ECON 147 Homework 6

No Due Date

Reading

- Please read the course material on the course website.

Review Questions

1. Figure 1 and 2 below show monthly log of stock prices and cc returns on Goldman Sachs Group (GS) and American International Group (AIG). The sample period is from May 2005 to September 2013. (T=100 monthly observations)

Figure 1: log prices on GS and AIG

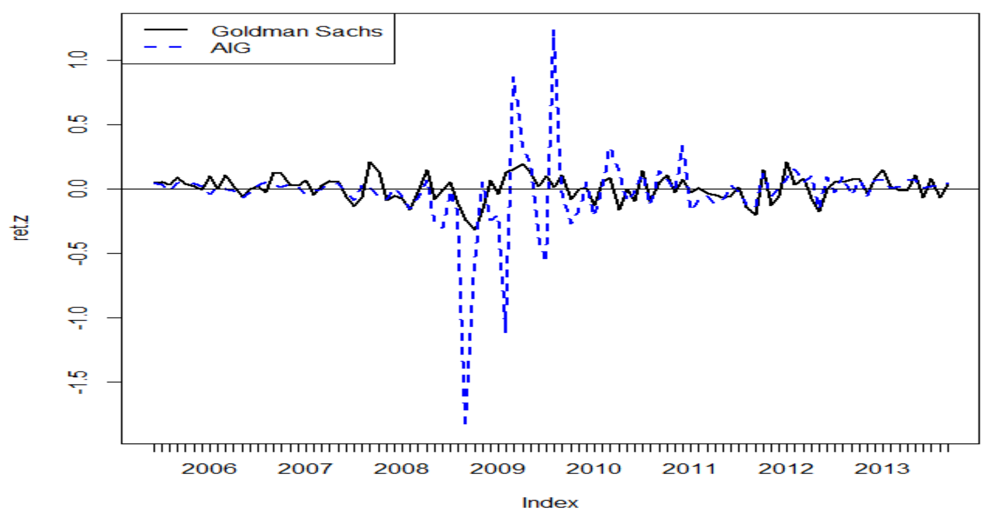


Figure 2: cc returns on GS and AIG

1.1. Do the monthly log of stock prices from AIG look like realizations from a covariance stationary stochastic process? Why or why not?

Answer: The AIG (log) stock prices does not look like a covariance stationary process. From Figure 1, we can clearly see that the mean of this process before 2008 is different from the mean of this process after 2010, which indicates that the mean of this process is not constant and changes with t. So it is not covariance stationary.

1.2. Do the monthly cc returns from AIG look like realizations from a covariance stationary stochastic process? Why or why not?

Answer: The monthly cc returns from AIG does not look like a covariance stationary process. From Figure 2, we can clearly see that the variance of this process before 2008 and after 2010 is much smaller than the variance of this process between 2008 and 2010, which indicates that the variance of this process is not constant and changes with t. So it is not covariance stationary.

1.3. In late 2008, AIG's credit rating was downgraded and it leads to AIG's liquidity crisis, meanwhile GS was maintaining sizeable profits (relatively). Comment on any common or distinctive features of the two stock prices and return series.

Answer: The liquidity crisis of AIG leads to decline of its stock price which decreases the market value/equity of this company. Therefore, the debt/equity ratio increases during the crisis which further enlarges the uncertainty of the stock price. The enlarged uncertainty is expressed in the large variation of the cc returns. This is the so called leverage effect in finance. For GS, its prices is much stable during the crisis and hence we do not see enlarged variance on its cc return.

