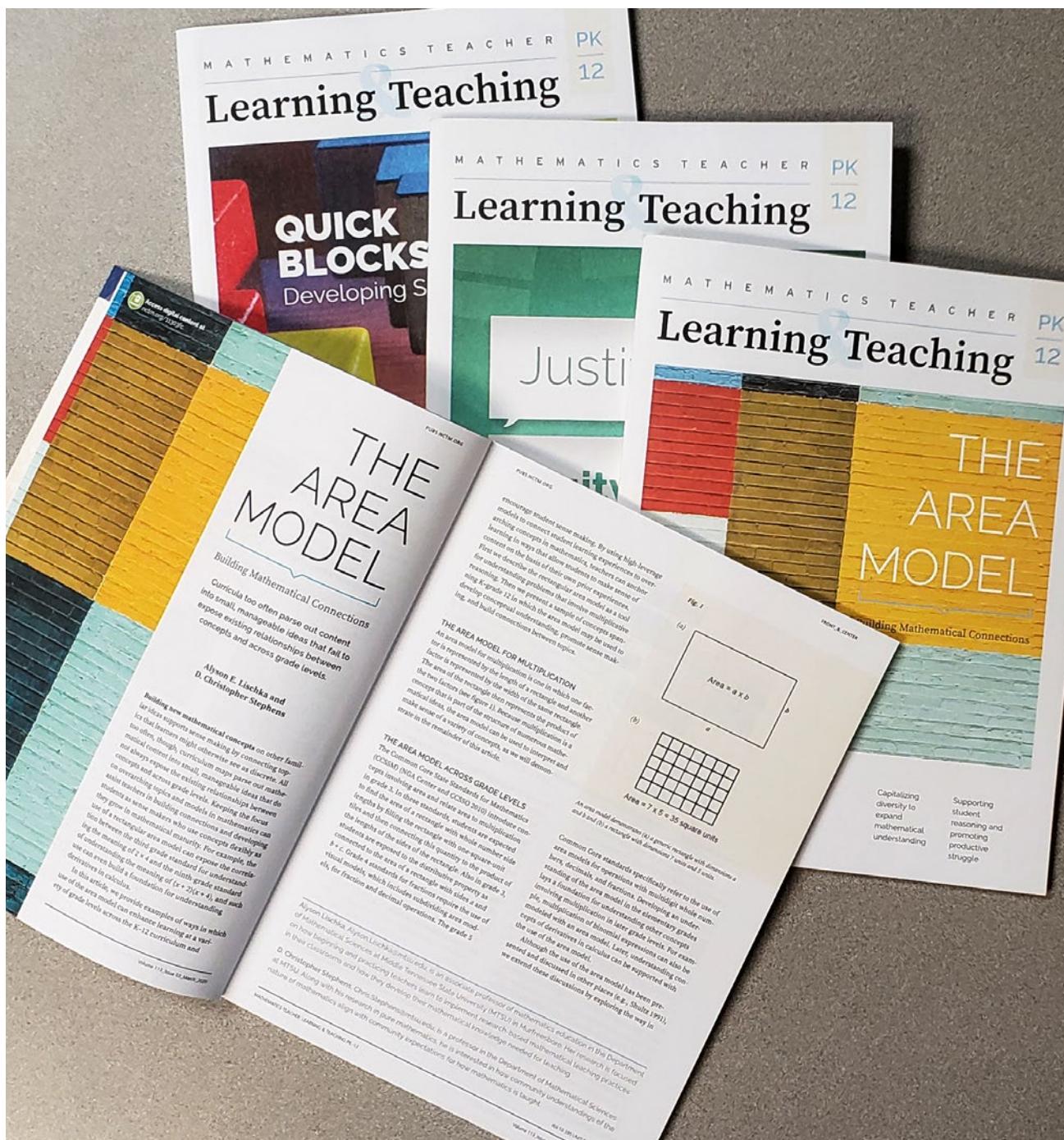


Learning & Teaching

Author Toolkit





Welcome

Welcome, *MTLT* author!

Thank you for your interest in *MTLT*, NCTM's digital-first practitioner journal offering useful and classroom-ready content for the elementary, middle, and high school grade bands as well as critical information on subjects across the PK-12 spectrum. Whether you're a PK-12 teacher or mathematics educator, we invite you to [submit to *MTLT*](#). By publishing in the journal, you will have a long-term impact on other teachers and their students through your teaching success stories. *MTLT* is practitioner journal by and for PK-12 mathematics teachers!

We realize that *MTLT* authors have little time to navigate the complexities of creating, submitting, and revising a journal manuscript. Our goal with this revised Author Toolkit is to explain the manuscript submission process step by step for interested authors. We do this by defining what we mean by **digital first**, describing manuscript types that the journal publishes, providing instructions on preparing and submitting a manuscript, and, finally, detailing the steps of peer review and the postacceptance workflow.

Thank you again for your interest in *MTLT*. We are looking forward to receiving your submission and hope that you are excited about NTCM's digital-first journal.

Sincerely,
Angela Barlow, PhD
Editor-in-Chief, *MTLT*



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WHAT IS MTLT?

Mathematics Teacher: Learning and Teaching PK-12 (MTLT) is NCTM's digital-first journal with the unique perspective of a practicing teacher of mathematics. The journal spans PK-12 and provides grade-band-specific articles.

Manuscripts in *MTLT* can vary in type and length. A **Front-and-Center** article has a message that appeals to *MTLT* readers working with students across all grades, PK-12. **Feature** and **Focus** articles emphasize one grade or grade band. Evidence of classroom implementation is expected. In contrast, an **Exploring Mathematics** article focuses on mathematical ideas that are of interest to teachers without requiring a connection to the PK-12 mathematics classroom. In addition to articles, **Departments** provide an opportunity to share your ideas.

Ready to Submit? Access the ScholarOne Submission and Peer Review system.

WHAT IS DIGITAL FIRST?

A digital-first journal takes advantage of 21st-century technology to go beyond the printed word. For example, instead of a static figure in a printed article, *MTLT* would prefer to include an embedded video that puts the reader inside the classroom where student-student and student-teacher interactions can be observed. Audio, apps, or interactive whiteboard files are just a few of the many opportunities that the new journal holds for authors and readers. Authors are encouraged to include such digital components as a short video clip, GeoGebra activity, Desmos exercise, Livescribe™ file, SMART Board™ file, or other form of multimedia to enhance their submission (however, such use does not imply NCTM endorsement of any product or developer).

In [appendix 1](#), we provide examples of digital assets in *MTLT* articles.



WRITE FOR MTLT

MTLT features four main types of articles:

- **Front-and-Center Article—Grade Band PK-12**

A Front-and-Center article has a message that appeals to *MTLT* readers working with students across all grades, PK-12. This can be achieved in many ways, including, but not limited to, writing on an issue of importance for all grades (providing relevant examples for different grade bands) or reframing the topic for each grade band in different sections of the article. Submissions should not exceed 5,000 words. See a sample in [appendix 2](#).

- **Feature Article—Grade Bands PK-2, 3-5, 6-8, 9-12, PK-5, 3-8, 6-12**

A Feature article emphasizes one grade or grade band. Evidence of classroom implementation is expected. Submissions should not exceed 3,500 words. See a sample in [appendix 3](#).

- **Focus Article—Grade Bands PK-2, 3-5, 6-8, 9-12, PK-5, 3-8, 6-12**

Focus articles share a single developed idea to be quickly read by the busy teacher. Teaching strategies, learning technologies, or tasks used for a lesson are but a few of the many possibilities. The evidence of classroom implementation will not be the main message of the article. Submissions should not exceed 1,500 words. See a sample in [appendix 4](#).

- **Exploring Mathematics—Grade Bands PK-2, 3-5, 6-8, 9-12, PK-5, 3-8, 6-12**

This article type focuses on mathematics content that appeals to PK-12 teachers and provides a forum for classroom teachers to discuss mathematics. Submissions can—

- pose and solve a novel or interesting mathematics problem;
- expand on connections among different mathematical topics;
- present a general method for describing a mathematical notion or solving a class of problems;
- elaborate on new insights into familiar school mathematics; or
- leave the reader with a mathematical idea to consider.

Submissions should not exceed 2,500 words and should be provided as an MS Word-based document (not LaTex) See a sample in [appendix 5](#).

MTLT offers Departments as well:

- **Ear to the Ground**

Ear to the Ground highlights voices working in various communities within the mathematics education world. Possible contributors include conference and webinar presenters, book



contributors (authors, editors and chapter authors), website creators, blog authors, and workshop facilitators. Contributions may be solicited or unsolicited and help keep *MTLT* readers up to date on current happenings in PK-12 mathematics education. Submissions should not exceed 750 words. See a sample in [appendix 6](#).

- **For the Love of Mathematics**

How do you show your love of mathematics? Submissions sent to the editor may include a photograph combined with a funny or engaging mathematical question; an original cartoon that is humorous or reflective; a puzzle; Math Circle prompts; a poem or vignette; original artwork; or general celebratory items, such as Pi Day or Metric Week. Submission of student work within these guidelines is also welcome. Submissions should include a brief description (no more than 150 words) that relates the creative aspect of the submission (e.g., photograph/figure, poem, song) to your love of mathematics. See a sample in [appendix 7](#).

- **From the Archives**

This department features reprinted articles from NCTM's legacy journals and considers them from a contemporary perspective. Although unsolicited submissions are not accepted, the online *MTLT* Journal Club on the first Tuesday of each month offers an avenue for individuals to participate in the selection and introductory writing for this department. See a sample in [appendix 8](#).

- **Growing Problem Solvers**

This department publishes manuscripts that show how one important mathematical idea can "grow" in mathematical complexity over the years. Manuscripts highlight a PK-12 learning trajectory, providing four high quality tasks that span PK-2, 3-5, 6-8, and 9-12 grade bands. The learning trajectory should be made explicit using a chart of the curriculum standards being addressed by each task. Tasks featured are low threshold, high ceiling tasks with multiple entry points, allowing for a variety of solution strategies and reaching many different learners. A teacher page highlights important features of the tasks, potential instructional strategies, and extension ideas. Manuscripts should be accompanied by four student-ready task sheets. Word count: 1500 plus four task sheets. See a sample in [appendix 9](#).

- **Problems to Ponder**

Problems to Ponder provides 28 classroom-ready mathematics problems that collectively span PK-12 and are arranged in grade-level order (problem 1 = prekindergarten; problem 28 = grade 12). Answers to the problems are available online. Individuals are encouraged to submit a problem or a collection of problems directly to mtlt@nctm.org. If accepted, authors of problems will be acknowledged. See a sample in [appendix 10](#).



- **Teaching Is a Journey!**

This department provides a space for PK-12 teachers of mathematics to connect with other teachers of mathematics through their stories that lend personal and professional support. Potential topics include, but are not limited to, these questions: What led you to the profession of teaching mathematics, and how have you developed in the profession? How have you come to incorporate effective teaching practices into your work? What difficulties have presented themselves? How have you persevered through challenges? What is your story with respect to realizing the vision of better mathematics for each and every student? Vulnerability is a component of our collective work. In what ways does vulnerability affect your professional trajectory? How do you practice self-care and care of colleagues? What strategies do you have for support within your professional learning community? Submissions should not exceed 1,600 words. See [appendix 11](#).

MTLT offers other article types:

- **Editorials**

Editorials are short articles that comment on issues of significance to mathematical thinking and learning. Authors should present the issue in a clear and constructive way and then add their critique or suggestion(s) on improving or expanding the idea. Alternatively, the author may present both sides of an issue and leave it up to readers to decide their personal point of view. Successful editorials will leave readers wanting to know more or will help them engage in the topic at hand. Authors are encouraged to include in their submission the invitation to continue the discussion on social media such as [myNCTM](#) (NCTM's discussion board) or Twitter (copy @nctm) and Facebook (tag @NCTM). Subject matter could include such topics as curriculum, pedagogy, assessment, educational philosophy, research implementation, or structure of the educational system. Submissions should not exceed 1,000 words. Should the article be accepted, it will be published online only.

- **Letter to the Editor**

A Letter to the Editor should be a brief response to an article, discussing only issues directly relevant to the original article's content. The response may be supportive or critical in nature. If the Letter to the Editor is accepted, the author(s) of the original article will have the opportunity to submit a response letter. Should the response letter be accepted, both letters will be published together online only. Submissions should not exceed 1,000 words.

Readers are encouraged to direct general comments (e.g., "I really enjoyed the article by _____") or personal reflections (e.g., "I used the activity mentioned in _____ in my classroom, and the children loved it!") on *MTLT* articles to social media sites such as Twitter and Facebook. When tweeting, be sure to tag @NCTM and use these hashtags: #MTLTPK12, #MTBoS, #iteachmath, #T2T, #math, #mathed. If posting on Facebook, tag @NCTM.



PREPARING YOUR MANUSCRIPT

Helpful Tools for Manuscript Preparation

- Digital Asset Types and Examples ([appendix 1](#))
- *MTLT* FAQs ([appendix 12](#)) for a list of common author questions
- Digital Assets and Figures: Instructions and Technical Specs ([appendix 13](#))
- Figure/Multimedia Permission Form—Adult Participants ([appendices 14 and 15](#))
- Figure/Multimedia Permission Form for Non-Adult Participants (those younger than 18 years old; requires parental completion and signature) ([appendices 16 and 17](#))
- Sample Manuscript Cover Letter, Revision Response Letter, and Manuscript Submission ([appendix 18](#)).

Helpful Webinars

- [How to Turn Your Presentation into an *MTLT* Article](#)

Discusses how meeting presenters can amplify their message by turning a “nugget” from their presentation into a submission for *MTLT*.

- [Writing for *MTLT*: Guiding Thoughts for Mathematics Teacher Educators](#)

Discusses how mathematics teacher educators can be valuable contributors to *MTLT*.

- [You've Got Stories: Now It's Time to Write!](#)

Shows teachers, particularly those new to writing, how to approach writing for a journal such as *MTLT*.

- [Navigating the *MTLT* Submissions Site](#)

Learn how to access and use the ScholarOne peer review submission system.

Ethical Guidelines

In preparing your manuscript, ensure that your materials are not under consideration for publication elsewhere or have not been published elsewhere.

Permissions/Consent from Individuals Shown in Videos or Figures

Adults appearing in videos or figures must complete a Video Permission (see [appendices 14 and 15](#)) form allowing NCTM to publish their image. **IMPORTANT:** For participants younger than 18 years, we require a completed **NCTM Release Form for Use of Student's Image or Written Work** ([appendices 16 and 17](#)) from each student's parent/legal guardian. The forms can be uploaded individually or combined into one PDF document for submission to ScholarOne.

Permissions for Student Written Work

Parental consent is also required for distinctive original written work created by students. For each such case, you will upload a completed **NCTM Release Form for Use of Student's Image or Written Work** ([appendices 16 and 17](#)) signed by the student's parent or legal guardian. Alternatively, you may combine the signed forms into one file for submission.



Reprinting or Adapting Materials from Other Sources

Authors are responsible for obtaining permissions for any copyrighted material included in their article. Copyrighted material typically includes adapted or reprinted tables, appendixes, or figures from existing publications or websites. Although permission-to-reprint documentation is required only if a manuscript is accepted, the process of securing permission can take time. Therefore, we encourage authors to contact the publisher of the original material early in the peer review process.

Word Count

The manuscript text, tables, figures, and references are included in the total word count of each submission; word counts are summarized here:

Article Type	Word-count limit
Front and Center	5000
Feature	3500
Focus	1500
Exploring Mathematics	2500
Ear to the Ground	750
For the Love of Mathematics	150
Growing Problem Solvers	1,500 (main text) + four one-page Task Sheets
Problems to Ponder	1500
Teaching Is a Journey	1600
Editorial	1000
Letter to the Editor	1000

Blinding

MTLT is a double-blind journal, meaning authors do not know the identity of reviewers and reviewers do not know the identity of authors. Associate editors and the editor-in-chief are *not* blinded to author names.

To preserve anonymity, author(s) names may appear *only* in an author cover letter that is separate from the manuscript file. Please ensure no author identification appears within the manuscript. Do not reference your own work or your school in a way that compromises the blind review process. For example:

Incorrect: We (Smith and Jones 2017) tested students' knowledge after implementation . . .

Correct: Author (2017) tested students' knowledge after implementation . . .

Use pseudonyms for students' or colleagues' names and blind project or grant information, acknowledgments, and conference presentations appearing in a manuscript. Links to personal or institutional websites should also be blinded. Reference list entries should be formatted as follows: Author (year); for example, Author (2017). No additional citation information for the reference should be provided.



Ensure all supplemental or multimedia materials, including links to websites, preserve author anonymity. For in-text links to websites that disclose author identity, please provide the links in a separately loaded file, which should be designated as a “supplemental file NOT for review.” In the manuscript text, provide a brief description of the website but do not include an active link.

General Writing Guidelines

- Order your manuscript text file as follows:
 - Anonymized title page
 - Manuscript body (include figures [with figure legends], video still images, and tables within the manuscript text, near their first mention)
 - Reference list
 - Appendixes
- Refer to the *Chicago Manual of Style*, 17th ed. (Chicago: University of Chicago Press, 2017) and the October 2021 [NCTM Style Guide](#) for complete style guidelines.
- Set text in Times New Roman, 12 point, double-spaced throughout, including quoted matter, lists, tables, notes, and references.
- Use formatted headings (up to four levels) to help organize your article for the reader.
- Please proofread and spell-check your manuscript *before* submission. Review it for grammar, completeness, mathematical correctness, and accuracy of references.
- Supplemental files that are intended for online-only publication should be carefully read and spell-checked prior to submission because they will not be copy edited or composed into proofs if the manuscript is accepted.
- Provide tables using the Microsoft Word table feature. Do not import an image-based table.
- Ensure all tables, figures, videos, or other multimedia material are mentioned in text and, on first mention, they are numerically ordered. Embed figures, tables, and video still images within the text, near their first mention. Multimedia files should maintain author anonymity.
- *MTLT* prefers that footnotes be eliminated. Please integrate them parenthetically into the main text instead.
- Ensure that all in-text citations are included in the reference list, and, likewise, that all reference list entries are cited in text.
- For quotations in text, include the quotation's page number(s) in the in-text citation.
- Use extended dialogue and direct quotes sparingly, incorporating the key ideas of conversations into the text wherever possible.



References

- View *MTLT* Reference Style Examples ([appendix 19](#))

MTLT uses the *Chicago Manual of Style* (17th ed.) for in-text citations and reference lists.

Figures and Multimedia Files

- When determining whether to add a figure, determine first whether the figure will add educational value to the text. For example, photographs of students working at their desks or the whiteboard often do not add appreciably to the content of an article. Include figures that advance the understanding of the article's main point(s).
- Information contained in tables, figures, videos, and digital assets should not be repeated in text. Rather, the text should be written under the assumption that the reader has read or viewed these items.
- During peer review, figures must be legible but do not have to be provided as high-resolution files. Should your manuscript be accepted, however, you will provide high-resolution images, which will be used for production. The Digital Assets and Figures: Instructions and Technical Specs ([appendix 13](#)) provide instructions on required specifications and image quality for final figure files.
- Within the manuscript text file, figure legends—labeled by figure number—should be placed within the manuscript body, beneath the figures they describe.
- Figures should be called out, on first mention, in numerical order within the manuscript text. For example, figure 2 should not be cited in text before figure 1.

Mathematical Characters and Equations

- All variables should be set in italics.
- Numbers, parentheses, and mathematical operators should not be set in italics.
- Points, as in “segment *AB*,” should be italicized, and labels in geometric figures must also be italicized.
- Do not use MS Word Equation Editor for mathematical expressions.
- Most of the mathematics in *MTLT* can be written in regular Microsoft Word characters and symbols, using italics, superscripts, and subscripts as necessary. For example:

$$(a + b)^2 = a^2 + 2ab + b^2$$

$$k(x) = 4 - y$$

$$|2 - x| \leq 6$$

$$A_1, A_2, \dots, A_n$$

- Fractions should be written in Microsoft Word without any formatting or stacking; they will be formatted during layout as needed. For example:

$$\frac{1}{2}x + \frac{3}{4}y = 4$$

$$4\frac{1}{2} - 1\frac{1}{8} = 3\frac{3}{8}$$

- Use MathType sparingly for expressions and equations that cannot be typed using the keyboard, such as in the examples below.
- Since most expressions set in MathType will require their own line in the journal, that should be a guiding principle for authors; **if the expression is not important enough to be broken out on its own line, then do not use MathType.**
- For more complex equations (e.g., the two equations below) that require MathType, authors should be sure to include versions within their manuscript that show how the equations should be arranged.

