ECON 147 MIDTERM EXAM

YOUR NAME:

Winter 2018, Feb 13th
3:30pm - 4:45pm

Instruction

- This is a closed book and closed note exam. All necessary information will be provided and all calculation will be done by hands, so no cheat sheet or calculator is needed.
- Try to answer all questions and write all answers **within this exam sheets**. You will hand in this exam sheets. Please write legibly.
- Total points are 100. Use your time wisely!
- *Examination Rules from Department Policy will be strictly followed.*
- The following results may be useful:

$$\begin{array}{ll} \Pr(Z \leq 0.5) = 0.69, & \Pr(Z \leq -1.1) = 0.136, \\ \Pr(Z > -1.96) = 0.975, & \Pr(Z \leq -0.6) = 0.274, \\ \Pr(Z \leq -1.64) = 0.05, & \Pr(Z \leq -0.70) = 0.242, \\ \Pr(Z \leq -0.35) = 0.363, & e^{-2.3} = 0.100, \\ e^{-1.2} = 0.30, & e^{-1.9} = 0.15, \\ e^{-3.2} = 0.041, & e^{-2.9} = 0.055, \end{array}$$

  where Z is a standard normal random variable.

1. **(10 pts)** The daily cc returns r_t on a stock are independent and normally distributed with mean 0.001 and standard deviation 0.01. Suppose you buy \$1000 worth of this stock.

1.1. What is the probability that after one trading day your investment is worth less than \$990? [**5 pts**] [**Hint:** use the fact that $\log(e^{r_t}) = r_t$ is a normal random variable to transfer the event in the question to an event about the standard normal random. You should use $\log(1+x) \simeq x$ for small x.]

Answer: Let $W_1 = W_0 \exp(r_1)$ (**1 pt**). Therefore

$$\begin{aligned}
\Pr(W_1 \leq 990) &= \Pr(1000 \exp(r_1) \leq 990) \\
&= \Pr(\exp(r_1) \leq 0.99) \ \ (\mathbf{1\ pt}) \\
&= \Pr(r_1 \leq -0.01) \ \ (\mathbf{1\ pt}) \\
&= \Pr\left(Z \leq \frac{-0.011}{0.01}\right) \ \ (\mathbf{1\ pt}) \\
&= \Pr(Z \leq -1.1) = 0.136 \ (\mathbf{1\ pt})
\end{aligned}$$

1.2. What is the probability that after four trading days your investment is worth less than \$990? [**5 pts**]

Answer: Let $W_4 = W_0 \exp(r_1 + r_2 + r_3 + r_4)$ (**1 pt**), where

$$r_1 + r_2 + r_3 + r_4 \sim N(0.004, 4 \cdot (0.01)^2) \ (\mathbf{1\ pt})$$

Therefore

$$\begin{aligned}
\Pr(W_4 \leq 990) &= \Pr(1000 \exp(r_1 + r_2 + r_3 + r_4) \leq 990) \\
&= \Pr(\exp(r_1 + r_2 + r_3 + r_4) \leq 0.99) \\
&= \Pr(r_1 + r_2 + r_3 + r_4 \leq -0.01) \ \ (\mathbf{1\ pt}) \\
&= \Pr\left(Z \leq \frac{-0.014}{0.02}\right) \ \ (\mathbf{1\ pt}) \\
&= \Pr(Z \leq -0.70) = 0.242 \ (\mathbf{1\ pt})
\end{aligned}$$

2 **(5 pts)** Let r_t be a cc monthly return. Suppose that $r_1, r_2, ...$ are independent and identically distributed normal random variables with mean 0.06 and variance 0.49. Let $W_0 = \$100$. Determine the 5% value-at-risk (VaR) over 9 months on the investment.

Answer: By definition, $r(9) \sim N(0.54, 0.49 * 9)$ (**1 pt**) and hence

$$L_1 = W_0(e^{r(9)} - 1) = W_0(e^{0.54+2.1Z} - 1) \text{ } (\mathbf{1\ pt})$$

where

$$Z = \frac{r(9) - 0.54}{\sqrt{0.49 * 9}} \text{ } (\mathbf{1\ pt})$$

is the standard normal random variable. Since $W_0(e^{0.54+2.1x} - 1)$ is a strictly increasing function of x,

$$VaR_\alpha = W_0(e^{0.54+2.1q_\alpha^Z} - 1). \text{ } (\mathbf{1\ pt})$$

Since $q_{0.05}^Z = -1.64$,

$$VaR_{0.05} = 100(e^{-2.9} - 1) = 100(0.0543 - 1) = -94.5. \text{ } (\mathbf{1\ pt})$$

3. **(30 pts)** Let Y_1, Y_2, Y_3, Y_4 and Y_5 be iid (μ, σ^2).

