Portfolio Theory

Econ 147

UCLA

Version 1.0

Introduction

- In practice, financial investment is typically done via portfolio
  - Market index (SP500) is a sort of portfolio
- Using the fundamental principle of (risk-averse) economic agent, we study portfolio theory
  - statistical and distributional properties of portfolio returns
  - financial risks of portfolio returns
  - portfolio frontier and efficient portfolio, mutual fund separation theorem

Introduction

- **Reading:** Course slides and Eric Zivot's Book Chapters on portfolio theory
- Optional: Chapter 5 (**Efficient Portfolios and Capital Asset Pricing Model**) in Fan and Yao's book
- Optional: Chapter 11 (**Portfolio Theory**) in Ruppert's book

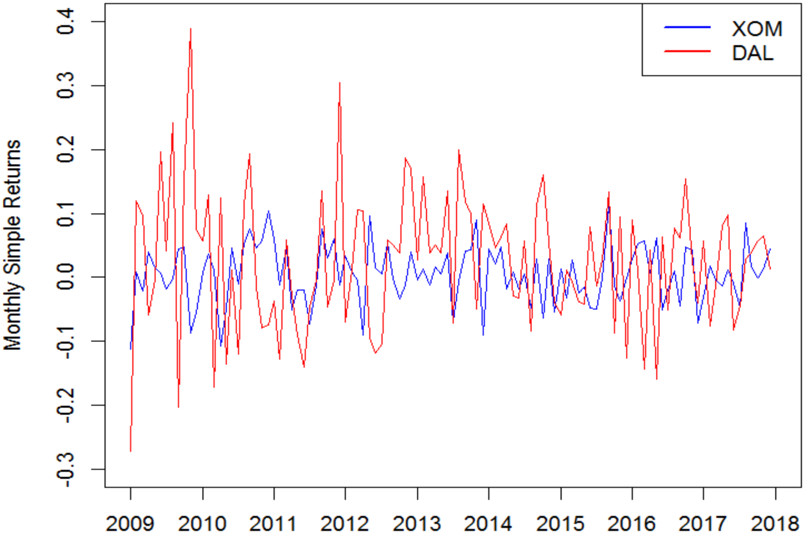
Introduction

- Investment in Two Risky Assets

$$
\begin{aligned}
R_A &= \text{ simple return on asset A} \\
R_B &= \text{ simple return on asset B} \\
W_0 &= \text{ initial wealth}
\end{aligned}
$$

- Assumptions:

$$
R_i \sim \text{iid } (\mu_i, \sigma_i^2), \; i = A, B
$$
$$
\text{cov}(R_A, R_B) = \sigma_{AB}, \; \text{cor}(R_A, R_B) = \rho_{AB}
$$



XOM
DAL
Monthly Simple Returns
0.4
0.3
0.2
0.1
0.0
-0.1
-0.2
-0.3
2009
2010
2011
2012
2013
2014
2015
2016
2017
2018

Introduction: underlying logics

- Let R_p denote the simple return of the portfolio
- Investors like high $E[R_p] = \mu_p$
- Investors dislike high $\mathrm{var}(R_p) = \sigma_p^2$
- Investment horizon is one period (e.g., one month or one year)

Note: Traditionally in portfolio theory, returns are simple and not continuously compounded

Introduction

- **Portfolios**

$$x_A = \text{share of wealth in asset A} = \frac{\$ \text{ in A}}{W_0}$$
$$x_B = \text{share of wealth in asset B} = \frac{\$ \text{ in B}}{W_0}$$

  - Long position

$$x_A > 0 \text{ or } x_B > 0$$

  - Short position

$$x_A < 0 \text{ or } x_B < 0$$

- Assumption: Allocate all wealth between assets A and B with $x_A + x_B = 1$, then the Portfolio Return is

$$R_p = x_A R_A + x_B R_B$$

Introduction

- Portfolio Distribution

$$\begin{aligned}\mu_p &= E[R_p] = x_A\mu_A + x_B\mu_B \\ \sigma_p^2 &= x_A^2\sigma_A^2 + x_B^2\sigma_B^2 + 2x_Ax_B\sigma_{AB} \\ &= x_A^2\sigma_A^2 + x_B^2\sigma_B^2 + 2x_Ax_B\rho_{AB}\sigma_A\sigma_B\end{aligned}$$

- End of Period Wealth

$$W_1 = W_0(1 + R_p) = W_0(1 + x_AR_A + x_BR_B)$$

Volatility Risk

- **Result for Portfolio Risk**: Portfolio's standard deviation (SD) is not a weighted average of asset SD unless $\rho_{AB} = 1$:

$$\begin{aligned}\sigma_p &= \left(x_A^2\sigma_A^2 + x_B^2\sigma_B^2 + 2x_Ax_B\rho_{AB}\sigma_A\sigma_B\right)^{1/2} \\ &\neq x_A\sigma_A + x_B\sigma_B \text{ for } \rho_{AB} \neq 1\end{aligned}$$

- If $\rho_{AB} = 1$ then

$$\sigma_{AB} = \rho_{AB}\sigma_A\sigma_B = \sigma_A\sigma_B$$

and

$$\begin{aligned}\sigma_p^2 &= x_A^2\sigma_A^2 + x_B^2\sigma_B^2 + 2x_Ax_B\sigma_A\sigma_B \\ &= (x_A\sigma_A + x_B\sigma_B)^2 \\ &\Rightarrow \sigma_p = x_A\sigma_A + x_B\sigma_B\end{aligned}$$

Examples:

- Data: Asset A (XOM) has lower expected return and risk than asset B (DAL)

$$
\begin{aligned}
\mu_A &= 0.00466,\ \ \mu_B = 0.02579 \\
\sigma_A^2 &= 0.00209,\ \ \sigma_B^2 = 0.01139 \\
\sigma_A &= 0.04566,\ \ \sigma_B = 0.10671 \\
\sigma_{AB} &= -0.00000713, \\
\rho_{AB} &= \frac{\sigma_{AB}}{\sigma_A \sigma_B} = -0.001463
\end{aligned}
$$

- The above numbers are estimators based on the monthly simple returns from 2009 Jan to 2018 Jan ($T = 108$).

Example: Long-Long Two-Asset Portfolio

- Consider an equally weighted portfolio with $x_A = x_B = 0.5$.
- The expected return ($\mu_A = 0.00466$, $\mu_B = 0.02579$), variance ($\sigma_A = 0.04566$, $\sigma_B = 0.10671$):

$$\mu_p = (0.5) \cdot (0.00466) + (0.5) \cdot (0.02579) = \mathbf{0.015227}$$

$$\sigma_p^2 = (0.5)^2 \cdot (0.00209) + (0.5)^2 \cdot (0.01139) + 2 \cdot (0.5)(0.5)(-0.00000713) = \mathbf{0.00336428}$$

$$\sigma_p = \sqrt{0.003633} = \mathbf{0.058}$$

  - This portfolio has expected return half-way between the expected returns on assets A and B,
  - but the portfolio standard deviation is less than half-way between the asset standard deviations, i.e.,

$$(0.5) \cdot (0.04566) + (0.5) \cdot (0.10671) = \mathbf{0.07619}$$

which reflects *risk reduction via diversification*.