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = 1 \quad \left(\frac{y_2 - y_1}{x_2 - x_1} \right)$$



SUBMITTING YOUR MANUSCRIPT TO THE SCHOLARONE PEER REVIEW SYSTEM

Helpful Tool for the ScholarOne System

- [How to Submit to ScholarOne](#)—A guided PowerPoint® provides an overview of the submission site.

All submissions for *MTLT* are handled through the ScholarOne peer review system, with the exception of the **Problems to Ponder** department (which should be emailed to mtlt@nctm.org).

Ready to Submit? Access the ScholarOne Submission and Peer Review system.

General Guidelines for the ScholarOne System

- You will be asked to assign copyright of your article to NCTM as part of the submission process. Please note that the copyright agreement is voided if your manuscript is not accepted for publication.
- If you have coauthors, you will enter each coauthor's name, their affiliation, address, and email into the system. Gathering this information in advance of submission can be helpful.
- For peer review, you will include a manuscript text file (MS Word) consisting of an anonymized title page; the manuscript body that includes embedded figures and tables near their respective first mentions in text; a reference list; and appendixes, in that order.
- If your submission contains figures or tables, you will embed them within the manuscript text file.
- If your submission contains media file(s) (e.g., video, GeoGebra, etc.), consider the following:
 - If possible, the media source file(s) should be uploaded to the peer review system, rather than adding a link within the text of your manuscript to an external website. For example, instead of providing a link in your manuscript to a self-authored YouTube video, we prefer that you upload the video source file to the ScholarOne system (however, if you are not the author of the video, an in-text link is appropriate). Please note that the ScholarOne submission system limits individual file sizes to 350 MB and 500 MB for the total of all files. If you have large videos, narrated PowerPoints, or other multimedia files, please contact the [editorial office](#) for alternative upload instructions.
 - Ensure all supplemental or multimedia materials, including links to websites, preserve author anonymity. If anonymity is not possible in files such as videos, please select the file type “supplemental file NOT for review” in the submission system. For in-text links to websites that disclose author identity, please provide the links in a separately loaded file, which should also be designated as a “supplemental file NOT for review.” In the manuscript text, provide a brief description of the website but do not include an active link.



A Note about Consent and/or Permission

Parental consent forms for students under 18 years who appear in figures or videos are not required at submission but will become required should your paper be accepted. It is best to gather these consent forms before photographing or recording students, however, to ensure you will be able to use the image(s) or footage you obtain. The parental consent form is available in [appendices 14 and 15 \(Spanish\)](#).

THE PEER REVIEW PROCESS

Once submitted to ScholarOne, your manuscript will be assigned to an associate editor (AE) and will undergo double-blind peer review by three reviewers. Reviewers are allowed 21 days to return their assessments.

After all reviews have been received, the AE will also read your manuscript and then integrate their comments with those of the reviewers to arrive at a recommended decision about the suitability of the manuscript for publication in *MTLT*.

After the AE has submitted a recommendation, your manuscript moves into the editor-in-chief's (EIC's) queue for final review. Once the EIC has entered their decision, the ScholarOne system will email you the decision letter. If a revision is requested, the AE will summarize the critical points from the reviewers' comments that you will need to focus on when revising your manuscript.

MTLT Decision Types

MTLT offers six decision types: (1) Accept, (2) Conditional Accept, (3) Minor Revision, (4) Major Revision, (5) Reject & Resubmit, and (6) Reject. Note that neither a major nor minor revision decision represents a commitment to accept. If you receive a revision decision, please attend carefully to all feedback in order to improve the chances of having your article accepted.

Accept: Your paper is accepted as is, without needing further edits. Editorial staff will contact you within two weeks after acceptance with a request for your final files.

Conditional Accept: A few minor or cosmetic changes are needed before your paper can be accepted.

Minor Revision: Some (typically) nonsubstantive issues need resolution or further clarification before a final decision can be rendered.

Major Revision: Your paper has merit, but significant issues must be addressed before a final decision can be considered. In this case, minor edits will likely be insufficient for addressing the editor and reviewer concerns, so a point-by-point response is recommended. Not thoroughly addressing all editor comments may result in a subsequent reject or reject and resubmit decision; two major revision decisions are rare.

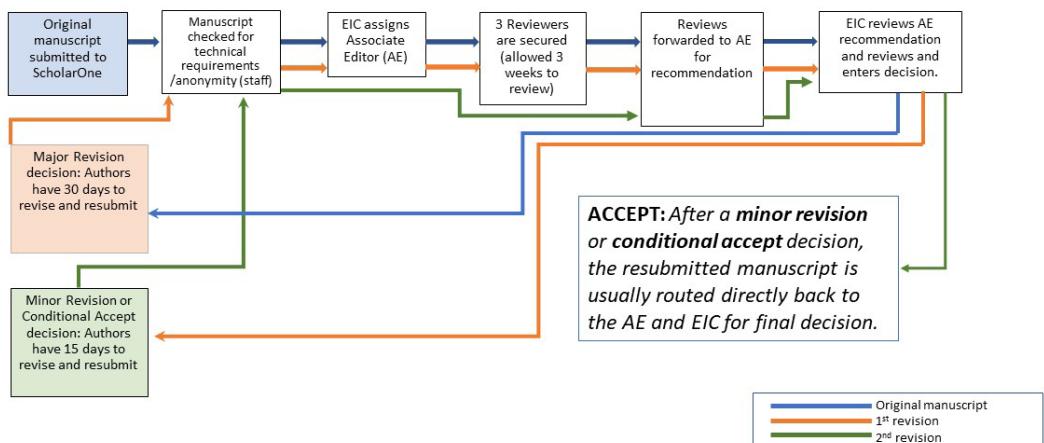
Reject with Encouragement to Resubmit: The concepts or activities in your paper are of interest, but the framing of your argument might need reworking, more classroom activity may be needed to support your conclusions, or other critical issues may need to be addressed before publication can be considered. Reject & Resubmit decisions essentially provide you with an opportunity to “revise” your paper under a new manuscript number. Upon resubmission, your paper will typically be assigned to the same Associate Editor but will have new reviewers.

Reject: Reject decisions are not uncommon. It is important to remember that a reject decision does not mean your paper is without merit or lacking a path toward publication. A reject decision is often not a reflection of the quality of the paper, but rather, for example, a lack of fit within the journal’s mission and scope, or the impression during review that the conclusions or assertions were not documented through descriptions of classroom implementation. At *MTLT*, we hope authors will view a reject decision as a challenge to improve their article, for potential new submission to *MTLT* or to another publication.

MTLT will consider previously rejected manuscripts when authors take into account the issues raised in peer review and create a new manuscript that reflects that feedback. If you submit a new manuscript that dovetails from a previous reject decision, you should cite the previous manuscript in your cover letter and explain how the new manuscript addresses critical feedback from the review process. You will upload the manuscript as a new submission in ScholarOne. Your paper will be assigned to a new associate editor and reviewers.

Below is a typical workflow for a new submission to *MTLT* that is eventually accepted:

Typical Workflow for Accepted Papers: Regular Articles





AFTER ACCEPTANCE

Should your manuscript be accepted, the ScholarOne system will move your manuscript into the “**Manuscripts Accepted for First Look**” area of your Author Dashboard soon thereafter. Approximately one or two weeks after acceptance, you will receive an email from the editorial office that provides instructions for uploading the files and forms that are needed for production. Please try to upload your final files within one week.

Production

Your accepted manuscript will be copyedited in accordance with [NCTM style guidelines](#), which are based on the *Chicago Manual of Style, 17th ed.* (2017). After copyediting, your manuscript will be composed into page proofs that represent how the published article will appear in print. You will receive a PDF copy of these proofs by email with instructions on returning your changes to the production vendor. Please be sure to adhere to the requested 48-hour turnaround time for review of these proofs. Please note that online supplemental material files are not handled by our compositors; they are published online as they were submitted. For that reason, page proofs for supplemental materials are not included in the article proofs sent to authors.

Appendix 1. Digital Asset Possibilities for MTLT

Background

Often, when we mention digital assets, we immediately think of classroom videos. Classroom videos can be very useful, but there are many other possibilities for representing classroom instruction and other digital assets in *MTLT*.

Below are some types of digital assets that could be included in an *MTLT* article. We also provide sample articles from *MTLT* showcasing each media type.

Examples of digital assets

Consider using a digital mathematics environment (DME), video(s) of students working and discussing, screencast/Livescribe student voices without faces while working and discussing, virtual manipulatives, myNCTM Discussion Board prompts, video or audio to pose a task, or innovative and dynamic navigation/organization.

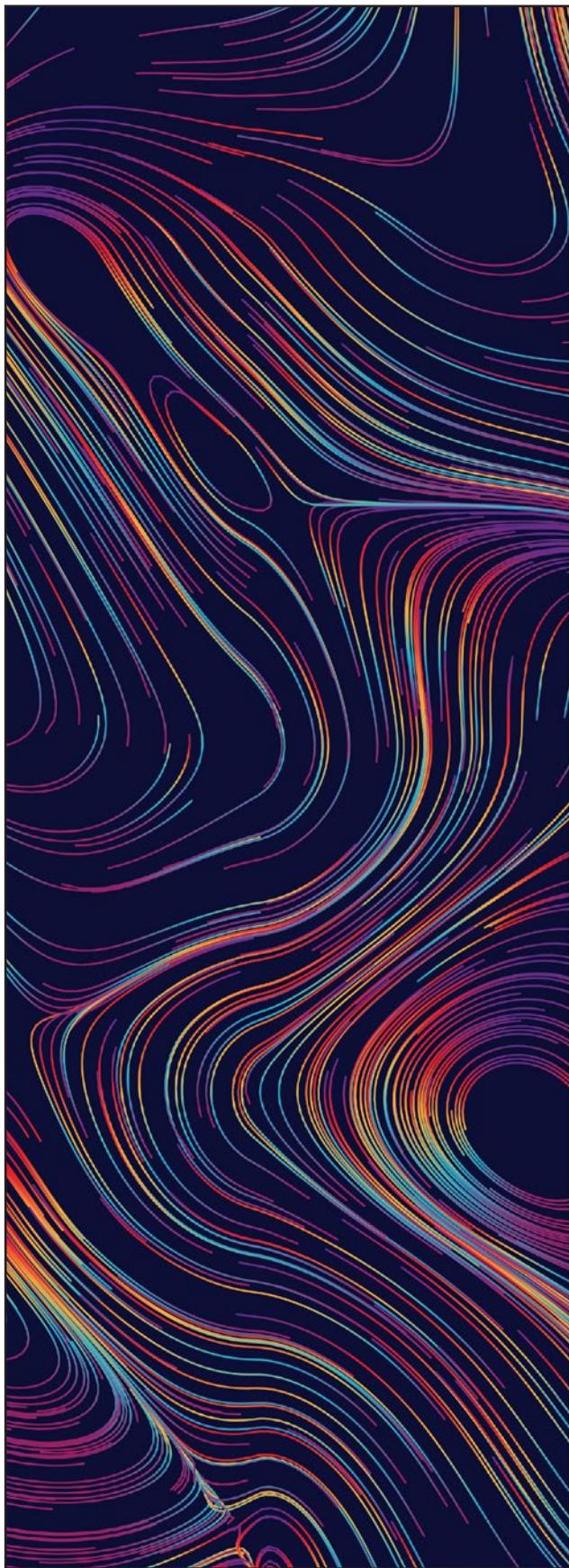
Digital Asset	Description of Example(s)	Example Article(s)
Video Clip	<p>Video modeling force</p> <p>Video with students engaged with activity in a computer lab</p> <p>Integrate YouTube video with gravity applied in space to make a realistic connection.</p>	<p>Exploring the Mathematics of Gravity</p> <p>Authors: Debasmita Basu, Nicole Panorkou, Michelle Zhu, Pankaj Lal, Bharath K. Samanthula</p> <p>Vol. 113, No. 1, January 2020</p> <p>doi: https://doi.org/10.5951/MTLT.2019.0130</p>
Audio File	Used as part of an activity	<p>Modeling a Bouncing Ball with Exponential Functions and Infinite Series</p> <p>Author: Tim Erickson</p> <p>Vol 113, No. 3, March 2020</p> <p>doi: https://doi.org/10.5951/MTLT.2019.0042</p>
Desmos Activity	Experiment with equations of sinusoidal graphs	<p>Envelope Curves Unify Sinusoidal Graphing</p> <p>Authors: Christopher Harrow, Nurfatimah Merchant</p> <p>Vol. 113, No. 4, April 2020</p> <p>doi: https://doi.org/10.5951/MTLT.2019.0129</p>



Appendix 1

(continued)

Digital Asset	Description of Example(s)	Example Article(s)
myNCTM Discussion Board Prompts	Engage with readers about your article by directing them to the myNCTM Discussion Board for continued conversation.	What's in a Name? Language Use as a Mirror into Your Teaching Practice Authors: Tracy E. Dobie, Miriam Gamoran Sherin Vol. 113, No. 5, May 2020 doi: https://doi.org/10.5951/MTLT.2019.0296
Interactive Simulation and Video	Use of Projectile Motion allows students to work with several variables that affect the launch of a cannonball, including changing the angle of launch, the height of the cannon, the speed of the ball, the mass of the cannonball, and the distance of the target on the ground from the cannon.	Let's Hit the Target Authors: Manouchehri Azita, Ozturk Ayse, Sanjari Azin Vol. 113, No. 5, May 2020 doi: https://doi.org/10.5951/MTLT.2019.0022
Virtual Manipulative or Game	Use of Mastermind, an online game in which the transition toward conditional reasoning is seen as a learning progression and not a cognitive leap.	Conditional Reasoning Online with Mastermind Authors: Sean P. Yee, George J. Roy, LuAnn Graul Vol. 113, No. 5, May 2020 doi: https://doi.org/10.5951/MTLT.2019.0093
Livescribe Pen Files (an Alternative to Video)	Use of Livescribe to demonstrate students' tendency to rely on familiar procedures and ideas instead of taking time to think about and analyze a problem situation.	Addressing the Hammer-and-Nail Phenomenon Author: Kien H. Lim Vol. 113, No. 6, June 2020 doi: https://doi.org/10.5951/MTLT.2019.0018



PUBS.NCTM.ORG

Access digital content at
nctm.org/mlt11405fc.

The Nature of Mathematics: Let's Talk about It

Teachers can offer opportunities for K–12 students to reflect on the nature of mathematics (NOM) as they learn.

Lucy A. Watson, Christopher T. Bonnesen,
and Jeremy F. Strayer

Most mathematics teachers have an answer for the following student question. Some teachers like answering it; others hate it: “Why do I need to know _____?” Fill in the blank with any mathematical concept or procedure, and we can all imagine students asking it. So, what is *your* answer? Some answers that teachers give focus on practical applications to science, technology, engineering, and mathematics education fields or different real-life scenarios. Other answers bring out the beauty and wonder in mathematics. Whatever your response, we, the authors, contend that every answer to this question is influenced by the answerer’s view of the *nature of mathematics* (NOM).

What should teachers help students come to understand about NOM? The standards, documents, and recommendations that guide the mathematics education profession are motivated by a relatively common view of NOM, but these documents do not clearly communicate what that view *is*. Even more, they do not give recommendations for what we should be teaching our students concerning NOM. In this article, we present a brief description of the different views of NOM, share a five-point view of NOM that undergirds our profession’s

guiding documents, and describe ways of providing opportunities for teachers and students to have conversations in the classroom that build understanding of NOM.

VIEWS OF THE NATURE OF MATHEMATICS

Historically, that different people have different views about mathematics is widely accepted. That is to say, if you posed the question, “What is mathematics?” to students, teachers, or mathematicians, you will likely receive different answers (Dossey 1992). From a teaching perspective, considering different views is critical, because a teacher’s view of NOM can influence instruction in the classroom and potentially students’ views of NOM. Thompson (1992) provided three descriptions of NOM as it relates to teachers’ views. First, teachers can view mathematics as a dynamic, problem-driven discipline in which NOM is defined as being creative, open to revision, and a product of creativity and inquiry. Second, teachers may hold the view that mathematics is a static-unified body of knowledge in which NOM is defined as being bound by truths that are discovered and never change. Finally, teachers may see mathematics as a bag of tools in which NOM is defined as a set of rules and facts to be memorized. Imagine three different classrooms, each with a different teacher, and each teacher with a different one of these three views of NOM. In these three classrooms, the approach to teaching mathematics will be different because these teachers hold different views of NOM. Also, what students are learning about NOM will be different because the students are exposed to very different approaches to mathematics. For example, a teacher who holds the view that mathematics is a bag of tools will likely use lecture, example, and practice as a main teaching strategy, producing students

who hold the belief that mathematics is computation. Whereas a teacher who holds the view that mathematics is a dynamic, problem-solving discipline will allow students to explore the mathematics, provide an argument, and then engage in discussion allowing students opportunities to build their understanding without being told exactly how to proceed. These different, often opposing, views of NOM produce different classrooms and different learning outcomes.

NATURE OF MATHEMATICS AND THE STANDARDS

Although individuals can hold different views regarding NOM, the mathematics education community can look to the professional standards for insight into how to attend to the nature of the subject we teach. While providing insight into the effective teaching and learning of the subject, mathematics education standards contain no explicit guidance for teaching and learning NOM. Perhaps it is because there was broad agreement on NOM from the start, or maybe it was an effort not to perpetuate the Math Wars, but mathematics educators decided not to give guidance on teaching NOM in the guiding documents that have shaped our profession for the last 30 years. We think this should change, and we detail what we see as a relatively common view of NOM in these documents.

The Five-Point View of the Nature of Mathematics

We can see how a particular view of NOM has influenced the field of mathematics education by exploring our field’s foundational documents: NCTM’s (2000) Process Standards, the Common Core’s (NGA Center and CCSSO 2010) Standards for Mathematical Practice (SMP), NCTM’s (2014) Mathematics Teaching Practices

Lucy A. Watson, lucy.watson@belmont.edu, is an assistant professor in the Mathematics and Computer Science Department at Belmont University in Nashville, Tennessee. She is interested in the nature of mathematics and how it influences the teaching and learning of mathematics for various populations.

Christopher T. Bonnesen, ctb4d@mtmail.mtsu.edu, is a doctoral student in the mathematics and science education PhD program at Middle Tennessee State University in Murfreesboro. He is interested in the nature of mathematics and teaching through problem solving.

Jeremy F. Strayer, jeremy.strayer@mtsu.edu, is a professor of mathematics education at Middle Tennessee State University in Murfreesboro. He has an interest in creating opportunities for K-16 teachers to reflect on the nature of mathematics as they implement standards-based teaching practices.

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(MTPs), the Association of Mathematics Teacher Educators (AMTE 2017) standards for preparing teachers, and the National Research Council's (NRC 2001) strands for mathematical proficiency. These guiding documents are based on research, describe functional ideas about mathematics, and are central in their influence of teachers and students. However, they focus on ideas about the teaching, learning, and the discipline of mathematics together. For example, the Process Standards and SMP focus on actions of students doing mathematics in a classroom, MTPs focus on strategies for teachers, AMTE's standards focus on programs, and the strands for mathematical proficiency relay outcomes for students who have learned mathematics successfully.

A further analysis of these documents by Watson (2019) resulted in a list of commonalities regarding NOM (see figure 1). We propose using this list as a foundation for providing opportunities for students to learn about NOM.

To understand the purpose of the five-point view of NOM, distinguishing it from the standards and professional documents from which they are derived is important. The standards documents offer details about how to effectively teach mathematics (MTPs; NCTM 2014), what students should be doing when learning mathematics (SMP; NGA Center and CCSSO 2010), characteristics that create a productive mathematics classroom (Process Standards, NCTM 2000), and characteristics of mathematically proficient students

(Strands for Mathematical Proficiency, NRC 2001). The five-point view combines what is implied within these standards documents to explicitly state the characteristics of mathematics as a discipline (i.e., NOM) and not necessarily how it is related to the teaching and learning of mathematics. See video 1 for more details about this distinction.

Although the five-point view is distinct from standards documents, noting the connections among them can be useful when considering how the five-point view can be used in classroom instruction. For example, let's specifically consider connections with SMP. When students *look for and express regularity in repeated*

Video 1 How Are the Nature of Mathematics and Professional Standards Different?



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Fig. 1

The Five-Point View of NOM

1. Mathematics is a product of the exploration of structure and patterns.
2. Mathematics uses multiple strategies and multiple representations to make claims.
3. Mathematics is critiqued and verified by people within particular cultures through justification or proof that is communicated to oneself and others.
4. Mathematics is refined over time as cultures interact and change.
5. Mathematics is worthwhile, beautiful, often useful, and can be produced by each and every person.

The five-point view of NOM is distilled from the field's foundational documents.

