Review of Probability Theory

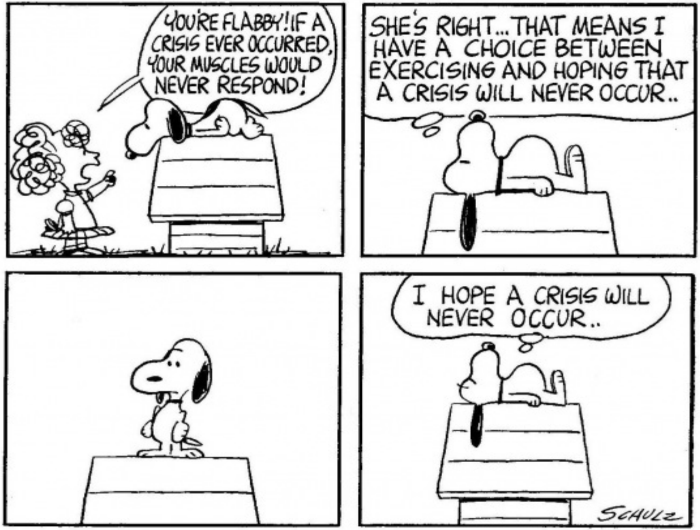
Econ 147

UCLA

Version 1.1

Outline

- **Probability, Random Variables and Distribution**
  - pdf, CDF, continuous and discrete r.v., quantiles.
- **Value at Risk**
  - introduction
- **Multivariate Distribution**
  - joint, marginal and conditional distribution, independence, covariance and correlation.
- **Reading:** E. Zivot's book chapter on probability concepts
- Optional: Chapter 5 (**Modeling Univariate Distributions**) and Chapter 7 (**Multivariate Statistical Models**) in Ruppert's book


YOU'RE FLABBY! IF A CRISIS EVER OCCURRED, YOUR MUSCLES WOULD NEVER RESPOND!
SHE'S RIGHT... THAT MEANS I HAVE A CHOICE BETWEEN EXERCISING AND HOPING THAT A CRISIS WILL NEVER OCCUR..
I HOPE A CRISIS WILL NEVER OCCUR..
SCHULZ

Random Variable

- **Motivation:** economic event in financial market is random (stochastic)
  - we do not know the outcome for sure, e.g., financial crisis, economic boom
  - at best, we could assign some probability to those events using some prior knowledge
- Realization of economic event in financial market can be summarized by realized financial data, such as stock price and returns.
  - we treat, for example, financial returns r_t or R_t as randomly changing objects
  - proper concepts are random variables and their distribution

Outline

- **Probability, Random Variables and Distribution**
  - Discrete r.v. and probability mass function (pmf)
  - Continuous r.v. and probability density function (pdf)
  - Cumulative Distribution Function (CDF)
  - Quantiles of r.v.
  - Normal Distribution
  - Shape characteristics
  - Linear functions of random variables

Random Variable

- **Definition:** A random variable (r.v.) X is a variable that can take on a given set of values, called the sample space (or the support) S_X, where the likelihood of the values in S_X is determined by the variable's probability mass/density function
  - we could also interpret X as a map between the outcome space Ω and S_X, which is a subset of $\mathbb{R}$ (real line):

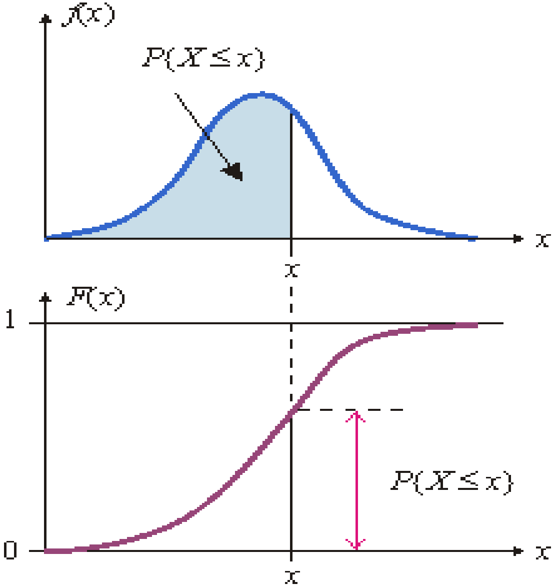
$$X : \Omega \mapsto S_X$$

  - Ω is usually called a population
  - probability mass/density function $f(x)$:

$$P(X \in A) = \begin{cases} \sum_{x \in A \cap S_X} f(x), & \text{discrete r.v.} \\ \int_{A \cap S_X} f(x) dx, & \text{continuous r.v.} \end{cases}$$

CDF

- Sometimes, especially for continuous r.v., we rather consider $\Pr(X \leq x)$ than $\Pr(X = x)$. Why?
- **Definition** The Cumulative Distribution Function (CDF), F, of a r.v. X is $F(x) = \Pr(X \leq x)$.


f(x)
P(X ≤ x)
x
x
F(x)
1
0
P(X ≤ x)
x
x

CDF

- **Properties:**
  - If $x_1 < x_2$, then $F(x_1) \leq F(x_2)$
  - $F(-\infty) = 0$ and $F(\infty) = 1$
  - $\Pr(X > x) = 1 - F(x)$
  - $\Pr(x_1 < X \leq x_2) = F(x_2) - F(x_1)$
  - $\frac{d}{dx}F(x) = f(x)$ if X is a continuous r.v.

CDF

- **Remark**: for a continuous r.v.

$$\Pr(X \leq x) = \Pr(X < x),$$
$$\Pr(X = x) = 0$$

Why?

Quantiles of a Distribution

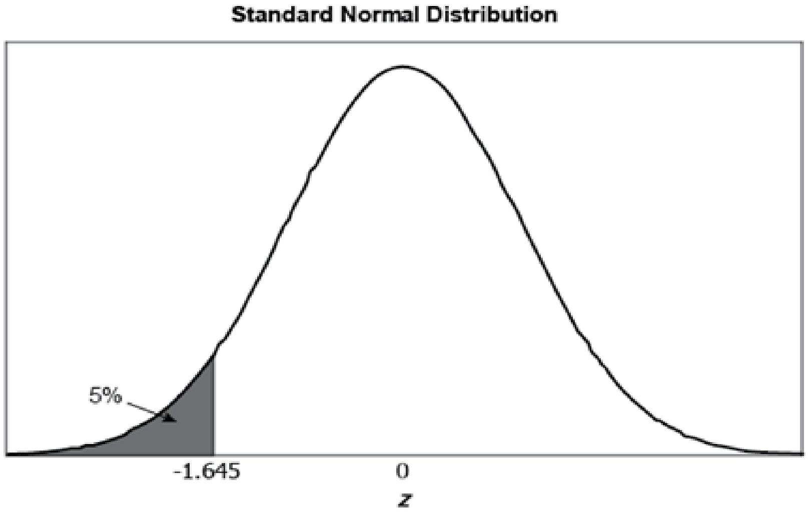
- X is a r.v. with continuous CDF $F_X(x) = \Pr(X \leq x)$
- **Definition**: The α-quantile of F_X for $\alpha \in (0, 1)$ is the **largest** value q_α such that

$$F_X(q_\alpha) = \Pr(X \leq q_\alpha) = \alpha$$

The area under the probability curve to the left of q_α is α. If the inverse CDF F_X^{-1} exists then

$$q_\alpha = F_X^{-1}(\alpha)$$

Note: F_X^{-1} is sometimes called the "quantile" function of X.