Example: Long-Short Two-Asset Portfolio

- Consider a long-short portfolio with $x_A = -\frac{1}{2}$ and $x_B = \frac{3}{2}$. In this portfolio, asset A is sold short and the proceeds of the short sale are used to leverage the investment in asset B. The portfolio characteristics are

$$\mu_p = (-0.5) \cdot (0.00466) + (1.5) \cdot (0.02579) = \mathbf{0.03635}$$

$$\sigma_p^2 = (-0.5)^2 \cdot (0.00209) + (1.5)^2 \cdot (0.01139) + 2 \cdot (-0.5) \cdot (1.5)(-0.00000713) = \mathbf{0.02615}$$

$$\sigma_p = \sqrt{0.02615} = \mathbf{0.16171}$$

This portfolio has both a higher expected return and standard deviation than asset B

Portfolio Value-at-Risk

- Assume an initial investment of $\$W_0$ in the portfolio of assets A and B
- By definition, the profit of the portfolio is

$$L_1 = W_1 - W_0 = W_0 R_p$$

- For $\alpha \in (0, 1)$, the $\alpha \times 100\%$ portfolio value-at-risk is

$$\mathrm{VaR}_{p,\alpha} = W_0 q_\alpha^{R_p}$$

  where $q_\alpha^{R_p}$ is the α quantile of R_p.
- Two possible estimators of $\mathrm{VaR}_{p,\alpha}$:
  - parametric estimator based on normal distribution on R_p
  - nonparametric estimator from the sample quantile of R_p

Portfolio Value-at-Risk

- Suppose $R_A \sim N(\mu_A, \sigma_A^2)$ and $R_B \sim N(\mu_B, \sigma_B^2)$. Then
$$R_P = x_A R_A + x_B R_B \sim N(\mu_p, \sigma_p^2),$$
$$\begin{aligned} \mu_p &= x_A \mu_A + x_B \mu_B, \\ \sigma_p^2 &= x_A^2 \sigma_A^2 + x_B^2 \sigma_B^2 + 2 x_A x_B \sigma_{AB} \end{aligned}$$
- Therefore,
$$\mathrm{VaR}_{p,\alpha} = W_0 q_\alpha^{R_p} = W_0 \cdot \mathbf{qnorm}(\alpha, \mu_p, \sigma_p)$$
- Then $\widehat{\mathrm{VaR}}_{p,\alpha}^{PARA} = W_0 \cdot \mathbf{qnorm}(\alpha, \widehat{\mu}_p, \widehat{\sigma}_p)$, where
$$\begin{aligned} \widehat{\mu}_p &= x_A \widehat{\mu}_A + x_B \widehat{\mu}_B, \\ \sigma_p^2 &= x_A^2 \widehat{\sigma}_A^2 + x_B^2 \widehat{\sigma}_B^2 + 2 x_A x_B \widehat{\sigma}_{AB} \end{aligned}$$
where $\widehat{\mu}_A$ and $\widehat{\mu}_B$ are the sample means, $\widehat{\sigma}_A^2$, $\widehat{\sigma}_B^2$ and $\widehat{\sigma}_{AB}$ are the sample variance and covariance.

Portfolio Value-at-Risk

- Given data $\{(R_{A,t}, R_{B,t})\}_{t=1}^{T}$, we can calculate the sample $\{R_{p,t}\}_{t=1}^{T}$ where

$$R_{p,t} = x_A R_{A,t} + x_B R_{B,t} \text{ for } t = 1, \ldots, T.$$

- Order the sample of the portfolio simple return from the smallest to the largest

$$\left\{ R_{p,[1]}, \ldots, R_{p,[T]} \right\}$$

- Then $\widehat{\mathrm{VaR}}_{p,\alpha}^{NONP} = W_0 \cdot R_{p,[T\alpha]}$, where $[T\alpha]$ is the largest integer which is smaller than $T\alpha$.

Example: Long-Long Two-Asset Portfolio

- Consider $W_0 = 10000$ and $x_A = x_B = 0.5$.
- The expected return and standard deviation:

$$\mu_p = \mathbf{0.015227} \text{ and } \sigma_p = \mathbf{0.058}$$

- Therefore

$$\widehat{\mathrm{VaR}}_{p,0.1}^{PARA} = 10000 \cdot \textbf{qnorm}(0.1, 0.015227, 0.058) = -591.0299$$

- Moreover,

$$\widehat{\mathrm{VaR}}_{p,\alpha}^{NONP} = W_0 \cdot R_{p,[T\alpha]} = 10000 \cdot R_{p,[10]} = -589.712$$

Example: Long-Short Two-Asset Portfolio

- Consider $W_0 = 10000$ and $x_A = -\frac{1}{2}$ and $x_B = \frac{3}{2}$
- The expected return and standard deviation:

$$\mu_p = \mathbf{0.03635} \text{ and } \sigma_p = \mathbf{0.16171}$$

- Therefore

$$\widehat{\text{VaR}}_{p,0.1}^{PARA} = 10000 \cdot \mathbf{qnorm}(0.1, 0.03635, 0.16171) = -1708.897$$

- Moreover,

$$\widehat{\text{VaR}}_{p,\alpha}^{NONP} = W_0 \cdot R_{p,[T\alpha]} = 10000 \cdot R_{p,[10]} = -1773.06$$

Portfolio Frontier

- We know that

$$\mu_p = E[R_p] = x_A\mu_A + x_B\mu_B = \mu_B + x_A(\mu_A - \mu_B)$$

and

$$\begin{aligned}\sigma_p^2 &= x_A^2\sigma_A^2 + x_B^2\sigma_B^2 + 2x_Ax_B\rho_{AB}\sigma_A\sigma_B \\ &= x_A^2\sigma_A^2 + (1-x_A)^2\sigma_B^2 + 2(1-x_A)x_A\rho_{AB}\sigma_A\sigma_B\end{aligned}$$

- Therefore,

$$x_A = (\mu_p - \mu_B)/(\mu_A - \mu_B)$$

and hence

$$\begin{aligned}\sigma_p^2 &= \left(\frac{\mu_p - \mu_B}{\mu_A - \mu_B}\right)^2\sigma_A^2 + \left(\frac{\mu_p - \mu_A}{\mu_A - \mu_B}\right)^2\sigma_B^2 \\ &\quad - 2\frac{(\mu_p - \mu_A)(\mu_p - \mu_B)}{(\mu_A - \mu_B)^2}\rho_{AB}\sigma_A\sigma_B\end{aligned}$$

