Understanding Financial Returns

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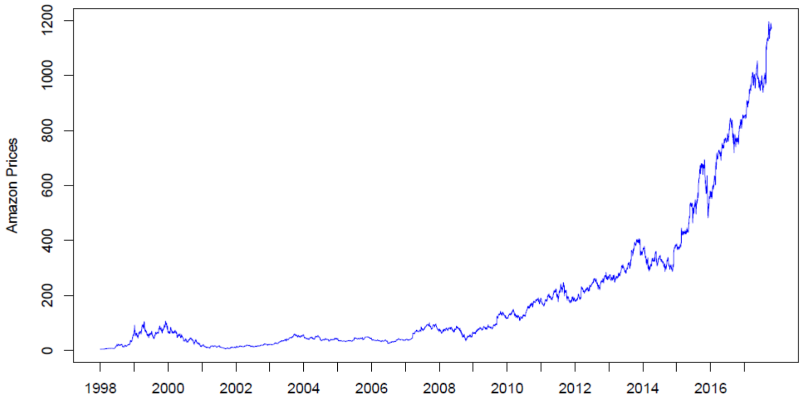
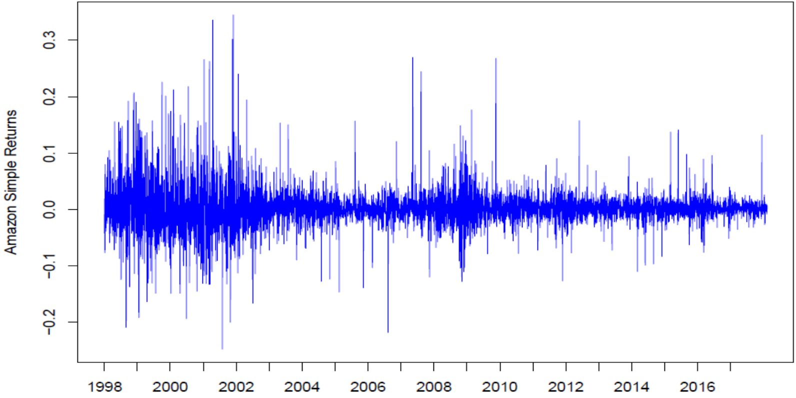
Motivation

- We hear about stock price more often than stock returns, e.g., from news or other media.
- Both price and returns are used in financial data analysis
- **Why not just price but returns?**
  - return is a relative concept, i.e., how much we gain (future value) from our initial investment (current value)
  - another important reason is that, return has more standard stochastic property than price, so easier to study using standard prob/stat theory

Daily Prices of Amazon (01/01/1998 – 12/26/2017)



Daily Simple Returns of Amazon (01/01/1998 – 12/26/2017)



Outline

- **Introductory concept: the time value of money**
  - Future value
  - Compounding
  - Effective annual rate
- **Asset return calculations**
  - Simple returns: multi-periods returns, portfolio returns and real returns (inflation adjustment), adjusting for dividends, annualizing and averaging
  - Continuously compounding (cc) returns: multi periods returns, portfolio returns and real returns (inflation adjustment)
- **Reading:** E. Zivot's book chapter on return calculation
- Optional: Chapter 1 (**Asset Returns**) in Fan and Yao's book
- Optional: Chapter 1 (**Returns**) and Chapter 2 (**Fixed Income Securities**) in Ruppert's book

The Time Value of Money

- **Future Value:**
  - $\$V$ invested for n years at simple interest rate R per year (annual rate)
  - Compounding of interest occurs at end of year (1 time per year)[1]

$$FV_n = \$V \cdot (1 + R)^n$$

where FV_n is future value after n years

[1] compound: to pay interest on both an amount of money and the interest it has already earned

Example

- Consider putting \$1000 in an interest checking account that pays a simple annual percentage rate of 3%. The future value after $n = 1$, 5 and 10 years is, respectively,

$$
\begin{aligned}
FV_1 &= \$1000 \cdot (1.03)^1 = \$1030, \\
FV_5 &= \$1000 \cdot (1.03)^5 = \$1159.27, \\
FV_{10} &= \$1000 \cdot (1.03)^{10} = \$1343.92.
\end{aligned}
$$