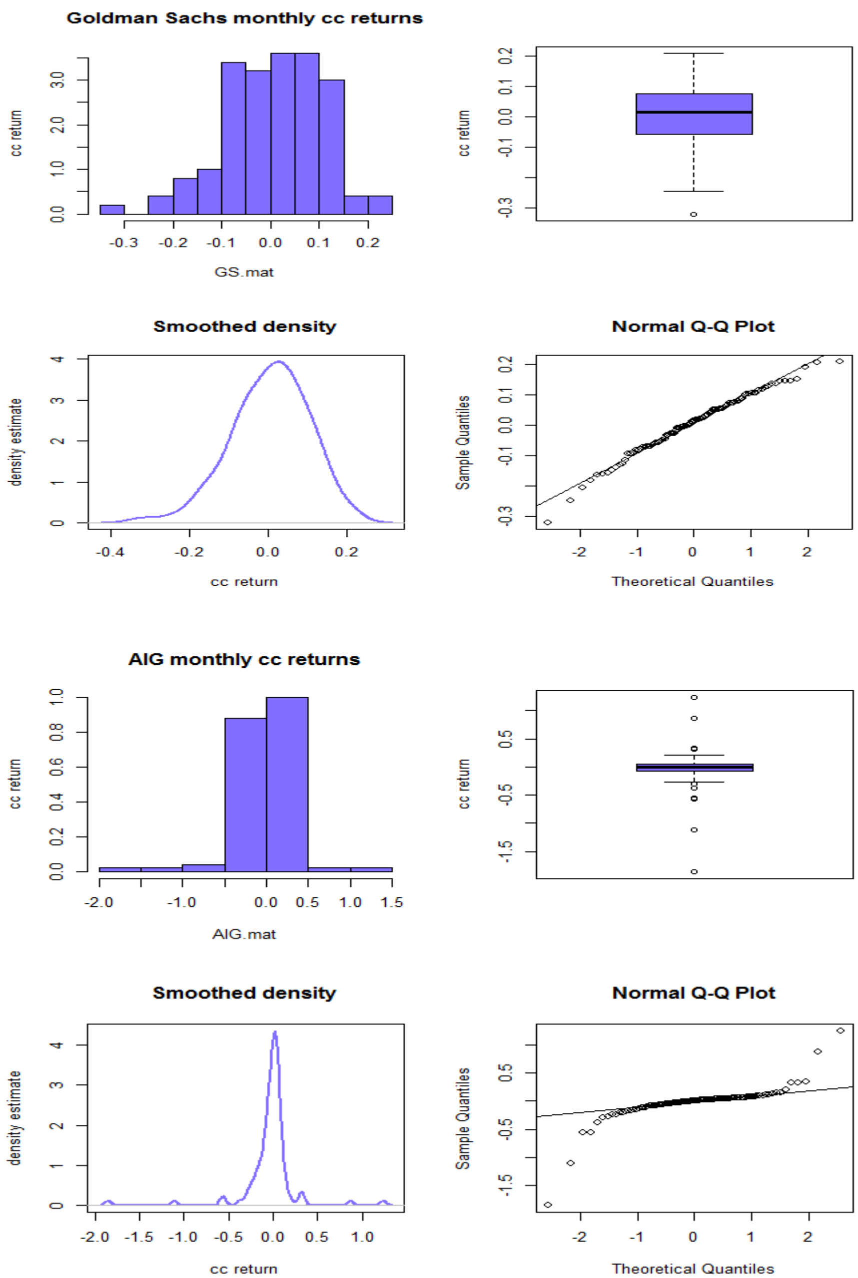


Figure 3: Descriptive Statistics on GS and AIG

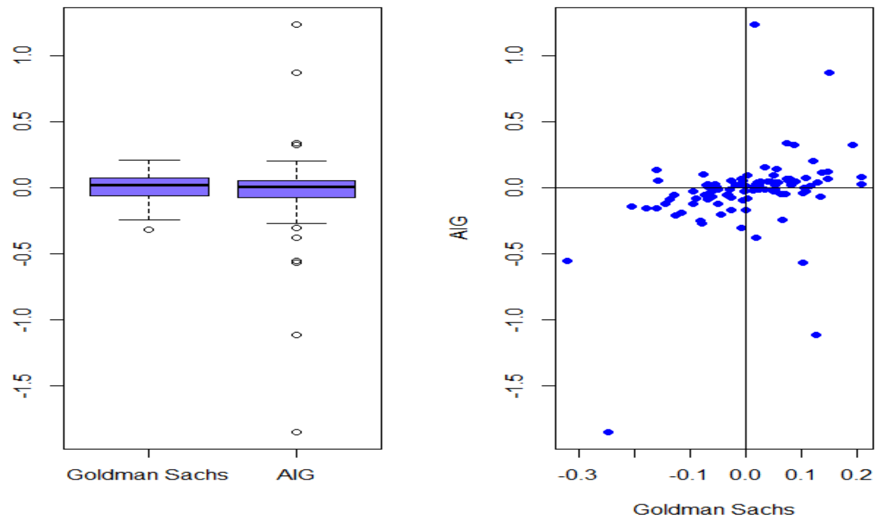


Figure 4: Box Plots and Scatter Plots on GS and AIG

1.4. Based on Figure 3 & 4, do the returns on AIG and GS look normally distributed? Briefly justify your answer.

Answer: Both cc returns do not follow closely the normal distribution, although the cc return of the GS is more close to be normal. From the histogram and the smoothed density, the GS cc return is slightly left tailed. That is due to the outliers from the left tail as we can see from the box plot and the QQ plot. For AIG, its cc return has heavy tails on both sides. There are more outliers in its cc return as we can see from the box plot and the QQ plot.

