ECON 147 Homework 5

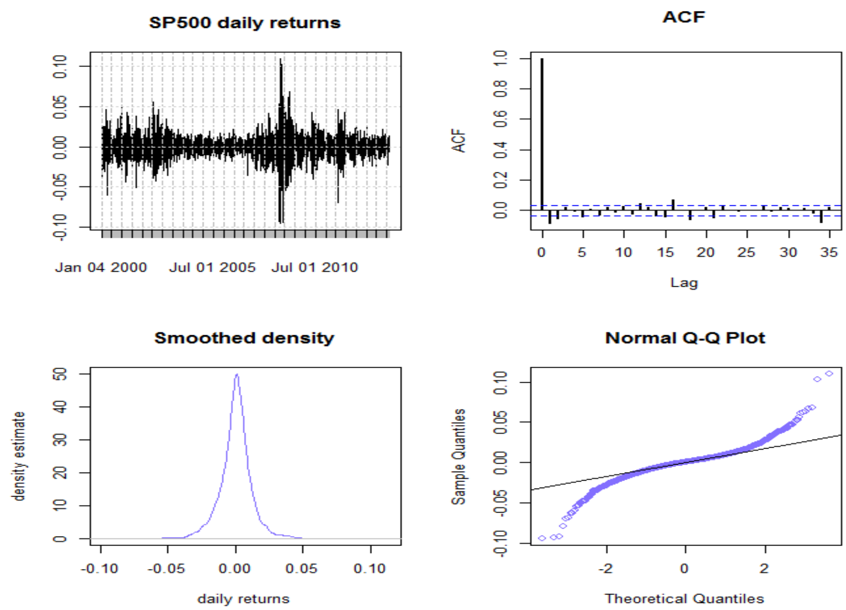
Due: 3:30 pm, March 6th

Reading and Program Downloads

- Please read the course material on the course website.
- For R exercises, the R code (econ147lab5_Hint.r) will be helpful. Make sure you understand the output rather than just following the hint.

Review Questions

1. The followings are descriptive statistics of daily cc returns on S&P 500 index (Jan 3, 2000 - Feb 21, 2014: T=3,555).



(a) Describe the stylized facts on the return series based on the above information.

Answer: From the first panel of the graph, we see that: i) the daily cc return is variating around its mean which is close to zero; ii) the volatility of the daily cc return is dependent or clustering. From the ACF, we see that the daily cc return is (almost) an uncorrelated process since its autocorrrelation coefficient function is close to zero. The smoothed density of the cc return shows that the density of the cc return is bell-shaped but the QQ plot in the last panel shows that it is not normally distributed and has heavier tails than the normal distribution.

(b) **(ARCH(1) Model)** Assuming that we use the following model:

$$\begin{aligned} r_t &= \sigma_t e_t,\ e_t \sim iid\ N(0,1),\ t = 1, ..., T \\ \sigma_t^2 &= \omega + \alpha_1 r_{t-1}^2,\ \omega > 0 \text{ and } \alpha_1 \geq 0. \end{aligned}$$

Show that this model can generate the stylized facts you explained in (a).

Answer: (1) We first show that $\{r_t\}_t$ has mean zero and is an uncorrelated process. Since $\sigma_t^2 = \omega + \alpha_1 r_{t-1}^2$ is known at time $t-1$,

$$\begin{aligned} E\left[r_t|\mathcal{F}_{t-1}\right] &= E\left[\sigma_t e_t|\mathcal{F}_{t-1}\right] && \text{by the definition of } r_t \\ &= \sigma_t E\left[e_t|\mathcal{F}_{t-1}\right] && \text{by } \sigma_t^2 \text{ is known at } t-1 \\ &= 0 && \text{by } e_t \sim iid\ N(0,1). \end{aligned}$$

This shows that $\{r_t\}_t$ is an m.d.s. and hence mean zero and uncorrelated process.

(2) We show the volatility clustering of $\{r_t\}_t$. Define

$$v_t = r_t^2 - \sigma_t^2.$$

Then

$$\begin{aligned} & E\left[v_t|\mathcal{F}_{t-1}\right] \\ = & E\left[r_t^2 - \sigma_t^2|\mathcal{F}_{t-1}\right] && \text{by the definition of } v_t \\ = & E\left[r_t^2|\mathcal{F}_{t-1}\right] - E\left[\sigma_t^2|\mathcal{F}_{t-1}\right] && \text{by the linearity of the conditional expectation} \\ = & E\left[\sigma_t^2 e_t^2|\mathcal{F}_{t-1}\right] - E\left[\sigma_t^2|\mathcal{F}_{t-1}\right] && \text{by the definition of } r_t \\ = & \sigma_t^2 E\left[e_t^2|\mathcal{F}_{t-1}\right] - \sigma_t^2 && \text{by } \sigma_t^2 \text{ is known at } t-1 \\ = & \sigma_t^2 E\left[e_t^2\right] - \sigma_t^2 && \text{by } e_t \sim iid\ N(0,1) \\ = & 0 && \text{by } E\left[e_t^2\right] = 1 \end{aligned}$$