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reasoning, they have the opportunity to understand that mathematics is a product of the exploration of structure and patterns and is refined over time as cultures interact and change. When students *make sense of problems and persevere in solving them*, they have the opportunity to develop the idea that mathematics is worthwhile, beautiful, and often useful and can be produced by each and every person. When students *model with mathematics*, they have the opportunity to understand that mathematics uses multiple strategies and representations to make claims and that mathematics is critiqued and verified by people within particular cultures through justification or proof that is communicated to oneself and others. In fact, modeling tasks usually have opportunities to promote all five of the characteristics of NOM.

We believe that teachers can use the five-point view of NOM to help each and every student refine and deepen their understanding of NOM. In the remainder of this article, we share a teaching strategy for using this list while teaching with existing rigorous mathematical activities to help students build well-rounded understandings of NOM. As we describe these lessons, we offer a shared language for discussing NOM with the hope that teachers will use these strategies to draw students' attention to NOM and create opportunities for them to refine and deepen their understandings of NOM.

TEACHING AND LEARNING THE NATURE OF MATHEMATICS

We begin this section by describing a four-part strategy (see figure 2) for how teachers and students can reflect on NOM while completing mathematics tasks. Then we show how that strategy can be enacted during mathematics lessons at four different grade levels. The focus in all instances is on the strategy for explicitly reflecting on NOM.

When using this teaching strategy, the teacher must first choose a rich mathematics task with the potential to bring out aspects of the five-point view categories and that aligns with their mathematical goals. Prior to the lesson or unit, the teacher should make notes regarding characteristics of the five-point view that the task allows the teacher to emphasize. Second, at the beginning of a lesson or unit, the teacher provides an opportunity for students to explicitly reflect on their own views of NOM through a prompt (such as a think-pair-share or an entrance ticket). Third, as students complete different mathematical tasks, the teacher may wish to prompt students periodically to reflect on their views of NOM. Only prompt NOM reflection if it is natural—do not force it—as students can tire of NOM conversation if it is a constant focus. Fourth and finally, the teacher should share or reshare the five-point view of NOM and ask students to reflect on how, if at all, their views of NOM have deepened or changed since the beginning of the lesson.

Fig. 2

Four-Part Strategy for Teaching NOM

1. Select mathematical tasks that will promote one or more aspects of the Five-Point View of NOM.
2. Provide students with an opportunity to reflect on and communicate their personal views of NOM at the beginning of a lesson or unit. Consider sharing the Five-Point View here if students need a starting point.
3. Launch the lesson or unit, monitor students' progress as you normally would, and note statements and student work that relates to NOM. If a natural moment arises, ask students to reflect on their NOM views as they complete tasks.
4. At the close of a lesson or unit, share the five-point view and provide students with an opportunity to reflect on and refine their views of NOM.

This four-part strategy for teaching NOM is useful for providing opportunities for students to reflect on NOM.

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or unit. As in the beginning of the strategy, students should share their thinking through a discussion, exit ticket, or some other means.

Research in science education has taught us that students must have opportunities to explicitly reflect on aspects of the nature of science while engaging in science practices to develop understandings of the nature of science (Schwartz, Lederman, and Crawford 2004). Incorporating the language of NOM into our everyday, common, good teaching practices is important because we are bringing a shared language to students and teachers alike in what otherwise would remain unnamed. Although this four-part strategy seems similar to everyday, good, common teaching practices, the intentional addition of NOM and the importance of *explicit reflection* set it apart. Furthermore, the four-part strategy emphasizes the time needed for students to deepen their understanding of NOM—it can start with a lesson, but it should be spoken of and refined over multiple lessons, units, semesters, or years.

We now briefly share examples of how this four-part strategy may be enacted during lessons at several grade levels. Any high-quality task can be used for these lessons, and we chose four from NCTM's (2020) collection of activities with rigor and coherence (ARC): counting strategies (K), equivalent fractions (grade 3), discovering area relationships (grade 6), and absolute value (high school algebra). For more mathematical details about each activity, follow the provided links. We present short descriptions of the lessons, drawing on our collective experiences with teachers and students to provide insights into how those enactments may unfold in varying grade levels. Although four examples are provided, the reader should focus on the example(s) that are of interest based on grade level. It is not necessary to read all four.

- For an example of this NOM teaching strategy in a PK–2 lesson, go to the section below.
- For an example of this NOM teaching strategy in a 3–5 lesson, go to p. 357.
- For an example of this NOM teaching strategy in a middle school lesson, go to p. 357.
- For an example of this NOM teaching strategy in a high school lesson, go to p. 358.

Counting in Kindergarten ARC Activity

The mathematical goal of this lesson is to have students make connections among their understandings of quantity and numeral. As teachers plan, they may notice how even the idea of a number is like a *pattern*.

If one has four snap cubes, four crayons, four shells, and so on, then in all cases, the quantity itself is like a pattern that points to mathematical *structure* (statement 1 of the five-point view), regardless of substance or shape. Likewise, the fact that each number can be represented by a quantity of several objects, or even as a point on a number line points to *multiple representations* (statement 2 of the five-point view). Moreover, students and the teacher can work collaboratively in this lesson to *critique* and *verify* claims, such as explicitly counting out loud to prove which symbol goes with which quantity, in alignment with statement 3. Finally, this activity is accessible to *all students* because of its concrete, hands-on nature, in alignment with statement 5. Therefore, this lesson is ideal for explicitly reflecting on NOM with students.

The second part of the strategy is to offer students a prompt to reflect on NOM just prior to completing the activity in class. For kindergartners, we want to keep in mind that students will respond in concrete ways. When the teacher begins by asking, “What is math?” students may respond by saying numbers, counting, telling time, matching patterns, or by drawing on other subjects they are learning. We can see these ideas expressed in an interview with a six-year-old student when we asked her, “What is math?” (see video 2). She provides a concrete idea of what she thinks mathematics is by stating, “Math is like this, two minus five

Video 2 A Student Explaining, “What Is Math?”



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equals what?" She also describes mathematics as patterns when she tells us about the upper- and lower-case letters. Though this student is at the beginning of her mathematics journey, she is able to describe what she thinks mathematics is, and more importantly, she is being asked that question. If teachers continue to ask her the same question, what is mathematics, over the years, she will have abundant opportunities to reflect on and refine her view of NOM.

Next, the teacher implements the activity, giving students an opportunity to engage in mathematical thinking so they can achieve the goals of the lesson. Finally, the teacher will debrief with students at the end of the lesson by helping them see how they engaged in aspects of the five-point view using concrete terms that their kindergartners can understand. They may help students verbalize that their experiences showed that mathematics is about numbers, counting, finding how many, seeing patterns, and other ideas. In this way, teachers and students can form the habit of thinking "What is this that I'm doing?" at a very young age.

- For a grades 3–5 example, go to the section below.
- For a middle school example, go to the middle of the next column.
- For a high school example, go to p. 358.
- To go to the discussion, go to p. 359.

Equivalent Fractions in Grade 3 ARC Activity

This four-part lesson focuses on the concept of fraction, comparing fractions, and determining equivalent fractions. When planning to use this lesson for reflecting on NOM, the teacher may first observe several connections to the concept of *structure*. Students learn about the overall structure of a fraction and explore the meanings of and relationships among the numerator, denominator, and whole unit. Students also use Cuisenaire® Rods to better understand how the size of a fraction is always relative to the whole unit, and they learn how to connect the Cuisenaire Rods to the symbols for fractions and the number line. Taken together, the experiences in this lesson offer opportunities for students to develop understanding of the *structure* of fractions (statement 1 of the five-point view), use *multiple strategies* and *multiple representations* (statement 2), and collaborate in a hands-on setting to *critique and justify* their claims (statement 3) while *producing their own mathematical knowledge* (statement 5). Therefore, this activity is a great opportunity for students to reflect on NOM.

Just prior to beginning this activity, students will need to consider a prompt such as "In your own words, what is mathematics?" This would be a great prompt for a written mathematics journal so that students could revisit their responses and keep a record of how their thinking has changed from time to time throughout the year. Next, students complete the activity in class. The teacher may wish to pick and choose parts of this thorough, four-part activity, on the basis of students' prior knowledge and the teacher's instructional goals. At the conclusion of the activity, the teacher displays and reads the list of NOM statements, asks students to consider their personal NOM statements from prior to the lesson, and asks students to write down how their views of what mathematics is have changed. Ideally, this process will engender sophisticated views of NOM beyond the idea that mathematics is about numbers, formulas, following steps, getting answers, and the like.

- For a PK–2 example, go to p. 356.
- For a middle school example, go to the section below.
- For a high school example, go to p. 358.
- To go to the discussion, go to p. 359.

Discovering Area Relationships in Middle School ARC Activity

This lesson uses several examples of rectangles, triangles, and parallelograms to help students see the underlying *structure* and *patterns* that give rise to the area formulas for those shapes, in accordance with statement 1 of the five-point view. Additionally, *multiple representations* are emphasized in the lesson, which prompts students to find areas using many different strategies, in support of statement 2 of the five-point view. Finally, the lesson provides opportunities for students to *communicate their ideas* to others as they work in groups and contribute during whole-class discussions (statement 3 of the five-point view). In these ways, *all students have the potential to produce mathematical knowledge* due to the hands-on and concrete nature of the activity, especially during the initial phases (statement 5 of the five-point view).

In part 2 of the teaching strategy, the teacher should offer students an opportunity to reflect on their view of what mathematics is. The teacher may ask for a response to the following question through an entrance ticket: *Different people describe mathematics in different ways. In your own words, answer the question "What is*

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mathematics?" Students may draw on their own experiences when initially asked, so potential answers may include the following: numbers, shapes, solving problems, or equations.

In part 3 of the strategy, the teacher must encourage students to explore the conceptually connected tasks in small groups and participate in brief whole-class discussions after each task. By using multiple strategies, patterns, and representations, students will have opportunities to build a deep understanding of how the area formulas for triangles and rectangles relate. This will prepare them to solve area problems with all types of triangles, including those with heights that are difficult to detect. Finally, students can consider all of these strategies together to explore the area formula for parallelograms.

At the conclusion of the lesson, the teacher may choose to share the five-point view statements with the class and give students an opportunity to briefly discuss their reactions. One possible ending is to use another prompt through an exit ticket: *Think again about how you would describe what mathematics is in your own words. What would you change, add to, or further clarify in what you wrote at the beginning of class? Expand on your thoughts.*

We enacted an adaptation of the grade 6 lesson in a geometry course for preservice elementary school teachers. You may view this lesson and student responses in video 3.

- For a PK–2 example, go to p. 356.
- For a grades 3–5 example, go to p. 357.
- For a high school example, go to the top of the next column.
- To go to the discussion, go to p. 359.

Video 3 An Adapted Grade 6 Lesson for a Geometry Class



 Watch the full video online.

Absolute Value in High School ARC Activity

When going through part 1 of the four-part strategy, the teacher will see that this lesson on absolute value and functions highlights the categories of *patterns, structure, multiple strategies, justification, beauty, and multiple representations*, in alignment with statements 1, 2, 3, and 5 of the five-point view of NOM. The lesson begins by asking students to use a Dynagraph applet (which shows different linear and absolute value function relationships by dynamically linking inputs to outputs on two separate number lines) to determine, express, and justify their mathematical description of those functions. Students continue to discover *patterns* and develop a deeper understanding of the *structure* of the idea of absolute value by comparing several graphs. The lesson ends by asking students to build understanding of what it means to solve absolute value equations for x by using Dynagraphs, traditional function notation, and traditional graphs. The lesson gives students opportunities to use *multiple representations* that beautifully connect function representations to build understandings of the similarities and differences between several forms of linear and absolute value functions. Therefore, this activity is excellent for exploring NOM through the focus of the five-point view.

In part 2 of the NOM strategy, the teacher should offer an opportunity for students to reflect on their view of what mathematics is. Because this lesson involves functions, graphs, and equations, it may be advisable to ask the more specific entrance prompt "*In your own words, describe what algebra is.*" Students often draw on their recent experiences when initially answering NOM prompts, so potential answers may include *mathematics with letters in it, equations, or solving for x*.

The teacher should implement the lesson, encouraging students to explore and share their mathematical thinking in the third part of the NOM teaching strategy. In part 4, the teacher should ask students to reflect on the five-point view and then communicate their thoughts during a final prompt such as *Think again about how you would describe what algebra is in your own words. What would you change, add to, or further clarify in what you wrote at the beginning of class? Expand on your thoughts.* The teacher may see that students expand their notion of algebra to include *relationships between inputs and outputs, connections between formulas and graphs, or multiple representations of functions*. In this way, NOM prompts can be more specific, focusing on the nature of a particular subject within mathematics.

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- For a PK–2 example, go to p. 356.
- For a grades 3–5 example, go to p. 357.
- For a middle school example, go to p. 357.
- To go to the discussion, go to the section below.

DISCUSSION

We believe that each of the preceding lessons representing existing NCTM materials can promote aspects of NOM, though they do not currently do so explicitly. Our goal is not to reinvent the wheel but to show teachers and the education community at large that rigorous, meaningful tasks can allow students to easily attend to aspects of NOM. These tasks not only allow students to work together, explore the mathematics, and share their thinking but also offer good questions teachers can ask students that will help promote ideas about NOM. To illustrate this idea further, in figure 3, we present a few of the questions that are included in the lesson plans from the ARCs we used in the earlier tasks (K, grade 3, grade 6, and high school) and connect them to how these questions can open space in a classroom for reflection on NOM. Each of these questions serves as an example of the type of question that can help advance one's views of NOM because the questions draw attention to specific characteristics from the five-point view of NOM.

Asking the types of questions in the center column of figure 3 alone is not enough to guarantee that students advance their view of NOM, just as students engaging in the SMP is insufficient on its own. The combination of these already high-quality tasks coupled with the practice of sharing the language of NOM and asking students to explicitly reflect on NOM can bring about refinement of students' ideas regarding the NOM.

CONCLUSION

In this article, we demonstrate how to teach lessons that give students opportunities to refine and deepen their understandings of NOM while they learn mathematics at multiple grade levels. All that teachers need is the five-point view of NOM, a high-quality mathematics task to implement (e.g., ARCs), and prompts for explicit reflection on NOM (e.g., entrance and exit slips). For NOM reflections to meaningfully affect students' understandings of mathematics will require repetition over time—preferably over multiple years. As the student reflections from the grade 6 lesson showed earlier, progress is incremental and unfolds slowly.

We note here that as teachers and students engage in these NOM reflective practices, they may discover the need to add to the five-point view because NOM means additional things within their learning community. Classes' understandings of NOM should reflect what mathematics means according to their experiences and contexts. Teachers can use additions judiciously to further help students understand that mathematics is a growing, vibrant, and creative space that is open to each and every person.

Finally, as students refine and deepen their understandings of NOM, perhaps they will ask the question “Why do I need to know _____?” less frequently. Or, perhaps they will develop their own insightful answers to this question on the basis of a deepened understanding that exploration, structure, strategy, justification, communication, refinement, and beauty are at the heart of mathematics. Understanding that mathematics is about more than application to other fields will serve students well. Regardless, we hope that this article helps everyone in the mathematics education community take up a shared language with which to discuss and build understanding together of NOM. [_____](#)

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Fig. 3

Lesson	Question Asked in ARC	Connection to the Five-Point View
Kindergarten	<p>A. If we rearrange the cubes and count again, what will happen to the total?</p> <p>B. What number is one more (one less) than this? How do you know?</p> <p>C. What is the connection between the last number name said and the number of objects counted?</p>	<p>A. Mathematics is a product of the exploration of structure and patterns.</p> <p>B. Mathematics is critiqued and verified by people within particular cultures through justification or proof that is communicated to oneself and others.</p> <p>C. Mathematics is a product of the exploration of structure and patterns.</p>
Grade 3	<p>A. What fraction name can we give each piece? Why do we give it that name?</p> <p>B. How do the unit fractions we created using the rods help us understand numbers less than one whole? Will these numbers help us solve problems in our everyday lives?</p> <p>C. What numbers did you place with the one whole? Why? What do you notice about the structure of fractions at the one whole mark?"</p>	<p>A. Mathematics is refined over time as cultures interact and change.</p> <p>B. Mathematics is worthwhile, beautiful, often useful, and can be produced by each and every person.</p> <p>C. Mathematics is critiqued and verified by people within particular cultures through justification or proof that is communicated to oneself and others.</p>
Grade 6	<p>A. How do the areas of resulting shapes compare to the area of the original shape?</p> <p>B. Can there be a formula for the area of a parallelogram that is only in terms of length of the sides (length times width)? Why?</p>	<p>A. Mathematics uses multiple strategies and multiple representations to make claims.</p> <p>B. Mathematics is a product of the exploration of structure and patterns.</p>
High School	<p>A. What do you notice about the graphs where $x > 0$? Where $x < 0$?</p> <p>B. Is there similar behavior when graphing other types of parent functions?</p> <p>C. How is a solution to a system of equations represented graphically? How can we find solutions algebraically?</p>	<p>A. Mathematics is a product of the exploration of structure and patterns.</p> <p>B. Mathematics is critiqued and verified by people within particular cultures through justification and/or proof that is communicated to oneself and others.</p> <p>C. Mathematics uses multiple strategies and multiple representations to make claims.</p>

Each ARC activity contains questions that provide opportunities to connect to the five-point view of NOM.

REFERENCES

- Association of Mathematics Teacher Educators. 2017. "Standards for Preparing Teachers of Mathematics." amte.net/standards.
- Dossey, John. 1992. "The Nature of Mathematics: Its Role and Its Influence." In *Handbook of Research in Mathematics Teaching and Learning*, edited by Douglas A. Grouws, pp. 39–48. New York: Macmillan.
- National Council of Teachers of Mathematics (NCTM). 2000. *Principles and Standards for School Mathematics*. Reston, VA: NCTM.
- National Council of Teachers of Mathematics (NCTM). 2014. *Principles to Actions: Ensuring Mathematical Successes for All*. Reston, VA: NCTM.
- National Council of Teachers of Mathematics (NCTM). 2020. "Activities with Rigor and Coherence." <https://www.nctm.org/ARCs/>.
- National Governors Association Center for Best Practices (NGA Center) and Council of Chief State School Officers (CCSSO). 2010. *Common Core State Standards for Mathematics*. Washington, DC: NGA Center and CCSSO. <http://www.corestandards.org>.
- National Research Council (NRC). 2001. *Adding It up: Helping Children Learn Mathematics*. Washington, DC: National Academies Press.
- Schwartz, Renee, Norman Lederman, and Barbara Crawford. 2004. "Developing Views of Nature of Science in an Authentic Context: An Explicit Approach to Bridging the Gap between Nature of Science and Scientific Inquiry." *Science Education* 88, no. 4 (May): 610–45.
- Thompson, Alba. 1992. "Teachers' Beliefs and Conceptions: A Synthesis of Research." In *Handbook of Research on Mathematics Teaching and Learning*, edited by Douglas A. Grouws, pp. 127–46. New York: Macmillan.
- Watson, Lucy Anna. 2019. "Elementary Prospective Teachers and the Nature of Mathematics: An Explanatory Phenomenological Study." PhD diss., Middle Tennessee State University.

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FEATURE

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Eight Unproductive Practices in Developing Fact Fluency

Basic fact fluency has always been of interest to elementary school teachers and is particularly relevant because a wide variety of supplementary materials of varying quality exist for this topic. This article unpacks eight common unproductive practices with basic facts instruction and assessment.

Gina Kling and Jennifer M. Bay-Williams

Ensuring students master their basic facts is a shared priority among teachers, parents, and administrators. As a result, the number of available resources for basic facts instruction, from concrete to virtual, is vast and continues to grow. Yet, many resources do not reflect the vision expressed in *Catalyzing Change in Early Childhood and Elementary Mathematics* (NCTM 2020) for ensuring each and every student has access to mathematics. In line with NCTM recommendations, basic

fact fluency can and must be developed in mathematics environments that emphasize “curiosity, flexibility, and wonder” (NCTM 2020, p. 82). Long-standing methods for teaching basic facts have not been effective for far too many students. Now, more than ever, it is important not to succumb to the allure of “quick fix” programs like rote drill online applications. Instead, it is time to acknowledge common unproductive practices and look to effective alternatives.

UNPRODUCTIVE PRACTICE 1

Not encouraging every student—including those who already know their facts—to learn reasoning strategies

Example. A student uses a finger-trick to recall their 9s facts but does not learn a strategy such as subtracting a group from the related 10s fact (e.g., $9 \times 7 = 10 \times 7 - 7$).