1.5. Which asset appears to be riskier? Briefly justify your answer.

Answer: AIG looks riskier, since it has larger interquartile range and its cc return is more widely spread as we can see from Figure 4.

1.6. Based on the scatterplot of returns, does there appear to be any linear dependence between the returns on GS and AIG? Briefly justify your answer.

Answer: The two returns is positively correlated. As we can see from the scatterplot in Figure 4, the are much more scatter points in the north-east and south-west sections of the (0, 0), which indicates positive correlation.

2. Figure 5 below shows US quarterly GDP time series data.

Figure 5: US GDP

Many empirical researchers used to model US GDP time series Y_t using the following Unit Root model:

$$Y_t = \mu + Y_{t-1} + \varepsilon_t,\ \varepsilon_t \sim iid\ N(0, \sigma_\varepsilon^2)$$

2.1. Do the US GDP data looks like realizations from a covariance stationary stochastic process? Why or why not?

Answer: The US GDP does not look like a covariance stationary process. From Figure 5, it is clear that the mean of this process changes across time and hence is not constant.

2.2. When $Y_0 = 0$, using recursive substitution, show that

$$Y_t = \mu t + \sum_{j=1}^{t} \varepsilon_j,$$

and find $E[Y_t]$ and $Var\,(Y_t)$. Is Y_t covariance stationary?

Answer: It is clear that

$$\begin{aligned}
Y_t &= \mu + Y_{t-1} + \varepsilon_t \\
&= \mu + \mu + Y_{t-2} + \varepsilon_{t-1} + \varepsilon_t \\
&= \mu + \mu + \mu + Y_{t-3} + \varepsilon_{t-2} + \varepsilon_{t-1} + \varepsilon_t \\
&= \cdots \\
&= \mu t + Y_0 + \varepsilon_1 + \cdots + \varepsilon_{t-1} + \varepsilon_t \\
&= \mu t + \sum_{j=1}^{t} \varepsilon_j.
\end{aligned}$$

Using the above expression, we have

$$\begin{aligned}
E[Y_t] &= E\left[\mu t + \sum_{j=1}^{t} \varepsilon_j\right], \text{ by the expression } Y_t = \mu t + \sum_{j=1}^{t} \varepsilon_j \\
&= \mu t + \sum_{j=1}^{t} E[\varepsilon_j], \text{ by the linearity of expectation} \\
&= \mu t + \sum_{j=1}^{t} 0, \text{ by } E[\varepsilon_j] = 0 \\
&= \mu t
\end{aligned}$$

and

$$
\begin{aligned}
Var\left(Y_t\right) &= Var\left(\mu t + \sum_{j=1}^{t} \varepsilon_j\right), \text{ by the expression } Y_t = \mu t + \sum_{j=1}^{t} \varepsilon_j \\
&= Var\left(\sum_{j=1}^{t} \varepsilon_j\right), \text{ by } \mu t \text{ is a constant} \\
&= \sum_{j=1}^{t} Var(\varepsilon_j), \text{ by } \{\varepsilon_t\} \text{ is an independent process} \\
&= \sum_{j=1}^{t} \sigma_\varepsilon^2, \text{ by } \varepsilon_t \sim iid\ N(0, \sigma_\varepsilon^2) \\
&= t\sigma_\varepsilon^2.
\end{aligned}
$$

It is clear that both the mean and variance of Y_t may change with t. So $\{Y_t\}$ is not a covariance stationary process.

3. Consider the constant expected return model for two stocks (Boeing and Microsoft)

$$R_i = \mu_i + \epsilon_i \text{ for } i = 1, 2 \text{ (Boeing and Msft, respectively)}$$

where $\epsilon_i \sim N(0, \sigma_i^2)$, $\text{cov}(\epsilon_1,\ \epsilon_2) = \sigma_{12}$ and $\text{cor}(\epsilon_1,\ \epsilon_2) = \rho_{12}$.