Portfolio Frontier

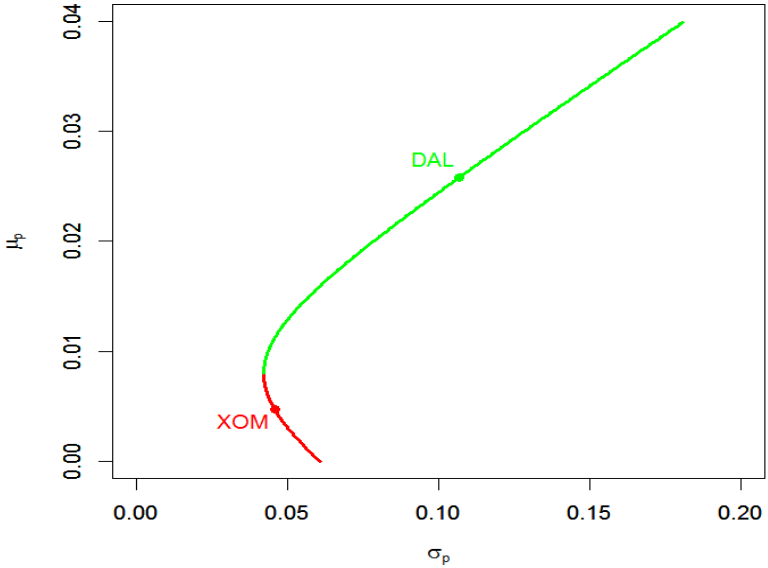
- **Portfolio Frontier:**

$$\sigma_p^2 = \left(\frac{\mu_p - \mu_B}{\mu_A - \mu_B}\right)^2 \sigma_A^2 + \left(\frac{\mu_p - \mu_A}{\mu_A - \mu_B}\right)^2 \sigma_B^2 - 2\frac{(\mu_p - \mu_A)(\mu_p - \mu_B)}{(\mu_A - \mu_B)^2}\sigma_{AB}$$

- Recall

$$\mu_A = 0.00466,\ \mu_B = 0.02579$$
$$\sigma_A^2 = 0.00209,\ \sigma_B^2 = 0.01139$$
$$\sigma_{AB} = -0.00000713$$

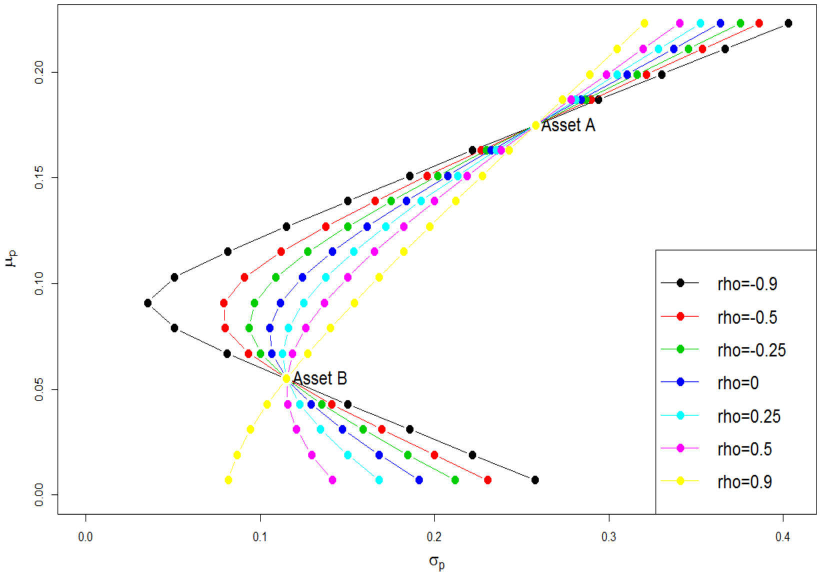
- Plot (σ_p, μ_p)



0.04
0.03
0.02
0.01
0.00
μ_p
DAL
XOM
0.00
0.05
0.10
0.15
0.20
σ_p

Portfolio Frontier

- Shape of portfolio frontier depends on correlation between assets A and B
- If $\rho_{AB} = -1$ then there exists x_A and x_B s.t. $\sigma_p^2 = 0$
- If $\rho_{AB} = 1$ then no benefit from diversification



Asset A
Asset B
rho=-0.9
rho=-0.5
rho=-0.25
rho=0
rho=0.25
rho=0.5
rho=0.9
μ_p
σ_p
0.00
0.05
0.10
0.15
0.20
0.0
0.1
0.2
0.3
0.4

Efficient Portfolio

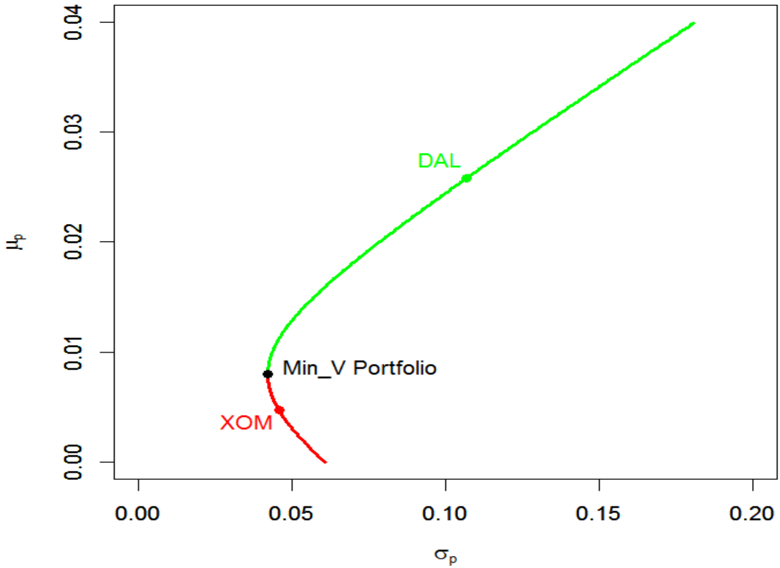
- **Definition**: Portfolios with the highest expected return for a given level of risk, as measured by portfolio standard deviation, are efficient portfolios
- If investors like portfolios with high expected returns and dislike portfolios with high return standard deviations then they will want to hold efficient portfolios
- Which efficient portfolio an investor will hold depends on their risk preferences
  - Very risk averse investors dislike volatility and will hold portfolios near the global minimum variance portfolio. They sacrifice expected return for the safety of low volatility
  - Risk tolerant investors don't mind volatility and will hold portfolios that have high expected returns. They gain expected return by taking on more volatility.