Why it is unproductive. First, tricks like the finger patterns may get to an answer, but they deny students an opportunity to reason quantitatively. Learning the Subtract-a-Group strategy not only helps students learn their 9s facts but also extends to different types of numbers. For example—

- larger values: $99 \times 7 \rightarrow$ think 100 groups of 7 is 700 and just subtract a group of 7 to get 693.
- decimal values: $7.9 \times 5 \rightarrow$ think $8 \times 5 = 40$ and just subtract 0.5 (one-tenth of 5) to get 39.5.

In both of these examples, the standard algorithm for multiplication is less efficient than reasoning.

Second, *procedural fluency* includes three components: (1) flexibility, (2) efficiency, and (3) accuracy (NCTM 2014b; NRC 2001). When basic facts are learned through tricks or rote practice and memorization, students do not develop flexibility or efficiency; hence, they do not develop basic fact *fluency*.

Instead. Focus instruction on strategy development. Students need opportunities to reason and access to a variety of efficient strategies. This sets them up for success with procedural fluency for all operations and number types. Table 1 shows how just two addition fact strategies extend to other numbers. Given how useful these two strategies are, what a significant mistake it is for students to not engage with these strategies as they learn their basic facts!

UNPRODUCTIVE PRACTICE 2

Telling a strategy rather than providing explicit strategy instruction

Example. A teacher introduces the Making 10 strategy by telling students to take some from the smaller addend to give to the larger addend before adding. For example, for $8 + 7$, take 2 from the 7 to give to the 8 to make 10, and then add the leftover 5 to get 15.

Why it is unproductive. You cannot simply tell a student to understand. Many studies show that students who use strategies with understanding outperform their peers (Baroody et al. 2016; Locuniak and Jordan 2008; Purpura et al. 2016; Tournaki 2003). Students need time and experiences to see number relationships and develop understanding, exploring representations and connecting them to abstract reasoning (Clements, Fuson, and Sarama 2017; NRC 2009; Van de Walle, Karp, and Bay-Williams 2019).

Instead. Use visuals and contexts in ways that lead students to be the ones discovering and explaining the strategies (how they work and why they work). For each and every child to understand and be able to use strategies, instruction must be explicit. Merriam-Webster.com defines *explicit* as “fully revealed or expressed without vagueness.” One way to reveal a strategy is to use Quick Looks that nudge children to develop a particular strategy (Bay-Williams and Kling 2019). A Quick Look activity involves showing an image briefly (two to three seconds) and then hiding it, asking children to use visualization strategies to describe what they saw. After providing a second look, the class discusses both the number of dots shown and most importantly, *how students saw it*. Carefully pairing Quick Looks can help students use known facts to employ a strategy. Figure 1 shows two examples of paired Quick Looks, one that works for both Making 10 and Pretend-a-10 (addition) and one for Doubling (multiplication). Please see video 1 (link online) for a demonstration on using Quick Looks.

Gina Kling, gina.garza-kling@wmich.edu, is an elementary school mathematics educator at Western Michigan University in Kalamazoo. Kling has a commitment to improving the teaching and assessing of basic fact fluency.

Jennifer M. Bay-Williams, j.baywilliams@louisville.edu, is a professor at the University of Louisville, Kentucky. Bay-Williams has a commitment to improving the teaching and assessing of basic fact fluency.

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Table 1 Basic Fact Strategies Grow into General Reasoning Strategies

Basic Fact Reasoning Strategy	Extended to Multidigit Numbers	Extended to Rational Numbers
<p>Making 10: $9 + 6 \rightarrow 10 + 5$ (Move a quantity from one addend to the other to make a 10.)</p>	<p>Make Tens: $69 + 58 = 70 + 57$ $= 127$ Make Hundreds: $395 + 784 = 400 + 779$ $= 1,179$</p>	<p>Make a whole: $4\frac{7}{8} + 3\frac{5}{8} = 5 + 3\frac{4}{8}$ $= 8\frac{1}{2}$ $8.9 + 6.2 = 9 + 6.1$ $= 15.1$</p>
<p>Pretend-a-10/use 10: $9 + 6 \rightarrow 10 + 6$ $16 - 1 = 15$ (Pretend the 9 is a 10, add, take one off the answer because 9 is one less than 10.)</p>	<p>Compensation: Adjust both numbers: $39 + 28 \rightarrow 40 + 30 - 3$ $\rightarrow 70 - 3 \rightarrow 67$ Adjust one number: $398 + 514 \rightarrow 400 + 514 - 2$ $\rightarrow 914 - 2 \rightarrow 912$</p>	<p>Compensation: Adjust one number: $5.9 + 6.47 \rightarrow 6 + 6.47 - 0.1$ $\rightarrow 12.47 - 0.1$ $\rightarrow 12.37$ Adjust both numbers: $7\frac{7}{8} + 12\frac{7}{8} \rightarrow 8 + 13 - 2\frac{1}{8}$ $\rightarrow 20\frac{3}{4}$</p>

Adapted from SanGiovanni and colleagues (2022).

We have found that when we use Quick Looks, the thinking behind the strategy emerges from students as they describe how they determined the total. As teachers, we can attend to that strategy and invite students to try it on a new Quick Look. After several Quick Looks with a similar theme, the teacher can close the discussion by engaging the class in summarizing and classifying the strategy (i.e., “All of our cards had one ten-frame close to a 10, which made it easier to make 10. So, our Making 10 strategy works well when an addend is close to 10”). In this way, we make strategies

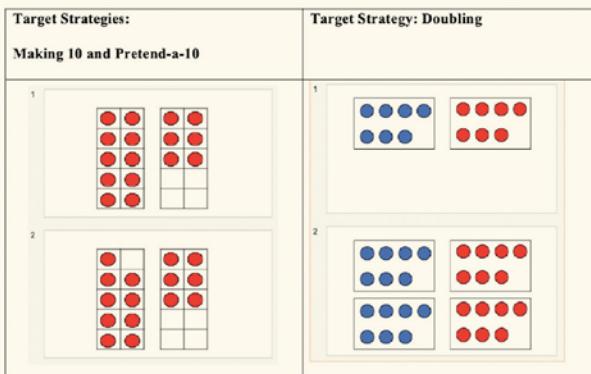
explicit while still preserving the ownership of the strategy by the students.

UNPRODUCTIVE PRACTICE 3

Teaching facts in order of addend or factor size (0s, then 1s, then 2s, etc.)

Example. Having a mastery progress chart, with columns 0–10 is an example of this unproductive practice.

Fig. 1



This figure shows paired Quick Looks to reveal strategies.

This is how most adults remember learning multiplication facts!

Why it is unproductive. The traditional order treats facts as isolated objects, does not build on students' strengths and prior knowledge, and can result in lower achievement (Brendefur et al. 2015; Henry and Brown 2008; Steinberg 1985; Thornton 1978).

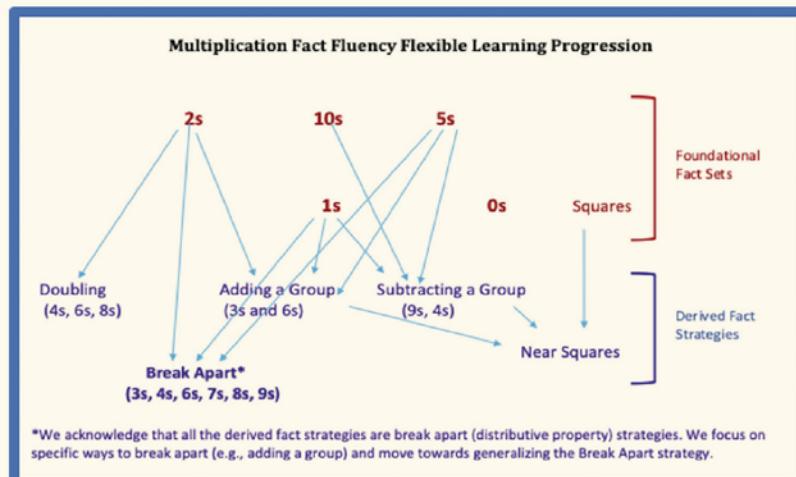
Video 1 A Quick Looks Demonstration



 Watch the full video online.

Instead. Use a research-based, strategy-focused progression (e.g., Baroody 2006; Brendefur et al. 2015; Heege 1985; Steinberg 1985; Thornton 1978). Figure 2 illustrates such a learning progression for multiplication. It begins with two fact groups most familiar to students and also vital to the strategies they will learn: the 2s and 10s. Next are the 5s. Although students can skip-count by 5s, learning instead to take half of the related 10s fact can better encourage fluency. Then with groups understood, 1 group (and 0 groups) make sense, and the 1s and 0s can be mastered with meaning. Next introduce the squares, which are not even represented in a traditional go-in-numerical-order approach, but are a very helpful fact group to know because some of the toughest facts to learn (e.g., 7×8 and 6×7) are close to squares, and squares are useful for later work in algebra, geometry, and measurement. Note the trajectory below is divided into "foundational facts" and "derived facts." The most important aspect of this progression is that foundational facts must be mastered before engaging students in making sense of the Derived Fact strategies. We invite the reader to examine figure 3 and consider how foundational facts are used in the given samples of student work from end-of-year third graders. Such examples illustrate the necessity of mastering foundational facts for developing reasoning strategies.

Fig. 2



This learning progression for multiplication facts demonstrates a research-based, strategy-focused progression. Adapted from Bay-Williams and Kling 2019.

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UNPRODUCTIVE PRACTICE 4

Not using a coherent, multigrade approach to facts instruction

Examples. A first-grade teacher presses for automaticity within 10 at the expense of working on reasoning strategies within 20. A third-grade class focuses on rote memorization as a priority over learning strategies.

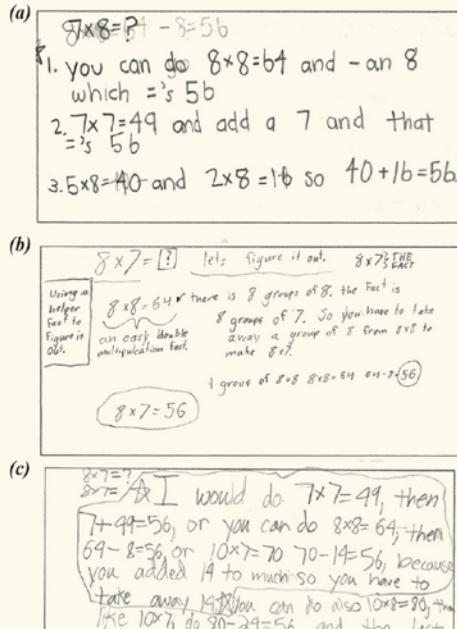
Why it is unproductive. This approach does not work! Too often, teachers at a particular grade level feel pressured to ensure their students have mastered their basic facts by the end of the school year. And yet year after year, *facts memorized through rote techniques do not tend to stick*. As a result, “many educators find that children, even in the upper grades, continue to draw tally marks and count by ones as

their dominant solution approach in solving problems” (NCTM 2020, p. 82).

Instead. Be the tortoise in the fable of the tortoise and the hare. It is better for students’ future learning and their mathematical identities to focus on learning strategies and not feel pressured to memorize. Ensuring students have developed automaticity with foundational facts before moving on to strategy instruction is an important aspect of pacing. Key to this is collaboration across grade levels to create a coherent facts mastery plan grounded in research-based learning progressions (see figure 2 for multiplication and figure 4 for addition). A K-2 plan might elaborate on when the elements of the addition learning progression are developed and when we may reasonably expect automaticity. For example, you may have the goal of “automaticity of all foundational facts by the end of grade 1.” Such a goal helps set up children for success in mastering strategies throughout grade 2 for both addition and subtraction, with the goal of automaticity with addition facts by the end of grade 2.

Mastering multiplication facts often appears as a third-grade standard—a tall order to be sure! To help ease this challenge, children can actually begin exploring multiplication at the end of second grade, learning multiplication through story contexts that involve equal groups (e.g., *A box of granola bars contains five bars. If there are three boxes on the shelf, how many granola bars are there?*). Spending time in grade 2 solving story problems as well as on Quick Looks involving equal groups and arrays (see figure 1 and video 1 [link online]) can help children gain a healthy, developmentally appropriate jump on their learning of multiplication facts.

Fig. 3



These sample responses from third graders to the question, “If your friend didn’t know the answer to 7×8 , how could they figure it out?” illustrate the necessity of mastering foundational facts in developing reasoning strategies.

UNPRODUCTIVE PRACTICE 5

Teaching only the think addition strategy for subtraction facts

Example. Some teachers instruct students to change every subtraction fact to the related addition fact (e.g., change $15 - 9 = ?$ to $9 + ? = 15$) and ask them to recall the addition fact.

Why it is unproductive. Other subtraction strategies warrant attention because they can transfer to computation beyond facts (Steinberg 1985).

Instead. Teach Think Addition and other strategies. There is no question that the inverse relationship between addition and subtraction is a powerful tool to use when developing computational fluency. Research suggests that children tend to be more accurate with addition than subtraction (NRC 2001). Thus, thinking of the related addition facts does build on students' strengths. But balancing this with other strategies supports reasoning and prepares students for work beyond facts. Let's revisit the example of $15 - 9$ and look at other options:

1. Use compensation: Start with $15 - 10 = 5$ then compensate by adding 1 back to get 6.
2. Break apart the subtrahend to go down under 10: $15 - 5 = 10$ and then $10 - 4 = 6$.
3. Break apart the minuend and subtract from 10: $10 - 9 = 1$ and then $5 + 1 = 6$.

Such strategies prepare students to use similar methods with other numbers. For example, for $132 - 99$, a student may think $132 - 100$ and compensate by adding 1, or they may break apart the minuend to $100 + 32$, subtract 99 from 100 (1) and add that to 32 to get

33. Just as with addition facts, providing students with opportunities to learn a variety of subtraction strategies has value far beyond fact work.

UNPRODUCTIVE PRACTICE 6

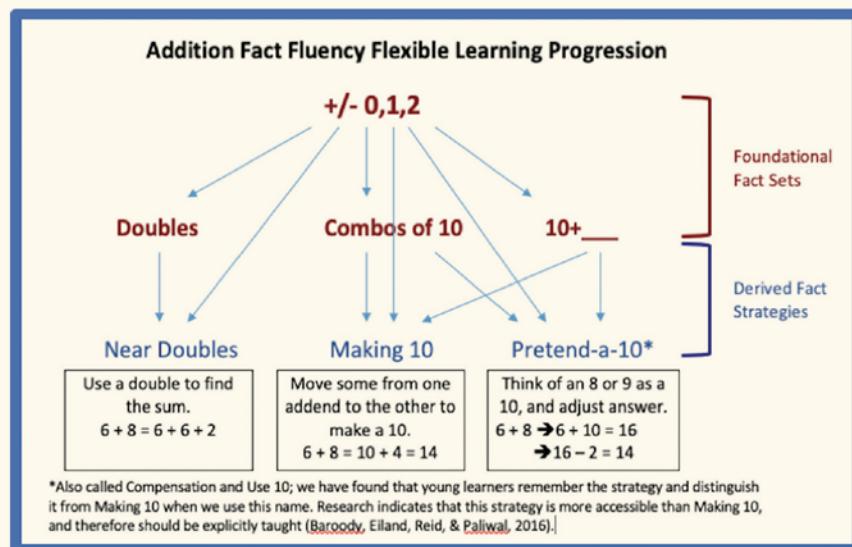
Using traditional forms of fact practice

Example. Using activity sheets that focus on a fact group (e.g., multiplying by 6).

Why it is unproductive. Activity sheets assess only accuracy and thus do not focus on fluency. The message such a practice sends to students is that the memorization of isolated facts is what is important, rather than developing, discussing, and applying strategies. Furthermore, such a practice is distasteful to students. Very few students want to do 30+ problems on a page. That bad taste leads many students to decide they do not like mathematics.

Instead. Fact mastery can be achieved by meaningful experiences with physical games and interactive activities.

Fig. 4



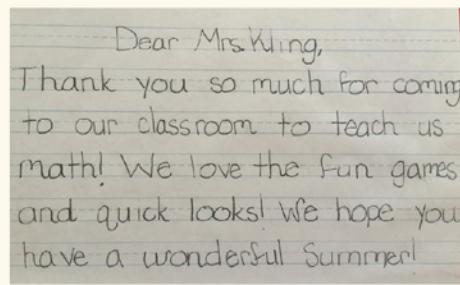
This addition learning progression represents a facts mastery plan grounded in research-based learning progressions. Adapted from Bay-Williams and Kling 2019.

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Unlike activity sheets, computer applications, or flash cards, games offer valuable opportunities for discussion and reflection. Games can focus on a single fact set (e.g., combinations of 10) or a strategy (e.g., doubling), providing developmental support for students. Teachers can use observation during game play to gather more authentic assessment data that address all components of fluency: efficiency, flexibility, accuracy, and appropriate strategy use. Playing facts games with classmates or family members is one of the most beneficial ways to encourage children to master their basic facts and can greatly increase their taste for mathematics, as shown in figure 5.

Fig. 5



An end-of-year thank you note from second graders expresses appreciation for the author's use of facts games.

UNPRODUCTIVE PRACTICE 7

Pressing for speed

Example. Practicing facts by doing a board race or other game in which speed is a factor, including many online applications.

Why it is unproductive. Requiring children to generate answers quickly before they have sufficiently developed and practiced strategies has a tendency to drive fluency development in the opposite direction (i.e., back to counting or skip counting) because mentally implementing a strategy initially takes more time and concentration. In summary, “Fluency does not equal speed” (NCTM 2020, p. 89).

Instead. Choose games that have each student solving a different problem as they take turns. For example, first graders can play a modified version of the classic game Go Fish, in which players look for combinations of two cards that sum to 10 instead of matching pairs. This widely-used game is highly enjoyable for students and provides low-stress, meaningful practice of an important facts group. For multiplication, TRIOS (see figure 6) is one of our favorite games and is easily adapted to other multiplication fact sets or even addition combinations (Bay-Williams and Kling 2019). Note that when you use the TRIOS game board, you are also

Fig. 6

TRIOS: Multiples of Six

18	54	30	0	42
36	24	12	18	48
6	42	0	48	30
48	18	36	12	24
24	6	54	42	36

Materials: 10-sided die [you can also use cards], game board, two different colored counters (or two colors of marker).

Goal: Get as many 3-in-a-row as possible (they can overlap).

How to Play: Player 1 rolls the die, multiplies it by 6 and chooses a square to claim on the game board. Player 2 takes a turn. Players can go for their own ‘trios’ or block their opponent. Play until gameboard is full (or time is up). Players count up their ‘trios’, multiply that number by 6, and high score wins.

This is an example of the TRIOS game with no speed element.

infusing ideas of division as students think of the factor they hope to draw or roll.

UNPRODUCTIVE PRACTICE 8

Using timed testing to assess fluency

Example. Asking third graders to solve 100 multiplication problems in three minutes can be counterproductive.

Why it is unproductive. Timed testing has a long history in American mathematics education, and it is likely that one does not need to look far to find a classroom that still uses it. Despite the history, many reasons exist for why this practice must be eradicated. First of all, if schools value true fluency as defined earlier in this article, one must immediately recognize that timed testing cannot possibly assess flexibility or strategy use. That is not to say children are *not* flexible, or that they are *not* using an appropriate strategy; a timed test simply does not assess these things.

Second, the two parts of fluency that timed tests are supposed to measure (efficiency and accuracy) in truth are not reliably measured. There is nothing stopping students from quickly counting, even though it is not an efficient method. Furthermore, fine motor capabilities in young children affect how quickly they can record answers, thus interfering with efficiency and completion. Finally, the anxiety that many students experience when taking a timed test can hamper their abilities to think clearly, causing them to underperform in such settings (Boaler, Williams, and Confer 2015). A growing body of research suggests that mathematics anxiety starts as early as first grade and can have permanent impacts, including mathematics avoidance in adults (Choe et al. 2019; Ramirez et al. 2013).

Instead. Many assessment options do not have the negative impact of timed testing and actually offer better assessment data. They include observations, interviews, writing samples, strategy sorts, and self-assessments (Bay-Williams and Kling 2019; Kling and Bay-Williams 2014). Interviewing is a particularly powerful (and underused) tool, despite an abundance of advocacy to focus on children's mathematical thinking (Jacobs, Lamb, and Philipp 2010; NCTM 2014a; NRC 2009). At its most basic level, interviewing involves asking a student two simple questions:

"What is $[8 \times 7]?$ "

"How did you figure it out?"

