Econ 147 (Spring 2018)
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Supplemental Notes on Time Series Econometrics

1 AR(1)

A time series $\{Y_t\}_{t=-\infty}^{+\infty}$ is called a first order autoregressive (AR(1)) process if it satisfies

$$Y_t - \mu = \phi(Y_{t-1} - \mu) + \varepsilon_t \tag{1}$$

where $\{\varepsilon_t\}_{t=-\infty}^{+\infty}$ is an i.i.d.$(0, \sigma_\varepsilon^2)$ process and $|\phi| < 1$. We first show that

$$Y_t = \mu + \sum_{k=0}^{\infty} \phi^k \varepsilon_{t-k} \tag{2}$$

is the solution of the first-order difference equation (1). By definition,

$$\phi(Y_{t-1} - \mu) = \sum_{k=0}^{\infty} \phi^{k+1} \varepsilon_{t-1-k} = \sum_{k=1}^{\infty} \phi^k \varepsilon_{t-k}$$

Therefore,

$$\begin{aligned} & (Y_t - \mu) - \phi(Y_{t-1} - \mu) \\ = \ & \sum_{k=0}^{\infty} \phi^k \varepsilon_{t-k} - \sum_{k=1}^{\infty} \phi^k \varepsilon_{t-k} = \varepsilon_t \end{aligned}$$

which verifies (1).

We next show that Y_t is covariance stationary using the representation in (2). The dominated convergence theorem (DCT) will be used in the proof. Let

$$B = |\mu| + \sum_{k=0}^{\infty} \left| \phi^k \varepsilon_{t-k} \right|. \tag{3}$$

Then

$$
\begin{aligned}
E[B] &= |\mu| + E\left[\sum_{k=0}^{\infty}\left|\phi^k \varepsilon_{t-k}\right|\right] \\
&= |\mu| + \sum_{k=0}^{\infty} E\left[\left|\phi^k \varepsilon_{t-k}\right|\right] \\
&= |\mu| + \sum_{k=0}^{\infty} |\phi|^k E\left[|\varepsilon_{t-k}|\right] \\
&\leq |\mu| + \sum_{k=0}^{\infty} |\phi|^k \left(E\left[|\varepsilon_{t-k}|^2\right]\right)^{1/2}
\end{aligned}
$$

where the second equality is by the monotone convergence theorem (MCT), the inequality is by Lyapunov's inequality. Since $E[|\varepsilon_{t-k}|^2] = \sigma_\varepsilon^2$, we deduce that

$$
E[B] \leq |\mu| + \sigma_\varepsilon \sum_{k=0}^{\infty} |\phi|^k = |\mu| + \frac{\sigma_\varepsilon}{1-|\phi|} < \infty. \tag{4}
$$

Moreover,

$$
\begin{aligned}
E[B^2] &\leq 2\mu^2 + 2E\left[\sum_{k=0}^{\infty}\left|\phi^k \varepsilon_{t-k}\right| \sum_{j=0}^{\infty}\left|\phi^j \varepsilon_{t-j}\right|\right] \\
&= 2\mu^2 + 2E\left[\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\left|\phi^{k+j} \varepsilon_{t-k}\varepsilon_{t-j}\right|\right] \\
&= 2\mu^2 + 2\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\left|\phi^{k+j}\right| E\left[|\varepsilon_{t-k}\varepsilon_{t-j}|\right] \\
&\leq 2\mu^2 + 2\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\left|\phi^{k+j}\right| (E\left[\varepsilon_{t-k}^2\right])^{1/2} (E\left[\varepsilon_{t-j}^2\right])^{1/2}
\end{aligned}
$$

where the first inequality is by the simple inequality $(a+b)^2 \leq 2a^2 + 2b^2$, the last inequality is by Hölder's inequality. Since $E\left[\varepsilon_{t-k}^2\right] = \sigma_\varepsilon^2 = E[\varepsilon_{t-j}^2]$, we deduce that

$$
\begin{aligned}
E[B^2] &\leq 2\mu^2 + 2\sigma_\varepsilon^2 \sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\left|\phi^{k+j}\right| \\
&= 2\mu^2 + 2\sigma_\varepsilon^2 \sum_{k=0}^{\infty} |\phi|^k \sum_{j=0}^{\infty} |\phi|^j \\
&= 2\mu^2 + \frac{2\sigma_\varepsilon^2}{(1-|\phi|)^2} < \infty. \qquad (5)
\end{aligned}
$$

We will use B as the dominating variable for Y_t, and B^2 as the dominating variable for $Y_t Y_{t-j}$ for any t and for any j.