Global Minimum Variance Portfolio

- The portfolio with the smallest possible variance is called the *global minimum variance portfolio*.
- This portfolio is chosen by the most risk averse individuals
- To find this portfolio, one has to solve the following *constrained minimization problem*

$$\min_{x_A, x_B} \ \sigma_p^2 = x_A^2 \sigma_A^2 + x_B^2 \sigma_B^2 + 2 x_A x_B \sigma_{AB}$$

$$s.t. \ x_A + x_B = 1$$



0.04
0.03
0.02
0.01
0.00
μ_p
DAL
Min_V Portfolio
XOM
0.00
0.05
0.10
0.15
0.20
σ_p

Global Minimum Variance Portfolio

- **Calculus Solution**

Minimization problem

$$\min_{x_A, x_B} \ \sigma_p^2 = x_A^2 \sigma_A^2 + x_B^2 \sigma_B^2 + 2 x_A x_B \sigma_{AB}$$

$$s.t. \ x_A + x_B = 1$$

Use substitution method with

$$x_B = 1 - x_A$$

to give the univariate minimization

$$\min_{x_A} \ \sigma_p^2 = x_A^2 \sigma_A^2 + (1 - x_A)^2 \sigma_B^2 + 2 x_A (1 - x_A) \sigma_{AB}$$

Global Minimum Variance Portfolio

- First order conditions

$$
\begin{aligned}
0 &= \frac{d}{dx_A}\sigma_p^2 \\
&= \frac{d}{dx_A}\left(x_A^2\sigma_A^2 + (1-x_A)^2\sigma_B^2 + 2x_A(1-x_A)\sigma_{AB}\right) \\
&= 2x_A\sigma_A^2 - 2(1-x_A)\sigma_B^2 + 2\sigma_{AB}(1-2x_A) \\
&= x_A(2\sigma_A^2 + 2\sigma_B^2 - 4\sigma_{AB}) - 2\sigma_B^2 + 2\sigma_{AB} \\
&\Rightarrow x_A^{\min} = \frac{\sigma_B^2 - \sigma_{AB}}{\sigma_A^2 + \sigma_B^2 - 2\sigma_{AB}},\ x_B^{\min} = 1 - x_A^{\min}
\end{aligned}
$$

Portfolio with a Risk Free Asset

- Risk Free Asset
  - Asset with fixed and known rate of return over investment horizon
  - Usually use U.S. government T-Bill rate (horizons < 1 year) or T-Note rate (horizon > 1 yr)
- Note: T-Bill or T-Note rate is only nominally risk free

Portfolio with a Risk Free Asset

- Property of risk-free asset

$$
\begin{aligned}
R_f &= \text{return on risk-free asset} \\
E[R_f] &= r_f = \text{ constant} \\
\operatorname{var}(R_f) &= 0 \\
\operatorname{cov}(R_f, R_i) &= 0,\ R_i = \text{ return on any asset}
\end{aligned}
$$

- Portfolios of Risky Asset and Risk Free Asset

$$
\begin{aligned}
x_f &= \text{ share of wealth in T-Bills} \\
x_B &= \text{ share of wealth in asset B} \\
x_f + x_B &= 1 \\
x_f &= 1 - x_B
\end{aligned}
$$

Portfolio with a Risk Free Asset

- Portfolio return

$$\begin{aligned} R_p &= x_f r_f + x_B R_B \\ &= (1 - x_B) r_f + x_B R_B \\ &= r_f + x_B (R_B - r_f) \end{aligned}$$

- Portfolio excess return

$$R_p - r_f = x_B (R_B - r_f)$$

- Portfolio Distribution

$$\begin{aligned} \mu_p &= E[R_p] = r_f + x_B (\mu_B - r_f) \\ \sigma_p^2 &= \mathrm{var}(R_p) = x_B^2 \sigma_B^2 \\ \sigma_p &= x_B \sigma_B \end{aligned}$$

Portfolio with a Risk Free Asset

- **Risk Premium**

$$\begin{aligned} \mu_B - r_f &= \text{excess expected return on asset B} \\ &= \text{expected return on risky asset over return on safe asset} \end{aligned}$$

- For the portfolio of T-Bills and asset B

$$\begin{aligned} \mu_p - r_f &= x_B(\mu_B - r_f) \\ &= \text{expected portfolio return over T-Bill} \end{aligned}$$

- The risk premia is an increasing function of the amount invested in asset B.

Portfolio with a Risk Free Asset

- **Leveraged Investment**

$$x_f < 0, \ x_B > 1$$

  Borrow at T-Bill rate to buy more of asset B

- Result: Leverage increases portfolio expected return and risk

$$\mu_p = r_f + x_B(\mu_B - r_f)$$

$$\sigma_p = x_B \sigma_B$$

$$x_B \uparrow \ \Rightarrow \mu_p \ \& \ \sigma_p \uparrow$$

Portfolio with a Risk Free Asset

- **Determining Portfolio Frontier**

Goal: Plot μ_p vs. σ_p

$$\sigma_p = x_B \sigma_B \Rightarrow x_B = \frac{\sigma_p}{\sigma_B}$$

$$\begin{aligned}\mu_p &= r_f + x_B(\mu_B - r_f) \\ &= r_f + \frac{\sigma_p}{\sigma_B}(\mu_B - r_f) \\ &= r_f + \left(\frac{\mu_B - r_f}{\sigma_B}\right)\sigma_p\end{aligned}$$

where

$$\left(\frac{\mu_B - r_f}{\sigma_B}\right) = \mathrm{SR}_B = \text{Asset B's } \textit{Sharpe Ratio}$$

$$= \text{excess expected return per unit risk}$$

Portfolio with a Risk Free Asset

- The Sharpe Ratio (SR) is commonly used to rank assets.
- Assets with high Sharpe Ratios are preferred to assets with low Sharpe Ratios

William F. Sharpe, Nobel Prize in 1990

- William Sharpe was born on June 16, 1934 in Boston, Massachusetts.
- He enrolled at the *University of California at Berkeley* planning to pursue a degree in medicine. However, in the first year he decided to change his focus and moved to the **University of California at Los Angeles** to study Business Administration.
- Finding that he was not interested in accounting, Sharpe had a further change in preferences, finally majoring in Economics.
- From UCLA, he earned a B.A. in 1955, a M.A. in 1956 and a Ph.D in 1961 with a thesis on a **single factor model** of security prices, also including an early version of **the security market line**.