3.1. Let $W_1 = \frac{1}{8}Y_1 + \frac{1}{8}Y_2 + \frac{1}{2}Y_3 + \frac{1}{8}Y_4 + \frac{1}{8}Y_5$. Find the mean and variance of W_1? [**4 pts**]

Answer: By definition

$$\begin{aligned} E[W_1] &= \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{2} + \frac{1}{8} + \frac{1}{8}\right)\mu = \mu, \text{ } (\mathbf{2\ pts}) \\ Var(W_1) &= \left(\frac{1}{64} + \frac{1}{64} + \frac{1}{4} + \frac{1}{64} + \frac{1}{64}\right)\sigma^2 = \frac{5\sigma^2}{16} \text{ } (\mathbf{2\ pts}) \end{aligned}$$

3.2. Let $W_2 = \frac{1}{4}Y_1 + \frac{1}{4}Y_2 + \frac{1}{5}Y_3 + \frac{1}{4}Y_4 + \frac{1}{4}Y_5$. Find the mean and variance of W_2? [**4 pts**]

Answer: By definition

$$\begin{aligned} E[W_2] &= \left(\frac{1}{4} + \frac{1}{4} + \frac{1}{5} + \frac{1}{4} + \frac{1}{4}\right)\mu = \frac{6}{5}\mu, \text{ } (\mathbf{2\ pts}) \\ Var(W_2) &= \left(\frac{1}{16} + \frac{1}{16} + \frac{1}{25} + \frac{1}{16} + \frac{1}{16}\right)\sigma^2 = \frac{29\sigma^2}{100} \text{ } (\mathbf{2\ pts}) \end{aligned}$$

3.3. Which estimator of μ (W_1 or W_2) do you prefer? Fully justify your answer. [**9 pts**]

Answer: By definition

$$\begin{aligned} MSE(W_1) &= \frac{5}{16}\sigma^2 \ (\textbf{2 pts}) \\ MSE(W_2) &= \frac{\mu^2}{25} + \frac{29}{100}\sigma^2 \ (\textbf{2 pts}) \end{aligned}$$

Therefore

$$\begin{aligned} MSE(W_1) - MSE(W_2) &= \frac{125-116}{400}\sigma^2 - \frac{\mu^2}{25} \\ &= \frac{9}{400}\sigma^2 - \frac{\mu^2}{25} \ (\textbf{2 pts}) \end{aligned}$$

If $\frac{9}{400}\sigma^2 > \frac{\mu^2}{25}$, we prefer W_2. If $\frac{9}{400}\sigma^2 < \frac{\mu^2}{25}$, we prefer W_1. If $\frac{9}{400}\sigma^2 = \frac{\mu^2}{25}$, there is no difference between these two estimators. (**3 pts**)

3.4. An estimator W_a is called a linear estimator of μ if it takes the following form

$$W = a_1Y_1 + a_2Y_2 + a_3Y_3 + a_4Y_4 + a_5Y_5.$$

Among all possible unbiased linear estimators of μ based on Y_1, Y_2, Y_3, Y_4 and Y_5, does there exist a best (unbiased) estimator which has smallest mean square error? Fully justify your answer. [Hint: the inequality $\left(\sum_{i=1}^5 a_i\right)^2 \le 5\sum_{i=1}^5 a_i^2$ will be useful.] [**9 pts**]

Answer: for any linear unbiased estimator,

$$\begin{aligned} Var(W) &= \left(\sum_{i=1}^5 a_i^2\right)\sigma^2 \ (\textbf{1 pt}) \\ E[W] &= \mu \ (\textbf{1 pt}) \\ MSE(W) &= \left(\sum_{i=1}^5 a_i^2\right)\sigma^2 \ (\textbf{1 pt}\text{, or 3 pts if directly write MSE of W}). \end{aligned}$$

While for the sampe average

$$\begin{aligned}
Var\left(\bar{Y}\right) &= \frac{\sigma^2}{n}\ (\mathbf{1\ pt}) \\
E[\bar{Y}] &= \mu\ (\mathbf{1\ pt}) \\
MSE(\bar{Y}) &= \frac{\sigma^2}{n}\ (\mathbf{1\ pt}\text{, or 3 pts if directly write MSE of } \bar{Y}).
\end{aligned}$$

Since $\frac{\sigma^2}{5} \leq \left(\sum_{i=1}^{5} a_i^2\right)\sigma^2$ by the inequality in hint,

$$Var\left(\bar{Y}\right) \leq Var\left(W_a\right)$$

so $\bar{Y}$ is the *best* in MSE sense (**2 pts**).

