ECON 147 Homework 2

Answer Keys

1 Review Questions

1. Suppose X is a uniform random variable over [0,1] (i.e., $X \backsim U[0,1]$) and Y is a Bernoulli random variable with a success probability $\Pr(Y = 1) = 0.5, i.e.,$

$$Y = \begin{cases} 1, & \text{with probability } 0.5 \\ 0, & \text{with probability } 0.5 \end{cases}.$$

Compute the following (You will not need any program for this problem, just solve by hands after your readings.)

- $\Pr(X < 0.1)$ and $\Pr(Y < 0.1)$:

$$\begin{aligned} \Pr(X &< 0.1) = \int_0^{0.1} 1dx = 0.1; \\ \Pr(Y &< 0.1) = \Pr(Y = 0) = 0.5. \end{aligned}$$

- $E(X), Var(X)$, $E(Y)$ and $Var(Y)$:

$$\begin{aligned} E(X) &= \int_0^1 xdx = \left.\frac{x^2}{2}\right|_0^1 = 1/2; \\ Var(X) &= E[X^2] - (E(X))^2 = \int_0^1 x^2dx - \frac{1}{4} = \left.\frac{x^2}{3}\right|_0^1 - \frac{1}{4} = \frac{1}{12}; \\ E(Y) &= \frac{1}{2} \times 1 + \frac{1}{2} \times 0 = \frac{1}{2}; \\ Var(Y) &= E[Y^2] - (E(Y))^2 = \frac{1}{2} \times 1^2 + \frac{1}{2} \times 0^2 - \frac{1}{4} = \frac{1}{4}. \end{aligned}$$

- $E(0.3X+0.7Y)$ and $E(0.5X+0.5Y)$:

$$\begin{aligned} E(0.3X+0.7Y) &= 0.3E(X)+0.7E(Y)=\frac{0.3}{2}+\frac{0.7}{2}=\frac{1}{2}; \\ E(0.5X+0.5Y) &= 0.5E(X)+0.5E(Y)=\frac{0.5}{2}+\frac{0.5}{2}=\frac{1}{2}. \end{aligned}$$

- For any $\alpha \in [0,1]$, $E(\alpha X+(1-\alpha)Y)$:

$$E(\alpha X+(1-\alpha)Y)=\alpha E(X)+(1-\alpha)E(Y)=\frac{\alpha}{2}+\frac{1-\alpha}{2}=\frac{1}{2}.$$

- Now suppose X and Y represent (statistically independent) outcomes of two lotteries, and you would like to invest your \$100 to these lotteries. Assume you only care about the mean (preferably high) and variance (preferably low) of your investment. How do you want to distribute your \$100, i.e., how do you choose your $\alpha \in [0,1]$ for $\alpha X+(1-\alpha)Y$? Justify your answer.
  - *Since* $E(\alpha X+(1-\alpha)Y)=1/2$ *for all* α, *we can just compute variance,*

$$\begin{aligned} Var(\alpha X+(1-\alpha)Y) &= \alpha^2 Var(X)+(1-\alpha)^2 Var(Y) \\ &= \alpha^2\frac{1}{12}+(1-\alpha)^2\frac{1}{4} \\ &= \frac{1}{3}\alpha^2-\frac{1}{2}\alpha+\frac{1}{4}, \end{aligned}$$

    and it is easy to show the variance is minimum at $\alpha=3/4$ *from the first order condition*

$$\frac{d\{Var(\alpha X+(1-\alpha)Y)\}}{d\alpha}=\frac{2}{3}\alpha-\frac{1}{2}=0.$$

    (check the second order conditions). Therefore, you will choose $\alpha=3/4$ *to minimize the variance.*