The follow-up question allows the interviewer to note which strategies the student is using and to communicate to the student that *their thinking matters*. A short list of problems (10 or fewer) can give you enough data to assess strategy use, flexibility, accuracy, and efficiency. Automaticity, which is often defined as within three seconds, can also be assessed as interviewers silently count in their heads to determine whether the student is automatic without them knowing they are being timed. Teachers who have experienced interviews see that they are more effective and affirming, as the following quotes (collected by the first author) illustrate:

- Because if you just gave them a timed test, like you wouldn't know, um, where their struggle was at least when you're working with them you can kind of see, can they apply a strategy or if they're trying to apply a strategy, where is it going wrong?
- So it was really exciting to see what was happening and then know, here's that entry point right now to help that student.
- I just like, I absolutely love it. I just, I just love seeing the, the strategies that kids use and love, it's just so cool to see and I never would have gotten to see that had I not taken the time to pull the kids, you know, in the hallway to talk to them about it.

CONCLUDING REMARKS

We (the authors) are not "glass-half-empty" people. Yet, we have seen how these unproductive moves have had tragic consequences, denying students access to research-based, effective teaching practices; inhibiting their attainment of reasoning strategies; and shaping their mathematical identities in negative ways. As a result, society loses potential mathematicians, scientists, engineers, and more. We hope that our "instead" ideas present steps in the right direction. Inviting parents into the conversation is also important—share why you are avoiding the traditional practices above and discuss the value of learning strategies with respect to long-term mathematical reasoning and their child's emotional connection to mathematics. Use learning progressions and diagnostic assessment practices that will lead to efficient teaching of the basic facts. As you make thoughtful decisions about your next steps in supporting your students, consider the vital importance of basic fact fluency, with an eye on the long-term goal of creating thoughtful, creative, and flexible mathematical thinkers. —

REFERENCES

- Baroody, Arthur. 2006. "Why Children Have Difficulties Mastering the Basic Number Combinations and How to Help Them." *Teaching Children Mathematics* 13, no. 1 (August): 22–23.
- Baroody, Arthur, David Purpura, Michael Eiland, Erin Reid, and Veena Paliwal. 2016. "Does Fostering Reasoning Strategies for Relatively Difficult Basic Combinations Promote Transfer by K–3 Students?" *Journal of Educational Psychology* 108, no. 4 (May): 576–91.
- Bay-Williams, Jennifer, and Gina Kling. 2019. *Math Fact Fluency: 60+ Games and Assessment Tools to Support Learning and Retention*. Alexandria, VA: Association for Supervision and Curriculum Development; Reston, VA: National Council of Teachers of Mathematics.
- Boaler, Jo, Cathy Williams, and Amanda Confer. 2015. "Fluency without Fear: Research Evidence on the Best Ways to Learn Math Facts." *YouCubed*. <https://www.youcubed.org/evidence/fluency-without-fear/>.
- Brendefur, Jonathan, Sam Strother, Keith Thiede, and Sarah Appleton. 2015. "Developing Multiplication Fact Fluency." *Advances in Social Sciences Research Journal* 2, no. 8 (August): 142–54.
- Choe, Kyoungh, Jalisha B. Jenifer, Christopher S. Rozek, Marc G. Berman, and Sian L. Beilock. 2019. "Calculated Avoidance: Math Anxiety Predicts Math Avoidance in Effort-Based Decision-Making." *Science Advances* 5, no. 11 (November): eaay1062–69.
- Clements, Douglas H., Karen C. Fuson, and Julie Sarama. 2017. "What Is Developmentally Appropriate Teaching?" *Teaching Children Mathematics* 24, no. 3 (November/December): 178–88.
- Heege, Hans Ter. 1985. "The Acquisition of Basic Multiplication Skills." *Educational Studies in Mathematics* 16, no. 4 (November): 375–88.
- Henry, Valerie, and Richard Brown. 2008. "First-Grade Basic Facts: An Investigation into Teaching and Learning of an Accelerated, High-Demand Memorization Standard." *Journal for Research in Mathematics Education* 39, no. 2 (March): 153–83.
- Jacobs, Victoria R., Lisa L. C. Lamb, and Randolph A. Philipp. 2010. "Professional Noticing of Children's Mathematical Thinking." *Journal for Research in Mathematics Education* 41, no. 2 (March): 169–202.
- Kling, Gina, and Jennifer M. Bay-Williams. 2014. "Assessing Basic Fact Fluency." *Teaching Children Mathematics* 20, no. 8 (April): 488–97.
- Locuniak, Maria N., and Nancy C. Jordan. 2008. "Using Kindergarten Number Sense to Predict Calculation Fluency in Second Grade." *Journal of Learning Disabilities* 41, no. 5 (September/October): 451–59.
- Merriam-Webster's Collegiate Dictionary*. 2003. 11th ed. Springfield, MA: Merriam-Webster. Continually updated at <https://www.merriam-webster.com/>.
- National Council of Teachers of Mathematics (NCTM). 2020. *Catalyzing Change in Early Childhood and Elementary Mathematics*. Reston, VA: NCTM.
- National Council of Teachers of Mathematics (NCTM). 2014a. *Principles to Actions: Ensuring Mathematical Success for All*. Reston, VA: NCTM.
- National Council of Teachers of Mathematics (NCTM). 2014b. "Procedural Fluency in Mathematics: A Position of the National Council of Teachers of Mathematics." Reston, VA: NCTM. <https://www.nctm.org/Standards-and-Positions/Position-Statements/Procedural-Fluency-in-Mathematics/>.
- National Research Council. 2001. *Adding It Up: Helping Children Learn Mathematics*, edited by Jeremy Kilpatrick, Jane Swafford, and Bradford Findell (NRC), Mathematics Learning Study Committee, Center for Education, Division of Behavioral and Social Sciences and Education. Washington, DC: National Academies Press.
- National Research Council. 2009. *Mathematics in Early Childhood: Paths toward Excellence and Equity*, edited by Christopher T. Cross, Taniesha A. Woods, and Heidi Schweingruber (NRC). Washington, DC: National Academies Press.
- Purpura, David J., Arthur J. Baroody, Michael D. Eiland, and Erin E. Reid. 2016. "Fostering First Graders' Reasoning Strategies with Basic Sums: The Value of Guided Instruction." *The Elementary School Journal* 117, no. 1 (September): 72–100.
- Ramirez, Gerardo, Elizabeth A. Gunderson, Susan C. Levine, and Sian L. Beilock. 2013. "Math Anxiety, Working Memory, and Math Achievement in Early Elementary School." *Journal of Cognition and Development* 14, no. 2 (January): 187–202.
- SanGiovanni, John J., Jennifer M. Bay-Williams, and Rosalba Serrano. 2022. *Figuring Out Fluency with Whole Number Addition and Subtraction*. Thousand Oaks, CA: Corwin.
- Steinberg, Ruth. 1985. "Instruction on Derived Facts Strategies in Addition and Subtraction." *Journal for Research in Mathematics Education* 16, no. 5 (November): 337–55.
- Thornton, Carol. 1978. "Emphasizing Thinking Strategies in Basic Fact Instruction." *Journal for Research in Mathematics Education* 9, no. 3 (May): 214–27.
- Tournaki, Nelly. 2003. "The Differential Effects of Teaching Addition through Strategy Instruction versus Drill and Practice to Students with and without Learning Disabilities." *Journal of Learning Disabilities* 36, no. 5 (September/October): 449–58.
- Van de Walle, John A., Karen S. Karp, and Jennifer M. Bay-Williams. 2019. *Elementary and Middle School Mathematics: Teaching Developmentally*. 10th ed. New York: Pearson.

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Using Shared Story Reading in Mathematics

Teachers can use shared story reading with interdisciplinary lessons to simultaneously advance students' mathematics, literacy, and social-emotional competencies. In this article, we use the book, *Two of Everything*, to illustrate how this routine can be used in K–2 classrooms.

Erin Smith, Jo Hawkins-Jones, Shelby Cooley, and R. Alex Smith

Making connections to students' lives while addressing content standards within the constraints of the school day is one of elementary school teachers' greatest challenges. One way to address these challenges is to use an instructional routine such as a shared story reading (Courtade et al. 2013). A shared story reading can support teachers' instruction across content areas, create connections to students' experiences, promote students' social-emotional learning, and maximize instructional time. When used in mathematics lessons, a shared story reading can increase the quantity of mathematical talk (Hojnoski, Polignano, and Columba 2016), promote student engagement, and foster language development (Whitney et al. 2017). A range of stories could be used with a shared story reading; however, folktales offer an appealing way for students to learn mathematics because they can create opportunities for students to engage with concepts in nonthreatening ways that can foster students' mathematical identities (Furner 2017; Rezvi, Han, and Larnell 2020).

USING SHARED STORY READING WITH *TWO OF EVERYTHING*

One of our favorite books for a shared story reading with children in PK-grade 2 is *Two of Everything* (Hong 1992) (link online) by Lily Toy Hong. This storybook

explores a Chinese folktale in which a poor farmer, Mr. Haktak, discovers a brass pot while digging in his garden. The farmer and his wife discover the pot is magical and doubles every object that is placed inside. We like this book for three main reasons. First, the mathematical content spans counting, addition, multiplication, and early algebra, easily providing opportunities to use this book across multiple grades. For instance, we have used it to facilitate first and second graders' understanding of addition and double facts, multiplication, even/odd numbers (SMP 7: *Look for and make use of structure*; NGA Center and CCSSO 2010), and algebra (e.g., input/output). Second, the context of the story provides a rich opportunity for students to develop a mathematical understanding of "double," which connects to the first Standards for Mathematical Practice (SMP 1: *Make sense of problems*). Third, the story can advance students' social emotional learning, particularly the domains of self- and social-awareness and decision-making, as they consider the Haktaks' situation, and the ethical decisions related to it as well as consider what they might use the pot for in their own lives.

A shared story reading has three parts: (1) *before*, (2) *during*, and (3) *after* reading, with each part having different goals. To illustrate what this routine can look like in an interdisciplinary mathematics lesson, we describe our experiences with first and second graders.

Before Reading

The first part of this routine focuses on introducing students to the book and its text features. To foster students' understanding of "two," we pointed to the title and talked about how the number two is represented on the cover (e.g., there are two people, two birds, but one pot) and asked such questions as "How does the illustration help you understand the number two?" and "What do you think this story is about?" When we discussed this latter question, students' predictions drew on their own experiences (e.g., cooking and holiday traditions), which promoted their self-awareness. We also used this stage of the routine to tell students this story is a Chinese folktale, ask students about their prior experiences with folktales (inside and outside of our shared experiences), and remind students of the purpose of many folktales (i.e., teach a moral lesson or describe a tradition connected to a specific culture).

During Reading

The second part of this routine focuses on students' reading comprehension, understanding the mathematical relation of the pot, and introducing the term *double*. It is important at this stage that students (1) identify that the pot doubles items and (2) determine total quantities after items are doubled. To facilitate this discussion, we used a large pot to model and asked questions: "What happens when you put objects into the pot?" "How many ___ are there?" "How can we represent what happens when you put objects in the pot?" In these discussions, we represented the mathematical relation of the pot in a number of ways, including

counting cubes, base-ten blocks, diagrams, drawings, and equations, while emphasizing the terms *double* and *groups of*.

We also used this stage as an opportunity to foster students' social awareness and decision-making. For example, we discussed the emotions and behaviors of Mr. and Mrs. Haktak after they discovered the pot and began to put different items inside. We imagined what life may have been like for the Haktaks and discussed points of conflict and resolution, such as when Mrs. Haktak falls into the pot!

After Reading

The last part of this routine focuses on asking students to verify their initial story predictions and retell the story in their own words using ordinal language (i.e., *first* and *second*). At this time, we also gave students a range of mathematical tasks in which they identified, applied, and represented strategies they used to calculate the number of items that doubled when put into the pot. For instance, they solved problems similar to the following:

I put (8) (toys) into the pot. Now, I have (16) (toys).

Jae put some cars into the pot. Jae now has 18 cars. How many cars did Jae put into the pot?

We like these kinds of tasks because students can use a range of strategies (e.g., direct modeling, double facts, and multiplication). We also asked students to create their own rule for a magical pot and a

Erin Smith (she/her), erin.marie.smith@usm.edu, is an assistant professor of mathematics education at the University of Southern Mississippi in Hattiesburg. Her research focuses on increasing access to high-quality mathematics instruction for every student and facilitating collaborations with parents/guardians in school mathematics.

Jo Hawkins-Jones (she/her), joyarketta.hawkins@usm.edu, is a teaching professor of literacy education at the University of Southern Mississippi in Hattiesburg. Her interests are in culturally relevant teaching and service-learning, bridging her passion for providing literacy and mathematics support to community families and children.

Shelby Cooley (she/her), shelby.cooley0421@gmail.com, teaches lower elementary grades at Thomas L. Reeves Elementary School in Long Beach, Mississippi. She is interested in multisensory learning and the creation of interdisciplinary lessons.

R. Alex Smith (he/him), r.alexander.smith@usm.edu, is an assistant professor of special education at the University of Southern Mississippi in Hattiesburg. He is interested in supporting students identified for special education services in the general education classroom and improving content area literacy instruction.

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corresponding problem to solve. (See our PowerPoint® in the supplementary materials [link online] for examples of student work.) We asked students who needed additional challenge to represent and solve problems for a pot that will triple or quadruple items.

After students shared their problem-solving strategies and the rules they created, we ended the class by discussing what they would put in a magic pot to double. We found that students' experiences influenced what they said. For instance, students said they would put pumpkin seeds, food, schoolwork, Target®, and money into a pot. These discussions were highly engaging and provided opportunities to foster students' decision-making, self-awareness, and social awareness as they considered what they would do with such a pot and why, their peers' ideas, and the ramifications of those decisions. When time has allowed, or in the next day's lessons, we have followed up this lesson by making connections to and comparing the moral or ethical lessons of other folktales that students have read (i.e., contentment in *Two of Everything*, greed in *Two Greedy Bears* [Ginsburg 1998], honesty in *The Empty Pot* [Demi 1990], and fairness in *One Grain of Rice* [Demi 1997]). Additionally, students compared items they would put into the pot with those of the Haktaks' as a way to understand the Haktaks' experiences.

Reflecting on the Lesson

Across the shared reading and problem-solving activities, students demonstrated high levels of enthusiasm. For instance, they were excited to share their story predictions and raised questions of the logistics of having such a pot (e.g., Could you put a house in the pot? Could you make a husband for the second Mrs. Haktak?). The range of students' strategies and representations (e.g., drawings and equations) led to rich mathematical discussions and situations where students tried peers' more sophisticated strategies.

BENEFITS OF SHARED STORY READING

We enjoy using a shared story reading with folktales in interdisciplinary lessons because it provides a structure that can facilitate students' mathematics, literacy, and social-emotional competencies while maximizing instructional time. Furthermore, folktales present a unique entry into mathematics while tapping into student interests and increasing the relevancy of mathematics to students' lives. Moreover, it can challenge students' notions of what it means to do mathematics, as one student stated after a lesson, "I thought we were doing math today!" —

REFERENCES

- Bray, Wendy S. 2009. "The Power of Choice." *Teaching Children Mathematics* 16, no. 3 (October): 178–83.
- Courtade, Ginevra R., Amy S. Lingo, Karen S. Karp, and Todd Whitney. 2013. "Shared Story Reading: Teaching Mathematics to Students with Moderate and Severe Disabilities." *Teaching Exceptional Children* 45, no. 3 (January/February): 34–44.
- Demi. 1997. *One Grain of Rice: A Mathematical Folktale*. New York: Scholastic.
- Demi. 1990. *The Empty Pot*. New York: Henry Holt and Company.
- Furner, Joseph. 2017. "Teachers and Counselors: Building Math Confidence in Schools." *European Journal of STEM Education* 2, no. 2 (August): 1–10.
- Ginsburg, Mirra. 1998. *Two Greedy Bears*. New York: Aladdin Paperbacks, Simon and Schuster Children's Publishing Division.
- Hojnoski, Robin, Joy Polignano, and Helen Lynn Columba. 2016. "Increasing Teacher Mathematical Talk during Shared Book Reading in the Preschool Classroom: A Pilot Study." *Early Education and Development* 27, no. 5 (December): 676–91.
- Hong, Lily Toy. 1992. *Two of Everything*. Park Ridge, IL: Albert Whitman and Company.
- Rezvi, Sara, Abreum Han, and Gregory V. Larnell. 2020. "Mathematical Mirrors, Windows, and Sliding Glass Doors: Young Adult Texts as Sites for Identifying with Mathematics." *Journal of Adolescent and Adult Literacy* 63, no. 5 (February): 589–92.
- Whitney, Todd, Amy S. Lingo, Justin Cooper, and Karen Karp. 2017. "Effects of Shared Story Reading in Mathematics for Students with Academic Difficulty and Challenging Behaviors." *Remedial and Special Education* 38, no. 5 (May): 284–96.

Sibling Longevity

We explore the statistical likelihood of one, or more, siblings in a family of nine surviving to 100 years of age.

Alisan Boes, Duane C. Boes, and Nichola Hillis

On May 17, 1932, William Michael Boes married Laura Engel in Wall Lake, Iowa. Their marriage produced nine children—six boys and three girls—all of whom are alive today. After the death of their mother, the nine children were gathered, and one of the children, Duane Boes (co-author), wondered whether any of them was likely to live to be 100. For this question, we do not mean one specific child in particular. What we are looking for is at least a 50-50 chance that one or more of the nine would survive to 100. Table 1 lists the birth order and current age for Duane Boes and his siblings. Throughout the article, we will refer to these nine people as children.

Many applications of statistics involve time-to-failure analysis: new cars, packaged food, medical devices, and even lifespans can be looked at this way. We are going to look at several methods to attempt to solve the problem. We start by considering the problem through a simple geometric distribution; next, by using a distribution in which we adjust the probability of survival each year; and finally, by analyzing the problem using actuarial tables.

Table 1 Birth Order with Age and Gender

Child	Current Age	Gender
1	87	F
2	86	M
3	85	M
4	83	M
5	82	M
6	81	M
7	77	F
8	75	M
9	68	F

Alisan Boes, she/her, is a teacher at Tesoro High School in Las Flores, California. She is interested in problem solving and making mathematics accessible to all. She is part of a family of mathematicians, including her co-authors.

Duane C. Boes is professor emeritus at Colorado State University and father of Alisan Boes.

Nichola Hillis is a former student at Tesoro High School and daughter of Alisan Boes, granddaughter of Duane Boes.

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Exploration One

For our first exploration, we are going to assume a geometric distribution. A geometric distribution has the following features: trials continue until success, there are only two outcomes (success/failure) for each trial, the probability of success does not change, and trials are independent. An example of this would be tossing a coin until you get heads or rolling a die until you roll a 1.

In this exploration, we are going to use the geometric distribution and assume the probability of death in the coming year for people who are over 65 is always 1/6. We are then going to simulate the lifetimes of each of the 9 children by rolling dice. As there are only two possible outcomes, and we assume independence and that the probability of death stays the same, we stop rolling the die when death occurs. To simulate, we roll a die for each of the 9 children. The value of “1” on the die represents death in the coming year; the values “2-6” represent survival. For example, when rolling for child one, we rolled 2, 3, 5, 5, 1. In this case, child one does not survive to be 100. Instead, child one lives to be 88, 89, 90, 91 but not 92. Table 2 displays a sample simulation. Notice that in this sample none of the children lives to 100 and child 4 does not live to be 84.

Table 2 – Sample Simulation

Child	Age	Rolls Outcomes	Ages Survived
1	87	2, 3, 5, 5, 1	88, 89, 90, 91
2	86	4, 6, 3, 1	87, 88, 89
3	85	5, 2, 2, 1	86, 87, 88
4	83	1	
5	82	6, 4, 1	83, 84
6	81	5, 5, 3, 2, 6, 1	82, 83, 84, 85, 86
7	77	4, 6, 2, 4, 6, 1	78, 79, 80, 81, 82
8	75	5, 4, 2, 4, 4, 2, 6, 2, 6, 3, 4, 5, 3, 4, 6, 2, 3, 1	76, 77, 78, 79, 80, 81, 82, 83, 84, 85, 86, 87, 88, 89, 90, 91, 92
9	68	5, 2, 3, 2, 3, 1	69, 70, 71, 72, 73

If multiple simulations are performed, we can estimate the probability that one, or more, of the children live to 100. For example, if we run the simulation 30 times and 2 of the 30 have one or more children live to 100, we find that there is a $2/30=0.06$ or a 6% chance that one or more survive. We can improve the accuracy of the estimate for this model by using the Law of Large numbers and running many more than 30 simulations.

What might be wrong with our model? Does the probability of death stay constant? Does surviving one year have no influence on whether or not we survive the next year (the assumption of independence)? Is the probability of death as high as 1/6 to begin with? With a geometric distribution, we calculate the expected mean value as $1/p$ thus the probability of death of 1/6 results in an expected mean value of 6 years. That is, in the long-run, we would expect the children to live an average of six years and six years might be too few.