which implies that $\{v_t\}_t$ is an m.d.s. Then

$$r_t^2 = \sigma_t^2 + r_t^2 - \sigma_t^2 = \omega + \alpha_1 r_{t-1}^2 + v_t$$

which implies that $\{r_t^2\}_t$ is an AR(1) process and hence dependent when $\alpha_1 \neq 0$. This shows the volatility clustering of $\{r_t\}_t$.

(3) We show the heavy tail of $\{r_t\}_t$.

$$\begin{aligned} & E\left[r_t^4|\mathcal{F}_{t-1}\right] \\ = & E\left[\sigma_t^4 e_t^4|\mathcal{F}_{t-1}\right] && \text{by the definition of } v_t \\ = & \sigma_t^4 E\left[e_t^4|\mathcal{F}_{t-1}\right] && \text{by } \sigma_t^4 \text{ is known at } t-1 \\ = & \sigma_t^4 E\left[e_t^4\right] && \text{by } e_t \sim iid\ N(0,1) \\ = & 3\sigma_t^4 && \text{by } E\left[e_t^4\right] = 3 \end{aligned}$$

We will use the Jensen's inequality: $E[X^2] \geq (E[X])^2$ for any random variable X. Therefore

$$
\begin{aligned}
& E[r_t^4] \\
= \;& 3E[\sigma_t^4] \qquad\qquad \text{by } E\left[r_t^4|\mathcal{F}_{t-1}\right] = 3\sigma_t^4 \\
= \;& 3E[(E\left[r_t^2|\mathcal{F}_{t-1}\right])^2] \;\; \text{by } \sigma_t^2 = E\left[r_t^2|\mathcal{F}_{t-1}\right] \\
\geq \;& 3(E[E\left[r_t^2|\mathcal{F}_{t-1}\right]])^2 \;\; \text{by the Jensen's inequality with } X = E\left[r_t^2|\mathcal{F}_{t-1}\right] \\
= \;& 3(E[r_t^2])^2 \qquad\quad \text{by the property of the conditional expectation}
\end{aligned}
$$

which shows that

$$\frac{E[r_t^4]}{(E[r_t^2])^2} \geq 3$$

i.e., the kurtosis of r_t is larger than 3.

(c) **(GARCH(1,1) Model)** Now we turn to the following model:

$$
\begin{aligned}
r_t &= \sigma_t e_t, \; e_t \sim iid\ N(0,1), \; t = 1, ..., T \\
\sigma_t^2 &= \omega + \alpha_1 r_{t-1}^2 + \beta_1 \sigma_{t-1}^2, \; \omega > 0, \alpha_1 \geq 0 \text{ and } \beta_1 \geq 0.
\end{aligned}
$$

Show that this model can be interpreted as ARMA (1,1) model for squared returns.

Answer: (1) We first show that $\{r_t\}_t$ has mean zero and is an uncorrelated process. Since $\sigma_t^2 = \omega + \alpha_1 r_{t-1}^2 + \beta_1 \sigma_{t-1}^2$ is known at time $t-1$,

$$
\begin{aligned}
E\left[r_t|\mathcal{F}_{t-1}\right] &= E\left[\sigma_t e_t|\mathcal{F}_{t-1}\right] \qquad \text{by the definition of } r_t \\
&= \sigma_t E\left[e_t|\mathcal{F}_{t-1}\right] \qquad \text{by } \sigma_t^2 \text{ is known at } t-1 \\
&= 0 \qquad\qquad\qquad \text{by } e_t \sim iid\ N(0,1).
\end{aligned}
$$

This shows that $\{r_t\}_t$ is an m.d.s. and hence mean zero and uncorrelated process.

(2) We show the volatility clustering of $\{r_t\}_t$. Define

$$v_t = r_t^2 - \sigma_t^2.$$

Then

$$\begin{aligned}
& E\left[v_t|\mathcal{F}_{t-1}\right] \\
= \; & E\left[r_t^2-\sigma_t^2|\mathcal{F}_{t-1}\right] && \text{by the definition of } v_t \\
= \; & E\left[r_t^2|\mathcal{F}_{t-1}\right]-E\left[\sigma_t^2|\mathcal{F}_{t-1}\right] && \text{by the linearity of the conditional expectation} \\
= \; & E\left[\sigma_t^2 e_t^2|\mathcal{F}_{t-1}\right]-E\left[\sigma_t^2|\mathcal{F}_{t-1}\right] && \text{by the definition of } r_t \\
= \; & \sigma_t^2 E\left[e_t^2|\mathcal{F}_{t-1}\right]-\sigma_t^2 && \text{by } \sigma_t^2 \text{ is known at } t-1 \\
= \; & \sigma_t^2 E\left[e_t^2\right]-\sigma_t^2 && \text{by } e_t \sim iid\ N(0,1) \\
= \; & 0 && \text{by } E\left[e_t^2\right]=1
\end{aligned}$$