Standard Normal Distribution
5%
-1.645
0
z

Quantiles of a Distribution

- Example:

$$
\begin{aligned}
1\% \text{ quantile} &= q_{.01} \\
5\% \text{ quantile} &= q_{.05} \\
50\% \text{ quantile} &= q_{.5} = \text{ median}
\end{aligned}
$$

Quantiles of a Distribution

- Example: Quantile function of uniform dist'n on [0,1]

$$
\begin{aligned}
F_X(x) = x \Rightarrow q_\alpha &= \alpha \\
q_{.01} &= 0.01 \\
q_{.5} &= 0.5
\end{aligned}
$$

The Standard Normal Distribution

- Let Z be a r.v. such that $Z \backsim N(0,1)$. Then

$$f(z) = \phi(z) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}z^2\right), \quad -\infty \leq z \leq \infty$$

$$\Phi(z) = \Pr(Z \leq z) = \int_{-\infty}^{z} \phi(u)du$$

The Standard Normal Distribution

- **Shape Characteristics**
- Centered at zero
- Bell-shaped
- Symmetric about zero (same shape to left and right of zero)

$$\Pr(-1 \leq Z \leq 1) = \Phi(1) - \Phi(-1) = 2\Phi(1) - 1 \simeq 0.67$$
$$\Pr(-2 \leq Z \leq 2) = \Phi(2) - \Phi(-2) = 2\Phi(2) - 1 \simeq 0.95$$
$$\Pr(-3 \leq Z \leq 3) = \Phi(3) - \Phi(-3) = 2\Phi(3) - 1 \simeq 0.99$$

The Standard Normal Distribution

- Finding Areas under the Normal Curve
- $\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx = 1$
- $\Pr(a < Z < b) = \int_a^b \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx = \Phi(b) - \Phi(a)$, cannot be computed analytically!
- Special numerical algorithms are used to calculate $\Phi(z)$

The Standard Normal Distribution

- R functions (will be used in HW's)

1. `pnorm(z)` computes $\Pr(Z \leq z) = \Phi(z)$
2. `qnorm(`α`)` computes the quantile $q_{\alpha}^{Z} = \Phi^{-1}(\alpha)$
3. `dnorm(z)` computes the density $\phi(z)$

The Standard Normal Distribution

- Some Tricks for Computing Area under Normal Curve

$N(0,1)$ is symmetric about 0; total area $= 1$

$$\Pr(Z \leq z) = 1 - \Pr(Z \geq z)$$
$$\Pr(Z \geq z) = \Pr(Z \leq -z)$$
$$\Pr(Z \geq 0) = \Pr(Z \leq 0) = 0.5$$

The Standard Normal Distribution

- **Example** In R use

$$\texttt{pnorm(2)} - \texttt{pnorm(-1)} = 0.81860$$

The 1%, 2.5%, 5% quantiles are

$$\text{R} : \texttt{qnorm(0.010)} = -2.326348$$

$$\text{R} : \texttt{qnorm(0.025)} = -1.959964$$

$$\text{R} : \texttt{qnorm(0.050)} = -1.644854$$

Shape Characteristics

- Expected Value or Mean - Center of Mass
  - location information: on average, how much we expect.
- Variance and Standard Deviation - Spread about mean
  - scale information: how volatile it is.
- Skewness - Symmetry about mean
  - measure of symmetry
- Kurtosis - Tail thickness
  - measure of extreme (tail) events frequency

Shape Characteristics: Expected Value (Mean)

- **Expectation of a Function of X**

Definition: Let $g(X)$ be some function of the r.v. X. Then

$$E[g(X)] = \sum_{x \in S_X} g(x) \cdot f(x), \text{ Discrete case}$$

$$E[g(X)] = \int_{S_X} g(x) \cdot f(x) dx, \text{ Continuous case}$$

Shape Characteristics: Expected Value (Mean)

- **Expected Value - Discrete r.v.**

$$g(X) = X$$

$$\mu_X = E[X] \begin{cases} \sum_{x \in S_X} xf(x), & \text{discrete r.v.} \\ \int_{S_X} xf(x)dx, & \text{continuous r.v.} \end{cases}$$

Shape Characteristics: Variance and Standard Deviation

- **Variance and Standard Deviation**

$$
\begin{aligned}
g(X) &= (X - E[X])^2 = (X - \mu_X)^2 \\
\sigma_X^2 &= \mathrm{Var}(X) = E[(X - \mu_X)^2] = E[X^2] - \mu_X^2 \\
\mathrm{SD}(X) &= \sigma_X = \sqrt{\mathrm{Var}(X)}
\end{aligned}
$$

Note: $\mathrm{Var}(X)$ is in squared units of X, and $\mathrm{SD}(X)$ is in the same units as X. Therefore, $\mathrm{SD}(X)$ is easier to interpret.

Shape Characteristics: Variance and Standard Deviation

- **Computation of** $\mathrm{Var}(X)$ **and** $\mathrm{SD}(X)$

$$\begin{aligned}\sigma_X^2 &= E[(X-\mu_X)^2] \\ &= \sum_{x\in S_X} (x-\mu_X)^2 \cdot f(x) \text{ if } X \text{ is a discrete r.v.} \\ &= \int_{-\infty}^{\infty} (x-\mu_X)^2 \cdot f(x)dx \text{ if } X \text{ is a continuous r.v.} \\ \sigma_X &= \sqrt{\sigma_X^2}\end{aligned}$$

- **Remark**: For "bell-shaped" data, σ_X measures the size of the typical deviation from the mean value μ_X.

Shape Characteristics: Variance and Standard Deviation

- **Example**: $Z \backsim N(0,1)$.

$$\mu_Z = \int_{-\infty}^{\infty} z \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz = 0$$

$$\sigma_Z^2 = \int_{-\infty}^{\infty} (z-0)^2 \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz = 1$$

$$\sigma_Z = \sqrt{1} = 1$$

$\Rightarrow$ size of typical deviation from $\mu_Z = 0$ is $\sigma_Z = 1$

Shape Characteristics: Variance and Standard Deviation

- **The General Normal Distribution**

$$X \backsim N(\mu_X,\ \sigma_X^2)$$

$$f(x) = \frac{1}{\sqrt{2\pi\sigma_X^2}} \exp\left(-\frac{1}{2}\left(\frac{x-\mu_X}{\sigma_X}\right)^2\right),\ -\infty \le x \le \infty$$

$$E[X] = \mu_X = \text{ mean value}$$

$$\mathrm{Var}(X) = \sigma_X^2 = \text{ variance}$$

$$\mathrm{SD}(X) = \sigma_X = \text{ standard deviation}$$

Shape Characteristics: Normal Distribution

- **Remarks**
- $Z \backsim N(0,1)$: Standard Normal $\Longrightarrow \mu_X = 0$ and $\sigma_X^2 = 1$
- The pdf of the general Normal is completely determined by values of μ_X and σ_X^2

Some Codes

- **R Functions**
- simulate data: `rnorm(n, mean, sd)`
- compute CDF: `pnorm(q, mean, sd)`
- compute quantiles: `qnorm(p, mean, sd)`
- compute density: `dnorm(x, mean, sd)`

Normal Distribution: cc return and simple return

- **Note**: Why the normal distribution may not be appropriate for simple returns

$$R_t = \frac{P_t - P_{t-1}}{P_{t-1}} = \text{simple return}$$

$$\text{Assume } R_t \sim N(0.05, (0.50)^2)$$

Note: $P_t \geq 0 \implies R_t \geq -1$. However, based on the assumed normal distribution

$$\Pr(R_t < -1) = \texttt{pnorm(-1, 0.05, 0.50)} = 0.018$$

This implies that there is a 1.8% chance that the asset price will be smaller than -1. This is why the normal distribution may not be appropriate for simple returns.

