \text{by the definitions of } \varepsilon_t \text{ and } \varepsilon_{t-1} \\ = \ & E[e_t^2] E[e_{t-1}^4] E[e_{t-2}^2] - E[e_t^2] E[e_{t-1}^2] E[e_{t-1}^2] E[e_{t-2}^2], \quad \text{by the independence} \\ = \ & 3 - 1 = 2, \quad \text{by } E[e_t^2] = 1 \text{ and } E[e_t^4] = 3 \text{ for any } t. \end{aligned}$$

Note: the justfication should be 4 points.

4.2. If $\varepsilon_t \sim mds\ (0, \sigma_\varepsilon^2)$, then $\varepsilon_t \sim WN\ (0, \sigma_\varepsilon^2)$.

True False

Why?

Answer: True **(2 pts)**. Because for any $t \neq s$ (say $t > s$),

$$\begin{aligned} & E[\varepsilon_t \varepsilon_s] \\ = \ & E[E[\varepsilon_t \varepsilon_s|\mathcal{F}_{t-1}]], \text{ by the property of the conditional expectation} \\ = \ & E[\varepsilon_s E[\varepsilon_t|\mathcal{F}_{t-1}]], \text{ by the fact that } \varepsilon_s \text{ is known given } \mathcal{F}_{t-1}\ (t > s) \\ = \ & E[\varepsilon_s \cdot 0], \quad \text{by } E[\varepsilon_t|\mathcal{F}_{t-1}] = 0 \\ = \ & E[0] \\ = \ & 0. \ \mathbf{(4\ pts)} \end{aligned}$$

4.3. If $\{\varepsilon_t\}_t$ is strictly stationary, then it is also covariance stationary.

True False

Why?

Answer: False **(2 pts)**. If ε_t has infinite variance (e.g., student-t with 1 degree of freedom), then it is not covariance stationary. **(4 pts)**

4.4. Let Y_1 and Y_2 be iid random variables with mean μ and variance σ^2, and let $\hat{\mu}_1 = Y_1$ and $\hat{\mu}_2 = \frac{Y_1+Y_2}{2}$ are two different point estimators for μ. From the MSE criteria, we prefer to use $\hat{\mu}_1$ rather than $\hat{\mu}_2$.

True False

Why?

1. **Answer:** False **(2 pts)**. Since

$$E[\hat{\mu}_1] = E[Y_1] = \mu \text{ } (\mathbf{0.5 \ pt})$$

and

$$E[\hat{\mu}_2] = E[Y_1/2 + Y_2/2] = \frac{1}{2}E[Y_1] + \frac{1}{2}E[Y_2] = \mu, (\mathbf{0.5 \ pt})$$

both estimators are unbiased. To compare their MSE, it is therefore sufficient to compare their variances.

$$Var(\hat{\mu}_1) = Var(Y_1) = \sigma^2 \text{ } (\mathbf{0.5 \ pts})$$

and

$$Var[\hat{\mu}_2] = Var[Y_1/2 + Y_2/2] = \frac{1}{4}Var[Y_1] + \frac{1}{4}Var[Y_2] = \frac{\sigma^2}{2}. \text{ } (\mathbf{0.5 \ pts})$$

It is clear that $\hat{\mu}_2$ has smaller variance and hence smaller MSE (since it is also unbiased). We should prefer to use $\hat{\mu}_2$ rather than $\hat{\mu}_1$. **(2 pts)**

4.5. MA(1) process $Y_t = \mu + \varepsilon_t + \theta\varepsilon_{t-1}$, $\varepsilon_t \sim mds(0, \sigma_\varepsilon^2)$ is *not covariance stationary* when $|\theta| = 1$.

True False

Why?

Answer: False **(2 pts)**. We have show that Y_t is covariance stationary as long as θ is a finite number. **(4 pts)**

5. Bonus Questions **(5 pts, 1 puts each)**. Briefly explain what are the following R commands:

5.1. abline

Answer: add straights line to plot

5.2. tail

Answer: show last few rows of data object

5.3. rmvnorm

Answer: generate normal random variables

5.4. solve(A)

Answer: find the inverse of the matrix A

5.5. qnorm

Answer: find the quantile of normal random variable