4. **(30 pts)** Suppose X is a uniform random variable over [-1,1] (i.e., $X \backsim U[-1,1]$) and Y is a Bernoulli random variable with a success probability $\Pr(Y=1) = 0.6$, *i.e.*,

$$Y = \begin{cases} 1, & \text{with probability } 0.6 \\ -\frac{3}{2}, & \text{with probability } 0.4 \end{cases}.$$

Compute the following.

4.1. $\Pr(X < 0.1)$ and $\Pr(Y < 0.1)$. **[6 pts]**

Answer:

$$\begin{aligned}
\Pr(X &< 0.1) = \int_{-1}^{0.1} \frac{1}{2} dx = 0.55, (\mathbf{3\ pts}) \\
\Pr(Y &< 0.1) = \Pr(Y = -1) = 0.4. (\mathbf{3\ pts})
\end{aligned}$$

4.2. $E(X)$, $Var(X)$, $E(Y)$ and $Var(Y)$. **[10 pts]**

Answer:

$$\begin{aligned}
E(X) &= \int_{-1}^{1} \frac{x}{2} dx = \left[\frac{x^2}{4}\right]_{-1}^{1} = 0, (\mathbf{2\ pts}) \\
E(X^2) &= \int_{-1}^{1} \frac{x^2}{2} dx = \left[\frac{x^3}{6}\right]_{-1}^{1} = \frac{1}{3}, (\mathbf{2\ pts}) \\
Var(X) &= E(X^2) - (E(X))^2 = \frac{1}{3}, (\mathbf{1\ pts}\text{,or } \mathbf{3\ pts} \text{ if directly calculate the variance}) \\
E(Y) &= 1 \cdot 0.6 + (-3/2) \cdot 0.4 = 0, (\mathbf{2\ pts}) \\
E(Y^2) &= 1^2 \cdot 0.6 + (-3/2)^2 \cdot 0.4 = 1.5, (\mathbf{2\ pts}) \\
Var(Y) &= E(Y^2) - (E(Y))^2 = 1.5. (\mathbf{1\ pts}\text{,or } \mathbf{3\ pts} \text{ if directly calculate the variance})
\end{aligned}$$

4.3. For any $\alpha \in [0, 1]$, $E(\alpha X + (1 - \alpha)Y)$. [**4 pts**]

Answer:

$$\begin{aligned}
& E(\alpha X + (1 - \alpha)Y) \\
= \;& \alpha E(X) + (1 - \alpha)E(Y) \ (\mathbf{3\ pts}) \\
= \;& 0 \ (\mathbf{1\ pts}\text{,or } \mathbf{4\ pts\ without\ the\ above\ derivation})
\end{aligned}$$

4.4. Now suppose X and Y represent (statistically independent) outcomes of two lotteries, and you would like to invest your \$100 to these lotteries. Assume you only care about the mean (preferably high) and variance (preferably low) of your investment. How do you want to distribute your \$100, i.e., how do you choose your $\alpha \in [0, 1]$ for $\alpha X + (1 - \alpha)Y$? Justify your answer. [**10 pts**]

Answer:

$$\begin{aligned}
Var(\alpha X + (1 - \alpha)Y) &= \alpha^2 Var(X) + (1 - \alpha)^2 Var(Y) \ (\mathbf{3\ pts}) \\
&= \frac{\alpha^2}{3} + \frac{3(\alpha^2 - 2\alpha + 1)}{2} \\
&= \frac{11\alpha^2 - 18\alpha + 9}{6} \ (\mathbf{3\ pts})
\end{aligned}$$

which is minimized at $\alpha = 9/11$ (**3 pts**). Therefore, put \$900/11 on X and \$200/11 on Y (**1 pt**).