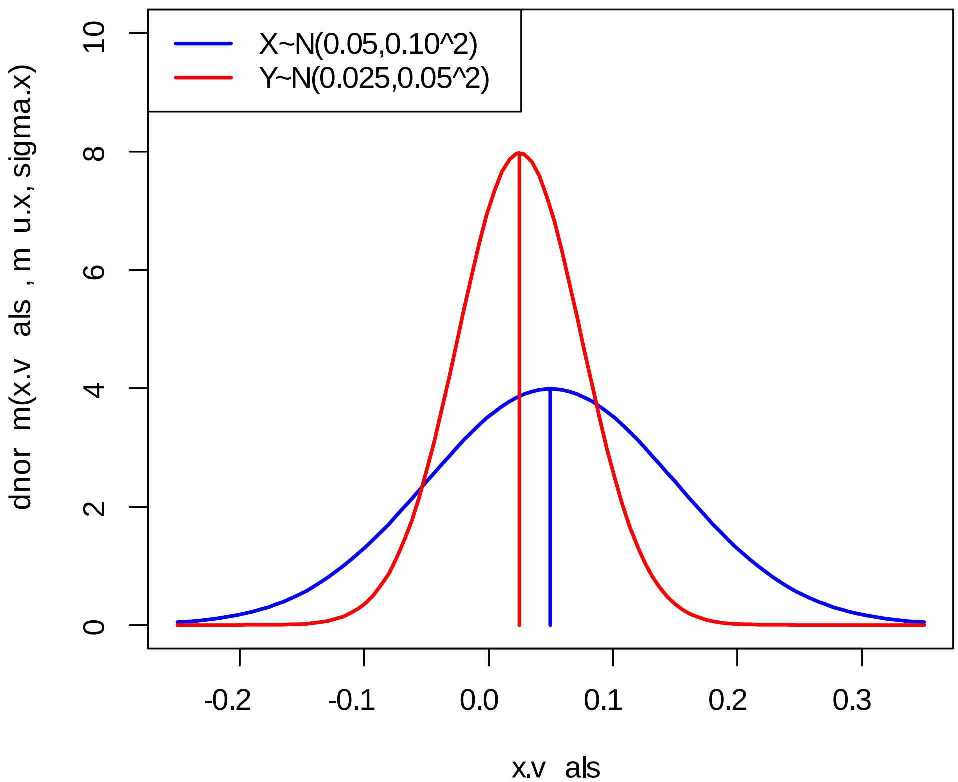
2. Suppose X is a normally distributed random variable with mean 0 and variance 1 (i.e., standard normal). Compute the following (Hint: you can use the R functions *pnorm* and *qnorm* to answer these questions).

   - $Pr(X<-1.96)$ =0.025, or 2.5% app. (approximately, hereafter)

- $Pr(X > 1.64)$ =0.0505, or 5% app.
- $Pr(-0.5 < X < 0.5)$ =0.3829, or 38.3% app.
- 1% quantile, $q_{.01}$ and 99% quantile, $q_{.99}$: -2.326 and 2.326, respectively.
- 5% quantile, $q_{.05}$ and 95% quantile, $q_{.95}$: -1.645 and 1.645, respectively.

3. . Let X denote the monthly return on Microsoft Stock and let Y denote the monthly return on Starbucks stock. Assume that $X \sim N(0.05, (0.10)^2)$ and $Y \sim N(0.025, (0.05)^2)$.

   - Using a grid of values between –0.25 and 0.35, plot the normal curves for X and Y. Make sure that both normal curves are on the same plot.



- Comment on the risk-return tradeoffs for the two stocks, which one do you want to invest? *X has a higher expected return (mean) but also is riskier (variance), so individual investor would decide which one to invest, based on his risk aversion.*

4. Let R denote the simple monthly return on Microsoft stock and let W_0 denote initial wealth to be invested over the month. Assume that $R \sim N(0.05, (0.12)^2)$ and that $W_0 = \$100{,}000$.

   - Determine the 1% and 5% value-at-risk (VaR) over the month on the investment. That is, determine the loss in investment value that may occur over the next month with 1% probability and with 5% probability.

$$\begin{array}{ll} 5\% \text{ VaR:} & -\$14,738 \\ 1\% \text{ VaR:} & -\$22,916 \end{array}$$

So with 5% probability, we could lose \$14,738 or more over the next month out of our initial \$100,000 investment; with 1% probability, we could lose \$22,916 or more.

5. Let r denote the continuously compounded monthly return on Microsoft stock and let W_0 denote initial wealth to be invested over the month. Assume that $r \sim iid\ N(0.05, (0.12)^2)$ and that $W_0 = \$100{,}000$.

   - Determine the 1% and 5% value-at-risk (VaR) over the month on the investment. That is, determine the loss in investment value that may occur over the next month with 1% probability and with 5% probability. (Hint: compute the 1% and 5% quantile from the Normal distribution for r and then convert continuously compounded return quantile to a simple return quantile using the transformation $R = e^r - 1$.)

$$\begin{array}{ll} 5\% \text{ VaR:} & -\$13,704 \\ 1\% \text{ VaR:} & -\$20,480 \end{array}$$

   - Determine the 1% and 5% value-at-risk (VaR) over the year on the investment. (Hint: to answer this question, you must determine the normal distribution that applies to the annual (12 month) continuously compounded return.) *In general, assume r_t is continuously compounded (cc) 1-month returns with $r_t \sim iid\ N(\mu, \sigma^2)$. Then the distribution of annualized cc returns $r_A = \sum_{j=0}^{11} r_{t+j}$ is distributed as $r_A \sim N(12\mu, 12\sigma^2)$ since*

$$\begin{aligned} Var\left(\sum_{j=0}^{11} r_{t+j}\right) &= \sum_{j=0}^{11} Var(r_{t+j}) \text{ (from independence assumption)} \\ &= 12\sigma^2 \text{ (from identical distribution assumption)} \end{aligned}$$