Before we leave this exploration, could we confirm our simulated values through a direct calculation?

To calculate the probability that one or more survives to 100, let us consider the following:

$$\begin{aligned}
 P(\text{one or more of the children live to 100}) \\
 &= 1 - P(\text{none of them does}) \\
 &= 1 - P(\text{all nine children die before 100}) \\
 &= 1 - P(\text{child 1 dies before 100}) P(\text{child 2 dies before 100}) \dots P(\text{child 9 dies before 100})
 \end{aligned}$$

For this last step, we must assume that these probability events are independent. We will discuss this further in the conclusion.

For child one, in order to live to be 100, she would need to live at least 13 more years. What is the probability that she lives at least 13 more years? $(5/6)^{13} \approx 0.0934$. So, the chance she dies before 100 is

$1 - (5/6)^{13} \approx 0.9065$. Table 3 shows this calculation for each of the 9 children. The product of these probabilities is 0.6762 and $1 - 0.6762 = 0.3238$. This results in a 32.38% chance that one or more live to 100, less than our 50% benchmark.

Table 3 – The probability that each child dies before 100 using the geometric distribution

Child	Current Age	Probability of Death
1	87	$1 - (5/6)^{13} \approx 0.9065$
2	86	$1 - (5/6)^{14} \approx 0.9221$
3	85	$1 - (5/6)^{15} \approx 0.9351$
4	83	$1 - (5/6)^{17} \approx 0.9549$
5	82	$1 - (5/6)^{18} \approx 0.9624$
6	81	$1 - (5/6)^{19} \approx 0.9687$
7	77	$1 - (5/6)^{23} \approx 0.9849$
8	75	$1 - (5/6)^{25} \approx 0.9895$
9	68	$1 - (5/6)^{32} \approx 0.9971$

Exploration Two

Suppose that we do not expect anyone to live past 110. We are going to assume that the chance an individual dies in the coming year is inversely proportional to how far they are from 110. Using this method, someone who is 107 has a $1/3$ chance of dying in the coming year, someone who is 99 has a $1/11$ chance, and someone who is 89 has a $1/21$ chance of dying in the coming year. This model suggests that the probability someone dies in the coming year gets larger as one ages. For child one, who is currently 87 years old, assume that she has a $1/23$ chance of dying this year; if she survives, the chance she dies becomes $1/22$, the next year it becomes $1/21$, and so on. What is the chance she survives to 100 or beyond then? Let us conduct this as another simulation. For child one, the first time we are generating random integers 1 to 23 where “1” represents death and “2-23”

represent living. Then we adjust by generating a random integer 1 to 22 where “1” represents death and “2-22” represent living. This continues until the integer “1” is generated. We repeat the process for each of the children keeping in mind that each child will have a different initial number of years from 110. If multiple simulations are performed, you can estimate the probability that one, or more, of the children live to 100, just as in the first exploration. In table 4, we show a sample simulation in which we see that child number six lives to 100.

Table 4: Simulation Using Inverse Proportionality

Child	Age	Random Integer Outcomes	Ages Survived
1	87	22, 20, 4, 11, 8, 14, 1	88, 89, 90, 91, 92, 93
2	86	9, 23, 5, 17, 20, 5, 7, 1	87, 88, 89, 90, 91, 92, 93
3	85	24, 3, 1	86, 87
4	83	15, 23, 25, 7, 7, 3, 2, 15, 1	84, 85, 86, 87, 88, 89, 90, 91
5	82	12, 25, 13, 15, 20, 6, 3, 18, 9, 13, 9, 7, 15, 8, 6, 1	83, 84, 85, 86, 87, 88, 89, 90, 91, 92, 93, 94, 95, 96, 97
6	81	3, 20, 4, 19, 17, 9, 21, 7, 11, 13, 15, 12, 14, 8, 2, 10, 4, 12, 10, 3, 7, 2, 1	82, 83, 84, 85, 86, 87, 88, 89, 90, 91, 92, 93, 94, 95, 96, 97, 98, 99, 100, 101, 102, 103
7	77	33, 3, 13, 26, 18, 20, 11, 26, 24, 14, 15, 11, 21, 5, 16, 17, 8, 12, 4, 2, 8, 1	78, 79, 80, 81, 82, 83, 84, 85, 86, 87, 88, 89, 90, 91, 92, 93, 94, 95, 96, 97, 98
8	75	34, 2, 23, 10, 7, 8, 18, 1	76, 77, 78, 79, 80, 81, 82
9	68	41, 35, 25, 8, 38, 27, 11, 9, 32, 4, 21, 31, 4, 11, 1	69, 70, 71, 72, 73, 74, 75, 76, 77, 78, 79, 80, 81, 82

Could we calculate each of these probabilities directly, without simulation? Once again, we look at child number one. The first year her chance of survival is 22/23, the next year it is 21/22, etc... Notice that the numerators and denominators cancel:

$$\frac{22}{23} \times \frac{21}{22} \times \frac{20}{21} \times \frac{19}{20} \times \frac{18}{19} \times \frac{17}{18} \times \frac{16}{17} \times \frac{15}{16} \times \frac{14}{15} \times \frac{13}{14} \times \frac{12}{13} \times \frac{11}{12} \times \frac{10}{11} = \frac{10}{23}.$$

This calculation gives her a 10/23 chance of surviving to 100 or $1 - \left(\frac{10}{23}\right)$ chance of dying before 100. It follows that each individual's probability of death before 100 could be calculated in the same manner by using this expression: $1 - \left(\frac{10}{\text{number of years to 110}}\right)$. These probabilities are displayed in table 5.

Table 5 – The probability that each child dies before 100 using the inverse proportions

Child	Current Age	Probability of Death
1	87	$1 - (10 / 23) \approx 0.5652$
2	86	$1 - (10 / 24) \approx 0.5833$
3	85	$1 - (10 / 25) \approx 0.6000$
4	83	$1 - (10 / 27) \approx 0.6296$
5	82	$1 - (10 / 28) \approx 0.6429$
6	81	$1 - (10 / 29) \approx 0.6552$
7	77	$1 - (10 / 33) \approx 0.6970$
8	75	$1 - (10 / 35) \approx 0.7143$
9	68	$1 - (10 / 42) \approx 0.7619$

The product of these is 0.0199, this results in a 98% chance that one or more of the children live to 100.

So far, our explorations have provided drastically different results! Why might this be the case? This method is probably an improvement upon the geometric distribution because the probability of death is no longer constant for every year of life. However, this method assumes someone who is 99 has a 10/11 chance of living for another year, which seems unreasonably high. Perhaps 110 is not the correct upper bound. We could try different bounds and then decide whether any of these boundaries will capture the true change in probability of death year upon year, which leads us now to our last exploration.

Exploration Three

Neither of the previous models accurately predicts the probability that one or more of the children live past 100. We must acknowledge that the probability of death at any age is more complex. If we worked for an insurance company and we were trying to estimate the residual lifespan for a customer, we would take into account variables such as their gender, the age of their parents at death, smoking status, weight, exercise habits, diet, race, and socioeconomic status. For this exploration, we turn to the actuarial life tables compiled by the US Social Security Administration. Looking at the data (Social Security Administration 2021), we note the following:

- Children under the age of 1 have a relatively high death rate compared to older children.
- The death probability is higher for males of a given age than for females of the same age.
- The death probability decreases slightly as age increases from 2 to 10.
- Beginning at the age of 10, death probability increases as age increases.

Using the life tables (Social Security Administration 2021) table 6 was created. Note that small adjustments to the life table data might be made over time which will result in minor changes to the following statistics.

Table 6 – Life Table Data

Age	Male		Female	
	Death Probability	Life Probability	Death Probability	Life Probability
0	0.006364	0.993636	0.005331	0.994669
1	0.000432	0.999568	0.000359	0.999641
:				
98	0.319421	0.680579	0.268378	0.731622
99	0.335392	0.664608	0.284481	0.715519

In table 6, death probability is the probability that an individual of that age will die before their next birthday. For ease of use, and to avoid having to interpolate the probabilities to the nearest age in days we will use the data as given and assume the children are at the beginning of these years. Life probability was calculated as 1 - (death probability) or the probability of surviving to the next birthday.

How can we use the life table data?

Let's define the following event:

$$A_n = \text{child one lives to be } n \text{ years old}$$

$$P(A_n | A_{n-1}) = \frac{P(A_n \cap A_{n-1})}{P(A_{n-1})} = \frac{P(A_n)}{P(A_{n-1})}$$

Notice that we can make the last step because she cannot be n years old without having been $n-1$ years old. Rearranging the equation we get:

$$P(A_n) = P(A_n | A_{n-1}) P(A_{n-1})$$

$$P(A_{100}) = P(A_{100} | A_{99}) P(A_{99})$$

$$P(A_{99}) = P(A_{99} | A_{98}) P(A_{98})$$

$$\text{So } P(A_{100}) = P(A_{100} | A_{99}) P(A_{99} | A_{98}) \dots P(A_{88} | A_{87})$$

Thus for a current age of c we have the general formula of: $P(A_{100}) = P(A_{100} | A_{99}) P(A_{99} | A_{98}) \dots P(A_{c+1} | A_c)$

To calculate the probability that child one survives to 100 we need to calculate the probability that she survives her 99th year. To do that, we take the product of the female life cells in the table from 87 to 99. That product has been calculated for each of the children using the appropriate gender and starting year and is displayed in table 7.

Table 7 – Probability of Death before 100 for each of the children, using the life table

Child	P(survives to 100)	P(dies before 100)
1	0.070080	0.929920
2	0.031561	0.968439
3	0.028512	0.971488
4	0.023990	0.976010
5	0.022292	0.977708
6	0.020868	0.979132
7	0.040663	0.959337
8	0.015714	0.984286
9	0.034102	0.965898

The probability they all die before 100 is the product of the last column in table 7: 0.7456. Therefore, using the actuarial tables, the probability that one, or more, of the children survive is $1 - 0.7456 = 0.2544$, or 25%.

There are many reasons to suggest that the actuarial tables have underestimated this probability for this set of children. The children are healthy, educated, and are financially stable. They exercise, maintain proper weights, do not smoke, and their parents lived into their 90's.

CONCLUSION

For each of the three explorations, we have assumed independence between the children's lifespans, which simplifies the calculations. The nine children here have different lifestyles and live in different areas of the country so are unlikely to fall to a singular event such as a natural disaster. However, because they are genetically related, the independence assumption might be suspect. They grew up in the same house, eating the same food, so were therefore exposed to similar environmental factors as children. They are

also genetically related so might be predisposed to the same illnesses. Would it be easier to assume independence if the children had been adopted, but still grew up in the same household?

Of the methods here, which one would you select for use? Perhaps the one that is easiest to calculate? But what if you were in charge of providing life insurance to these children? What other tactics would you use to solve this problem?

The authors hope to live healthfully well past 100, and we hope that you do as well! [—](#)

REFERENCE

Social Security Administration. 2019. "Actuarial Life Table." https://www.ssa.gov/OACT/STATS/table4c6_2016.html.

Mathematical Modeling as a Tool for Empowerment

Ear to the Ground features voices from several corners of the mathematics education world.

Elizabeth G. Arnold, Elizabeth A. Burroughs, Mary Alice Carlson, Elizabeth W. Fulton, and Megan H. Wickstrom

What are your long-term goals for your students? As teachers, when we think about this question, we tend to move away from specific topics and toward the ways big mathematical ideas relate to students' lives. We think about how students might use mathematics in their future careers, their mathematical habits of mind and ability to work through complex problems, their interactions with other human beings, and their sense of belonging in STEM-related (science, technology, engineering, and mathematics) fields. Mathematical modeling has a unique role in the school curriculum because it broadens students' understanding of what mathematics is, whom it is for, and how it applies in the world. Mathematical modeling in classrooms affirms students' dignity as creative—and mathematical—human beings. Mathematical modeling empowers students as learners and doers of mathematics and engages them in empathetic critical thinking skills.

Modeling is the process of using mathematics to understand and make decisions about the world. The modeling process in classrooms begins when students encounter an authentic problem that is important to them or their communities, which happens often! For example, we have worked with students who ask questions such as, "What is the best way to share snacks?", "Do we have enough pencils to last the school year?", "Where should the mobile library bus park each week?", "How should the cross-country coach choose runners for the state meet?", or "Is the SAT a fair criterion for college admissions?"

When engaged in modeling, students consider a problem from a mathematical perspective to understand if and how mathematics can be used to understand the situation. As they develop a model, they make assumptions, assign variables, and begin to craft

a solution to the problem. Along the way, they check in with classmates and the teacher to revise and refine their thinking, considering the benefits and limitations of different approaches and how their models capture the perspectives of different human beings.

Mathematical modeling is a powerful classroom practice that increases students' capacity to use mathematics as a tool to address complex problems that they encounter in their day-to-day experiences. Students can bring multiple facets of their identity and their experiences to understand a problem and inform their solution strategies. We have found it useful to phrase three "I can see" statements that encapsulate the perspective that modeling nurtures in students about themselves, mathematics, and their communities.

I can see mathematics in the world. Modeling gives students

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the opportunity to find mathematics in the world around them. Mathematics is one tool, of many, that students can draw on to become more powerful and informed decision makers.

I can see how and why my decisions matter. Modeling affirms the resources our students bring to the classroom and connects to issues they care about. It allows them to see that their solutions matter and empowers them to act on their findings.

I can see that different approaches to a problem have merit. Modeling allows students to see mathematics as a creative endeavor in which multiple solutions exist. Modeling also allows for community building, as students learn from one another to revise and refine their thinking.

If any of these ideas resonate with you and you are inspired to learn more about mathematical modeling, we encourage you to explore these new books recently published by the National Council of Teachers

of Mathematics: *Becoming a Teacher of Mathematical Modeling, Grades K–5* (Arnold et al. 2021a) and *Becoming a Teacher of Mathematical Modeling, Grades 6–12* (Arnold et al. 2021b).

These books will help you learn more about the process of modeling and conceptualize what modeling could look like in your classroom. We also invite you to watch two webinars related to modeling: Grades K–5 ([link online](#)) (Fulton and Wickstrom 2022) and Grades 6–12 ([link online](#)) (Arnold 2022). —

Elizabeth G. Arnold, she/her, liz.arnold@colostate.edu, is an assistant professor of mathematics education at Colorado State University in Fort Collins. Her research centers on the preparation and development of pre-service and in-service K–12 mathematics teachers, with a focus on mathematical modeling.

Elizabeth A. Burroughs, she/her, burroughs@montana.edu, is the department head and a professor in the Department of Mathematical Sciences at Montana State University in Bozeman. A former high school mathematics teacher, she finds joy in fostering empathy through mathematical modeling.

Mary Alice Carlson, she/her, mary.carlson5@montana.edu, is an associate professor of mathematics education at Montana State University in Bozeman. She is interested in K–12 mathematical modeling and enjoys working with teachers as they try out new practices in their own classrooms.

Elizabeth W. Fulton, she/her, elizabeth.fulton@montana.edu, is an assistant research professor of mathematics education in the Department of Mathematical Sciences at Montana State University in Bozeman and a former high school mathematics teacher. She enjoys working with current and future K–12 teachers.

Megan H. Wickstrom, she/her, megan.wickstrom@montana.edu, is an associate professor of mathematics education at Montana State University in Bozeman. Her research focuses on the teaching and learning of mathematical modeling in K–16 classrooms. She enjoys co-developing rich tasks that foster students' mathematical identities.

REFERENCES

- Arnold, Elizabeth G.. 2022. *Becoming a Teacher of Mathematical Modeling, Grades 6–12*, 1:01:37. NCTM Professional Development Webinar Series. January 27, 2022. <https://www.nctm.org/online-learning/Webinars/Details/601>.
- Arnold, Elizabeth G., Elizabeth A. Burroughs, Mary Alice Carlson, Elizabeth W. Fulton, and Megan H. Wickstrom. 2021a. *Becoming a Teacher of Mathematical Modeling, Grades K–5*. Reston, VA: National Council of Teachers of Mathematics.
- Arnold, Elizabeth G., Elizabeth A. Burroughs, Mary Alice Carlson, Elizabeth W. Fulton, and Megan H. Wickstrom. 2021b. *Becoming a Teacher of Mathematical Modeling, Grades 6–12*. Reston, VA: National Council of Teachers of Mathematics.
- Fulton, Elizabeth W., and Megan H. Wickstrom. 2022. *Becoming a Teacher of Mathematical Modeling, Grades K–5*, 58:17. NCTM Professional Development Webinar Series. February 23, 2022. <https://www.nctm.org/online-learning/Webinars/Details/600>.

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Golden Ratio Butterflies

Clarissa Grandi



For the Love of Mathematics, driven by reader suggestions and submissions, offers visual, engaging, and inspiring material for you, the teacher.

I have always been irresistibly drawn to color, symmetry, and pattern and have been lucky enough to pursue my interests in both mathematics and art.



The images featured here explore the use of the golden ratio, *phi*, as a proportioning guide. The “fractal packaging” nature of this ratio really intrigues me, and I have found that playing with it visually helps me better understand it mathematically. When I came across this beautiful origami swallowtail fold, designed by Evi Binzinger, I was compelled to make it from paper squares sized according to the ratio, with each smaller square approximately 0.618 times the previous one. I then began to play . . .



Clarissa Grandi, artfulmaths@gmail.com, Twitter: @complexnumber, Instagram: @clarissagrandi.art, is a Key Stage 5 (KS5; 12th grade) mathematics teacher at Thurston Community College, Suffolk, UK. Her interests include mathematical art and geometric origami. She blogs at www.artfulmaths.com and publishes the newsletters *Clarissa Grandi Art* (link online) and *Artful Maths Quarterly Newsletter* (link online).

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“Gerrymandering: When Equivalent Is Not Equal!”

From the Archives highlights articles from NCTM's legacy journals, as chosen by leaders in mathematics education.

Kevin J. Dykema

In “**Gerrymandering: When Equivalent Is Not Equal!**”, authors Farshid Safi, Sarah Bush, and Siddhi Desai provide a series of learning tasks designed to help students apply mathematics to the real world. Although this article was published not that long ago, the political and educational world has changed tremendously, and thus this article is worth taking a fresh look at. The world has become much more polarized, whether it is debating masks, vaccine effectiveness, what is being taught in schools (consider the bans some states have now implemented on critical race theory), or a host of other topics. Because of this polarization, we, as educators, must show—now more than ever before—how mathematics is used in decision-making.

Although the original article was written for middle school teachers, the ideas definitely

extend to other grade bands as well. The opportunity to incorporate social justice into the mathematics classroom is a key feature of this article. Discussing how those in power can create structures to help keep them in power and can use mathematics to do so can be eye opening for middle school students. In addition to the actual tasks, which are outstanding, this article can teach several other lessons.

MATHEMATICS IS USED TO MAKE DECISIONS

Many middle school students likely believe that mathematics is merely solving problems to get an answer. Through the tasks found in the article (available as supplementary files to this Introduction [link online]), they begin to see that mathematics is used to make decisions—and not always equitable ones. A great follow-up to these

tasks is to show pictures of actual congressional districts so that students see this is truly an issue. Doing an internet search will yield many real-life examples of gerrymandered districts. This task also provides a wonderful opportunity to work with your social studies colleagues to further student knowledge.

DISCOURSE IN MATHEMATICS CLASS IS IMPORTANT

These tasks also provide a wonderful opportunity to promote student-to-student discourse. Most middle schoolers do not have difficulties talking in mathematics class, but talking about mathematics in mathematics class can be challenging. The engaging nature of these tasks, especially the final gerrymandering one, can make *stopping* the mathematics conversations difficult! Students see why discussing

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FROM_THE_ARCHIVE

mathematics is so important and how doing so can help them discover other ideas and deepen their knowledge.

TASK SEQUENCING

This article provides a series of four tasks designed to help students explore the idea of equal versus equivalent. These tasks are strategically organized to best support students in developing the main ideas to build the necessary conceptual understanding. Careful attention was paid in this series of tasks to building from what students had previously done to help develop and deepen their understanding. The importance of thoughtfully sequencing tasks to support students is clearly illustrated in this article and should serve as an example that can be followed when developing other concepts.