Future Value

- FV function $FV_n = \$V \cdot (1+R)^n$ is a relationship between four variables: FV_n, V, R and n. Given three variables, you can solve for:
- **Present value**:
$$V = \frac{FV_n}{(1+R)^n}.$$
- **Compound annual return**:
$$R = \left(\frac{FV_n}{V}\right)^{1/n} - 1.$$
- **Investment horizon**:
$$n = \frac{\ln(FV_n/V)}{\ln(1+R)}.$$
  - in this course, both log and ln will be *natural* log

Multiple Compounding

- If compounding occurs m times per year:

$$FV_n^m = \$V \cdot \left(1 + \frac{R}{m}\right)^{m \cdot n},$$
$$\frac{R}{m} = \text{periodic interest rate.}$$

- If compounding occurs *continuously*[2]:

$$FV_n^\infty = \lim_{m \to \infty} \$V \cdot \left(1 + \frac{R}{m}\right)^{m \cdot n} = \$Ve^{R \cdot n},$$
$$e^1 = 2.71828.$$

[2] Exponential function: $e^x = \lim_{m \to \infty} \left(1 + \frac{x}{m}\right)^m$

Multiple Compounding

- **Example**: If the simple annual percentage rate is 10%, then the value of \$1000 at the end of one year ($n = 1$) for different values of m is given in the table below.

Compounding Frequency	Value of \$1000 at the end of 1 year ($R = 10\%$)
Annually ($m = 1$)	1100.00
Quarterly ($m = 4$)	1103.81
Weekly ($m = 52$)	1105.06
Daily ($m = 365$)	1105.16
Continuously ($m = \infty$)	1105.17

Effective Annual Rate

- Annual rate R_A that equates FV_n^m with FV_n; that is,

$$\$V\left(1+\frac{R}{m}\right)^{m\cdot n} = \$V(1+R_A)^n.$$

  - Recall R was annual rate with 1-time compounding
  - R_A: effective annual rate with m-times compounding
  - Solving for R_A gives

$$\left(1+\frac{R}{m}\right)^{m} = 1+R_A \Rightarrow R_A = \left(1+\frac{R}{m}\right)^{m} - 1.$$

Effective Annual Rate and CC Rate

- Effective annual rate R_A with ∞-times compounding

$$R_A = \lim_{m \to \infty} \left(1 + \frac{R}{m}\right)^m - 1 = e^R - 1$$

- *Continuous Compounding Rate*: an annual rate r corresponding with the above R_A

$$r = \ln\left(1 + R_A\right)$$

  - in lecture note R is used for r, but we use r to be consistent with cc return concept below

Multiple Compounding

- Example: *Compute effective annual rate with semi-annual compounding*
  - the effective annual rate associated with an investment with a simple annual rate $R = 10\%$ and semi-annual compounding ($m = 2$) is determined by solving

$$(1 + R_A) = \left(1 + \frac{0.10}{2}\right)^2$$

$$\Rightarrow R_A = \left(1 + \frac{0.10}{2}\right)^2 - 1 = 0.1025.$$

Multiple Compounding

- Effective annual rate, R_A, with above example:

Compounding Frequency	Value of \$1000 at end of 1 year $(R = 10\%)$	R_A
Annually $(m = 1)$	1100.00	10%
Quarterly $(m = 4)$	1103.81	10.38%
Weekly $(m = 52)$	1105.06	10.51%
Daily $(m = 365)$	1105.16	10.52%
Continuously $(m = \infty)$	1105.17	10.52%

- Thus *cc rate* $r = \ln(1 + R_A) = \ln(1.1052)$

Outline

- Simple returns
  - concept and multi-periods returns,
  - portfolio returns
  - some adjustment: real returns (inflation adjustment), adjusting for dividends, annualizing and averaging
- Continuously compounding (cc) returns
  - concept and multi periods returns,
  - portfolio returns
  - real returns (inflation adjustment)

Simple Return Calculations

- Now let's think of monthly rate of return, rather than annual rate
- Recall (for $n = 1$ period): $\$FV = \$V \cdot (1 + R)$
- From now on, $V = P_{t-1}$ and $FV = P_t$

