ECON 490 Homework 4

Answer Keys

Readings

- Course slides and lecture notes.
- Using the provided R coding, you will be able to simulate AR and MA processes. I highly recommend you to understand the plots, depending on the different values of θ and ϕ.
- Using the provided R coding, you will also be able to compute the descriptive statistics of real financial returns. Make sure you understand the output rather than just following the hint.

Review Questions

1. One popular concept on stock market price is called "martingale pricing", which is relevant to "rational expectations" or "efficient market hypothesis" in economic theories. A rough statement on this is that, a (log of) stock market price p_t (e.g., S&P 500 index) is supposed to reflect all the relevant information available up to time t, i.e.,

$$E\left[p_t|\mathcal{F}_{t-1}\right] = p_{t-1},$$

therefore $p_t = \log P_t$ is MG.

   (a) Under the martingale pricing, show that continuously compounded (cc) return for stock market price is mds.

   Answer: *By the definition of cc return,* $r_t = p_t - p_{t-1}$. Therefore

$$\begin{aligned} E[r_t|\mathcal{F}_{t-1}] &= E[p_t - p_{t-1}|\mathcal{F}_{t-1}] \\ &= E[p_t|\mathcal{F}_{t-1}] - E[p_{t-1}|\mathcal{F}_{t-1}] \\ &= E[p_t|\mathcal{F}_{t-1}] - p_{t-1} \text{ (since } E[p_{t-1}|\mathcal{F}_{t-1}] = p_{t-1}) \\ &= p_{t-1} - p_{t-1} \qquad \text{(since } E[p_t|\mathcal{F}_{t-1}] = p_{t-1} \text{ by the MG property)} \\ &= 0 \end{aligned}$$

   (b) In what follows, we assume cc return r_t is a covariance stationary process. Prove the following statements:

      i. If $r_t \sim iid\ (0, \sigma^2)$ (or indepedent white noise), then $r_t \sim mds\ (0, \sigma^2)$.

      Answer:

$$\begin{aligned} E\left[r_t|\mathcal{F}_{t-1}\right] &= E\left[r_t|r_{t-1}, r_{t-2}, ...\right] \\ &= E\left[r_t\right] \text{ (by independence)} \\ &= 0. \end{aligned}$$

      ii. If $r_t \sim mds\ (0, \sigma^2)$, then $r_t \sim WN\ (0, \sigma^2)$ (or weak white noise).

Answer: For any $s \leq t-1$,

$$\begin{aligned}
Cov\left(r_t, r_s\right) &= E\left[r_t r_s\right] \text{ (since } E\left[r_t\right]=0 \text{ and } E\left[r_s\right]=0\text{)} \\
&= E\left[E\left[r_t r_s | \mathcal{F}_{t-1}\right]\right] \text{ (from LTE)} \\
&= E\left[r_s E\left[r_t | \mathcal{F}_{t-1}\right]\right] \text{ (by definition of } \mathcal{F}_{t-1}, r_s \text{ is known given } \mathcal{F}_{t-1}\text{)} \\
&= E\left[r_s \cdot 0\right] \text{ (from mds property)} \\
&= 0.
\end{aligned}$$

(c) Prove the following statements:

i. If $\{r_t\}$ is i.i.d., then it is strictly stationary.

Answer: *For any r and any integers $a_1, a_2, \ldots, a_r$*, let $f_{a_1,a_2,\ldots,a_r}$ denote the joint probability density/mass function of $r_{a_1}, r_{a_2}, \ldots, r_{a_r}$, and let $f_{a_1}, f_{a_2}, \ldots, f_{a_r}$ denote the marginal probability density/mass functions of $r_{a_1}, r_{a_2}, \ldots, r_{a_r}$ respectively. Then *for any r and k*,

$$\begin{aligned}
f_{t_1-k,t_2-k,\ldots,t_r-k} &= f_{t_1-k} \cdot f_{t_2-k} \cdot \ldots \cdot f_{t_r-k} \text{ (from independence)} \\
&= \left(f_0\right)^r \text{ (from indentical destribution)} \\
&= f_{t_1} \cdot f_{t_2} \cdot \ldots \cdot f_{t_r} \text{ (from indentical destribution)} \\
&= f_{t_1,t_2,\ldots,t_r} \text{ (from independence).}
\end{aligned}$$

1. (a) i. If $\{r_t\}$ is strictly stationary and $E[r_1^2] < \infty$, then it is covariance stationary.

Answer: The covariances are finite since $E[r_t^2] = E[r_1^2] < \infty$ *for any t*. For any t and any j, since the joint distribution of r_t and r_{t-j} is the same as the joint distribution of r_0 and r_{-j},

$$Cov(r_t, r_{t-j}) = Cov(r_0, r_{-j})$$

where $Cov(r_0, r_{-j})$ is finite and does not depend on t. $Cov(r_0, r_{-j})$ is finite because we know that the absolute value of the correlation coefficient between r_0 and r_{-j} is bounded by 1, i.e.,

$$|Corr(r_t, r_{t-j})| \leq 1.$$

Since $Corr(r_t, r_{t-j}) = Cov(r_0, r_{-j}) / \sqrt{Var(r_0) \cdot Var(r_j)}$ and $Var(r_0) = Var(r_j)$ by strictly stationarity,

$$|Cov(r_0, r_{-j})| \leq Var(r_0) \leq E[r_0^2] < \infty.$$

2. Consider the AR(1) model

$$\begin{aligned} Y_t &= 5 - 0.55Y_{t-1} + \varepsilon_t, \\ \varepsilon_t &\sim mds(0, 1.2). \end{aligned}$$

(a) Is this process stationary? Why or why not?

Answer: *It is stationary since* $|0.55| < 1$.

(b) What is the mean of this process?

Answer:

$$E[Y_t] = \frac{5}{1 - (-0.55)}.$$

(c) What is the variance of this process?

Answer:

$$Var[Y_t] = \frac{1.2}{1 - (-0.55)^2}$$

(d) What is the auto-covariance function of this process?

Answer:

$$\gamma_j = \left(\frac{1.2}{1 - (-0.55)^2}\right)(-0.55)^j$$