which implies that $\{v_t\}_t$ is an m.d.s. Then

$$\begin{aligned}
r_t^2 &= \sigma_t^2+r_t^2-\sigma_t^2 \\
&= \omega+\alpha_1 r_{t-1}^2+\beta_1\sigma_{t-1}^2+v_t \\
&= \omega+\alpha_1 r_{t-1}^2+\beta_1(\sigma_{t-1}^2-r_{t-1}^2+r_{t-1}^2)+v_t \\
&= \omega+(\alpha_1+\beta_1)r_{t-1}^2-\beta_1(r_{t-1}^2-\sigma_{t-1}^2)+v_t \\
&= \omega+(\alpha_1+\beta_1)r_{t-1}^2-\beta_1 v_{t-1}+v_t
\end{aligned}$$

which implies that $\{r_t^2\}_t$ is an ARMA(1,1) process and hence dependent when $\alpha_1+\beta_1\neq 0$. This shows the volatility clustering of $\{r_t\}_t$.
(3) We show the heavy tail of $\{r_t\}_t$.

$$\begin{aligned}
& E\left[r_t^4|\mathcal{F}_{t-1}\right] \\
= \; & E\left[\sigma_t^4 e_t^4|\mathcal{F}_{t-1}\right] && \text{by the definition of } v_t \\
= \; & \sigma_t^4 E\left[e_t^4|\mathcal{F}_{t-1}\right] && \text{by } \sigma_t^4 \text{ is known at } t-1 \\
= \; & \sigma_t^4 E\left[e_t^4\right] && \text{by } e_t \sim iid\ N(0,1) \\
= \; & 3\sigma_t^4 && \text{by } E\left[e_t^4\right]=3
\end{aligned}$$

We will use the Jensen's inequality: $E[X^2]\geq(E[X])^2$ for any random variable X. Therefore

$$\begin{aligned}
& E[r_t^4] \\
= \; & 3E[\sigma_t^4] && \text{by } E\left[r_t^4|\mathcal{F}_{t-1}\right]=3\sigma_t^4 \\
= \; & 3E[(E\left[r_t^2|\mathcal{F}_{t-1}\right])^2] && \text{by } \sigma_t^2=E\left[r_t^2|\mathcal{F}_{t-1}\right] \\
\geq \; & 3(E[E\left[r_t^2|\mathcal{F}_{t-1}\right]])^2 && \text{by the Jensen's inequality with } X=E\left[r_t^2|\mathcal{F}_{t-1}\right] \\
= \; & 3(E[r_t^2])^2 && \text{by the property of the conditional expectation}
\end{aligned}$$

which shows that

$$\frac{E[r_t^4]}{(E[r_t^2])^2} \geq 3$$

i.e., the kurtosis of r_t is larger than 3.

R Exercises

The following questions require R. On the course website is the R script files econ147lab5_Hint.r. The file contains hints for completing this R exercises. Copy and paste all statistical results and graphs into a MS Word document (or your favorite word processor) while you work, and add any comments and answer all questions in this document. Start MS Word and open a blank document. You will save all of your work in this document.

In this lab, you will analyze continuously compounded monthly return data on the S&P 500 index (^GSPC) and Microsoft stock (MSFT). I encourage you to go to finance.yahoo.com and research these assets. The script file econ147lab5_Hint.r walks you through all of the computations for the lab. You will also need to install several packages.

1. Do the following exercises.

   (a) Make time plots of the return data from 2000-01-03 to 2014-02-21. Comment on any stylized fact on returns suggested by the plots.

   (b) For each return series, make a four panel plot containing a return plot, acf, density plot and normal QQ-plot. Do the return series look normally distributed?

   (c) Testing normality of each return distribution using Jarque-Bera test statistics.

   (d) Now estimate GARCH(1,1) model parameters (as in Review Questions) and report the estimated values of $\alpha_1 + \beta_1$. How do you interpret these results?

   (e) Plot the real cc returns and the fitted cc returns for MSFT and GSPC respectively. Comment on what you find.

   (f) For parameter α_1 and β_1 compute 95% and (asymptotic) confidence intervals.

   (g) Test $H_0 : \alpha_1 = 0$ with 95% confidence level for each returns. Do the test $H_0 : \beta_1 = 0.9$ as well.