Simple Return Calculations

- Simple Returns:

$$P_t = P_{t-1}\left(1 + R_t\right)$$

  - P_t = price at the end of month t on an asset that pays no dividends[3]
  - P_{t-1} = price at the end of month $t-1$

- The simple return R_t:

$$\begin{aligned} 1 + R_t &= \frac{P_t}{P_{t-1}} = \textit{gross return} \text{ over month } t, \\ R_t &= \frac{P_t - P_{t-1}}{P_{t-1}} = \left(\% \triangle P_t\right) = \textit{net return} \text{ over month } t \end{aligned}$$

[3] dividend: an amount of a company's profits that the company pays to people who own stock in the company

Simple Return Calculations

- **Example**. *One month investment in Amazon stock.*

Buy stock at end of month $t-1$ at $P_{t-1} = \$85$ and sell stock at end of next month for $P_t = \$90$. Assuming that *Amazon* does not pay a dividend between months $t-1$ and t, the one-month simple net and gross returns are

$$R_t = \frac{\$90 - \$85}{\$85} = \frac{\$90}{\$85} - 1 = 1.0588 - 1 = 0.0588,$$

$$1 + R_t = 1.0588.$$

The one month investment in *Amazon* yielded a 5.88% monthly return.

Simple Return Calculations

- **Multi-period Returns**
- Simple two-month return

$$R_t(2) = \frac{P_t - P_{t-2}}{P_{t-2}} = \frac{P_t}{P_{t-2}} - 1.$$

- Relationship to one month simple returns

$$R_t(2) = \frac{P_t}{P_{t-2}} - 1 = \frac{P_t}{P_{t-1}} \cdot \frac{P_{t-1}}{P_{t-2}} - 1 = (1 + R_t) \cdot (1 + R_{t-1}) - 1.$$

Simple Return Calculations

- Here

$$
\begin{aligned}
1 + R_t &= \text{one-month gross return over month } t, \\
1 + R_{t-1} &= \text{one-month gross return over month } t-1, \\
&\implies 1 + R_t(2) = (1 + R_t) \cdot (1 + R_{t-1}).
\end{aligned}
$$

  - two-month gross return = the product of two one-month gross returns

- Note: two-month returns are *not additive*:

$$
\begin{aligned}
R_t(2) &= R_t + R_{t-1} + R_t \cdot R_{t-1} \\
&\approx R_t + R_{t-1} \;\; \text{if } R_t \text{ and } R_{t-1} \text{ are small}
\end{aligned}
$$

Simple Return Calculations

- **Example**: *Two-month return on Amazon*
- Suppose that the price of *Amazon* in month $t-2$ is \$80 and no dividend is paid between months $t-2$ and t. The two-month net return is

$$R_t(2) = \frac{\$90 - \$80}{\$80} = \frac{\$90}{\$80} - 1 = 1.1250 - 1 = 0.1250,$$

or 12.50% per two months. The two one-month returns are

$$R_{t-1} = \frac{\$85 - \$80}{\$80} = 1.0625 - 1 = 0.0625$$
$$R_t = \frac{\$90 - 85}{\$85} = 1.0588 - 1 = 0.0588,$$

and the product of the two one-month gross returns is

$$1 + R_t(2) = 1.0625 \times 1.0588 = 1.1250.$$

Simple Return Calculations

- Simple *k*-months Return

$$
\begin{aligned}
R_t(k) &= \frac{P_t - P_{t-k}}{P_{t-k}} = \frac{P_t}{P_{t-k}} - 1 \\
1 + R_t(k) &= \frac{P_t}{P_{t-1}} \cdot \frac{P_{t-1}}{P_{t-2}} \cdot \cdots \cdot \frac{P_{t-k+1}}{P_{t-k}} \\
&= (1 + R_t) \cdot (1 + R_{t-1}) \cdot \cdots \cdot (1 + R_{t-k+1}) \\
&= \prod_{j=0}^{k-1} (1 + R_{t-j})
\end{aligned}
$$

- Note

$$
R_t(k) \neq \sum_{j=0}^{k-1} R_{t-j}
$$

hence *not additive*

Portfolio Simple Return Calculations

- Invest $\$V$ in two assets: A and B for 1 period
- x_A = share of $\$V$ invested in A; $\$V \times x_A = \$$ amount
- x_B = share of $\$V$ invested in B; $\$V \times x_B = \$$ amount
- Assume $x_A + x_B = 1$ (and this will always hold)
- Portfolio is defined by investment shares x_A and x_B