5. **(25 pts)** Consider the constant expected return model

$$r_{it} = \mu_i + \epsilon_{it} \quad t = 1, \cdots, T; \quad i = 1 \text{ (GS)}, \ 2 \text{ (AIG)},$$
$$\epsilon_{it} \sim \text{iid } N(0, \sigma_i^2), \ \text{cov}(\epsilon_{1t}, \ \epsilon_{2t}) = \sigma_{12}, \ \text{cor}(\epsilon_{1t}, \ \epsilon_{2t}) = \rho_{12}$$

for the monthly cc returns on GS (Goldman Sachs) and AIG (American International Group). The estimates (rounded for computations) are given ($T = 100$ months):

	GS	AIG	GS&AIG	
$\hat{\mu}_i$	0.03	-0.01	$\hat{\sigma}_{12}$	0.048
$\hat{\sigma}_i$	0.2	0.4		

5.1. For both GS and AIG cc returns, compute (asymptotic) 95% CI for μ_i and σ_i^2. [Hint: $q^Z_{0.975} \simeq 2$ and $\sqrt{2} \simeq 1.4$.] **[8 pts]**

Answers: use the formula:

$$\hat{\mu}_i \pm 2\frac{\hat{\sigma}_i}{\sqrt{T}} \text{ and } \hat{\sigma}_i^2 \pm 2\frac{\sqrt{2}\hat{\sigma}_i^2}{\sqrt{T}}$$

(**2 pts** if the formula of the CI of μ_i is correct and **2 pts** if the anwsers are correct. **2 pts** if the formula of the CI of σ_i^2 is correct and **2 pts** if the anwsers are correct)

5.2. Compute (asymptotic) 95% confidence interval for ρ_{12} (You will use $SE(\hat{\rho}_{12}) = \sqrt{(1 - \hat{\rho}_{12}^2)/T}$). [Hint: $q^Z_{0.975} \simeq 2$] **[5 pts]**

Answer: use $\hat{\rho}_{12} \pm 2\sqrt{\frac{1-\hat{\rho}_{12}^2}{T}}$.(**3 pts** if the formula of the CI of ρ_{12} is correct and **2 pts** if the anwsers are correct.)

	GS	AIG
μ	$(-0.01, 0.07)$	$(-0.09, 0.07)$
σ^2	$(0.029, 0.051)$	$(0.12, 0.20)$
ρ	$(0.44, 0.76)$	$(0.44, 0.76)$

5.3. Test the hypothesis (significance tests) for $i = 1, 2$, with 5% confidence level,

$$H_0 : \mu_i = 0 \quad \text{v.s.} \quad H_1 : \mu_i \neq 0.$$

Are expected returns of these assets (statistically) different from zero? Justify your answer. [Hint: $q_{0.975}^{T(99)} \simeq 2$.] **[8 pts]**

Answers: First calculate the test statistics:

$$\left| \frac{\hat{\mu}_i - 0}{\hat{\sigma}_i / \sqrt{T}} \right| = \begin{cases} 1.50, & \text{GS} \\ 0.25, & \text{AIG} \end{cases} .$$

Since the critical value is $q_{0.975}^{T(99)} \simeq 2$, the alternative hypothesis H_1 is rejected for both GS and AIG.

(**4 pts (2 pts for each i)** if the formula of the test statistic is correct, and **2 pts (1 pt for each i)** if the calculation is correct and **2 pts (1 pt for each i)** if the decisions are correct.)

5.4. Test the hypothesis for $i = 1, 2$,

$$H_0 : \sigma_i^2 = 0.0225 \quad \text{v.s.} \quad H_1 : \sigma_i^2 \neq 0.0225$$

with 5% confidence level. [**4 pts**].

Answers: First calculate the test statistics:

$$\left| \frac{\hat{\sigma}_i^2 - 0.0225}{\sqrt{2}\hat{\sigma}_i^2 / \sqrt{T}} \right| \approx \begin{cases} 3.1, & \text{GS} \\ 6.1, & \text{AIG} \end{cases} .$$

Since the critical value is $q_{0.975}^{Z} = 1.96$, the null hypothesis H_0 is rejected for both GS and AIG.

(**2 pts (1 pt for each i)** if the formula of the test statistic is correct, and **1 pt (0.5 pt for each i)** if the calculation is correct and **1 pts (0.5 pt for each i)** if the decisions are)