Normal Distribution: cc return and simple return

- **Example**: The normal distribution is more appropriate for cc returns

$$r_t = \ln(1 + R_t) = \text{cc return}$$
$$R_t = e^{r_t} - 1 = \text{ simple return}$$
$$\text{Assume } r_t \sim N(0.05, (0.50)^2)$$

Unlike R_t, r_t can take on values less than -1. For example,

$$r_t = -2 \implies R_t = e^{-2} - 1 = -0.865$$
$$\Pr(r_t < -2) = \texttt{pnorm(-2, 0.05, 0.50)} = 0.00002$$

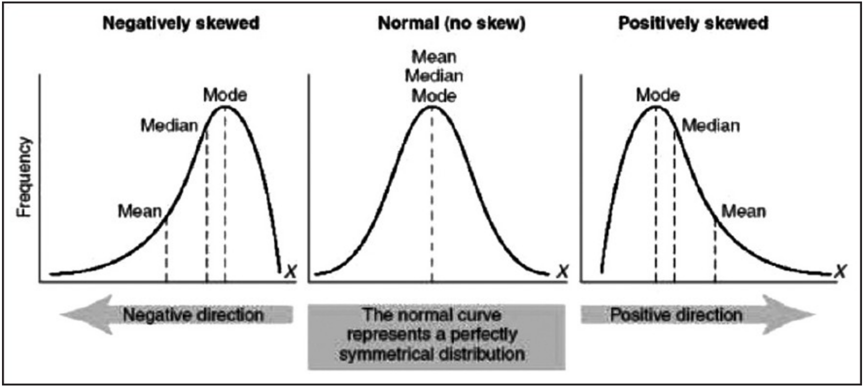
Skewness

- **Skewness - Measure of symmetry**

$$
\begin{aligned}
g(X) &= ((X-\mu_X)/\sigma_X)^3 \\
\mathrm{Skew}(X) &= E\left[\left(\frac{X-\mu_X}{\sigma_X}\right)^3\right] \\
&= \sum_{x \in S_X} \left(\frac{x-\mu_X}{\sigma_X}\right)^3 f(x) \text{ if } X \text{ is discrete} \\
&= \int_{S_X} \left(\frac{x-\mu_X}{\sigma_X}\right)^3 f(x) dx \text{ if } X \text{ is continuous}
\end{aligned}
$$

Skewness

- **Skewness:** Intuition
  - If X has a symmetric distribution about μ_X then $\mathrm{Skew}(X) = 0$
  - $\mathrm{Skew}(X) > 0 \implies$ pdf has long right tail, and median $<$ mean
  - $\mathrm{Skew}(X) < 0 \implies$ pdf has long left tail, and median $>$ mean


Negatively skewed
Normal (no skew)
Positively skewed
Frequency
Mode
Median
Mean
X
Mean
Median
Mode
X
Mode
Median
Mean
X
Negative direction
The normal curve represents a perfectly symmetrical distribution
Positive direction

- **Example: Probability Distribution for Annual Return**

Table 1: Discrete Distribution for Annual Return

State of Economy	S_X = Sample Space	$f(x) = \Pr(X = x)$
Depression	-0.30	0.05
Recession	0.00	0.20
Normal	0.10	0.50
Mild Boom	0.20	0.20
Major Boom	0.50	0.05

Skewness

- Example: Skewness for a discrete random variable

Table 1: Discrete Distribution for Annual Return

S_X :	-0.30	0.00	0.10	0.20	0.50
$f(x)$:	0.05	0.20	0.50	0.20	0.05

Using the discrete distribution for the return in Table 1, the results that $\mu_X = 0.1$ and $\sigma_X = 0.141$, we have

$$\begin{aligned} \mathrm{skew}(X) = {} & [(-0.3 - 0.1)^3 \cdot (0.05) + (0.0 - 0.1)^3 \cdot (0.20) \\ & + (0.1 - 0.1)^3 \cdot (0.5) + (0.2 - 0.1)^3 \cdot (0.2) \\ & + (0.5 - 0.1)^3 \cdot (0.05)] / (0.141)^3 = 0.0 \end{aligned}$$

Skewness

- Example: $X \backsim N(\mu_X, \sigma_X^2)$. Then

$$\mathrm{Skew}(X) = \int_{-\infty}^{\infty} \left(\frac{x-\mu_X}{\sigma_X}\right)^3 \frac{1}{\sqrt{2\pi\sigma_X^2}} \exp\left(^{-\frac{1}{2}\left(\frac{x-\mu_X}{\sigma_X}\right)^2}\right) dx = 0$$

Kurtosis

- **Kurtosis - Measure of tail thickness**

$$
\begin{aligned}
g(X) &= ((X-\mu_X)/\sigma_X)^4 \\
\mathrm{Kurt}(X) &= E\left[\left(\frac{X-\mu_X}{\sigma_X}\right)^4\right] \\
&= \sum_{x \in S_X} \left(\frac{x-\mu_X}{\sigma_X}\right)^4 f(x) \text{ if } X \text{ is discrete} \\
&= \int_{-\infty}^{\infty} \left(\frac{x-\mu_X}{\sigma_X}\right)^4 f(x)dx \text{ if } X \text{ is continuous}
\end{aligned}
$$

Kurtosis

- **Kurtosis:**
  - Intuition: values of x far from μ_X get blown up resulting in large values of kurtosis
  - Two extreme cases: fat tails (large kurtosis); thin tails (small kurtosis)

Kurtosis

- Example: Kurtosis for a discrete random variable

Table 1: Discrete Distribution for Annual Return

S_X :	-0.30	0.00	0.10	0.20	0.50
$f(x)$:	0.05	0.20	0.50	0.20	0.05

Using the discrete distribution for the return Table 1, the results that $\mu_X = 0.1$ and $\sigma_X = 0.141$, we have

$$\begin{aligned}\mathrm{Kurt}(X) = {} & [(-0.3-0.1)^4 \cdot (0.05) + (0.0-0.1)^4 \cdot (0.20) \\ & + (0.1-0.1)^4 \cdot (0.5) + (0.2-0.1)^4 \cdot (0.2) \\ & + (0.5-0.1)^4 \cdot (0.05)] / (0.141)^4 = 6.5\end{aligned}$$

Kurtosis

- Example: $X \backsim N(\mu_X, \sigma_X^2)$

$$\mathrm{Kurt}(X) = \int_{-\infty}^{\infty} \left(\frac{x-\mu_X}{\sigma_X}\right)^4 \frac{1}{\sqrt{2\pi\sigma_X^2}} e^{-\frac{1}{2}\left(\frac{x-\mu_X}{\sigma_X}\right)^2} dx = 3$$

Definition: Excess kurtosis $= \mathrm{Kurt}(X) - 3 =$ kurtosis value in excess of kurtosis of normal distribution.

- Excess kurtosis $(X) > 0 \Rightarrow X$ has fatter tails than normal distribution
- Excess kurtosis $(X) < 0 \Rightarrow X$ has thinner tails than normal distribution

Kurtosis

- **The Student-t Distribution**

A distribution similar to the standard normal distribution but with fatter tails, and hence larger kurtosis, is the Student-t distribution. If Z is a standard normal random variable and U_v is a Chi-square random variable with degree of freedom v, then

$$X = \frac{Z}{\sqrt{U_v / v}}$$

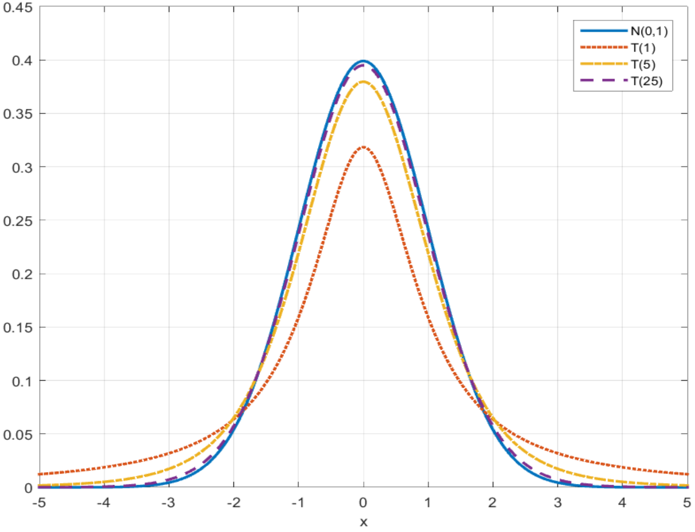
is a Student-t random variable with degrees of freedom v, denoted $X \sim t_v$.