Portfolio Simple Return Calculations

- At the end of the period, the investments in A and B grow to

$$
\begin{aligned}
\$V(1+R_{p,t}) &= \$V\left[x_A(1+R_{A,t}) + x_B(1+R_{B,t})\right] \\
&= \$V\left[x_A + x_B + x_A R_{A,t} + x_B R_{B,t}\right] \\
&= \$V\left[1 + x_A R_{A,t} + x_B R_{B,t}\right] \\
&\Rightarrow R_{p,t} = x_A R_{A,t} + x_B R_{B,t}
\end{aligned}
$$

- The simple portfolio return is a share weighted average of the simple returns on the individual assets: hence *additive* with proper weights.

Portfolio Simple Return Calculations

- **Example**: *Portfolio of Amazon and Starbucks stock*

Purchase ten shares of each stock at the end of month $t-1$ at prices

$$P_{amzn,t-1} = \$85,\ P_{subx,t-1} = \$30,$$

The initial value of the portfolio is

$$V_{t-1} = 10 \times \$85 + 10 \times \$30 = \$1,150.$$

The portfolio shares are

$$x_{amzn} = 850/1150 = 0.7391,\ x_{sbux} = 300/1150 = 0.2609.$$

The end of month t prices are $P_{amzn,t} = \$90$ and $P_{sbux,t} = \$28$.

Portfolio Simple Return Calculations

- Assuming *Amazon* and Starbucks do not pay a dividend between periods $t-1$ and t, the one-period returns are

$$R_{amzn,t} = \frac{\$90 - \$85}{\$85} = 0.0588$$
$$R_{sbux,t} = \frac{\$28 - \$30}{\$30} = -0.0667$$

The return on the portfolio is

$$R_{p,t} = (0.7391)(0.0588) + (0.2609)(-0.0667) = 0.02609$$

and the value at the end of month t is

$$FV_t = \$1,150 \times (1.02609) = \$1,180$$

Portfolio Simple Return Calculations

- In general, for a portfolio of n assets with investment shares x_i such that $x_1 + \cdots + x_n = 1$

$$
\begin{aligned}
1 + R_{p,t} &= \sum_{i=1}^{n} x_i (1 + R_{i,t}) \\
R_{p,t} &= \sum_{i=1}^{n} x_i R_{i,t} \\
&= x_1 R_{1t} + \cdots + x_n R_{nt}
\end{aligned}
$$

  - *additive* with proper weights.

Adjusting for Dividends

- In addition to the price change, if dividend is being paid

$$
\begin{aligned}
D_t &= \text{ dividend payment between months } t-1 \text{ and } t \\
R_t^{total} &= \frac{P_t + D_t - P_{t-1}}{P_{t-1}} = \frac{P_t - P_{t-1}}{P_{t-1}} + \frac{D_t}{P_{t-1}} \\
&= \text{capital gain return} + \text{dividend yield} \\
1 + R_t^{total} &= \frac{P_t + D_t}{P_{t-1}}
\end{aligned}
$$

- We call R_t^{total} as *total return*

Adjusting for Dividends

- **Example**. *Total return on Amazon stock.*

Buy stock in month $t-1$ at $P_{t-1} = \$85$ and sell the stock the next month for $P_t = \$90$. Assume *Amazon* pays a \$1 dividend between months $t-1$ and t. The capital gain, dividend yield and total return are then

$$\begin{aligned} R_t^{total} &= \frac{\$90 + \$1 - \$85}{\$85} = \frac{\$90 - \$85}{\$85} + \frac{\$1}{\$85} \\ &= 0.0588 + 0.0118 \\ &= 0.0707 \end{aligned}$$

The one-month investment in *Amazon* yields a 7.07% per month total return. The capital gain component is 5.88%, and the dividend yield component is 1.18%.