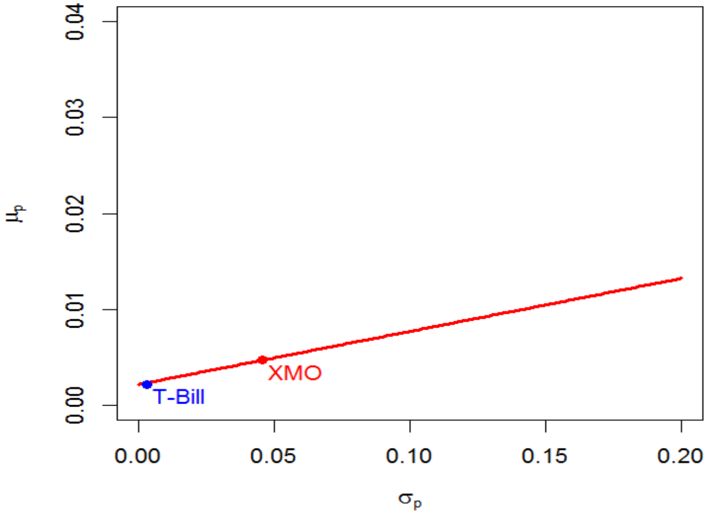
Portfolio Frontier

- 3 Month T-Bill and XOM

$$\begin{aligned}\mu_A &= 0.00466,\ \mu_f = 0.00214,\\ \sigma_A^2 &= 0.00209,\ \sigma_T^2 = 0.00000959 \approx 0,\\ \sigma_{AT} &= 0.00000599 \approx 0\end{aligned}$$

- Portfolio Frontier

$$\begin{aligned}\mu_p &= r_f + \left(\frac{\mu_A - r_f}{\sigma_A}\right)\sigma_p\\ &= 0.00214 + \frac{0.00466 - 0.00214}{\sqrt{0.00209}}\sigma_p\\ &= 0.00214 + 0.05516\sigma_p\end{aligned}$$


0.00
0.01
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0.04
μ_p
XMO
T-Bill
0.00
0.05
0.10
0.15
0.20
σ_p

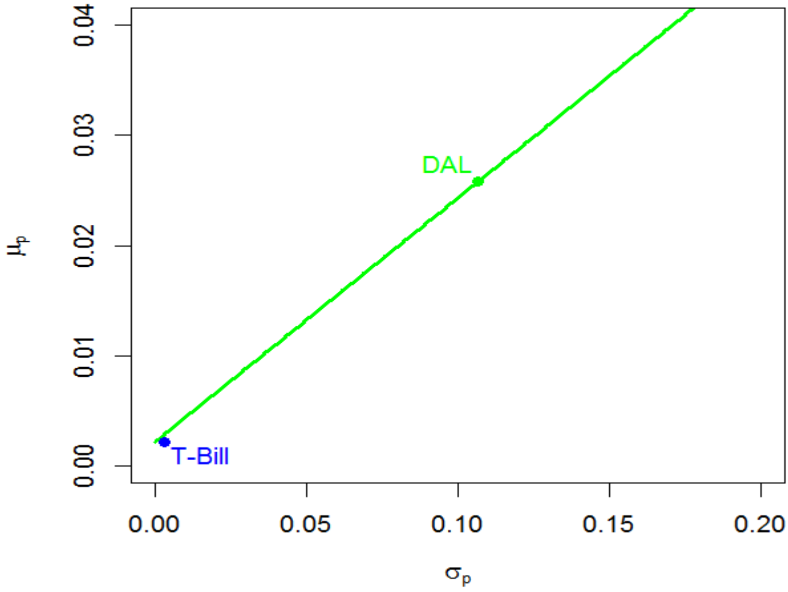
Portfolio Frontier

- 3 Month T-Bill and DAL

$$\begin{aligned}\mu_B &= 0.02579,\ \mu_f = 0.00214,\\ \sigma_B^2 &= 0.01139,\ \sigma_T^2 = 0.00000959 \approx 0,\\ \sigma_{BT} &= -0.00001759 \approx 0\end{aligned}$$

- Portfolio Frontier

$$\begin{aligned}\mu_p &= r_f + \left(\frac{\mu_B - r_f}{\sigma_B}\right)\sigma_p\\ &= 0.00214 + \frac{0.02579 - 0.00214}{\sqrt{0.01139}}\sigma_p\\ &= 0.00214 + 0.22165\sigma_p\end{aligned}$$


DAL
T-Bill
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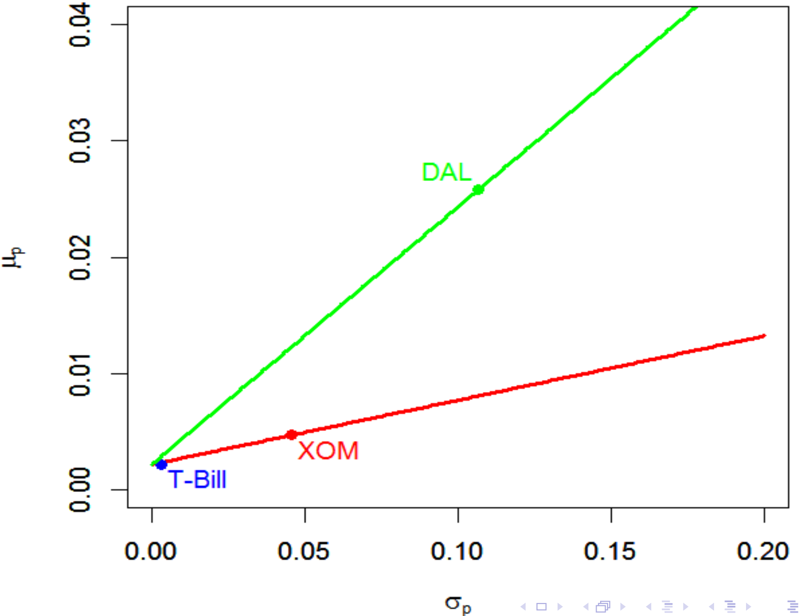
0.15
0.20
μ_p
σ_p



DAL
XOM
T-Bill
0.00
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μ_p
0.00
0.05
0.10
0.15
0.20
σ_p

Portfolio with 2 Risky Assets and a Risk Free Asset

- **Efficient Portfolios with 2 Risky Assets and a Risk Free Asset**

Investment in 2 Risky Assets and T-Bill

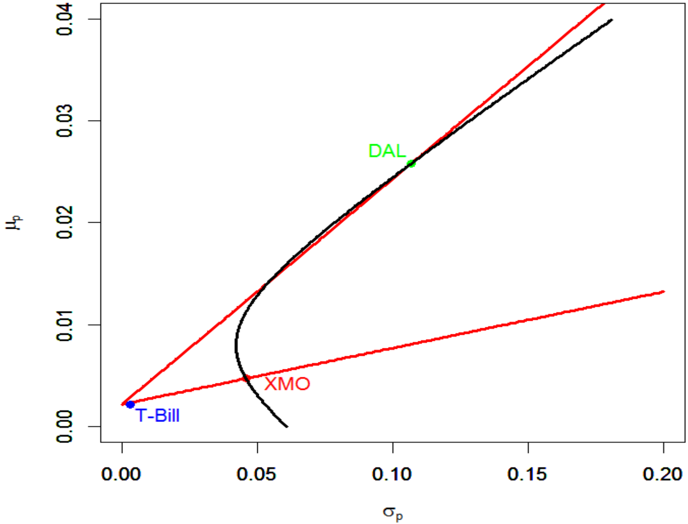
$$\begin{aligned} R_A &= \text{ simple return on asset A} \\ R_B &= \text{ simple return on asset B} \\ R_f = r_f &= \text{ return on T-Bill} \end{aligned}$$