Student-t Distribution

- It can be shown that

$$
\begin{aligned}
E[X] &= 0,\ \ v > 1 \\
\mathrm{var}(X) &= \frac{v}{v-2},\ \ v > 2, \\
\mathrm{skew}(X) &= 0,\ \ v > 3, \\
\mathrm{kurt}(X) &= \frac{6}{v-4} + 3,\ \ v > 4.
\end{aligned}
$$

The parameter v controls the scale and tail thickness of distribution. If v is close to four, then the kurtosis is large and the tails are thick. If $v < 4$, then $\mathrm{kurt}(X) = \infty$. As $v \to \infty$ the Student-t pdf approaches that of a standard normal random variable and $\mathrm{kurt}(X) = 3$.


N(0,1)
T(1)
T(5)
T(25)
0.45
0.4
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
-5
-4
-3
-2
-1
0
1
2
3
4
5
x

Some Codes

- **R Functions**
- simulate data: `rt(n, df)`
- compute CDF: `pt(q, df)`
- compute quantiles: `qt(p, df)`
- compute density: `dt(x, df)`

Here df is the degrees of freedom parameter v.

Linear Functions of a Random Variable

- Let X be a discrete or continuous r.v. with $\mu_X = E[X]$ and $\sigma_X^2 = \mathrm{Var}(X)$. Define a new r.v. Y to be a linear function of X :

$$Y = g(X) = a \cdot X + b$$

a and b are known constants

Then

$$\mu_Y = E[Y] = E[a \cdot X + b] = a \cdot E[X] + b = a \cdot \mu_X + b$$
$$\sigma_Y^2 = \mathrm{Var}(Y) = \mathrm{Var}(a \cdot X + b) = a^2 \cdot \mathrm{Var}(X) = a^2 \cdot \sigma_X^2$$
$$\sigma_Y = |a| \cdot \sigma_X$$

Linear Functions of a Random Variable

- **Linear Function of a Normal r.v.:**

Let $X \backsim N(\mu_X, \sigma_X^2)$ and define $Y = a \cdot X + b$. Then

$$Y \sim N(\mu_Y, \sigma_Y^2)$$

with

$$\begin{aligned} \mu_Y &= a \cdot \mu_X + b \\ \sigma_Y^2 &= a^2 \cdot \sigma_X^2 \end{aligned}$$

- Remarks:
  - Proof of result relies on change-of-variables formula for determining pdf of a function of a r.v.
  - Result may or may not hold for random variables whose distributions are not normal

Linear Functions of a Random Variable

- **Example** - Standardizing a Normal r.v.: let $X \sim N(\mu_X, \sigma_X^2)$. The standardized r.v. Z is created using

$$Z = \frac{X - \mu_X}{\sigma_X} = \frac{1}{\sigma_X} \cdot X - \frac{\mu_X}{\sigma_X} = a \cdot X + b,$$

with $a = \frac{1}{\sigma_X}$, $b = -\frac{\mu_X}{\sigma_X}$.

- Properties of Z :

$$E[Z] = \frac{1}{\sigma_X} E[X] - \frac{\mu_X}{\sigma_X} = \frac{1}{\sigma_X} \cdot \mu_X - \frac{\mu_X}{\sigma_X} = 0$$

$$\mathrm{Var}(Z) = \left(\frac{1}{\sigma_X}\right)^2 \cdot \mathrm{Var}(X) = \left(\frac{1}{\sigma_X}\right)^2 \cdot \sigma_X^2 = 1$$

hence, $Z \sim N(0, 1)$

Outline

- **Value at Risk**
  - concept and introduction
  - computation

Value at Risk (VaR)

- In financial risk management, we need to characterize some measure of probable risk
- **Motivation:** Consider a $\$10,000$ investment in *Amazon* for 1 month. Assume

$$R = \text{simple monthly return on } \textit{Amazon}$$
$$R \sim N(0.05, 0.01),\ \mu_R = 0.05,\ \sigma_R = 0.10$$

Goal: measure the profit of the investment with a given probability α

- That is given α, the VaR at α denoted as VaR_α is the largest value such that the profit of the investment is smaller than VaR_α with probability at most α.
- For example, $\mathrm{VaR}_{0.05}$ tells us that the probability that the profit of the investment is smaller than $\mathrm{VaR}_{0.05}$ is less than 0.05.
- Negative profit means loss

Value at Risk (VaR)

- **Questions:**

1. What is the profit of the investment?
2. What is the distribution of the profit?
3. What is the probability that the profit is smaller than $-1,000$?
4. What is the monthly value-at-risk (VaR) on the \$10,000 investment with 0.10 probability?
5. What is the monthly value-at-risk (VaR) on the \$10,000 investment with 0.05 probability?

Value at Risk (VaR)

1. **Question:** What is the profit of the investment?

Answer: The end of month wealth is $W_1 = \$10,000 \cdot (1 + R)$. Since the initial wealth is $W_0 = \$10,000$, the profit of the investment is

$$\begin{aligned} L_1 &= W_1 - W_0 \\ &= \$10,000 \cdot R \\ &= W_0 \cdot R \end{aligned}$$

Value at Risk (VaR)

2. **Question:** What is the distribution of the profit?

Answer: $L_1 = \$10,000 \cdot R$ is a linear function of R, and R is a normal r.v. Therefore, L_1 is normally distributed: $L_1 \sim N(500, 10^6)$ where

$$E[L_1] = 10,000 \cdot E[R] = 10,000 \cdot (0.05) = 500$$

$$\mathrm{Var}(L_1) = (10,000)^2 \mathrm{Var}(R) = (10,000)^2 \cdot (0.01) = 10^6$$

Value at Risk (VaR)

3. **Question:** What is the probability that the profit is smaller than $-1,000$?

Answer: Since $L_1 \sim N(500, 10^6)$,

$$\begin{aligned}
\Pr\left(L_1 \leq -1000\right) &= \Pr\left(\frac{L_1 - 500}{\sqrt{10^6}} \leq \frac{-1000 - 500}{\sqrt{10^6}}\right) \\
&= \Pr\left(Z \leq -1.5\right) \\
&= \text{pnorm(-1.5)} \\
&= 0.0668
\end{aligned}$$

Value at Risk (VaR)

Lemma 1. Let q_α denote the α quantile of L_1. Then $\mathrm{VaR}_\alpha = q_\alpha$.

Proof. By the definition of q_α, we know that q_α is the largest value such that

$$\Pr\left(L_1 \leq q_\alpha\right) = \alpha.$$

The above equation implies that $q_\alpha \leq \mathrm{VaR}_\alpha$.
Moreover if $\mathrm{VaR}_\alpha > q_\alpha$, we have

$$\Pr\left(L_1 \leq \mathrm{VaR}_\alpha\right) > \alpha$$

which violates the requirement that $\Pr\left(L_1 \leq \mathrm{VaR}_\alpha\right) \leq \alpha$.
Therefore, we must have $\mathrm{VaR}_\alpha = q_\alpha$.

Value at Risk (VaR)

4. **Question:** What is the monthly value-at-risk (VaR) on the \$10,000 investment with 0.10 probability?

Answer: Since $L_1 \sim N(500, 10^6)$, by Lemma 1

$$\mathrm{VaR}_{0.1} = q_{0.1} = \texttt{qnorm}(0.1,\ 500,\ 1000) = -781.5516$$

Value at Risk (VaR)

5. **Question:** What is the monthly value-at-risk (VaR) on the \$10,000 investment with 0.05 probability?