Adjusting for Inflation

- The computation of *real returns* on an asset is a two step process:
- Deflate the nominal price P_t of the asset by an index of the general price level CPI_t
- Compute returns in the usual way using the deflated prices

$$P_t^{\text{Real}} = \frac{P_t}{CPI_t}$$

$$R_t^{\text{Real}} = \frac{P_t^{\text{Real}} - P_{t-1}^{\text{Real}}}{P_{t-1}^{\text{Real}}} = \frac{\frac{P_t}{CPI_t} - \frac{P_{t-1}}{CPI_{t-1}}}{\frac{P_{t-1}}{CPI_{t-1}}}$$

$$= \frac{P_t}{P_{t-1}} \cdot \frac{CPI_{t-1}}{CPI_t} - 1$$

Adjusting for Inflation

- Alternatively, define inflation as

$$\begin{aligned} \pi_t &= \%\Delta CPI_t = \frac{CPI_t - CPI_{t-1}}{CPI_{t-1}}, \\ 1 + \pi_t &= \frac{CPI_t}{CPI_{t-1}} \end{aligned}$$

Then

$$\begin{aligned} R_t^{\text{Real}} &= \frac{1 + R_t}{1 + \pi_t} - 1, \\ 1 + R_t^{\text{Real}} &= \frac{1 + R_t}{1 + \pi_t} \end{aligned}$$

Adjusting for Inflation

- **Example**. *Compute real return on Amazon stock.*

Suppose the CPI in months $t-1$ and t is 1 and 1.01, respectively, representing a 1% monthly growth rate in the overall price level. The real prices of *Amazon* stock are

$$P_{t-1}^{\text{Real}} = \frac{\$85}{1} = \$85, \;\; P_t^{\text{Real}} = \frac{\$90}{1.01} = \$89.1089$$

The real monthly return is

$$R_t^{\text{Real}} = \frac{\$89.10891 - \$85}{\$85} = 0.0483$$

Adjusting for Inflation

- The nominal return and inflation over the month are

$$R_t = \frac{\$90 - \$85}{\$85} = 0.0588, \;\; \pi_t = \frac{1.01 - 1}{1} = 0.01$$

Then the real return is

$$R_t^{\text{Real}} = \frac{1.0588}{1.01} - 1 = 0.0483$$

Notice that simple real return is almost, but not exactly, equal to the simple nominal return minus the inflation rate

$$R_t^{\text{Real}} \approx R_t - \pi_t = 0.0588 - 0.01 = 0.0488$$

from the approximating formula

$$\frac{y}{1+x} \simeq y \text{ if } x \text{ is small}$$

Annualizing Returns

- Returns are often converted to an annual return to establish a standard for comparison
- **Example**: Assume same monthly return R_m for 12 months:

$$\begin{aligned} & \text{Compound annual gross return (CAGR)} \\ = \quad & 1 + R_A = 1 + R_t(12) = (1 + R_m)^{12} \end{aligned}$$

$$\text{Compound annual net return} = R_A = (1 + R_m)^{12} - 1$$

  - Note: We don't use $R_A = 12R_m$ because this ignores multiple compounding.

Average Returns

- For investments over a given horizon, it is often of interest to compute a measure of average return over the horizon
- Consider a sequence of monthly investments over the year with monthly returns

$$R_1, R_2, \ldots, R_{12}$$

The annual return is

$$R_A = R(12) = (1 + R_1)(1 + R_2) \cdots (1 + R_{12}) - 1$$

Average Returns

- Question: What is the average monthly return?
- **Two possibilities**

1. Arithmetic average (can be misleading)

$$\bar{R} = \frac{1}{12}(R_1 + \cdots + R_{12})$$

2. Geometric average (better measure of average return)

$$\begin{aligned}(1+\bar{R})^{12} &= (1+R_A) = (1+R_1)(1+R_2)\cdots(1+R_{12}) \\ \Rightarrow \bar{R} &= (1+R_A)^{1/12} - 1 \\ &= [(1+R_1)(1+R_2)\cdots(1+R_{12})]^{1/12} - 1\end{aligned}$$

Average Returns

- **Example**: Consider a two period investment with returns

$$R_1 = 0.5, \;\; R_2 = -0.5$$

$1 invested over two periods grows to

$$FV = \$1 \times (1 + R_1)(1 + R_2) = (1.5)(0.5) = 0.75$$

for a 2-period return of

$$R(2) = 0.75 - 1 = -0.25$$

Hence, the 2-period investment loses 25%!