3.1. Write down the optimization problem to determine the global minimum variance portfolio. For $i = 1, 2$, let m_i denote the portfolio weight in the global minimum variance portfolio on asset i.

Answer:

$$\min_{(m_1, m_2)} \ \sigma_p^2 = m_1^2\sigma_1^2 + m_2^2\sigma_2^2 + 2m_1m_2\sigma_{12}$$

$$s.t.\ m_1 + m_2 = 1.$$

3.2. Write down the optimization problem used to determine the tangency portfolio when the risk free rate is given by r_f. For $i = 1, 2$, let t_i denote the portfolio weight in the tangency portfolio on asset i. What does the following ratio represent,

$$\frac{t_1\mu_1 + t_2\mu_2 - r_f}{\left(t_1^2\sigma_1^2 + t_2^2\sigma_2^2 + 2t_1t_2\sigma_{12}\right)^{1/2}}$$

in financial economics?

Answer: The optimization problem is

$$\max_{t_1,\ t_2} \text{SR}_p = \frac{\mu_p - r_f}{\sigma_p} \text{ subject to}$$

$$\begin{aligned} \mu_p &= t_1\mu_1 + t_2\mu_2 \\ \sigma_p^2 &= t_1^2\sigma_1^2 + t_2^2\sigma_2^2 + 2t_1t_2\sigma_{12} \\ 1 &= t_1 + t_2. \end{aligned}$$

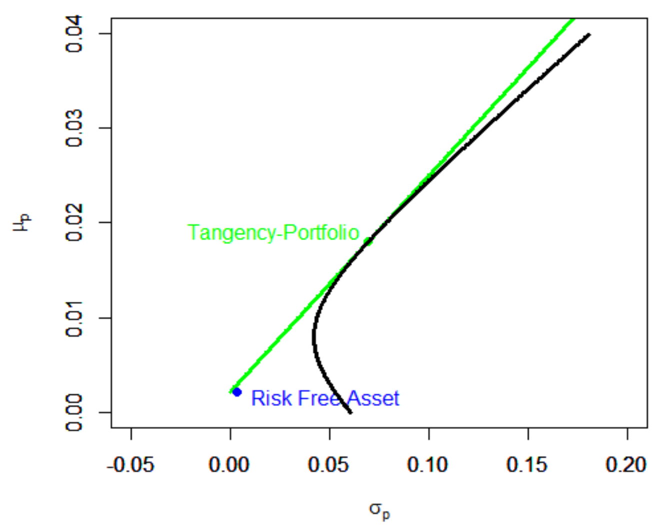
The maximizer of the above optimization problem is the portfolio weights of the tangency portfolio t_1 and t_2. The ratio

$$\frac{t_1\mu_1 + t_2\mu_2 - r_f}{\left(t_1^2\sigma_1^2 + t_2^2\sigma_2^2 + 2t_1t_2\sigma_{12}\right)^{1/2}}$$

is the Sharpe ratio of the tangency portfolio which measures the excess expected return of the tangency portfolio per unit risk.

3.3. State Mutual Fund Separation Theorem and draw a portfolio frontier based on the Theorem. Discuss the resulting weights determination (x_f for r_f and $x_{\tan}$ for $R_{\tan}$) according to an investor's risk preferences.

Answer: The Mutual Fund Separation Theorem states that the efficient portfolios are combinations of two portfolios: the risky free asset and the tangency portfolio. Therefore, All investors hold assets Boeing and Msft according to their proportions in the tangency portfolio regardless of their risk preferences.



The green solid line in the above picture is the efficient portfolio frontier which represents the expected mean and standard deviation of the simple returns of all efficient portfolios. The black solid line is the efficient portfolio frontier when there is no risk free asset. Since there is risk free asset, the consumer will choose portfolio according to the green solid line. The risk averse consumers tends to choose portfolios whose expected return and risk are closer to the risk free asset. Hence they will choose high x_f and low $x_{\tan}$. The less risk averse consumers will choose portfolios whose expected return and risk are far away from the risk free asset. Hence they will choose low x_f and high $x_{\tan}$.

3.4. Suppose that

$$r_f = 0.1,\ \mu_1 = 0.2,\ \mu_2 = 0.4,\ \sigma_1^2 = 1,\ \sigma_2^2 = 4 \text{ and } \sigma_{12} = 0. \qquad (1)$$

Find the global minimum variance portfolio and the tangency portfolio.