Answer: Since $L_1 \sim N(500, 10^6)$, by Lemma 1

$$\mathrm{VaR}_{0.05} = q_{0.05} = \mathtt{qnorm}(0.05,\ 500,\ 1000) = -1144.854$$

Value at Risk (VaR)

- Remarks:

1. In reality, α is usually set to 0.05, 0.025 or 0.01. Therefore, VaR_α is usually a negative number with such choices.
2. VaR_α usually represents the loss of a investment with small α
3. Because VaR_α represents a loss, it is often reported as a positive number. For example, $-\$1,144$ represents a loss of $\$1,144$. So the VaR is reported as $\$1,144$.

Value at Risk (VaR)

- **VaR for Continuously Compounded Returns**

$$r = \ln(1+R), \text{ cc monthly return}$$
$$R = e^r - 1, \text{ simple monthly return}$$

Assume

$$r \sim N(\mu_r, \sigma_r^2)$$
$$W_0 = \text{initial investment}$$

The profit of the investment

$$L_1 = W_0 \cdot R = W_0 \cdot (e^r - 1)$$

- What we need is to calculate the α quantile of L_1.

Value at Risk (VaR)

Lemma 2. Suppose that X is a random variable and $h(x)$ is a strictly increasing function of x. Define a new random variable $Y = h(X)$. Then

$$q_\alpha^Y = h(q_\alpha^X)$$

where q_α^Y and q_α^X denote the α quantiles of X and Y respectively.

With simple return, $L_1 = W_0 \cdot R$ and $W_0 \cdot R$ is a strictly increasing function of R, we know that

$$q_\alpha^{L_1} = W_0 \cdot q_\alpha^R$$

$q_\alpha^{L_1}$ and q_α^R denote the α quantiles of L_1 and R respectively. This gives us an alternative way of calculating the VaR with simple return.

Value at Risk (VaR)

Lemma 2. Suppose that X is a random variable and $h(x)$ is a strictly increasing function of x. Define a new random variable $Y = h(X)$. Then

$$q_\alpha^Y = h(q_\alpha^X)$$

where q_α^Y and q_α^X denote the α quantiles of X and Y respectively.

Since $L_1 = W_0 \cdot (e^r - 1)$ and $W_0 \cdot (e^r - 1)$ is a strictly increasing function of r, we know that

$$q_\alpha^{L_1} = W_0 \cdot (e^{q_\alpha^r} - 1)$$

where $q_\alpha^{L_1}$ and q_α^r denote the α quantiles of L_1 and r respectively.

Value at Risk (VaR)

- VaR_α computation for cc return
  - Compute α quantile of Normal Distribution for r:

$$q_\alpha^r = \mathsf{qnorm}(\alpha,\ \mu_r,\ \sigma_r)$$

  - Compute VaR_α using q_α^r:

$$\mathrm{VaR}_\alpha = W_0 \cdot (e^{q_\alpha^r} - 1)$$

Value at Risk (VaR)

- Example: Compute 5% VaR assuming

$$r_t \sim N(0.05, (0.10)^2), W_0 = \$10,000$$

  - The 5% cc return quantile is

$$q^r_{.05} = \texttt{qnorm}(0.05,\ 0.05,\ 0.1) = -0.1144854$$

  - The 5% VaR based on a \$10,000 initial investment is

$$\begin{aligned} \mathrm{VaR}_{.05} &= W_0 \cdot (e^{q^r_\alpha} - 1) \\ &= 10,000 \cdot (e^{-0.1144854} - 1) = -1081.75 \end{aligned}$$

Value at Risk (VaR)

- The probability that the profit of the investment L_1 is less than VaR_α is α

Value at Risk (VaR)

- The probability that the future value of the investment W_1 is less than $W_0 + \mathrm{VaR}_\alpha$ is α

Value at Risk (VaR)

- The probability that the future value of the investment W_1 is less than $W_0 + \mathrm{VaR}_\alpha$ is α
-

$$
\begin{aligned}
& \Pr\left(W_1 \leq W_0 + \mathrm{VaR}_\alpha\right) \\
= \ & \Pr\left(W_1 - W_0 \leq \mathrm{VaR}_\alpha\right) \\
= \ & \Pr\left(L_1 \leq \mathrm{VaR}_\alpha\right) = \alpha
\end{aligned}
$$

Value at Risk (VaR)

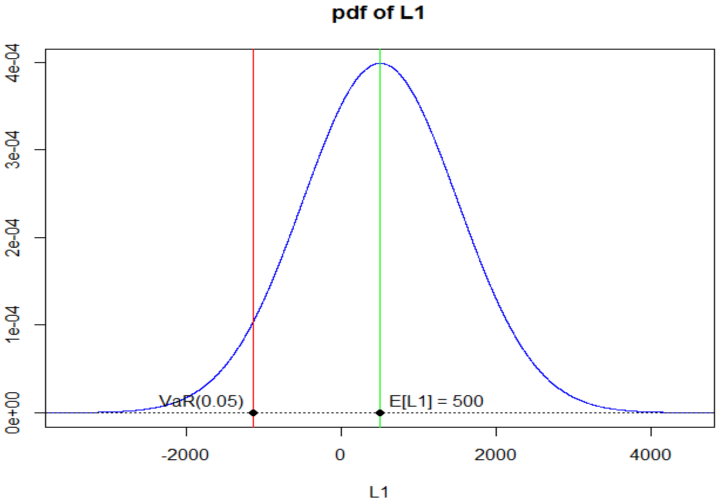
- $W_0 + \mathrm{VaR}_\alpha$ is the α quantile of W_1

Value at Risk (VaR)

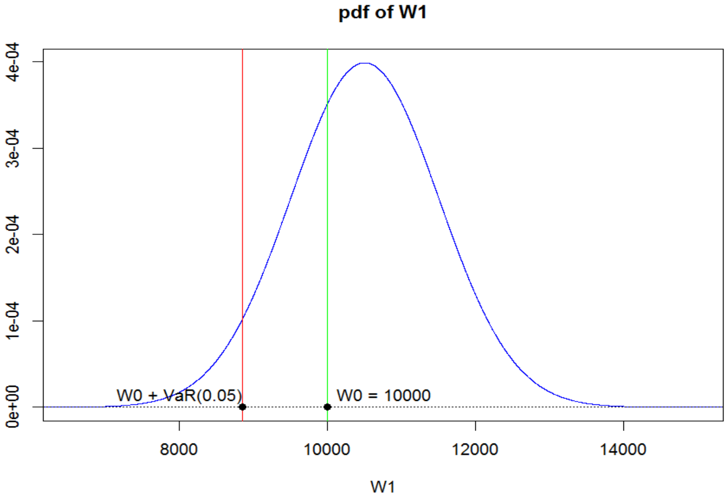
- $W_0 + \mathrm{VaR}_\alpha$ is the α quantile of W_1
-

$$
\begin{aligned}
& \Pr\left(W_1 \le W_0 + \mathrm{VaR}_\alpha\right) \\
= \; & \Pr\left(W_1 - W_0 \le \mathrm{VaR}_\alpha\right) \\
= \; & \Pr\left(L_1 \le \mathrm{VaR}_\alpha\right) = \alpha
\end{aligned}
$$

$$L_1 = W_0 \cdot R \text{ where } W_0 = 10,000 \text{ and } R \sim N(0.05, 0.01)$$



$$W_1 = W_0 \cdot (1 + R) \text{ where } W_0 = 10,000 \text{ and } R \sim N(0.05, 0.01)$$



Value at Risk (VaR)

- Value at Risk (VaR) is the most popular *risk measure* in finance
- Estimating VaR precisely requires a careful modelling of financial return distribution
  - key shape characteristics: mean, variance, skewness, kurtosis, etc.
  - some dynamic components
- We will get back to this topic after we cover some popular volatility modelling in financial econometrics

Outline

- **Multivariate Distribution**
  - joint, marginal and conditional distribution,
  - statistical independence,
  - covariance and correlation.