Average Returns

- **Example Cont'd:** The *arithmetic average* return is

$$\bar{R} = \frac{1}{2}(0.5 + (-0.5)) = 0$$

This is misleading because the actual investment lost money over the 2 period horizon. The compound 2-period return based on the arithmetic average is

$$(1 + \bar{R})^2 - 1 = 1^2 - 1 = 0$$

- The reason is that, as we have seen before, simple returns are not additive, so we should not use the additive average (arithmetic average)

Average Returns

- **Example Cont'd:** The *geometric average* is

$$[(1+0.5)(1-0.5)]^{1/2}-1=(0.75)^{1/2}-1=-0.1340$$

This is a better measure because it indicates that the investment eventually lost money. The compound 2-period return is

$$(1+\bar{R})^2-1=(0.867)^2-1=-0.25$$

Log and Exponential Functions

- $\log(0) = -\infty$, $\log(1) = 0$
- $e^{-\infty} = 0,\ e^{0} = 1,\ e^{1} = 2.7183$
- $\frac{d\log(x)}{dx} = \frac{1}{x},\ \frac{de^{x}}{dx} = e^{x}$
- $\log(e^{x}) = x,\ e^{\log(x)} = x$
- $\log(x \cdot y) = \log(x) + \log(y)$; $\log(\frac{x}{y}) = \log(x) - \log(y)$
- $\log(x^{y}) = y\log(x)$
- $e^{x}e^{y} = e^{x+y},\ e^{x}e^{-y} = e^{x-y}$
- $(e^{x})^{y} = e^{xy}$

Continuously compounded returns

- Notation: simple return R_t , cc return r_t
  - recall $P_t = P_{t-1} \cdot (1 + R_t)$ so $\frac{P_t}{P_{t-1}} = (1 + R_t)$
- Intuition: from the idea of continuous compounding,

$$
\begin{aligned}
P_t &= P_{t-1} \cdot \lim_{m \to \infty} \left(1 + \frac{r_t}{m}\right)^m \\
&= P_{t-1} \cdot e^{r_t}
\end{aligned}
$$

  - $e^{r_t} = \frac{P_t}{P_{t-1}}$
  - $r_t = \log\left(\frac{P_t}{P_{t-1}}\right)$: cc growth rate in prices between months $t-1$ and t
- Relation between simple return (R_t) and cc return (r_t)
  - $r_t = \log(1 + R_t)$: worth *remembering* this formula.

Continuously compounded returns

- For a function $f(x)$, a first order Taylor series expansion about $x = x_0$ is

$$f(x) = f(x_0) + \frac{d}{dx} f(x_0)(x - x_0) + \text{ remainder}$$

  - let $f(x) = \log(1 + x)$ and $x_0 = 0$. Note that

$$\frac{d}{dx} \log(1 + x) = \frac{1}{1 + x}, \ \frac{d}{dx} \log(1 + x_0) = 1$$

    Then

$$\log(1 + x) \approx \log(1) + 1 \cdot x = 0 + x = x$$

  - if x is small, $\log(1 + x) \approx x$
- From $r_t = \log(1 + R_t)$, when R_t is small: $r_t \approx R_t$ (but not exactly same)

Continuously compounded returns

- **Computational Trick:** from stock price to cc return

$$
\begin{aligned}
r_t &= \log\left(\frac{P_t}{P_{t-1}}\right) \\
&= \log(P_t) - \log(P_{t-1}) \\
&= p_t - p_{t-1} \\
&= \text{difference in log prices}
\end{aligned}
$$

where

$$p_t = \ln(P_t)$$

  - Mostly, lower case involves cc returns (hence log) while upper case used for simple returns.
  - again, log and ln are both natural logs, so same.