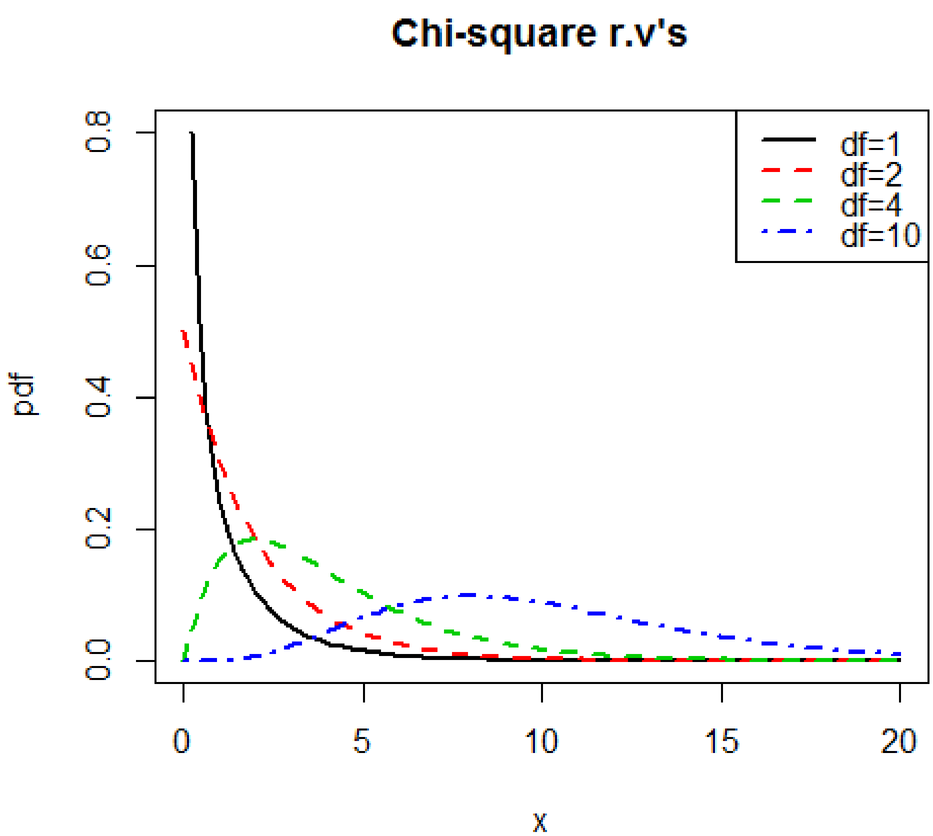
In this problem,

$$\begin{aligned} r_A &\sim N\left(12 \cdot 0.05, 12(0.12)^2\right) \\ &= N\left(12 \cdot 0.05, (\sqrt{12} \cdot 0.12)^2\right) \end{aligned}$$

5% VaR:	$-\$8,034$
1% VaR:	$-\$30,722$

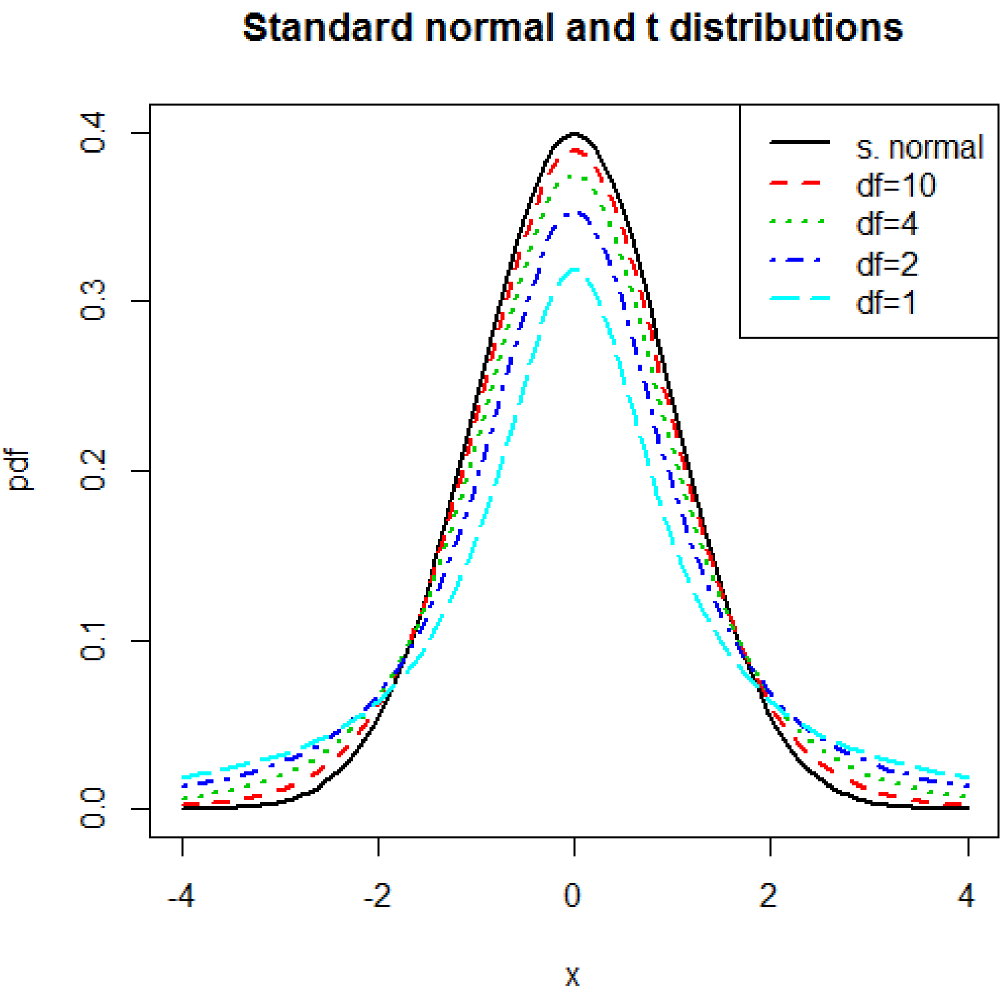
6. In this question, you will examine the chi-square and Student's t distributions.