Joint Probability Distribution

- **Motivation:** if we consider returns of multiple assets, or some portfolio, we need to understand "joint random behavior' of them
- The corresponding concept is multivariate rv's
- Concepts we review
  - joint distribution
  - marginal distribution, conditional distribution and independence
  - conditional mean and variance
  - covariance and correlation (linear dependence measure)

Joint Probability Distribution

- **Bivariate Probability Distribution**

Example - Two discrete rv's X and Y

Table 2: Bivariate PMF

		X				
	$S_Y \backslash S_X$	0	1	2	3	$\Pr(Y)$
Y	0	1/8	2/8	1/8	0	4/8
	1	0	1/8	2/8	1/8	4/8
	$\Pr(X)$	1/8	3/8	3/8	1/8	1

$$f(x,y) = \Pr(X = x, Y = y) = \text{values in table}$$
$$e.g.,\ f(0,0) = \Pr(X = 0, Y = 0) = \ 1/8$$

Joint Probability Distribution

- Properties of joint pmf $f(x, y)$

$$
\begin{aligned}
S_{XY} &= \{(0,0),\ (0,1),\ (1,0),\ (1,1),\\
&\qquad (2,0),\ (2,1),\ (3,0),\ (3,1)\}\\
f(x,y) &\geq 0 \text{ for } x, y \in S_{XY}\\
\sum_{x,y \in S_{XY}} f(x,y) &= 1
\end{aligned}
$$

Joint Probability Distribution

- Marginal pmfs

$$
\begin{aligned}
f_X(x) = \Pr(X = x) &= \sum_{y \in S_Y} f(x, y) \\
&= \text{sum over columns in joint table}
\end{aligned}
$$

$$
\begin{aligned}
f_Y(y) = \Pr(Y = y) &= \sum_{x \in S_X} f(x, y) \\
&= \text{sum over rows in joint table}
\end{aligned}
$$

Conditional Probability Distribution

- **Conditional Probability**

Suppose we know $Y = 0$. How does this knowledge affect the probability that $X = 0, 1, 2$ or 3? The answer involves conditional probability.

Example

$$\Pr(X = 0|Y = 0) = \frac{\Pr(X = 0, Y = 0)}{\Pr(Y = 0)}$$
$$= \frac{\text{joint probability}}{\text{marginal probability}} = \frac{1/8}{4/8} = 1/4$$

Remark

$$\Pr(X = 0|Y = 0) = 1/4 \neq \Pr(X = 0) = 1/8$$
$$\Longrightarrow X \text{ depends on } Y$$

The marginal probability, $\Pr(X = 0)$, ignores information about Y.

Conditional Probability Distribution

- **Definition - Conditional Probability**
- The conditional pmf of X given $Y = y$ is, for all $x \in S_X$,

$$f(x|y) = \Pr(X = x|Y = y) = \frac{\Pr(X = x, Y = y)}{\Pr(Y = y)}$$

- The conditional pmf of Y given $X = x$ is, for all values of $y \in S_Y$

$$f(y|x) = \Pr(Y = y|X = x) = \frac{\Pr(X = x, Y = y)}{\Pr(X = x)}$$

Conditional Probability Distribution

- **Conditional Mean and Variance**

$$\mu_{X|Y=y} = E[X|Y=y] = \sum_{x \in S_X} x \cdot \Pr(X=x|Y=y),$$
$$\mu_{Y|X=x} = E[Y|X=x] = \sum_{y \in S_Y} y \cdot \Pr(Y=y|X=x).$$

$$\begin{aligned}\sigma^2_{X|Y=y} &= \mathrm{var}(X|Y=y) \\ &= \sum_{x \in S_X} (x-\mu_{X|Y=y})^2 \cdot \Pr(X=x|Y=y), \\ \sigma^2_{Y|X=x} &= \mathrm{var}(Y|X=x) \\ &= \sum_{y \in S_Y} (y-\mu_{Y|X=x})^2 \cdot \Pr(Y=y|X=x).\end{aligned}$$

Table 2: Bivariate PMF

		X				
	$S_Y \backslash S_X$	0	1	2	3	$\Pr(Y)$
Y	0	1/8	2/8	1/8	0	4/8
	1	0	1/8	2/8	1/8	4/8
	$\Pr(X)$	1/8	3/8	3/8	1/8	1

$$E[X] = 0 \cdot 1/8 + 1 \cdot 3/8 + 2 \cdot 3/8 + 3 \cdot 1/8 = 3/2$$

$$E[X|Y=0] = 0 \cdot 1/4 + 1 \cdot 1/2 + 2 \cdot 1/4 + 3 \cdot 0 = 1,$$

$$E[X|Y=1] = 0 \cdot 0 + 1 \cdot 1/4 + 2 \cdot 1/2 + 3 \cdot 1/4 = 2,$$

Table 2: Bivariate PMF

		X				
	$S_Y \backslash S_X$	0	1	2	3	$\Pr(Y)$
Y	0	1/8	2/8	1/8	0	4/8
	1	0	1/8	2/8	1/8	4/8
	$\Pr(X)$	1/8	3/8	3/8	1/8	1

$$\begin{aligned}
\mathrm{var}(X) &= (0-3/2)^2 \cdot 1/8 + (1-3/2)^2 \cdot 3/8 \\
&\quad + (2-3/2)^2 \cdot 3/8 + (3-3/2)^2 \cdot 1/8 = 3/4, \\
\mathrm{var}(X|Y=0) &= (0-1)^2 \cdot 1/4 + (1-1)^2 \cdot 1/2 \\
&\quad + (2-1)^2 \cdot 1/4 + (3-1)^2 \cdot 0 = 1/2, \\
\mathrm{var}(X|Y=1) &= (0-2)^2 \cdot 0 + (1-2)^2 \cdot 1/4 \\
&\quad + (2-2)^2 \cdot 1/2 + (3-2)^2 \cdot 1/4 = 1/2.
\end{aligned}$$

Independence

- **Definition:** Let X and Y be discrete rvs with pmfs $f_X(x)$, $f_Y(y)$, sample spaces S_X, S_Y and joint pmf $f(x, y)$. Then X and Y are **independent** rv's if and only if

$$f(x, y) = f_X(x) \cdot f_Y(y)$$

for all values of $x \in S_X$ and $y \in S_Y$.

Independence

- **Property:** If X and Y are independent rv's, then

$$f(x|y) = f_X(x) \text{ for all } x \in S_X,\ y \in S_Y$$
$$f(y|x) = f_Y(y) \text{ for all } x \in S_X,\ y \in S_Y$$

- Intuition:
  - Knowledge of X does not influence probabilities associated with Y
  - Knowledge of Y does not influence probabilities associated with X

Bivariate Distributions - Continuous rv's

- The joint pdf of X and Y is a non-negative function $f(x,y)$ such that
$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dxdy = 1$$
- Let $[x_1, x_2]$ and $[y_1, y_2]$ be intervals on the real line. Then
$$\Pr(x_1 \leq X \leq x_2, y_1 \leq Y \leq y_2) = \int_{x_1}^{x_2}\int_{y_1}^{y_2} f(x,y)\,dxdy$$
$=$ volume under probability surface over the intersection of the intervals $[x_1, x_2]$ and $[y_1, y_2]$

Marginal and Conditional pdfs

- The **marginal pdf** of X is found by integrating y out of the joint pdf $f(x, y)$ and the marginal pdf of Y is found by integrating x out of the joint pdf:

$$f(x) = \int_{-\infty}^{\infty} f(x, y) dy,$$
$$f(y) = \int_{-\infty}^{\infty} f(x, y) dx.$$

Marginal and Conditional pdfs

- The **conditional pdf** of X given that $Y = y$, denoted $f(x|y)$, is computed as
$$f(x|y) = \frac{f(x,y)}{f(y)},$$
and the conditional pdf of Y given that $X = x$ is computed as
$$f(y|x) = \frac{f(x,y)}{f(x)}.$$

Marginal and Conditional pdfs

- The **conditional means** are computed as

$$\mu_{X|Y=y} = E[X|Y=y] = \int x \cdot f(x|y)dx,$$
$$\mu_{Y|X=x} = E[Y|X=x] = \int y \cdot f(y|x)dy$$

and the **conditional variances** are computed as

$$\sigma^2_{X|Y=y} = \mathrm{var}(X|Y=y) = \int (x-\mu_{X|Y=y})^2 f(x|y)dx,$$
$$\sigma^2_{Y|X=x} = \mathrm{var}(Y|X=x) = \int (y-\mu_{Y|X=x})^2 f(y|x)dy.$$

Marginal and Conditional pdfs

- **Definition**: Let X and Y be continuous random variables . X and Y are independent iff

$$f(x,y) = f(x)f(y)$$

  - To compute the joint pdf for two independent random variables: we simple compute the product of the marginal distributions.