Assumptions: R_A and R_B satisfy

$$R_i \sim iid\ (\mu_i, \sigma_i^2),\ i = A, B$$
$$\text{cov}(R_A, R_B) = \sigma_{AB},\ \text{corr}(R_A, R_B) = \rho_{AB}$$

Portfolio with 2 Risky Assets and a Risk Free Asset

- **Results**: The best portfolio of two risky assets and T-Bills is the *one with the highest Sharpe Ratio*
- Graphically, this portfolio occurs at the tangency point of a line drawn from R_f to the risky asset only frontier.
- The maximum Sharpe Ratio portfolio is called the "tangency portfolio"


0.00
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μ_p
DAL
XMO
T-Bill
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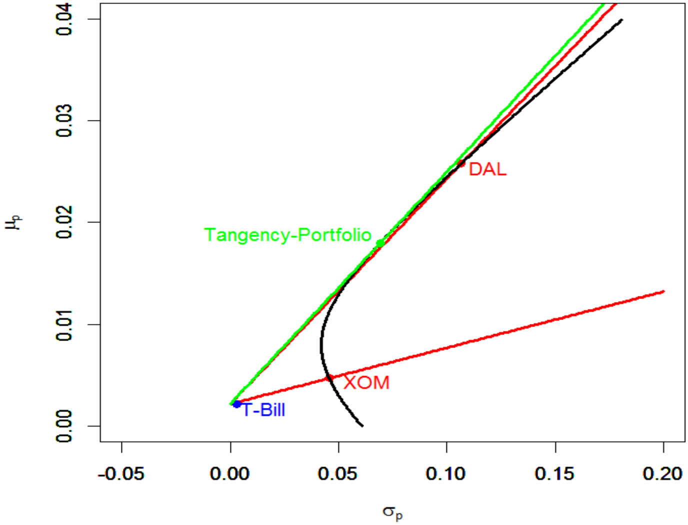
0.20
σ_p


0.04
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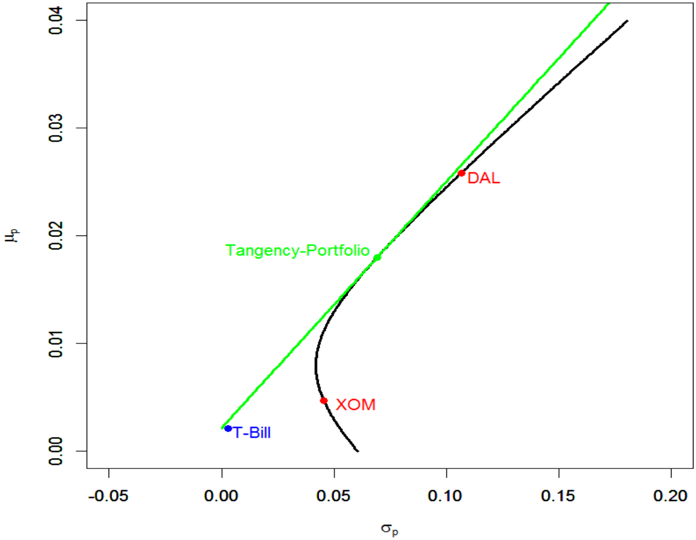
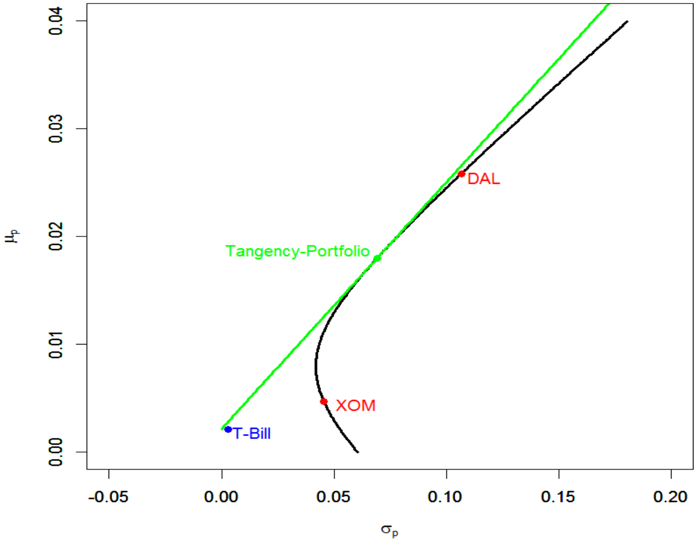
0.00
μ_p
DAL
Tangency-Portfolio
XOM
T-Bill
-0.05
0.00
0.05
0.10
0.15
0.20
σ_p



0.00
0.01
0.02
0.03
0.04
μ_p
-0.05
0.00
0.05
0.10
0.15
0.20
σ_p
DAL
Tangency-Portfolio
XOM
T-Bill

Portfolio with 2 Risky Assets and a Risk Free Asset

- **Mutual Fund Separation Theorem:** Efficient portfolios are combinations of two portfolios (mutual funds)
  - T-Bill portfolio
  - Tangency portfolio - portfolio of assets A and B that has the maximum Shape ratio
- Implication: All investors hold assets A and B according to their proportions in the tangency portfolio regardless of their risk preferences.



Tangency-Portfolio
DAL
XOM
T-Bill
μ_p
σ_p
0.00
0.01
0.02
0.03
0.04
-0.05
0.00
0.05
0.10
0.15
0.20

Portfolio with 2 Risky Assets and a Risk Free Asset

- **Finding the tangency portfolio**

$$\max_{x_A,\, x_B} \mathrm{SR}_p = \frac{\mu_p - r_f}{\sigma_p} \text{ subject to}$$

$$\begin{aligned} \mu_p &= x_A \mu_A + x_B \mu_B \\ \sigma_p^2 &= x_A^2 \sigma_A^2 + x_B^2 \sigma_B^2 + 2 x_A x_B \sigma_{AB} \\ 1 &= x_A + x_B \end{aligned}$$

- Solution can be found analytically or numerically.