    - On the same graph, plot the probability curves of chi-squared distributed random variables with 1, 2, 4 and 10 degrees of freedom. Use different colors and line styles for each curve.



    - On the same graph, plot the probability curves of Student's t distributed random variables with 1, 2, 4 and 10 degrees of freedom.

Also include the probability curve for the standard normal distribution. Use different colors and line styles for each curve.



- Without doing any actual calculation, which of 5% VaR will be greater (in absolute value) between standard normal and t distribution with d.f. 2? *5% VaR of t distribution with d.f. 2 will have a greater absolute value since its has fatter tails than standard normal. Therefore, assuming normality for financial asset return, that often has heavy tails, may result in "underestimation* *of risk".*

7. Consider the following joint distribution of X and Y:

Bivariate pdf

		Y		
	%	0	1	Pr(X)
	0	1/8	0	1/8
X	1	1/8	1/8	2/8
	2	1/8	3/8	4/8
	3	0	1/8	1/8
	Pr(Y)	3/8	5/8	1

- Find the marginal distributions of X and Y. Using these distributions, compute E[X], Var(X), SD(X), E[Y], Var(Y) and SD(Y). *See above for the marginal distribution.*

$$
\begin{aligned}
E(X) &= \frac{2}{8}+\frac{8}{8}+\frac{3}{8}=\frac{13}{8}, \\
E(X^2) &= \frac{2}{8}+\frac{16}{8}+\frac{9}{8}=\frac{27}{8}, \\
Var(X) &= E(X^2)-\{E(X)\}^2=\frac{27}{8}-\frac{169}{64}=\frac{47}{64}, \\
SD(X) &= \sqrt{\frac{47}{64}}.
\end{aligned}
$$

$$
\begin{aligned}
E(Y) &= \frac{5}{8}=E(Y^2), \\
Var(Y) &= \frac{5}{8}-\left(\frac{5}{8}\right)^2=\frac{15}{64}, \\
SD(Y) &= \sqrt{\frac{15}{64}}.
\end{aligned}
$$

- Compute COV(X,Y) and CORR(X,Y).

$$
\begin{aligned}
E(XY) &= \frac{1}{8}+\frac{6}{8}+\frac{3}{8}=\frac{10}{8}, \\
COV(X,Y) &= \frac{10}{8}-\frac{13}{8}\cdot\frac{5}{8}=\frac{15}{64}.
\end{aligned}
$$

- Are X and Y independent? Fully justify your answer. *No, since* $COV(X,Y) \neq 0$.
- $$f(X=0|Y=0) = \frac{f(X=0,Y=0)}{f(Y=0)} = \frac{1/8}{3/8} = 1/3$$

  and so on, so

  - $f(X=0|Y=0) = f(X=1|Y=0) = f(X=2|Y=0) = 1/3$,
  - $f(X=3|Y=0) = 0$
  - $f(Y=0|X=2) = 1/4, f(Y=1|X=2) = 3/4$,
- E(X|Y=0)=1 and E(Y|X=2)=3/4
- Var(X|Y=0)=5/3-1=2/3 and Var(Y|X=2)=3/4-9/16=3/16.