We next calculate the mean, variance and covariance of $\{Y_t\}_{t=-\infty}^{+\infty}$ using the representation in (2). First for any t,

$$\begin{aligned}
E[Y_t] &= E\left[\mu + \sum_{k=0}^{\infty} \phi^k \varepsilon_{t-k}\right] \\
&= E[\mu] + E\left[\sum_{k=0}^{\infty} \phi^k \varepsilon_{t-k}\right] \\
&= \mu + \sum_{k=0}^{\infty} E\left[\phi^k \varepsilon_{t-k}\right] \\
&= \mu + \sum_{k=0}^{\infty} \phi^k E[\varepsilon_{t-k}] \\
&= \mu + \sum_{k=0}^{\infty} \phi^k \cdot 0 = \mu
\end{aligned}$$

where the third equality is by $|Y_t| < B$, $E[B] < \infty$ and the dominated convergence theorem. This shows that the mean of Y_t is a constant μ which does not change with t.

Second for any t and any j,

$$\begin{aligned}
COV(Y_t, Y_{t-j}) &= E[(Y_t - \mu)(Y_{t-j} - \mu)] \\
&= E\left[\sum_{k=0}^{\infty} \phi^k \varepsilon_{t-k} \sum_{s=0}^{\infty} \phi^s \varepsilon_{t-j-s}\right] = E\left[\sum_{s=0}^{\infty}\sum_{k=0}^{\infty} \phi^{k+s} \varepsilon_{t-k}\varepsilon_{t-j-s}\right] \\
&= \sum_{s=0}^{\infty}\sum_{k=0}^{\infty} E\left[\phi^{k+s} \varepsilon_{t-k}\varepsilon_{t-j-s}\right] = \sum_{s=0}^{\infty}\sum_{k=0}^{\infty} \phi^{k+s} E[\varepsilon_{t-k}\varepsilon_{t-j-s}] \\
&= \sum_{s=0}^{\infty}\sum_{k=j+s} \phi^{k+s} E[\varepsilon_{t-k}\varepsilon_{t-j-s}] = \sum_{s=0}^{\infty} \phi^{2s+j} E\left[\varepsilon_{t-k}^2\right] \\
&= \sigma_\varepsilon^2 \sum_{s=0}^{\infty} \phi^{2s+j} = \frac{\phi^j \sigma_\varepsilon^2}{1-\phi^2}
\end{aligned}$$

where the third equality is by $|Y_t Y_{t-j}| < B^2$, $E[B^2] < \infty$ and the dominated convergence theorem. This shows that the variance of Y_t is

$$COV(Y_t, Y_t) = \frac{\phi^0 \sigma_\varepsilon^2}{1-\phi^2} = \frac{\sigma_\varepsilon^2}{1-\phi^2} < \infty$$

and the covariance $\{Y_t\}_{t=-\infty}^{+\infty}$ does not depend on t. Therefore $\{Y_t\}_{t=-\infty}^{+\infty}$ is a covariance stationary process.

The AR(1) process $\{Y_t\}_{t=-\infty}^{+\infty}$ has mean μ and autocovariance function

$$\gamma_j = COV(Y_t, Y_{t-j}) = \frac{\phi^j \sigma_\varepsilon^2}{1-\phi^2}$$

and autocorrelation function

$$\rho_j = \frac{\gamma_j}{\gamma_0} = \phi^j$$

as we have derived in class. The argument we used in class to derive the mean, autocovariance function and autocorrelation function of $\{Y_t\}_{t=-\infty}^{+\infty}$ is directly from the first-order difference equation (1). This can be justified only when one has shown that $\{Y_t\}_{t=-\infty}^{+\infty}$ is covariance stationary (which is what we have proved above).