Portfolio with 2 Risky Assets and a Risk Free Asset

- Using the substitution method it can be shown that

$$x_A^{\tan} = \frac{(\mu_A - r_f)\sigma_B^2 - (\mu_B - r_f)\sigma_{AB}}{(\mu_A - r_f)\sigma_B^2 + (\mu_B - r_f)\sigma_A^2 - (\mu_A - r_f + \mu_B - r_f)\sigma_{AB}}$$
$$x_B^{\tan} = 1 - x_A^{\tan}$$

- portfolio characteristics

$$\mu_p^{\tan} = x_A^{\tan}\mu_A + x_B^{\tan}\mu_B$$
$$\left(\sigma_p^{\tan}\right)^2 = \left(x_A^{\tan}\right)^2 \sigma_A^2 + \left(x_B^{\tan}\right)^2 \sigma_B^2 + 2x_A^{\tan}x_B^{\tan}\sigma_{AB}$$

Portfolio with 2 Risky Assets and a Risk Free Asset

- **Efficient Portfolios: tangency portfolio plus T-Bills**

$$
\begin{aligned}
x_{\text{tan}} &= \text{share of wealth in tangency portfolio} \\
x_f &= \text{share of wealth in T-bills} \\
x_{\text{tan}} + x_f &= 1 \\
R_p^e &= x_f r_f + x_{\text{tan}} R_p^{\text{tan}} = r_f + x_{\text{tan}} (R_p^{\text{tan}} - r_f) \\
\mu_p^e &= r_f + x_{\text{tan}} (\mu_p^{\text{tan}} - r_f) \\
\sigma_p^e &= x_{\text{tan}} \sigma_p^{\text{tan}}
\end{aligned}
$$

  Result: The weights x_{tan} and x_f are determined by an investor's risk preferences

- Risk averse investors hold mostly T-Bills
- Risk tolerant investors hold mostly tangency portfolio

Portfolio with 2 Risky Assets and a Risk Free Asset

- **Example:** for the two asset example, the tangency portfolio is

$$
\begin{aligned}
x_A^{\text{tan}} &= 0.36975,\ \ x_B^{\text{tan}} = 0.63025 \\
\mu_p^{\text{tan}} &= (.36975)\cdot(.00466) + (.63025)\cdot(.02579) = 0.01798 \\
\left(\sigma_p^{\text{tan}}\right)^2 &= (.36975)^2\cdot(.00209) + (.63025)^2\cdot(.01139) \\
&\quad + 2(.36975)\cdot(.63025)\cdot(-.0.00000713) = 0.0048 \\
\sigma_p^{\text{tan}} &= \sqrt{0.0048} = 0.06931
\end{aligned}
$$

Portfolio with 2 Risky Assets and a Risk Free Asset

- **Example (continued):** Efficient portfolios have the following characteristics

$$\begin{aligned}\mu_p^e &= r_f + x_{\tan}(\mu_p^{\tan} - r_f) \\ &= 0.00214 + x_{\tan} \cdot (0.01798 - 0.00214) \\ &= 0.00214 + x_{\tan} \cdot 0.01584\end{aligned}$$

and

$$\sigma_p^e = x_{\tan}\sigma_p^{\tan} = 0.06931 \cdot x_{\tan}$$

Portfolio with 2 Risky Assets and a Risk Free Asset

- **Problem**: Find the efficient portfolio that has the same risk (SD) as asset A? That is, determine $x_{\tan}$ and x_f such that

$$\sigma_p^e = \sigma_A = 0.04582 = \text{target risk}$$

Note: The efficient portfolio will have a higher expected return than asset A

Portfolio with 2 Risky Assets and a Risk Free Asset

- **Solution**:

$$0.04582 = \sigma_p^e = x_{\tan}\sigma_p^{\tan} = x_{\tan}(0.06931)$$
$$\Rightarrow x_{\tan} = \frac{0.04582}{0.06931} = 0.66$$
$$x_f = 1 - x_{\tan} = 0.34$$

Efficient portfolio with same risk as asset B has

$$(0.66)(0.36975) = 0.244 \text{ in asset A}$$
$$(0.66)(0.63025) = 0.416 \text{ in asset B}$$
$$0.34 \text{ in T-Bills}$$

The expected Return on efficient portfolio is

$$\mu_p^e = 0.00214 + 0.66 \cdot 0.01584 = 0.01259$$

compared with $\mu_A = 0.00466$

Portfolio with 2 Risky Assets and a Risk Free Asset

- **Problem**: find the efficient portfolio that has the same expected return as asset A. That is, determine $x_{\tan}$ and x_f such that

$$\mu_p^e = \mu_A = 0.00466 = \text{target expected return}$$

Note: The efficient portfolio will have a lower SD than asset B

Portfolio with 2 Risky Assets and a Risk Free Asset

- **Solution**:

$$0.00466 = \mu_p^e = 0.00214 + x_{\tan} \cdot 0.01584$$
$$x_{\tan} = \frac{0.00466 - 0.00214}{0.01584} = 0.1595$$
$$x_f = 1 - x_{\tan} = 0.8405$$

Efficient portfolio with same expected return as asset B has

$$(0.1595)(0.36975) = 0.059 \text{ in asset A}$$
$$(0.1595)(0.63025) = 0.100 \text{ in asset B}$$
$$0.841 \text{ in T-Bills}$$

The SD of the efficient portfolio is

$$\sigma_p^e = x_{\tan}(0.06931) = 0.1595 \cdot (0.06931) = 0.01106$$

compared with $\sigma_A = 0.04582$

Out of Sample Validation

- We have solved two portfolios: the first one has the same risk (SD) as asset A, but has large expected return; the second one has the same expected return as asset A but has smaller risk (SD).
- The calculation is based on the full sample (i.e., from 2009 Jan to 2018 Jan).
- We are more interested in checking how these two portfolios perform in the future.
- To do this, we will use the rolling window techniques we used for checking the VaR_α estimators in the previous lecture.

Out of Sample Validation

- There are 108 observations (9 year monthly data) in our sample
- At any time t (t $>$ 60), we use 60 previous (5 year) data to estimate μ_A, μ_B, σ_A^2, σ_B^2 and σ_{AB} then construct two
- portfolios: the first one track the risk (SD) of asset A, while the second one track the expected return of asset A
- At time 61, $\{(R_{A,1}, R_{B,1}, R_{T,1}), ..., (R_{A,60}, R_{B,60}, R_{T,60})\}$
- At time 62, $\{(R_{A,2}, R_{B,2}, R_{T,2}), ..., (R_{A,61}, R_{B,61}, R_{T,61})\}$
- ....
- At time 108, $\{(R_{A,48}, R_{B,48}, R_{T,48}), ..., (R_{A,107}, R_{B,107}, R_{T,107})\}$
- We consider investment of $W_0 = 10,000$ for one month

Out of Sample Validation

- The data at time 61: $\{(R_{A,1}, R_{B,1}, R_{T,1}), \ldots, (R_{A,60}, R_{B,60}, R_{T,60})\}$
- We calculate the estimators of μ_A, μ_B, σ_A^2, σ_B^2 and σ_{AB}, the tangency portfolio
$$(x^1_{A,\tan}, x^1_{B,\tan})$$
and the two target portfolios
$$(x^1_{1,\tan}, x^1_{1,T}) \text{ and } (x^1_{2,\tan}, x^1_{2,T})$$
- Using $(R_{A,61}, R_{B,61}, R_{T,61})$, we calculate the future (one month later) profits of the two portfolios and asset A
$$\begin{aligned} L_1^1 &= 10,000 \cdot (x^1_{1,\tan} R_{\tan,61} + x^1_{1,T} R_{T,61}) \\ L_2^1 &= 10,000 \cdot (x^1_{2,\tan} R_{\tan,61} + x^1_{2,T} R_{T,61}) \\ L_A^1 &= 10,000 \cdot R_{A,61} \end{aligned}$$
where $R_{\tan,61} = x^1_{A,\tan} R_{A,61} + x^1_{B,\tan} R_{B,61}$.