Marginal, Conditional pdfs and Independence

- **Property**:

Let X and Y be continuous random variables. X and Y are independent iff

$$f(x|y) = f(x), \text{ for } -\infty < x, y < \infty,$$
$$f(y|x) = f(y), \text{ for } -\infty < x, y < \infty.$$

Multivariate r.v's

- Example - Bivariate standard normal distribution: let $Z_1 \sim N(0,1)$, $Z_2 \sim N(0,1)$ and let Z_1 and Z_2 be independent. Then

$$f(z_1, z_2) = f(z_1)f(z_2) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z_1^2} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z_2^2}$$
$$= \frac{1}{2\pi} e^{-\frac{1}{2}(z_1^2+z_2^2)}.$$

- To find $\Pr(-1 < Z_1 < 1, -1 < Z_2 < 1)$ we must solve

$$\int_{-1}^{1}\int_{-1}^{1} \frac{1}{2\pi} e^{-\frac{1}{2}(z_1^2+z_2^2)} dz_1 dz_2$$

  - unfortunately, this does not have an analytical solution. Numerical approximation methods are required to evaluate the above integral. See R package mvtnorm.

Multivariate r.v's

- **Covariance and Correlation - Measuring linear dependence between two rv's**

Covariance: Measures direction but not strength of linear relationship between 2 rv's

$$
\begin{aligned}
\sigma_{XY} &= E[(X-\mu_X)(Y-\mu_Y)] \\
&= \sum_{x.y\in S_{XY}} (x-\mu_X)(y-\mu_Y)\cdot f(x,y) \quad \text{(discrete)} \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(x-\mu_X)(y-\mu_Y)f(x,y)dxdy \quad \text{(continuous)}
\end{aligned}
$$

Multivariate r.v's

Table 2: Bivariate PMF

		X				
	$S_Y \backslash S_X$	0	1	2	3	$\Pr(Y)$
Y	0	1/8	2/8	1/8	0	4/8
	1	0	1/8	2/8	1/8	4/8
	$\Pr(X)$	1/8	3/8	3/8	1/8	1

$$\sigma_{XY} = \mathrm{Cov}(X, Y) = (0 - 3/2)(0 - 1/2) \cdot 1/8$$
$$+(0 - 3/2)(1 - 1/2) \cdot 0 + \cdots$$
$$+(3 - 3/2)(1 - 1/2) \cdot 1/8 = 1/4$$

Multivariate r.v's

- **Properties of Covariance:**

$$
\begin{aligned}
\mathrm{Cov}(X, Y) &= \mathrm{Cov}(Y, X) \\
\mathrm{Cov}(aX, bY) &= a \cdot b \cdot \mathrm{Cov}(X, Y) = a \cdot b \cdot \sigma_{XY} \\
\mathrm{Cov}(X + Y, Z) &= \mathrm{Cov}(X, Z) + \mathrm{Cov}(Y, Z) \\
\mathrm{Cov}(X, X) &= \mathrm{Var}(X) \\
X, Y \text{ independent} \quad \Longrightarrow & \quad \mathrm{Cov}(X, Y) = 0 \\
\mathrm{Cov}(X, Y) &= 0 \not\Rightarrow X \text{ and } Y \text{ are independent} \\
\mathrm{Cov}(X, Y) &= E[XY] - E[X]E[Y]
\end{aligned}
$$

Multivariate r.v's

- **Correlation: Measures direction and strength of linear relationship between 2 rv's**

$$\rho_{XY} = \mathrm{Cor}(X, Y) = \frac{\mathrm{Cov}(X, Y)}{\mathrm{SD}(X) \cdot \mathrm{SD}(Y)}$$

$$= \frac{\sigma_{XY}}{\sigma_X \cdot \sigma_Y} = \text{ scaled covariance}$$

Multivariate r.v's

Table 2: Bivariate PMF

	$S_Y \backslash S_X$	X = 0	1	2	3	$\Pr(Y)$
Y	0	1/8	2/8	1/8	0	4/8
	1	0	1/8	2/8	1/8	4/8
	$\Pr(X)$	1/8	3/8	3/8	1/8	1

$$\rho_{XY} = \mathrm{Cor}(X, Y) = \frac{1/4}{\sqrt{(3/4) \cdot (1/2)}} = 0.577$$

Multivariate r.v's

- **Properties of Correlation:**

$$
\begin{aligned}
-1 \leq \rho_{XY} &\leq 1 \\
\rho_{XY} &= 1 \text{ if } Y = aX + b \text{ and } a > 0 \\
\rho_{XY} &= -1 \text{ if } Y = aX + b \text{ and } a < 0 \\
\rho_{XY} &= 0 \text{ if and only if } \sigma_{XY} = 0 \\
\rho_{XY} &= 0 \nRightarrow \; X \text{ and } Y \text{ are independent in general} \\
\rho_{XY} &= 0 \implies \text{independence if } X \text{ and } Y \text{ are normal}
\end{aligned}
$$

Multivariate r.v's

- **Linear Combination of 2 rv's:** let X and Y be rv's. Define a new r.v. Z that is a linear combination of X and Y : $Z = aX + bY$ where a and b are constants. Then

$$\begin{aligned}\mu_Z &= E[Z] = E[aX + bY] = aE[X] + bE[Y] \\ &= a \cdot \mu_X + b \cdot \mu_Y\end{aligned}$$

and

$$\begin{aligned}\sigma_Z^2 &= \mathrm{Var}(Z) = \mathrm{Var}(a \cdot X + b \cdot Y) \\ &= a^2 \mathrm{Var}(X) + b^2 \mathrm{Var}(Y) + 2a \cdot b \cdot \mathrm{Cov}(X, Y) \\ &= a^2 \sigma_X^2 + b^2 \sigma_Y^2 + 2a \cdot b \cdot \sigma_{XY}\end{aligned}$$

- If $X \sim N(\mu_X, \sigma_X^2)$ and $Y \sim N(\mu_Y, \sigma_Y^2)$ then $Z \sim N(\mu_Z, \sigma_Z^2)$

Multivariate r.v's

- Example: Portfolio returns
- R_A = return on asset A with $E[R_A] = \mu_A$ and $\mathrm{Var}(R_A) = \sigma_A^2$
- R_B = return on asset B with $E[R_B] = \mu_B$ and $\mathrm{Var}(R_B) = \sigma_B^2$
- $\mathrm{Cov}(R_A, R_B) = \sigma_{AB}$ and $\mathrm{Cor}(R_A,\ R_B) = \rho_{AB} = \frac{\sigma_{AB}}{\sigma_A \cdot \sigma_B}$
- Portfolio

x_A = share of wealth invested in asset A, x_B = share of wealth invested in asset B
$x_A + x_B = 1$ (exhaust all wealth in 2 assets)
$R_P = x_A \cdot R_A + x_B \cdot R_B$ = portfolio return

Multivariate r.v's

- Portfolio Problem: How much wealth should be invested in assets A and B?