Answer: We use the following formula to calculate the global minimum variance portfolio:

$$m_1 = \frac{\sigma_2^2 - \sigma_{12}}{\sigma_1^2 + \sigma_2^2 - 2\sigma_{12}},\ m_2 = 1 - m_1.$$

Therefore

$$m_1 = \frac{\sigma_2^2 - \sigma_{12}}{\sigma_1^2 + \sigma_2^2 - 2\sigma_{12}} = \frac{4 - (0)}{1 + 4 - 2 * (01)} = \frac{4}{5},\ m_2 = \frac{1}{5}.$$

We use the following formula to calculate the global minimum variance portfolio:

$$t_1 = \frac{(\mu_1 - r_f)\sigma_2^2 - (\mu_2 - r_f)\sigma_{12}}{(\mu_1 - r_f)\sigma_2^2 + (\mu_2 - r_f)\sigma_1^2 - (\mu_1 + \mu_2 - 2r_f)\sigma_{12}}$$
$$t_2 = 1 - t_1.$$

Therefore

$$\begin{aligned} t_1 &= \frac{(\mu_1 - r_f)\sigma_2^2 - (\mu_2 - r_f)\sigma_{12}}{(\mu_1 - r_f)\sigma_2^2 + (\mu_2 - r_f)\sigma_1^2 - (\mu_1 + \mu_2 - 2r_f)\sigma_{12}} \\ &= \frac{0.1 * 4 - 0.3 * 0}{0.1 * 4 + 0.3 * 1 - 0.4 * 0} = \frac{4}{7}, \\ t_2 &= \frac{3}{7}. \end{aligned}$$

3.5. Suppose we have $W_0 = 1000$. Find the value of risk at $\alpha = 0.05$ if we invest all the money in the global minimum variance portfolio found in 3.4. Find the value of risk at $\alpha = 0.05$ if we invest all the money in the tangency portfolio found in 3.4.

Answer: Let R_m denote the simple return of the global minimum variance portfolio. Since $m_1 = \frac{4}{5}$ and $m_2 = \frac{1}{5}$,

$$R_m = \frac{4}{5}R_1 + \frac{1}{5}R_2.$$

It is clear that

$$\begin{aligned}
\mu_m &= E[R_m] = \frac{4}{5}\mu_1 + \frac{1}{5}\mu_2 = 0.24, \\
\sigma_m^2 &= Var(R_m) = (\frac{4}{5})^2\sigma_1^2 + (\frac{1}{5})^2\sigma_2^2 + 2\frac{4}{5}\frac{1}{5}\sigma_{12} = 0.8.
\end{aligned}$$

Therefore,

$$R_m \sim N(0.24, 0.8).$$

Since

$$R_m = \mu_m + \sigma_m \frac{R_m - \mu_m}{\sigma_m} = \mu_m + \sigma_m Z$$

where Z is a normal random variable, the α-quantiles of R_m and Z satisfy

$$q_\alpha^{R_m} = \mu_m + \sigma_m q_\alpha^Z.$$

Therefore

$$\begin{aligned}
\text{VaR}_{m,0.05} &= W_0 q_{0.05}^{R_m} = W_0(\mu_m + \sigma_m q_{0.05}^Z) \\
&= 1000\left(0.24 - \sqrt{0.8} * 1.6449\right) \\
&= -1231.243.
\end{aligned}$$

Let R_t denote the simple return of the tangency portfolio. Since $t_1 = \frac{4}{7}$ and $t_2 = \frac{3}{7}$,

$$\begin{aligned}
\mu_t &= E[R_t] = \frac{4}{7}\mu_1 + \frac{3}{7}\mu_2 = \frac{2}{7}, \\
\sigma_t^2 &= \text{Var}(R_t) = (\frac{4}{7})^2\sigma_1^2 + (\frac{3}{7})^2\sigma_2^2 = \frac{52}{49}.
\end{aligned}$$

Therefore,

$$R_t \sim N(2/7, 52/49).$$

Since

$$R_m = \mu_m + \sigma_m \frac{R_m - \mu_m}{\sigma_m} = \mu_m + \sigma_m Z$$

where Z is a normal random variable, the α-quantiles of R_m and Z satisfy

$$q_\alpha^{R_m} = \mu_m + \sigma_m q_\alpha^Z.$$

Therefore

$$\begin{aligned}
\mathrm{VaR}_{m,0.05} &= W_0 q_{0.05}^{R_m} = W_0(\mu_m + \sigma_m q_{0.05}^{Z}) \\
&= 1000\left(\frac{2}{7} - \sqrt{\frac{52}{49}} * 1.6449\right) \\
&= -1408.792.
\end{aligned}$$

4. Let $\{X_t\}_t$ be a time series generated by

$$X_t = u_t + \theta_1 u_{t-1}, \text{ where } \{u_t\}_t \sim iid(0, \sigma^2),$$

where θ_1 is a finite real number. Let $\gamma(j)$ and $\rho(j)$ denote the auto-covariance function and the auto-correlation function of $\{X_t\}_t$ respectively.