2 ARMA(1,1)

A time series $\{Y_t\}_{t=-\infty}^{+\infty}$ is called an ARMA(1,1) process if it satisfies

$$Y_t = \phi Y_{t-1} + \varepsilon_t + \theta\varepsilon_{t-1} \tag{6}$$

where $\{\varepsilon_t\}_{t=-\infty}^{+\infty}$ is an i.i.d.$(0, \sigma_\varepsilon^2)$ process, $|\phi| < 1$ and $|\theta| < \infty$. Define

$$u_t = \varepsilon_t + \theta\varepsilon_{t-1}. \tag{7}$$

Then $\{u_t\}_{t=-\infty}^{+\infty}$ is an MA(1) process. In class, we have show that $\{u_t\}_{t=-\infty}^{+\infty}$ is covariance stationary with mean 0 and autocovariance function

$$\gamma_{u,j} = COV(u_t, u_{t-j}) = \left\{ \begin{array}{cc} (1+\theta^2)\sigma_\varepsilon^2, & j = 0 \\ \theta\sigma_\varepsilon^2, & |j| = 1 \\ 0, & |j| > 1 \end{array} \right. . \tag{8}$$

Therefore we can write

$$Y_t = \phi Y_{t-1} + u_t. \tag{9}$$

By the same arguments of showing that (2) is the solution of (1), we can show that the solution of (9) is

$$Y_t = \sum_{k=0}^{\infty} \phi^k u_{t-k}. \tag{10}$$

Define $B_u = \sum_{k=0}^{\infty} \left|\phi^k u_{t-k}\right|$. Using similar arguments in showing (4) and (5), we can show that

$$E[B_u] < \infty \text{ and } E[B_u^2] < \infty. \tag{11}$$

We next calculate the mean, variance and covariance of $\{Y_t\}_{t=-\infty}^{+\infty}$ using the representation in (10).

First for any t,

$$\begin{aligned} E\left[Y_t\right] &= E\left[\sum_{k=0}^{\infty} \phi^k u_{t-k}\right] = \sum_{k=0}^{\infty} E\left[\phi^k u_{t-k}\right] \\ &= \sum_{k=0}^{\infty} \phi^k E\left[u_{t-k}\right] = \sum_{k=0}^{\infty} \phi^k \cdot 0 = 0 \end{aligned}$$

where the second equality is by $|Y_t| < B_u$, $E[B_u] < \infty$ and the dominated convergence theorem. This shows that the mean of Y_t is a constant μ which does not change with t.

Second for any t and any $j > 0$,

$$
\begin{aligned}
COV(Y_t, Y_{t-j}) &= E\left[Y_t Y_{t-j}\right] = E\left[\sum_{k=0}^{\infty} \phi^k u_{t-k} \sum_{s=0}^{\infty} \phi^s u_{t-j-s}\right] \\
&= E\left[\sum_{k=0}^{\infty}\sum_{s=0}^{\infty} \phi^{k+s} u_{t-k} u_{t-j-s}\right] \\
&= \sum_{s=0}^{\infty}\sum_{k=0}^{\infty} \phi^{k+s} E\left[u_{t-k} u_{t-j-s}\right] \\
&= \sum_{s=0}^{\infty}\sum_{k=0}^{\infty} \phi^{k+s} \gamma_{u,j+s-k} \\
&= \sum_{s=0}^{\infty} (\phi^{j+s+s}\gamma_{u,0} + \phi^{j+s-1+s}\gamma_{u,1} + \phi^{j+s+1+s}\gamma_{u,-1}) \qquad (12)
\end{aligned}
$$

where the fourth equality is by $|Y_t Y_{t-j}| < B_u^2$, $E[B_u^2] < \infty$ and the dominated convergence theorem, the last equality is by $\gamma_{u,j} = 0$ for any $|j| > 1$. Since

$$\gamma_{u,0} = (1+\theta^2)\sigma_\varepsilon^2 \text{ and } \gamma_{u,1} = \theta\sigma_\varepsilon^2 = \gamma_{u,-1},$$