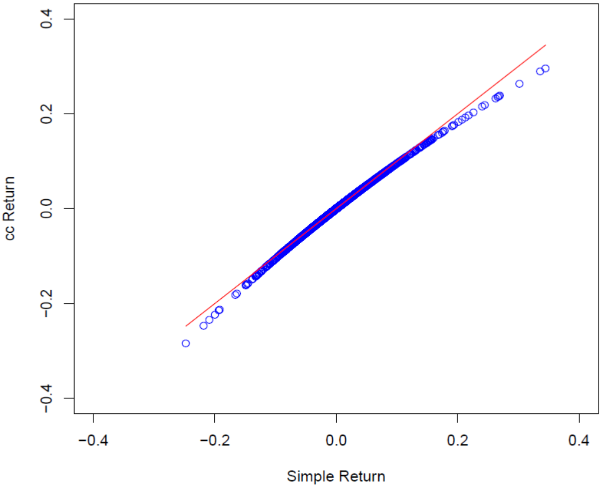
Continuously compounded returns

- Example. *Compute cc return*
- Let $P_{t-1} = 85$, $P_t = 90$ and $R_t = 0.0588$. Then the cc monthly return can be computed in two ways:

$$r_t = \log(1.0588) = 0.0571$$
$$r_t = \log(90) - \log(85) = 4.4998 - 4.4427 = 0.0571.$$

Notice that r_t is slightly smaller than R_t.

Simple Return v.s. cc Return (Amazon)



Continuously compounded returns

- **Multi-period cc Returns**

$$
\begin{aligned}
r_t(2) &= \log(1 + R_t(2)) \\
&= \log\left(\frac{P_t}{P_{t-2}}\right) \\
&= p_t - p_{t-2}
\end{aligned}
$$

- Note that

$$
e^{r_t(2)} = e^{\log(P_t/P_{t-2})} = \frac{P_t}{P_{t-2}}
$$

$$
\Rightarrow P_{t-2}e^{r_t(2)} = P_t
$$

$\Longrightarrow r_t(2) =$ cc growth rate in prices between months $t-2$ and t

Continuously compounded returns

- **Property**: cc returns are additive (UNLIKE simple returns)

$$
\begin{aligned}
r_t(2) &= \ln\left(\frac{P_t}{P_{t-1}} \cdot \frac{P_{t-1}}{P_{t-2}}\right) \\
&= \ln\left(\frac{P_t}{P_{t-1}}\right) + \ln\left(\frac{P_{t-1}}{P_{t-2}}\right) \\
&= r_t + r_{t-1}
\end{aligned}
$$

  - $r_t =$ cc return between months $t-1$ and t,
  - $r_{t-1} =$ cc return between months $t-2$ and $t-1$

Continuously compounded returns

- **Example**. *Compute cc two-month return*

Suppose $P_{t-2} = 80$, $P_{t-1} = 85$ and $P_t = 90$. The cc two-month return can be computed in two equivalent ways: (1) take difference in log prices

$$r_t(2) = \ln(90) - \ln(80) = 4.4998 - 4.3820 = 0.1178.$$

(2) sum the two cc one-month returns

$$\begin{aligned} r_t &= \ln(90) - \ln(85) = 0.0571 \\ r_{t-1} &= \ln(85) - \ln(80) = 0.0607 \\ r_t(2) &= 0.0571 + 0.0607 = 0.1178. \end{aligned}$$

Notice that $r_t(2) = 0.1178 < R_t(2) = 0.1250$.

Continuously compounded returns

- **General Result**

$$
\begin{aligned}
r_t(k) &= \ln(1 + R_t(k)) = \ln(\frac{P_t}{P_{t-k}}) \\
&= \ln(\frac{P_t}{P_{t-1}} \cdot \frac{P_{t-1}}{P_{t-2}} \cdot \cdots \cdot \frac{P_{t-k+1}}{P_{t-k}}) \\
&= \ln(\frac{P_t}{P_{t-1}}) + \ln(\frac{P_{t-1}}{P_{t-2}}) + \cdots + \ln(\frac{P_{t-k+1}}{P_{t-k}}) \\
&= r_t + r_{t-1} + \cdots + r_{t-k+1} \\
&= \sum_{j=0}^{k-1} r_{t-j}
\end{aligned}
$$

Portfolio cc returns

- Note that

$$R_{p,t} = \sum_{i=1}^{n} x_i R_{i,t}$$

$$r_{p,t} = \ln(1 + R_{p,t}) = \ln(1 + \sum_{i=1}^{n} x_i R_{i,t}) \neq \sum_{i=1}^{n} x_i r_{i,t}$$

$$\Rightarrow \text{ portfolio returns are not additive}$$

- If $R_{p,t} = \sum_{i=1}^{n} x_i R_{i,t}$ is not too large, then $r_{p,t} \approx R_{p,t}$ otherwise, $R_{p,t} > r_{p,t}$.