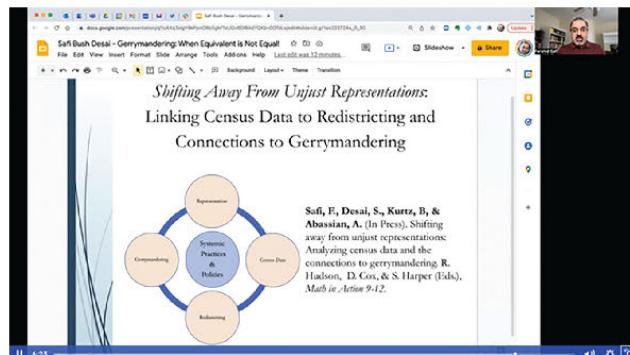
MULTIPLE ANSWERS ARE POSSIBLE IN MATHEMATICS

We know that many people believe that each problem in mathematics

has only one correct answer. This article clearly shows how a single problem might have many potential answers; the districts could be configured in many ways to yield the same results. Students may be accustomed to solving a problem in multiple ways and yet the answer remains the same. The gerrymandering task has multiple answers as well as multiple ways to do the problem.

Perhaps this task will enable you to provide a spark for some students to begin to see the joy, beauty, and wonder of mathematics. Enjoy the rich conversations and deep learning that will occur from these tasks! In video 1 (link online), Farshid Safi reflects on the original article from *Mathematics Teaching in the Middle School* (MTMS). —

Video 1 Farshid Safi Discussing the Legacy Article from MTMS



 Watch the full video online.

REFERENCE

Safi, Farshid, Sarah B. Bush, and Siddhai Desai. 2018. "Gerrymandering: When Equivalent Is Not Equal!" *Mathematics Teaching in the Middle School* 24, no. 2 (October): 82–89.

Kevin J. Dykema has taught eighth-grade mathematics for more than 25 years and is currently a district math specialist while serving his term as President-Elect for NCTM. He is interested in helping each and every student best learn mathematics.

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GROWING_PROBLEM_SOLVERS

 Access digital content at
nctm.org/matl11507gps

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GPS: How Long? Estimating Length and Time

Growing Problem Solvers provides four original, related, classroom-ready mathematical tasks, one for each grade band. Together, these tasks illustrate the trajectory of learners' growth as problem solvers across their years of school mathematics.

S. Asli Özgün-Koca and Monica G. McLeod

The focus of this month's Growing Problem Solvers is on using measurement to support number sense. These tasks invite students to conceptualize and discuss measurements, which we can think of as measurement sense! The tasks shared here purposefully focus on both physical measurement estimation and time estimation, so that the connections might support students'

reasoning. Rather than focusing on specific units in the task, unit choice is left open for the students to negotiate and discuss. If needed, the teacher might prompt students to identify the unit of measurement and consider different unit options. At each grade band, teachers can launch and end the task by prompting student discussion with the following questions:

Table 1 Associated Common Core State Standards

Standard	Description
1.MD.A.1	Order three objects by length; compare the lengths of two objects indirectly by using a third object.
2.MD.A.3	Estimate lengths using units of inches, feet, centimeters, and meters.
3.MD.A.1	Tell and write time to the nearest minute and measure time intervals in minutes. Solve word problems involving addition and subtraction of time intervals in minutes, e.g., by representing the problem on a number line diagram.
4.MD.A.3	Apply the area and perimeter formulas for rectangles in real world and mathematical problems. <i>For example, find the width of a rectangular room given the area of the flooring and the length, by viewing the area formula as a multiplication equation with an unknown factor.</i>
6.G.A.1	Find the area of right triangles, other triangles, special quadrilaterals, and polygons by composing into rectangles or decomposing into triangles and other shapes; apply these techniques in the context of solving real-world and mathematical problems.
HSG.GMD.A.1	Give an informal argument for the formulas for the circumference of a circle, area of a circle, volume of a cylinder, pyramid, and cone. <i>Use dissection arguments, Cavalieri's principle, and informal limit arguments.</i>
HSG.MG.A.1	Use geometric shapes, their measures, and their properties to describe objects (e.g., modeling a tree trunk or a human torso as a cylinder).

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GROWING_PROBLEM_SOLVERS

- When might it be reasonable to estimate a measurement?
- When is it better to get a precise measurement?
- How would you decide if your estimate is close enough?
- As a class, how do you decide if an estimation is acceptable? (What is your margin for error?)
- In the context of this problem, when would you rather underestimate? Overestimate?

Teachers might also consider including a discussion of comparison or reference objects (a hand, the size of a whiteboard, etc.) to support individuals or groups of students.

- What could you use as a reference when estimating?

- What other reference or comparison might be useful?
- What additional information do you need to make your estimate?

Table 1 states the Common Core State Standards for Mathematics (NGA Center and CCSSO 2010) addressed by the four tasks. The progression of the standards begins with one-dimensional measurements of length for the very early grades. Then area and perimeter (as an expansion on length) are introduced and developed in upper elementary grades and middle school grades. A special focus on various shapes is emphasized in the middle grades. Although the introductory conceptualizations of volume begin in the fifth grade and are further developed in the middle grades, this sequence of tasks focuses on three-dimensional shapes in the high school grade band.

S. Asli Özgün-Koca, she/her, aokoca@wayne.edu, teaches mathematics and mathematics education courses at Wayne State University in Detroit, Michigan. She is interested in effective and appropriate use of technology in secondary mathematics classrooms along with research in mathematics teacher education.

Monica G. McLeod, she/her, monicamcleod@wayne.edu, is a mathematics instructional coach in Detroit, Michigan. She is interested in strengthening teachers' mathematical knowledge for teaching, so that we can better serve all students.

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PK–2

In the PK–2 task (link online), the teacher begins the task by gathering a collection of small objects whose predominant attribute is length from around the classroom. These could be pencils, paper, markers, crayons, paintbrushes, paper clips, and so forth. This task could be completed by student pairs, with each pair having three objects. Students may intuitively consider wider objects (such as a piece of paper or a marker) as being longer. The teacher might need to emphasize the longer dimension through physical gestures and direct comparison of objects. The teacher might lead the students through a discussion using the following prompts:

- Which object do you think is the longest?
- Which object is the shortest?
- What makes you think so?
- How could you find out for sure?

Once the pairs answer these subquestions, the teacher would display another object and position it so that students could not line it up with their own objects. The aim here is for students to make estimations. Students may benefit from using a reference object, such as the teacher's hand, to form their estimations. Sharing various strategies would allow students to reflect on others' strategies and perhaps use

those strategies later. Questions to lead this portion of the exploration include these:

- Is the teacher's object longer or shorter than your longest object?
- What makes you think so?
- How could you find out for sure?



As mentioned above, each task has one physical and one time estimation. We ask these young students to estimate how long it would take to count all the objects. Many students might think it would take the same number of seconds as the number of objects. The teacher could explore the students' reasoning with question such as the following:

- How long do you think it would take us to count all the objects we have gathered here?
- What makes you think so?
- How could we find out for sure?

3–5

The upper elementary school task (link online) uses rectangular objects in the classroom, with a focus on perimeter. Students might confound perimeter and area, so the teacher may want to refer to the difference when launching the task. Some students might envision perimeter as one continuous length by picturing a rope going around the rectangular object and estimating the length all at once. Through shared strategies and questioning, the teacher might want to encourage the use of an estimation strategy that decomposes the rectangular object into tiles of smaller area, or by considering linear estimations of width and length first. For this task (and middle and high school tasks), students record their estimations, their estimation strategy, and their personal reference object (if they used one). Students then measure to find the actual perimeter and calculate the difference between

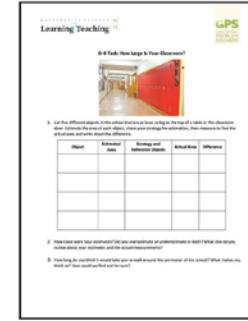
their estimation and the actual measurement. This idea was inspired by the NCTM Illuminations Area Contractor lesson plan (Healy 2008). Keeping track of their estimations and the actual measurements could help students answer the second question and reflect on their estimation patterns. The time estimation for this task asks students to estimate how long it would take to walk around the perimeter of their classroom. The teacher and the class can also discuss how one could walk the "exact" perimeter of the classroom.



6–8

The task structure and questions for the middle school task (link online) are very similar to the upper-elementary task. For this grade band, the task focuses on area. Students will record their estimations and actual measurements to study their estimation patterns. With area estimation, students might estimate the length and width and then the area, or some students might use a reference object to tile the shape. Again, sharing and comparing different strategies might help students to consider other options for estimation. For the time estimation, middle

schoolers are asked to estimate how long it would take to walk around the perimeter of the school. Even though this is not an estimation of time regarding area, it challenges middle school students to estimate the perimeter of the school.



9–12

The high school task (link online) focuses on volume and follows the same structure of the other tasks. High school students might estimate the length, width, and height to estimate the volume of an object, or they can think about filling a space with a reference object. This task asks students how long it would take to fill the volume of the largest object with water. This time estimation was inspired by Dan Meyer's (n.d.) Water Tank task. Watching Dan Meyer's video could help students adjust their time estimations.



REFERENCES

- Healy, Julie. 2008. "Area Contractor." *Illuminations*. <https://www.nctm.org/Classroom-Resources/Illuminations/Lessons/Area-Contractor/>.
- Meyer, Dan. n.d. "Water Tank." *Dan Meyer's Three-Act Math Tasks*. Accessed April 18, 2022. <https://mrmeyer.com/threeacts/watertank/>.
- National Governors Association Center for Best Practices (NGA Center) and Council of Chief State School Officers (CCSSO). 2010. *Common Core State Standards for Mathematics*. Washington, DC: NGA Center and CCSSO. <http://www.corestandards.org>.

Appendix 9 (continued)

Examples of the Growing Problem Solvers online supplementary files available for download.

MATHEMATICS TEACHER PK 12
Learning Teaching



PK–2 Task: Whose Pencil Is Longer?



the longest? _____

the shortest? _____

MATHEMATICS TEACHER PK 12
Learning Teaching

3–5 Task: How Big Is the Board in Your Classroom?



1. List five different rectangular objects in the classroom. Only one can be small. Estimate the perimeter of each object, share your strategy for estimation, then measure and write down the difference.

Object	Estimated Perimeter	Strategy and Reference Objects	Actual Perimeter

MATHEMATICS TEACHER PK 12
Learning Teaching



6–8 Task: How Large Is Your Classroom?



- Find five different objects in the school that are at least as big as the top of a table or the classroom. Estimate the area of each object, share your strategy for estimation, then measure to find the actual area and write down the difference.

Object	Estimated Area	Strategy and Reference Objects	Actual Area	Difference

MATHEMATICS TEACHER PK 12
Learning Teaching



9–12 Task: How Big Is Your School?



1. List five different objects in the school that are at least as big as a trash can. Estimate the volume of each object, share your strategy for estimation, then measure to find the actual volume and write down the difference.

Object	Estimated Volume	Strategy and Reference Objects	Actual Volume	Difference

Were your estimates? Did you overestimate or underestimate or both? What else do you notice about your estimates and the actual measurements?

Do you think it would take you to walk around the perimeter of the school? What makes you think so? How could we find out for sure?

2. How close were your estimates? Did you overestimate or underestimate or both? What else do you notice about your estimates and the actual measurements?

3. How long do you think it would take you to fill the volume of your largest object with water? What makes you think so? How could we find out for sure?

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PROBLEMS_TO_PONDER

 Answer key available at
nctm.org/matl11411p2p.

Problems to Ponder

Problems to Ponder provides 28 varying, classroom-ready mathematics problems that collectively span PK-12, arranged in the order of the grade level. Answers to the problems are available online. Individuals are encouraged to submit a problem or a collection of problems directly to mtlt@nctm.org. If published, the authors of problems will be acknowledged.

Megan Holmstrom

1

Show the number four in four ways.

2

What can we find in our classroom that is a square?
A circle?

For further thought: What other shapes can you find?

3

Here are some fish swimming in a circle. How many fish are there?



4

Andrew and Tyler see some fish. How many fish do they see?



PROBLEMS_TO_PONDER

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5

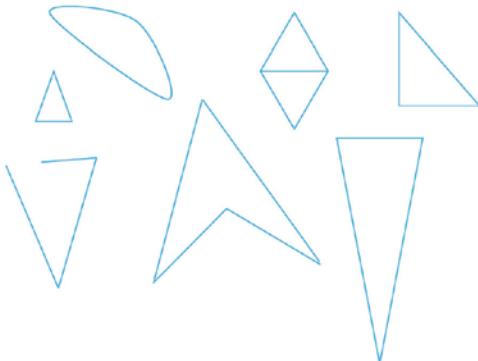
Look at the animals, which animals have more than two feet? How many feet does each one have?



For further thought: How many animal feet altogether are in the picture?

6

Which of the following are triangles? Why or why not? Let's name the attributes of all triangles.



7

Ryoji is setting up his schedule for the school week. Here are some times and activities. Match activities with times to create a schedule for Ryoji.

5:30 p.m.	Breakfast
12:00 p.m.	School day ends
7:00 a.m.	Recess
9:15 a.m.	Lunch
3:15 p.m.	Soccer practice begins
For further thought: Create two other times and activities that fit into Ryoji's day.	

8

Bryson and Eliza are using a hundred chart to add numbers. Bryson starts on 14 and adds 10. What number is he on now? Eliza starts on 38 and adds 20 more. What number is she on now? What are some ways Eliza might add 20 more?

9

Draw some examples of quadrilaterals. What are some nonexamples? What attributes are true for *all* quadrilaterals?

Megan Holmstrom, megan.holmstrom@gmail.com, is an independent consultant covering the scope of PK–8 mathematics. Her time is shared currently among several schools throughout Europe, the Middle East, and Africa.

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PROBLEMS_TO_PONDER

10

Ms. Nagle's third-grade class is going on a field trip to the aquarium.

- The cost of the trip is \$215.
- The school will pay \$100 toward the trip.
- The class earns money by selling school cookbooks. Each cookbook sells for \$15.

How much more money does the third-grade class still need to earn to pay for their trip? How many cookbooks will the class need to sell to pay for the trip?

For further thought: Write an equation to represent how you solved the problem.

11

Hulsing School is adding four more bookshelves to its library. The librarian would like to arrange them in order from shortest to tallest. Arrange the list of bookshelves in order from shortest to tallest.

Bookshelf 1	4 feet
Bookshelf 2	1 1/2 yards
Bookshelf 3	60 inches
Bookshelf 4	3 1/4 feet

For further thought: Show each bookshelf's measurement first in total feet and then in total inches.

12

Show what you know about solving the following problem: $5,396 + 2,814$

Concrete or pictorial:

--	--	--	--

Between what two numbers will the sum be on the number line?



Show what you know using the standard algorithm:

$$\begin{array}{r} 5,396 \\ + 2,814 \\ \hline \end{array}$$

13

Write a story that matches $\frac{3}{4} \times 3$ and ask a question that leads to the solution.

How can you model your story? What visual can you add to represent the fractions in your story?

14

Karina and Santiago go for walks each night. They kept track of the miles they walked for two weeks.

Daily miles: 6, 6.5, 3, 5, 4, 6, 5, 5, 3.5, 4, 5.5, 4, 7, 4
Draw a line plot to represent the data.

For further thought: What other measures do you notice? What do you wonder about the data?

15

In a bag of red and blue marbles, the ratio of blue to red marbles is 5:7. What is the ratio of total marbles to blue ones? How many marbles might be in the bag?

16

Notice and Wonder are following a cookie recipe which uses $\frac{2}{3}$ cup of nuts for 1 batch of cookies. How many batches of cookies can be made with 4 cups of nuts? Justify your answer using a diagram.



17

Two number cubes, each with faces labeled 1 through 6, are rolled at the same time. What is the probability that the sum of the number cubes will be a multiple of 3?

18

The scale drawing of a rectangular rug has dimensions 4 inches \times 5 inches. The length of the longer side of the actual rug is 15 feet. What is the area of the rug?

[19](#)

Which expressions are equivalent to $(5^4 \times 5^{-6})^{-2}$?

- a. $5^{-8} \times 5^{12}$
- b. $5^2 \times 5^{-8}$
- c. 625
- d. $1/625$

For further thought: Write another equivalent expression.

[20](#)

The following table represents a linear function.

x	y
4	-2
7	0.25
12	4

Write an equation to represent the linear function.

[21](#)

Which of the numbers below when multiplied by $7\sqrt{3}$ will result in a rational number?

- a. $-1\sqrt{3}$
- b. $7\sqrt{27}$
- c. $6 + 2\sqrt{3}$
- d. $\sqrt{\frac{16}{3}}$

[22](#)

In January, the Gonzales family starts saving for a trip to Mexico in September. The vacation is estimated to cost \$4,850. They start with \$675 in a savings account and will deposit 25 percent more each month than the previous month in the savings account. Will they have enough money for their vacation in September? Justify your reasoning.

For further thought: How much money would the family need to start with to have \$4,850 in August?

[23](#)

The Denver Metro area population was 582,000 in 1953 and 1,054,000 in 1970. What was the average yearly increase in the population during the years from 1953 to 1970?

[24](#)

Lava coming from the eruption of a volcano follows a parabolic path. The height h in feet of a piece of lava t seconds after it is ejected from the volcano is given by $h(t) = -t^2 + 16t + 936$. After how many seconds does the lava reach its maximum height? What is the maximum height of the lava?

[25](#)

Classify the polygon formed by connecting the points $(-3, -2)$, $(5, 3)$, $(9, 9)$, and $(1, 4)$.

[26](#)

In a game, you roll a single six-sided number cube numbered with one, two, three, four, five, and six. You earn three points if a six comes up; nine points if a two, four, or five come up; and nothing otherwise. What is the expected value?

[27](#)

An airplane leaves the runway and climbs at an 18 degree angle with a speed of 275 feet per second. Find the altitude of the plane after one minute.

[28](#)

A rocket is launched from a platform 150 feet above the ground. A function that models this situation is given by $h = -16t^2 + 96t + 150$ with t measured in seconds and h as the height in feet above the ground.

- a. What is the domain of the function given this context?
- b. What is the maximum height reached by the rocket?
- c. After how many seconds is the rocket 100 above the ground?
- d. After how many seconds does the rocket hit the ground?

TEACHING_IS_A_JOURNEY

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Teaching Is a Journey From Then to Now

This department provides a space for current and past PK–12 teachers of mathematics to connect with other teachers of mathematics through their stories that lend personal and professional support.

Kevin J. Dykema

I never wanted to be a middle school mathematics teacher. I knew I wanted to teach mathematics but was sure I wanted to be a high school teacher . . . until the day I was offered an eighth-grade mathematics teaching job. It was August, and I wanted (truthfully, needed) a job, interviewed in a district, and was offered the option of teaching eighth grade or being one of three finalists for the high school opening. I opted for the sure thing, thinking that I would move to the high school as soon as the opportunity arose. Fast forward 26 years, and I am still teaching eighth grade and loving every (or almost every) minute of it!

I am sure everyone says their grade is the best to teach, but teaching eighth grade is absolutely amazing! I have the privilege to see how much students mature; an end-of-the-year eighth grader is so incredibly different than a beginning-of-the-year eighth grader. They still want to be people pleasers yet assert their independence at times. One of my goals each year is to help them appreciate and enjoy mathematics more than they have in the past. Every day is a new adventure with them!

As I reflect on my growth as a practitioner, I realize how much I have learned about teaching and learning and how much more learning I have to do! I have made some definite shifts from early-career teacher to now as I strive to meet the needs of each and every one of my students. I will share four of my biggest shifts below that help illustrate this journey to better meet the needs of my students.

FROM TEACHER CENTERED TO STUDENT CENTERED

I have switched my focus from what I teach to what my students are learning. As an early-career teacher, I know there were times that I said to myself, “I taught it—they should have paid better attention, or they should have asked more questions or asked for help.” I now realize that just because I thought my lesson was good does not make it effective; it was not truly good unless the students learned the concepts.

Formative assessment plays a much greater role than in my early career. I need to be consistently checking for student understanding and adjusting the tasks and my plans to meet their needs, which can be a very difficult thing to do. To meet the needs of each and every learner, I must know where each one is. For those who need some extra time to develop a deep understanding of the mathematics currently being learned, I need to work with them individually or in small groups to help meet their needs.

FROM CONTENT FOCUSED TO STUDENT FOCUSED

I realized after a few years (probably longer than it should have taken) that I teach students, not teach mathematics. At the start of my career, I focused on

Appendix 11. Teaching is a Journey (Department) Article

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TEACHING_IS_A_JOURNEY

ways to make the mathematics content clear and to help students see connections between topics. I thought my passion for mathematics would inspire them to want to learn more mathematics. Although that passion probably has helped, I realized that it was not enough—I needed to focus more time and energy on building relationships with my students.

I had heard, “They don’t care how much you know until they know how much you care,” but I did not really believe it. But I also heard from some of my colleagues, “Don’t let them see you smile until Thanksgiving.” Looking back at my early career, I wish I had paid more attention to the first saying. Establishing appropriate relationships with students to help them learn is so incredibly vital. I work hard to get to know each student as an individual, rather than merely a mathematics student of mine. Asking about weekend plans, siblings, and after-school activities has become a regular part of my conversations with students. I often jot notes so that I remember to follow up after a student shares about an upcoming event. The smile on their faces when I ask how their game was or how their recital went is priceless.