4.1. Find the auto-covariance function $\gamma(j)$.

Answer: By definition,

$$\begin{aligned}
\gamma_0 &= Var(X_t), \quad \text{by the knowledge that } \{X_t\}_t \text{ is covariance stationary} \\
&= Var(u_t + \theta_1 u_{t-1}), \quad \text{by } X_t = u_t + \theta_1 u_{t-1} \\
&= Var(u_t) + \theta_1^2 Var(u_{t-1}), \text{by the independence between } u_t \text{ and } u_{t-1} \\
&= (1+\theta_1^2)\sigma^2. \quad \text{by } \{u_t\}_t \sim iid(0, \sigma^2)
\end{aligned}$$

Moreover

$$\begin{aligned}
& E[X_t] \\
=\ & E[u_t + \theta_1 u_{t-1}], \quad \text{by } X_t = u_t + \theta_1 u_{t-1} \\
=\ & E[u_t] + \theta_1 E[u_{t-1}], \quad \text{by the linearity of expectation} \\
=\ & 0 + \theta_1 \cdot 0, \quad \text{by } E[u_t] = 0 \text{ and } E[u_{t-1}] = 0 \\
=\ & 0.
\end{aligned}$$

$$\begin{aligned}
& E[X_t X_{t-1}] \\
=\ & E[(u_t + \theta_1 u_{t-1})(u_{t-1} + \theta_1 u_{t-2})], \quad \text{by } X_t = u_t + \theta_1 u_{t-1} \\
=\ & E[u_t u_{t-1}] + \theta_1 E[u_{t-1}^2] + \theta_1 E[u_t u_{t-2}] + \theta_1^2 E[u_{t-1} u_{t-2}], \text{ by the linearity of expectation} \\
=\ & \theta_1 E[u_{t-1}^2], \quad \text{by } \{u_t\}_t \sim iid(0, \sigma^2),\ E[u_t] = 0 \text{ and } E[u_{t-1}] = 0 \\
=\ & \theta_1 \sigma^2.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\gamma_1 &= Cov(X_t, X_{t-1}), \text{ by the knowledge that } \{X_t\}_t \text{ is covariance stationary} \\
&= E[X_t X_{t-1}] - E[X_t]E[X_{t-1}], \quad \text{by the the definition of covariance} \\
&= \theta_1 \sigma^2, \quad \text{by } E[X_t X_{t-1}] = \theta_1 \sigma^2,\ E[X_t] = 0 \text{ and } E[X_{t-1}] = 0
\end{aligned}$$

for any $j > 1$, we see that X_t and X_{t-j} are independent, since X_t depends on u_t and u_{t-1} while X_{t-j} depends on u_{t-j} and u_{t-j-1} and (u_t, u_{t-1}) is independent with respect to (u_{t-j}, u_{t-j-1}). Therefore,

$$\gamma_j = Cov(X_t, X_{t-j}) = 0 \text{ for } j > 1.$$

The auto-covariance function $\gamma(j)$ is

$$\gamma(j) = \begin{cases} (1+\theta_1^2)\sigma^2, & j = 0 \\ \theta_1\sigma^2, & |j| = 1 \\ 0, & |j| > 1 \end{cases}.$$

4.2. Find the auto-correlation function $\rho(j)$.

Answer: Since $\rho(j) = \gamma(j)/\gamma(0)$, from the auto-covariance function $\gamma(j)$ we found above, we get

$$\rho(j) = \begin{cases} 1, & j = 0 \\ \frac{\theta_1}{1+\theta_1^2}, & |j| = 1 \\ 0, & |j| > 1 \end{cases}.$$