we have

$$
\begin{aligned}
COV(Y_t, Y_{t-j}) &= \gamma_{u,0}\phi^j \sum_{s=0}^{\infty} \phi^{2s} + \gamma_{u,1}\phi^{j-1} \sum_{s=0}^{\infty} \phi^{2s} + \gamma_{u,1}\phi^{j+1} \sum_{s=0}^{\infty} \phi^{2s} \\
&= \frac{\phi^j \gamma_{u,0}}{1-\phi^2} + \frac{\phi^{j-1}\gamma_{u,1}}{1-\phi^2} + \frac{\phi^{j+1}\gamma_{u,1}}{1-\phi^2} \\
&= \frac{1+\theta^2+\theta\phi+\theta\phi^{-1}}{1-\phi^2}\phi^j\sigma_\varepsilon^2 \\
&= \frac{(\phi+\theta)(1+\theta\phi)}{1-\phi^2}\phi^{j-1}\sigma_\varepsilon^2
\end{aligned}
$$

for any $|j| > 1$. By similar aguments in showing (12), for any t,

$$
\begin{aligned}
COV(Y_t, Y_t) &= \sum_{s=0}^{\infty}\sum_{k=0}^{\infty}\phi^{k+s}\gamma_{u,s-k} \\
&= \sum_{k=0}^{\infty}\phi^{k}\gamma_{u,-k} + \sum_{s=1}^{\infty}\sum_{k=0}^{\infty}\phi^{k+s}\gamma_{u,s-k} \\
&= \gamma_{u,0} + \phi^{1}\gamma_{u,-1} \\
&\quad + \sum_{s=1}^{\infty}(\phi^{s+s}\gamma_{u,0} + \phi^{s+1+s}\gamma_{u,-1} + \phi^{s-1+s}\gamma_{u,1}) \\
&= \gamma_{u,0}\sum_{s=0}^{\infty}\phi^{2s} + \gamma_{u,-1}\phi\sum_{s=0}^{\infty}\phi^{2s} + \gamma_{u,1}\phi^{-1}\sum_{s=1}^{\infty}\phi^{2s} \\
&= \frac{\gamma_{u,0} + \gamma_{u,-1}\phi + \gamma_{u,1}\phi}{1-\phi^2} \\
&= \frac{1+\theta^2+2\phi\theta}{1-\phi^2}\sigma_{\varepsilon}^2 < \infty.
\end{aligned}
$$

Since $Var(Y_t) < \infty$, the mean and covariance of $\{Y_t\}_{t=-\infty}^{+\infty}$ do not depend on t. Therefore $\{Y_t\}_{t=-\infty}^{+\infty}$ is a covariance stationary process.

3 Some Auxiliary Lemmas

Theorem 1 (Hölder's Inequality) *For any $p \geq 1$,*

$$E\left[|XY|\right] \leq (E\left[|X|^p\right])^{1/p}(E\left[|Y|^q\right])^{1/q}$$

where $q = p/(p-1)$ if $p > 1$, and $q = \infty$ if $p = 1$.

Theorem 2 (Lyapunov's Inequality) *If $p > q > 0$, then $(E\left[|X|^p\right])^{1/p} \geq (E\left[|X|^q\right])^{1/q}$.*

Theorem 3 (Monotone Convergence Theorem) *Let $\{f_n\}_n$ be an increasing sequence of non-negative measurable functions on a measure space $(\Omega, \mathcal{F}, P)$. Define the function $f : \Omega \to [0, \infty]$ by $f_n(x) \to f(x)$ almost surely. Then*

$$E[f] = \lim_{n\to\infty} E[f_n]$$

where $E[\cdot]$ denotes the expectation operator induced by the probability measure P.

Theorem 4 (Dominated Convergence Theorem) *Let $\{f_n\}_n$ be a sequence of measurable functions on a measure space $(\Omega, \mathcal{F}, P)$. Suppose that the sequence converges pointwise to a function f and is dominated by some integrable function g in the sense that*

$$|f_n(x)| \leq g(x) \text{ for any } x \in \Omega \text{ and any } n, \text{ and } E[|g|] < \infty.$$

Then $E[|f|] < \infty$ *and*

$$E[f] = \lim_{n \to \infty} E[f_n]$$

where $E[\cdot]$ *denotes the expectation operator induced by the probability measure* P*.*