Out of Sample Validation

- The data at time 62: $\{(R_{A,2}, R_{B,2}, R_{T,2}), \ldots, (R_{A,61}, R_{B,61}, R_{T,61})\}$
- We calculate the estimators of μ_A, μ_B, σ_A^2, σ_B^2 and σ_{AB}, the tangency portfolio

$$(x_{A,\tan}^2, x_{B,\tan}^2)$$

and the two target portfolios

$$(x_{1,\tan}^2, x_{1,T}^2) \text{ and } (x_{2,\tan}^2, x_{2,T}^2)$$

- Using $(R_{A,62}, R_{B,62}, R_{T,62})$, we calculate the future (one month later) profits of the two portfolios and asset A

$$\begin{aligned}
L_1^2 &= 10,000 \cdot (x_{1,\tan}^2 R_{\tan,62} + x_{1,T}^2 R_{T,62}) \\
L_2^2 &= 10,000 \cdot (x_{2,\tan}^2 R_{\tan,62} + x_{2,T}^2 R_{T,62}) \\
L_A^2 &= 10,000 \cdot R_{A,62}
\end{aligned}$$

where $R_{\tan,62} = x_{A,\tan}^2 R_{A,62} + x_{B,\tan}^2 R_{B,62}$.

Out of Sample Validation

- ....
- ....
- ....

Out of Sample Validation

- The data at time 62: $\{(R_{A,48}, R_{B,48}, R_{T,48}), \ldots, (R_{A,107}, R_{B,107}, R_{T,107}$
- We calculate the estimators of μ_A, μ_B, σ_A^2, σ_B^2 and σ_{AB}, the tangency portfolio

$$(x_{A,\tan}^{48}, x_{B,\tan}^{48})$$

and the two target portfolios

$$(x_{1,\tan}^{48}, x_{1,T}^{48}) \text{ and } (x_{2,\tan}^{48}, x_{2,T}^{48})$$

- Using $(R_{A,108}, R_{B,108}, R_{T,108})$, we calculate the future (one month later) profits of the two portfolios and asset A

$$\begin{aligned} L_1^{48} &= 10,000 \cdot (x_{1,\tan}^{48} R_{\tan,108} + x_{1,T}^{48} R_{T,108}) \\ L_2^{48} &= 10,000 \cdot (x_{2,\tan}^{48} R_{\tan,108} + x_{2,T}^{48} R_{T,108}) \\ L_A^{48} &= 10,000 \cdot R_{A,108} \end{aligned}$$

where $R_{\tan,108} = x_{A,\tan}^{48} R_{A,108} + x_{B,\tan}^{48} R_{B,108}$.

- At the end, we get

$$\left(L_1^1, L_1^2, \ldots, L_1^{48}\right) \text{ and } \left(L_2^1, L_2^2, \ldots, L_2^{48}\right)$$

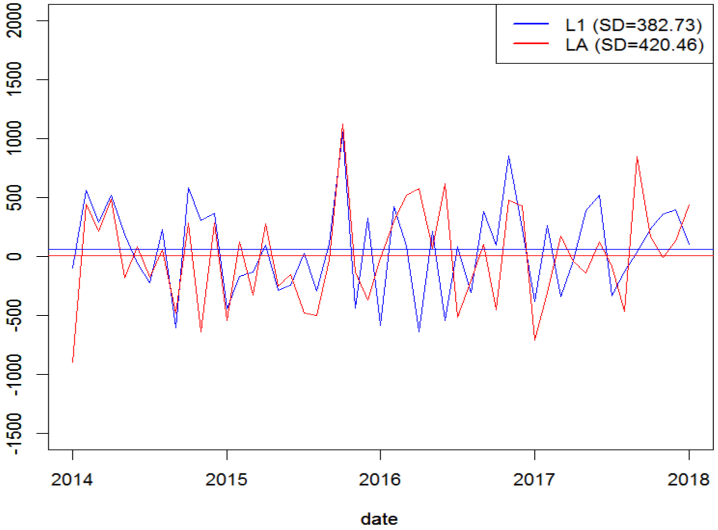
which are the future profits of the two portfolios
- We also get

$$\left(L_A^1, L_A^2, \ldots, L_A^{48}\right)$$

which are the future profits if we invest all the money W_0 in asset A.
- We can compare $\left(L_1^1, L_1^2, \ldots, L_1^{48}\right)$ against $\left(L_A^1, L_A^2, \ldots, L_A^{48}\right)$, and $\left(L_2^1, L_2^2, \ldots, L_2^{48}\right)$ against $\left(L_A^1, L_A^2, \ldots, L_A^{48}\right)$ to see if there
- is any benefit of applying the efficient portfolio theory.

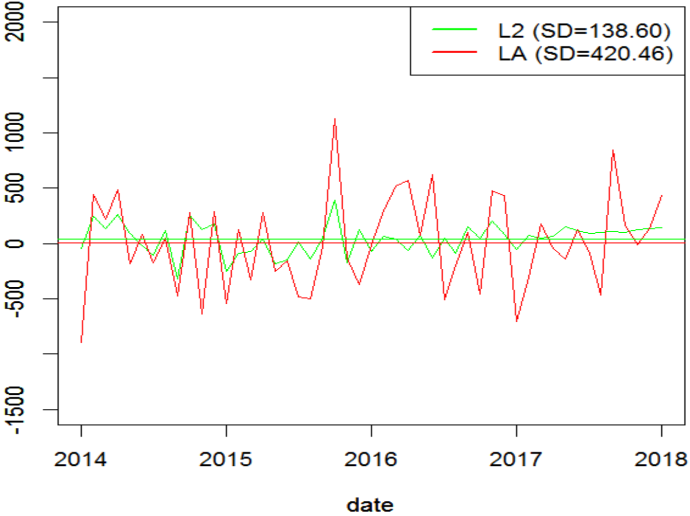
Out of Sample Validation

- The efficient portfolio 1 is to track the risk of Asset A
- Hence we expect it to have high expected return than asset A


L1 (SD=382.73)
LA (SD=420.46)
2000
1500
1000
500
0
-500
-1000
-1500
2014
2015
2016
2017
2018
date

Out of Sample Validation

- The efficient portfolio 2 is to track the expected return of Asset A
- Hence we expect it to have smaller risk than asset A


L2 (SD=138.60)
LA (SD=420.46)
2000
1000
500
0
-500
-1500
2014
2015
2016
2017
2018
date

What's next

- We have investigated the portfolio theory with two risky assets and one riskless asset
- If we consider more assets, we will have better portfolio frontier (in theory) which may help build better portfolio in practice
- Our next topic is on the portfolio theory with many risky assets
- Some knowledge on matrix theory will be required