Portfolio expected return (gain from investing)

$$\begin{aligned} E[R_P] = \mu_P &= E[x_A \cdot R_A + x_B \cdot R_B] \\ &= x_A E[R_A] + x_B E[R_B] \\ &= x_A \mu_A + x_B \mu_B \end{aligned}$$

Portfolio variance (risk from investing)

$$\mathrm{Var}(R_P) = \sigma_P^2 = \mathrm{Var}(x_A R_A + x_B R_B)$$

$$= x_A^2 \mathrm{Var}(R_A) + x_B^2 \mathrm{Var}(R_B) + 2 \cdot x_A \cdot x_B \cdot \mathrm{Cov}(R_A,\ R_B)$$

$$= x_A^2 \sigma_A^2 + x_B^2 \sigma_B^2 + 2 x_A x_B \sigma_{AB}$$

$$\mathrm{SD}(R_P) = \sqrt{\mathrm{Var}(R_P)} = \sigma_P = \left(x_A^2 \sigma_A^2 + x_B^2 \sigma_B^2 + 2 x_A x_B \sigma_{AB}\right)^{1/2}$$

Multivariate r.v's

- Linear Combination of N rv's.

Let $X_1,\ X_2, \cdots, X_N$ be rv's and let $a_1, a_2, \ldots, a_N$ be constants. Define

$$Z = a_1 X_1 + a_2 X_2 + \cdots + a_N X_N = \sum_{i=1}^{N} a_i X_i$$

Then

$$\begin{aligned}\mu_Z = E[Z] &= a_1 E[X_1] + a_2 E[X_2] + \cdots + a_N E[X_N] \\ &= \sum_{i=1}^{N} a_i E[X_i] = \sum_{i=1}^{N} a_i \mu_i\end{aligned}$$

Multivariate r.v's

- For the variance,

$$\begin{aligned}\sigma_Z^2 = \mathrm{Var}(Z) = {} & a_1^2\mathrm{Var}(X_1) + \cdots + a_N^2\mathrm{Var}(X_N) \\ & + 2a_1a_2\mathrm{Cov}(X_1,\ X_2) + \cdots + 2a_1a_N\mathrm{Cov}(X_1,\ X_N) \\ & + 2a_2a_3\mathrm{Cov}(X_2,\ X_3) + \cdots + 2a_2a_N\mathrm{Cov}(X_2,\ X_N) \\ & + \cdots \\ & + 2a_{N-1}a_N\mathrm{Cov}(X_{N-1}, X_N)\end{aligned}$$

  Note: N variance terms and $N(N-1) = N^2 - N$ covariance terms. If $N = 100$, there are $100 \times 99 = 9900$ covariance terms!

- Result: If $X_1,\ X_2, \cdots, X_N$ are each normally distributed random variables then

$$Z = \sum_{i=1}^{N} a_i X_i \sim N(\mu_Z, \sigma_Z^2)$$

Multivariate r.v's

- Example: Portfolio variance with three assets
- R_A, R_B, R_C are simple returns on assets A, B and C
- x_A, x_B, x_C are portfolio shares such that $x_A + x_B + x_C = 1$
- $R_p = x_A R_A + x_B R_B + x_C R_C$
- Portfolio variance

$$\sigma_P^2 = x_A^2\sigma_A^2 + x_B^2\sigma_B^2 + x_C^2\sigma_C^2 \\ + 2x_A x_B \sigma_{AB} + 2x_A x_C \sigma_{AC} + 2x_B x_C \sigma_{BC}$$

Note: Portfolio variance calculation may be simplified using matrix layout

$$\begin{array}{cccc} & x_A & x_B & x_C \\ x_A & \sigma_A^2 & \sigma_{AB} & \sigma_{AC} \\ x_B & \sigma_{AB} & \sigma_B^2 & \sigma_{BC} \\ x_C & \sigma_{AC} & \sigma_{BC} & \sigma_C^2 \end{array}$$

Multivariate r.v's

- Example: Multi-period continuously compounded returns

$$r_t = \ln(1 + R_t) = \text{ monthly cc return}$$
$$r_t \sim N(\mu,\ \sigma^2) \quad \text{for all } t$$
$$\mathrm{Cov}(r_t, r_s) = 0 \text{ for all } t \neq s$$

Annual return

$$r_t(12) = \sum_{j=0}^{11} r_{t-j} = r_t + r_{t-1} + \cdots + r_{t-11}$$

Then

$$E[r_t(12)] = \sum_{j=0}^{11} E[r_{t-j}] = \sum_{j=0}^{11} \mu \qquad (E[r_t] = \mu \text{ for all } t)$$
$$= 12\mu \qquad (\mu = \text{mean of monthly return})$$

Multivariate r.v's

- For variance

$$
\begin{aligned}
\mathrm{Var}(r_t(12)) &= \mathrm{Var}\left(\sum_{j=0}^{11} r_{t-j}\right) \\
&= \sum_{j=0}^{11} \mathrm{Var}(r_{t-j}) = \sum_{j=0}^{11} \sigma^2 \\
&= 12 \cdot \sigma^2 \quad (\sigma^2 = \text{monthly variance}) \\
\mathrm{SD}(r_t(12)) &= \sqrt{12} \cdot \sigma \text{ (square root of time rule)}
\end{aligned}
$$

- Then

$$
r_t(12) \sim N(12\mu, 12\sigma^2)
$$

Multivariate r.v's

- For example, suppose

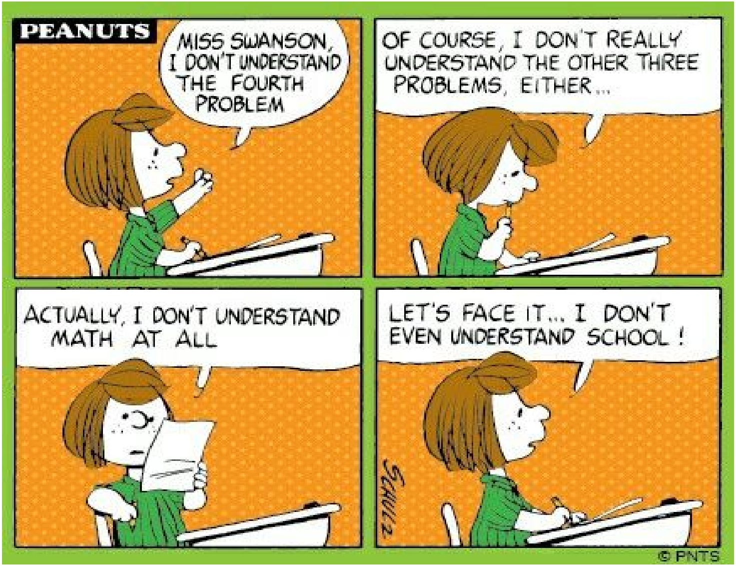
$$r_t \sim N(0.01, (0.10)^2)$$

Then

$$\begin{aligned} E[r_t(12)] &= 12 \times (0.01) = 0.12 \\ \mathrm{Var}(r_t(12)) &= 12 \times (0.10)^2 = 0.12 \\ \mathrm{SD}(r_t(12)) &= \sqrt{0.12} = 0.346 \\ r_t(12) &\sim N(0.12, (0.346)^2) \end{aligned}$$

Final Remarks and Next Topic

- Tools from probability theory are useful to describe financial market randomness
- In reality, the underlying "model" driving the financial market randomness is unknown
- The pdf of the financial returns is unknown in practice
- We don't really know the shape features of the pdf of the financial returns, e.g., variance, skewness, kurtosis and etc.
- We even do not know the mean of the financial returns


PEANUTS
MISS SWANSON, I DON'T UNDERSTAND THE FOURTH PROBLEM
OF COURSE, I DON'T REALLY UNDERSTAND THE OTHER THREE PROBLEMS, EITHER...
ACTUALLY, I DON'T UNDERSTAND MATH AT ALL
LET'S FACE IT... I DON'T EVEN UNDERSTAND SCHOOL!
SCHULZ
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Final Remarks and Next Topic

- Given "realized" random variables (data), how do we analyze the results?
  - we need tools from statistical theory
- Next topic: review of statistical concepts and applications