Portfolio cc returns

- **Example**. *Compute cc return on portfolio*

Consider a portfolio of *Amazon* and Starbucks stock with

$$
\begin{aligned}
x_{amzn} &= 0.25, \;\; R_{amzn,t} = 0.0588, \\
x_{sbux} &= 0.75, \;\; R_{sbux,t} = -0.0503 \\
R_{p,t} &= x_{amzn} R_{amzn,t} + x_{sbux,t} R_{sbux,t} = -0.02302
\end{aligned}
$$

The cc portfolio return is

$$
r_{p,t} = \ln(1 - 0.02302) = \ln(0.977) = -0.02329
$$

Note

$$
\begin{aligned}
r_{amzn,t} &= \ln(1 + 0.0588) = 0.05714 \\
r_{sbux,t} &= \ln(1 - 0.0503) = -0.05161 \\
x_{amzn} r_{amzn} + x_{sbux} r_{sbux} &= -0.02442 \neq r_{p,t}
\end{aligned}
$$

Adjusting for Inflation (cc returns)

- The cc one period real return is

$$r_t^{\text{Real}} = \ln(1 + R_t^{\text{Real}})$$
$$1 + R_t^{\text{Real}} = \frac{P_t}{P_{t-1}} \cdot \frac{CPI_{t-1}}{CPI_t}$$

It follows that

$$\begin{aligned} r_t^{\text{Real}} &= \ln\left(\frac{P_t}{P_{t-1}} \cdot \frac{CPI_{t-1}}{CPI_t}\right) = \ln\left(\frac{P_t}{P_{t-1}}\right) + \ln\left(\frac{CPI_{t-1}}{CPI_t}\right) \\ &= \ln(P_t) - \ln(P_{t-1}) + \ln(CPI_{t-1}) - \ln(CPI_t) \\ &= r_t - \pi_t^{cc} \end{aligned}$$

where

$$r_t = \ln(P_t) - \ln(P_{t-1}) = \text{nominal cc return}$$
$$\pi_t^{cc} = \ln(CPI_t) - \ln(CPI_{t-1}) = \text{ cc inflation}$$

Adjusting for Inflation (cc returns)

- **Example**. *Compute cc real return*

Suppose:

$$R_t = 0.0588$$
$$\pi_t = 0.01$$
$$R_t^{\text{Real}} = 0.0483$$

The real cc return is

$$r_t^{\text{Real}} = \ln(1 + R_t^{\text{Real}}) = \ln(1.0483) = 0.047.$$

Equivalently, $r_t^{\text{Real}} = r_t - \pi_t^{cc} = \ln(1.0588) - \ln(1.01) = 0.047$

Selected Summary

	simple returns $\mathbf{R}_t$	cc returns $\mathbf{r}_t$
Def'n	$1 + R_t = \frac{P_t}{P_{t-1}}$	$r_t = \log\left(\frac{P_t}{P_{t-1}}\right)$ $= \log(1 + R_t)$
Multi-	$1 + R_t(2) = (1 + R_t)(1 + R_{t-1})$; NOT additive over t's	$r_t(2) = r_t + r_{t-1}$; additive over t's
Real-	$1 + R_t^{\text{Real}} = \frac{1+R_t}{1+\pi_t}$	$r_t^{\text{Real}} = \ln(1 + R_t^{\text{Real}})$
	portfolio $\mathbf{R}_{p,t}$	portfolio $\mathbf{r}_{p,t}$
Def'n	$R_{p,t} = x_A R_{A,t} + x_B R_{B,t}$; additive across i's	$r_{p,t} = \ln(1 + R_{p,t})$; NOT additive across i's

What's next?

- Returns (R_t and r_t) are not constant over time but *random*
- Probabilistic thinking is a major tool to analyze financial returns
- Quick review on probability and statistics
  - basic knowledge (from prerequisites) will be assumed