4.3. What are the largest and smallest possible values for $\rho(1)$?

Answer: Since $1 + \theta_1^2 \geq 2\,|\theta_1|$, we have

$$\frac{1}{1+\theta_1^2} \leq \frac{1}{2\,|\theta_1|}.$$

Therefore,

$$\rho(1) = \frac{\theta_1}{1+\theta_1^2} \leq \frac{\theta_1}{2\,|\theta_1|} = \frac{1}{2} \text{ if } \theta_1 > 0$$

and

$$\rho(1) = \frac{\theta_1}{1+\theta_1^2} \geq \frac{\theta_1}{2\,|\theta_1|} = -\frac{1}{2} \text{ if } \theta_1 < 0.$$

When $\theta_1 = 0$, we have $\rho(1) = 0$. In sum, then smallest possible value for $\rho(1)$ is $-\frac{1}{2}$ which is achieved when $\theta_1 = -1$, and the largest possible value for $\rho(1)$ is $\frac{1}{2}$ which is achieved when $\theta_1 = 1$.

4.4. Suppose that $\sigma^2 = 1$. Find the value of θ_1 such that $\gamma(0) = 2$, $\gamma(1) < 0$ and $\gamma(h) = 0$ for $|h| > 1$.

Answer: Since $\gamma(0) = (1 + \theta_1^2)\sigma^2$ which together with $\sigma^2 = 1$ and $\gamma(0) = 2$ implies that

$$1 + \theta_1^2 = 2.$$

Solving for θ_1 from the above equation, we get two possible solutions: 1 and -1. Since $\gamma(1) = \theta_1\sigma^2$, $\gamma(1) < 0$ implies that $\theta_1 < 0$. Hence the value of θ_1 should be -1.

5 One of the recent Nobel laureate in Economics, Eugene Fama, has been advocating the efficient-market hypothesis. One testable implication of this hypothesis is called martingale (MG) pricing:

$$E\left[p_t|\mathcal{F}_{t-1}\right] = p_{t-1} \text{ for all } t,$$

where p_t is a log of stock price index (e.g., S&P 500 index) and $\mathcal{F}_{t-1}$ represents all available information up to time $t-1$. Answer the following questions.

5.1. Briefly discuss why MG pricing is one supporting evidence for efficient-market hypothesis.

Answer: Since $E\left[p_t|\mathcal{F}_{t-1}\right]$ is the best prediction of the price p_t at time t given all possible information at time $t-1$, the MG pricing implies that the best prediction of the future price is the price at today. This means that the market is efficient in the sense that it absorbs all the information about the future price when determining the equilibrium price today.

5.2. Under MG pricing, show cc return r_t is mds, i.e., $E\left[r_t|\mathcal{F}_{t-1}\right] = 0$.

Answer:

$$\begin{aligned}
& E[r_t|\mathcal{F}_{t-1}] \\
= \;& E\left[p_t - p_{t-1}|\mathcal{F}_{t-1}\right], \quad \text{by } r_t = p_t - p_{t-1} \\
= \;& E\left[p_t|\mathcal{F}_{t-1}\right] - E\left[p_{t-1}|\mathcal{F}_{t-1}\right], \text{ by the linearity of the conditional expectation} \\
= \;& p_{t-1} - p_{t-1}, \text{ by } E\left[p_t|\mathcal{F}_{t-1}\right] = p_{t-1} \text{ (MG) and } E\left[p_{t-1}|\mathcal{F}_{t-1}\right] = p_{t-1} \\
= \;& 0.
\end{aligned}$$

5.3. The definition of mds allows a flexible specification of the second moment. For example, $r_t \sim mds\left(0, \sigma_t^2\right)$ where σ_t^2 can be time varying (hence nonstationary). Briefly discuss the benefit of this specification over CER model.

Answer: This specification allows the variance of r_t to be a random variable indexed by t. Therefore, it can be used to model the time series which time varying variance and dependence of the time varying variance (or volatility clustering). Such features are consistent with our observation from real data. But the CER model assumes non-random and constant variance which fails to capture these important facts in reality.

6. Indicate whether the following statements are true or false (circle one). Briefly discuss why it is so.

6.1. If $\varepsilon_t \sim mds\ (0, \sigma_\varepsilon^2)$, then $\varepsilon_t \sim iid\ (0, \sigma_\varepsilon^2)$.

True False

Why?

Answer: False. For example, let $\varepsilon_t = e_t e_{t-1}$, where $e_t \sim iid\ N(0, \sigma^2)$. Then

$$E[\varepsilon_t|\mathcal{F}_{t-1}] = E[e_t e_{t-1}|\mathcal{F}_{t-1}] = e_{t-1}E[e_t|\mathcal{F}_{t-1}] = e_{t-1} \cdot 0 = 0$$

and $E[\varepsilon_t^2] = E\left[e_t^2 e_{t-1}^2\right] = E[e_t^2]\,E\left[e_{t-1}^2\right] = 1$, which implies that $\varepsilon_t \sim iid$ $(0, 1)$. However, ε_t is not independent because

$$\begin{aligned}
& Cov(\varepsilon_t^2, \varepsilon_{t-1}^2) \\
= \;& E[\varepsilon_t^2\varepsilon_{t-1}^2] - E[\varepsilon_t^2]E[\varepsilon_{t-1}^2], \quad \text{by the definition of covariance} \\
= \;& E[e_t^2 e_{t-1}^4 e_{t-2}^2] - E[e_t^2 e_{t-1}^2]E[e_{t-1}^2 e_{t-2}^2], \quad \text{by the definitions of } \varepsilon_t \text{ and } \varepsilon_{t-1} \\
= \;& E[e_t^2]E[e_{t-1}^4]E[e_{t-2}^2] - E[e_t^2]E[e_{t-1}^2]E[e_{t-1}^2]E[e_{t-2}^2], \quad \text{by the independence} \\
= \;& 3 - 1 = 2, \quad \text{by } E[e_t^2] = 1 \text{ and } E[e_t^4] = 3 \text{ for any } t.
\end{aligned}$$

6.2. If $\varepsilon_t \sim mds\ (0, \sigma_\varepsilon^2)$, then $\varepsilon_t \sim WN\ (0, \sigma_\varepsilon^2)$.

True False

Why?

Answer: True. Because for any $t \neq s$ (say $t > s$),

$$\begin{aligned}
& E[\varepsilon_t\varepsilon_s] \\
= \;& E[E[\varepsilon_t\varepsilon_s|\mathcal{F}_{t-1}]], \text{ by the property of the conditional expectation} \\
= \;& E[\varepsilon_s E[\varepsilon_t|\mathcal{F}_{t-1}]], \text{ by the fact that } \varepsilon_s \text{ is known given } \mathcal{F}_{t-1}\ (t > s) \\
= \;& E[\varepsilon_s \cdot 0], \quad \text{by } E[\varepsilon_t|\mathcal{F}_{t-1}] = 0 \\
= \;& E[0] \\
= \;& 0
\end{aligned}$$

6.3. If $\{\varepsilon_t\}_t$ is strictly stationary, then it is also covariance stationary.

True False

Why?

Answer: False. If ε_t has infinite variance (e.g., student-t with 1 degree of freedom), then it is not covariance stationary.

6.4. Let Y_1 and Y_2 be iid r.v's from (μ, σ^2), and let $\hat{\mu}_1 = Y_1$ and $\hat{\mu}_2 = \frac{Y_1+Y_2}{2}$ are two different point estimators for μ. From MSE criteria, we prefer to use $\hat{\mu}_1$ rather than $\hat{\mu}_2$.

True False

Why?

Answer: False. Since

$$E[\hat{\mu}_1] = E[Y_1] = \mu$$

and

$$E[\hat{\mu}_2] = E[Y_1/2 + Y_2/2] = \frac{1}{2}E[Y_1] + \frac{1}{2}E[Y_2] = \mu,$$

both estimators are unbiased. To compare their MSE, it is therefore sufficient to compare their variances.

$$Var(\hat{\mu}_1) = Var(Y_1) = \sigma^2$$

and

$$Var[\hat{\mu}_2] = Var[Y_1/2 + Y_2/2] = \frac{1}{4}Var[Y_1] + \frac{1}{4}Var[Y_2] = \frac{\sigma^2}{2}.$$

It is clear that $\hat{\mu}_2$ has smaller variance and hence smaller MSE (since it is also unbiased). We should prefer to use $\hat{\mu}_2$ rather than $\hat{\mu}_1$.

6.5. MA(1) process $Y_t = \mu + \varepsilon_t + \theta\varepsilon_{t-1}$, $\varepsilon_t \sim mds(0, \sigma_\varepsilon^2)$ is *not covariance stationary* when $|\theta| = 1$.

True False

Why?

Answer: False. We have show that Y_t is covariance stationary as long as θ is a finite number.