FROM “STRUGGLE IS BAD” TO “STRUGGLE IS NECESSARY”

As a new teacher, I worked hard to help my students avoid experiencing much struggle. I would give detailed step-by-step directions with clearly worked-out examples for them to follow and mimic. I created multiple different explanations of topics, so that reteaching did not mean “slower and louder.” After a few years, I realized I was doing most of the intellectual work, and my students were often not actively engaged. I also found myself becoming frustrated that they quickly forgot what I had “taught” them.

Around the same time, I started taking graduate classes in mathematics education and was introduced to inquiry learning and teaching. I realized that students must wrestle with the content to truly understand it. Implementing this in my classroom had some rocky moments, but I noticed my students becoming much more actively engaged in mathematics class and

starting to retain things longer. We were making the transition from “math by memorizing” to “math by understanding.”

I realized that we often recognize and appreciate the role of struggle in learning new skills, such as riding a bike, learning to swim, and playing a musical instrument. This same appreciation should occur when learning mathematics. Students must struggle and wrestle through the content to truly learn it, and one of my roles is to help support them through it. Some days are definitely better in my class than others with this. Years later, NCTM put a name to this with “productive struggle” in *Principles to Actions: Ensuring Mathematical Success for All* (NCTM 2014).

FROM “MY DISTRICT IS RESPONSIBLE FOR MY PROFESSIONAL DEVELOPMENT” TO “I AM RESPONSIBLE”

In my early career, I assumed the professional development I received from my school would be enough to help me develop my skills to help meet the needs of each and every one of my students. I was fortunate to attend some great sessions, but little was focused on teaching mathematics, so I worked with my mathematics colleagues to apply what was presented to our classroom situations. I began my masters in mathematics education coursework after several years of teaching and enjoyed the conversations and professional growth that was occurring. After graduating, I realized I would not naturally have those conversations and opportunities to think deeply about teaching mathematics unless I remained focused on it.

I made the conscious choice to become more involved in my state affiliate, which helped me grow professionally and eventually led to involvement with NCTM. I also realized I should be reading books about teaching mathematics, even though my school would not pay for them. Consistently reading, hearing, and reflecting on ideas about teaching mathematics are important for continuously developing new skills to try to meet the needs of each and every student. For an overview of my journey, please view video 1 (link online).

Kevin J. Dykema, kdykema@mattawanschools.org, Twitter: @kdykema, teaches eighth-grade mathematics at Mattawan Middle School in Mattawan, Michigan, and is the President-Elect for NCTM. He is interested in helping each and every student develop a conceptual understanding of mathematics.

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Appendix 11. Teaching is a Journey (Department) Article

TEACHING_IS_A_JOURNEY

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FUTURE GROWTH

Although I have highlighted only these four major shifts in my thinking and learning throughout my career so far, I have learned many other things along the way; and many more could help me learn and

improve to better meet the needs of each and every student. Each year, I determine an area I want to improve on and often share this with my students and encourage them to help keep me accountable. The following are two areas that I want to grow in.

1. Be more vocal in work around equity. I have been more of a quiet reflector and supporter in the past than an activist. I want to become more vocal in this work to continue challenging us to think about what must be done to help better meet the needs of each and every student and then take action. Through our collective efforts and consistently pushing one another, change is more likely to occur.
2. Improve my lesson closure. Often I don't do much at the end of the lesson other than dismiss students when the bell rings or have them work on some problems to help solidify their understanding. I want to improve on the closure because I know it is really more for the students' benefit than for mine.

If we want our students to appreciate and love learning, we must model that by continuing to grow professionally. I look forward to learning from and along with you in these next four years! —

Video 1 Kevin Dykema Shares His Teaching Journey



 Watch the full video online.

REFERENCE

National Council of Teachers of Mathematics (NCTM). 2014. *Principles to Actions: Ensuring Mathematical Success for All*. Reston, VA: NCTM.



Appendix 12. MTLT Frequently Asked Questions

Do I need to submit my digital assets at the same time as my manuscript?

Yes. Digital assets can be submitted as supplementary files. If the digital asset reveals the identity of the author, it should be submitted as a "supplementary file not for review."

Does every article require a digital asset?

No. At this time, we do not require the inclusion of a digital asset. However, digital assets are strongly encouraged.

Where can I find examples of digital-first articles?

[Appendix 1](#) in this Author Toolkit contains examples of articles taken from previous volumes of *MTLT* that showcase a variety of digital assets.

How long should my video recording be? Audio?

Typically, videos should be no longer than two or three minutes; otherwise you risk losing the attention of viewers. If you have longer videos, consider breaking the video into several smaller videos. Technical specifications for videos and other digital media can be found in [appendix 13](#).

What permissions do I need? (e.g. multimedia, parents/students, other)

Any time you capture video, audio, or written work of individuals, you need written permission. This is true for adults and children (parental permission needed for minors). Permission forms are available in [appendices 14, 15, 16, and 17](#).

How do I know when I have achieved a digital-first submission?

Digital-first submissions contain digital assets that help the reader be immersed in your content. Examples of various digital assets can be found in [appendix 1](#).

I do not have experience with digital-first publications. How do I work with MTLT to write?

The editor-in-chief is always happy to speak with potential authors and brainstorm ideas regarding how to use digital assets in preparing a manuscript. In addition, as a manuscript moves through the revision process, the associate editor will make recommendations for the inclusion of digital assets.

The associate editor has asked that I include a digital component with my revised manuscript. Is there financial support available for this?

No, this is not a service that NCTM can provide. We suggest conferring with your IT professionals or tech-savvy colleagues for help.

Which companies have contracts with NCTM as good resources regarding digital assets for my manuscript?

There are no companies that have contracts with NCTM for the purpose of developing digital assets. However, thus far, tech groups, such as Desmos and GeoGebra, have been willing to work with authors to feature their apps in *MTLT*.

How will my digital assets be referenced in my manuscript?

You should refer to your digital assets in the manuscript just as you would refer to a table or a figure. Statements such as, "See video 1 for a classroom excerpt," are best.



Appendix 12 (continued)

If I have a video, should I also include the dialogue from the video in the manuscript? I am thinking that some readers may not want to watch the video.

No. As the author, you should write your manuscript assuming that the reader watches the video. This assumption aligns with the [digital-first philosophy](#) of *MTLT*.

What multimedia sources are off limits?

We encourage creativity and innovative approaches to the use of appropriate multimedia.

Can I link to my YouTube channel (or other media source like this)?

In general, we would prefer to house all digital assets associated with *MTLT* articles on the journal's platform. We recognize, though, that this is not always possible. In these instances, authors are encouraged to work with the editor-in-chief.

I have an idea for a technology, but I do not know how to implement it. How do I get help?

We suggest conferring with your IT professionals or tech-savvy colleagues for help.

Appendix 13. Digital Assets and Figures: Instructions and Technical Specs

Video files*

As a digital-first journal, *MTLT* encourages authors to include video content that enhances the message in their manuscript. All multimedia content should be cited within the main text of the manuscript. In addition, a still image (and image description) from a relevant part of the video is needed for use in print.

A note about large file sizes: the ScholarOne system accepts files up to 350 MB in size. Should your multimedia or figure files exceed this limit, please contact the [editorial office](#) for additional instructions.

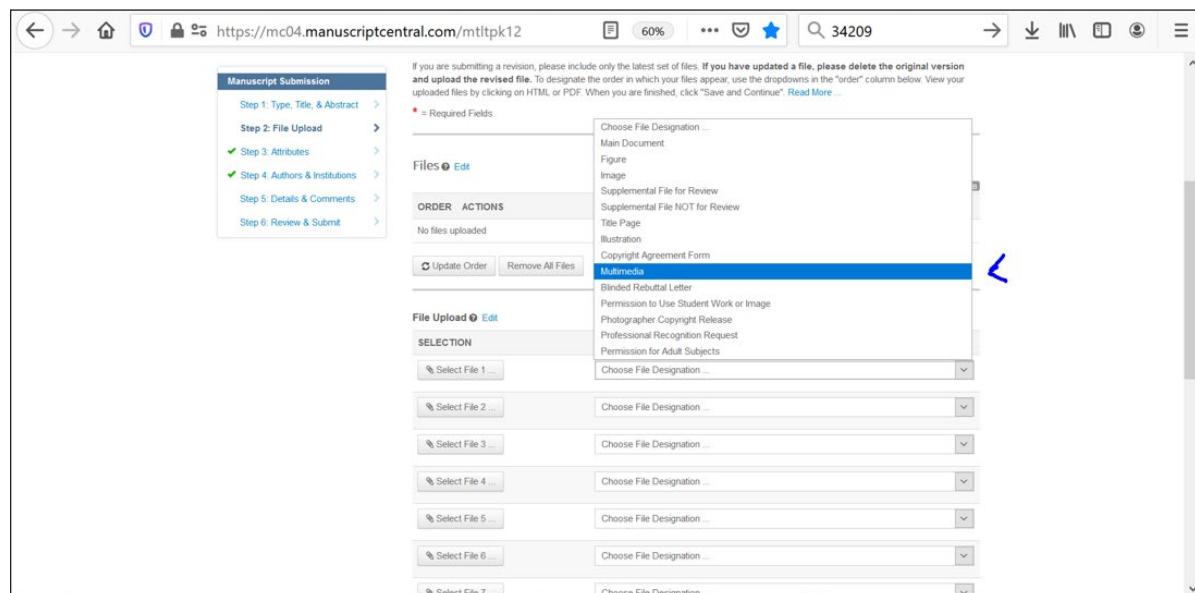


Specifications for video files:

- Duration: four minutes or less
- Formats accepted: MOV, MPG, AVI, FLV, F4V, MP4, M4V, ASF, WMV, VOB, MOD, 3GP, MKV, DIVX, XVID, WEBM
- Preferred file size: No more than 100 MB

Important! Permission to publish a video is required from all participants who are featured; to ensure that all legal requirements are met, the NCTM permission form ([appendix 14 or 15](#)) should be used when obtaining adult participants' signatures. For student participants who are under 18 years old, a parent/legal guardian permission form ([appendix 16 or 17](#)) is required instead. The signed permission releases should be uploaded into ScholarOne and the file type labeled as "Supplemental file not for review" during submission of the original manuscript. Permission to reproduce is also required from videographers and photographers involved in a video's creation.

When submitting to ScholarOne, during **Step 2: File Upload**, select "Multimedia," as shown here, for audio, video, or other digital-only file:



The screenshot shows the "Manuscript Submission" interface on a web browser. The left sidebar lists steps: Step 1: Type, Title, & Abstract, Step 2: File Upload (selected), Step 3: Attributes, Step 4: Authors & Institutions, Step 5: Details & Comments, and Step 6: Review & Submit. The main area has a heading "Files" with a "Edit" link. Below it is a table titled "ORDER ACTIONS" with one row "No files uploaded". There are buttons for "Update Order" and "Remove All Files". A note at the top says: "If you are submitting a revision, please include only the latest set of files. If you have updated a file, please delete the original version and upload the revised file. To designate the order in which your files appear, use the dropdowns in the 'order' column below. View your uploaded files by clicking on HTML or PDF. When you are finished, click 'Save and Continue'." A blue arrow points to the "Multimedia" option in a dropdown menu on the right side of the screen.

Appendix 13 (continued)

The still image from the video should be embedded in your manuscript file and have a legend accompanying it. If your paper is accepted, you will be asked to provide the still image as a separate file.

Choose File Designation ...
Main Document
Figure
Image
Supplemental File for Review
Supplemental File NOT for Review
Title Page
Illustration
Copyright Agreement Form
Multimedia
Blinded Rebuttal Letter
Permission to Use Student Work or Image
Photographer Copyright Release
Professional Recognition Request
Permission for Adult Subjects

When creating a still image, here are a few pointers:

- Try to capture a still image that gets the point of your video across. For example, if you are showcasing use of an app with PK-2 students, take a still image of the output from that app or capture an image of a child using the app.
- Avoid using a personal headshot for your still image.
- The legend explaining the image can be provided within the manuscript text file, placed below the still image.
- After Acceptance: Please label the image file in a way that links it to the video—e.g., "Still_Image_Video_1.eps"

YouTube Videos *

Videos featured on YouTube are welcome. For self-authored videos on an author's own channel, the journal strongly prefers that the video source file be provided

during submission, rather than as a link to the YouTube channel in the manuscript file. Authors can retain copyright on their YouTube videos if they specify this during submission; the source video file is generally required. If the video is from another channel, authors may instead provide the YouTube link within the text of their manuscript file.

A note about large file sizes: the ScholarOne system accepts files up to 350 MB in size. Should your multimedia or figure files exceed this limit, please contact the [editorial office](#) for additional instructions.

GeoGebra

If the GeoGebra is self-authored, please provide the original, raw .ggb file. If sourced from another user, please provide the correct link.

Desmos

Please provide the original Desmos link.



Appendix 13

(continued)

X

Geometer's Sketchpad

Please provide the raw file.

Other Apps *

(Such as PowerPoint, LiveScribe, simulations, and so on) Please provide the source file(s) when you upload your manuscript to ScholarOne.

A note about large file sizes: the ScholarOne system accepts files up to 350 MB in size. Should your multimedia or figure files exceed this limit, please contact the [editorial office](#) for additional instructions.

Images and Figures

At acceptance, images/figures must be provided in an acceptable format (see Artwork Guidelines on the next page for more information) at a resolution of *at least* 300 dpi and a width of *at least* 3 inches.

* A note about large file sizes: the ScholarOne submission system limits individual file sizes to 350 MB and 500 MB for the total of all files. Should your multimedia or figure files exceed this limit, please contact the [editorial office](#) for additional instructions.

An Author's Guide for Submitting Artwork

This brief overview highlights the main points to be aware of when submitting final production-quality artwork. When preparing figures, refer to printed copies of NCTM publications to get a sense of general size and style. The quality of the reproductions in your book can never be better than the original material you submit to us, so providing the best quality imagery is imperative.

If we judge a piece of artwork to be substandard for printing, we will ask you to provide a suitable replacement or to eliminate the image entirely.

BEFORE SUBMITTING YOUR DIGITAL ART

DO check the resolution of your files to be certain they meet NCTM requirements.

DO submit labeled printouts of all files that correspond to their respective placement in layout. You *may* embed art in the Word document, but you **must** also provide a JPEG, BMP, TIFF, or EPS file.

MISTAKES TO AVOID

DO NOT assume that a file that looks good on a computer screen is acceptable for print reproduction.

DO NOT submit digital images as PowerPoint files.

DO NOT edit or re-save JPEG files (see "A NOTE ON JPEGS" sidebar).

DO NOT enlarge substandard files (see "A NOTE ON RESOLUTION" sidebar).

RESOLUTION REQUIREMENTS FOR DIGITAL ART

Digital art renders images as a finite number of pixels (ppi), or dots, per inch. The resolution of a file, measured in ppi, limits the size at which a piece of digital art can be reproduced.

BASIC REQUIREMENTS FOR DIGITAL ART

Continuous-Tone Images – 300 Pixels per Inch



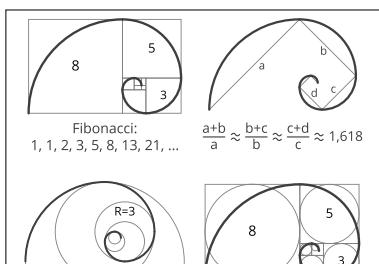
An example of a continuous-tone (raster) image

In continuous-tone images, or raster art, each pixel can vary in color and tonality; transitions from light to dark appear smooth and realistic.

To ensure quality reproduction of continuous-tone images, files must have a resolution of at least 300 ppi. For example, a continuous-tone image with dimensions of 900 × 1500 pixels can be reproduced no larger than 3" × 5".

$$\frac{900 \text{ pixels}}{300 \text{ ppi}} \times \frac{1500 \text{ pixels}}{300 \text{ ppi}} = 3" \times 5"$$

Bitonal Images – 1200 Pixels per Inch



An example of a bitonal (vector) image

In bitonal images, or **vector art**, each pixel will be one of two values: 100 percent black or 100 percent white. Such pieces of art require higher resolution to ensure quality reproduction.

The file must have a resolution of at least 1200 ppi. For example, a bitonal scan with dimensions of 3600 × 6000 pixels can be reproduced no larger than 3" × 5".

$$\frac{3600 \text{ pixels}}{1200 \text{ ppi}} \times \frac{6000 \text{ pixels}}{1200 \text{ ppi}} = 3" \times 5"$$

HOW TO CHECK FILE RESOLUTION ON A PC

Right-Click on the file and select "Properties" from the menu. Click the "Summary" tab at the top of the "Properties" dialog box. Click the "Advanced" button in the Summary window to display the width and height of the file in pixels.

HOW TO CHECK FILE RESOLUTION ON A MAC

Press the "Control" key, click on the file, and select "Get Info" from the drop-down menu. An "Info" dialog box will appear, and the dimensions of the file will be listed under the "More Info" section of the dialog box.



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A NOTE ON JPEGS

JPEG files compress their data to achieve a smaller, more portable file size. This compression is accomplished by discarding some of the data that comprises the image. Each time a JPEG is opened and re-saved in the JPEG file format, the image deteriorates. If you acquire an image from a library, museum, or stock photo agency, request EPS or TIFF format. If JPEGs are the only file format available, do not edit or re-save the image before submitting it. Still, when dealing with JPEGs, do make a copy of the original file as a back-up.

To ensure image integrity, never re-save a JPEG file. If you need to rename a JPEG, right-click on the file and select "Rename" from your menu options. DO NOT rename a JPEG file by opening it and using the "Save As" option.

A NOTE ON RESOLUTION

Some computer programs will allow you to artificially add resolution to a digital file. This will not improve the quality of the image. On the contrary, the image will become fuzzy and pixilated. If you find yourself tempted to enlarge a file, it is a sign that the file is substandard and should not be used.

A NOTE ON IMAGES FOUND ONLINE

Authors must secure rights for any imagery found on the internet that is intended for use in the book.

Computer screens display only 72 pixels per inch. Most images on the web are sized accordingly and are unacceptable for print publication. An image measuring 216 pixels × 360 pixels may appear to be 3" × 5" at 72 ppi on your computer screen, but at the 300 ppi standard required for print reproduction, the maximum size of the image is only about 3/4" × 1".



Appendix 14. Figure/Multimedia Permission Form—Adult Participants



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RELEASE FORM FOR VIDEO AND/OR PHOTOGRAPHS

I, _____ (print name), hereby give my permission to be photographed, videotaped, and/or audio taped. I also authorize these photographs, videos, audio recordings, and/or work samples to be used in all formats and methods of delivery by the National Council of Teachers of Mathematics (NCTM).

I hereby grant permission to the rights of my image, likeness, and sound of my voice as recorded on audio or video tape without payment, attribution, or any other consideration. I understand that my image may be edited, copied, exhibited, published or distributed and waive the right to inspect or approve the finished product wherein my likeness appears.

Additionally, I waive any right to any and all compensation arising or related to the use of my image or recording. I also understand that this material may be used in diverse educational settings within an unrestricted geographic area.

I hereby release NCTM, and any of its associates or affiliates, their directors, officers, agents, employees and customers from all claims of every kind on account of such use.

Full Name(s): _____

Address: _____

Signature(s): _____

Date: _____



Appendix 15. Figure/Multimedia Permission Form—Adult Participants (in Spanish)



NATIONAL COUNCIL OF
TEACHERS OF MATHEMATICS

AUTORIZACIÓN PARA VIDEO O FOTOGRAFÍAS

Yo, _____ (nombre en letra de imprenta), por medio de este documento, doy mi autorización para que se me fotografíe, grabe en video o audio. También autorizo que el National Council of Teachers of Mathematics (NCTM) utilice estas fotografías, videos, grabaciones de audio o muestras de trabajo en todos los formatos y métodos de entrega.

Por medio de este documento, otorgo autorización sobre los derechos de mi imagen, retrato y sonido de mi voz conforme se registrasen en audio o video sin ningún pago, atribución o cualquier otra contraprestación. Comprendo que mi imagen se puede editar, copiar, exhibir, publicar o distribuir y renuncio al derecho de inspeccionar o aprobar el producto finalizado donde mi imagen apareciera.

Adicionalmente, renuncio a cualquier derecho a toda compensación que surgiese o se relacionase con el uso de mi imagen o grabación. También comprendo que este material puede utilizarse en diversos entornos educativos dentro de un área geográfica no restringida.

Por medio de este documento, exono a NCTM y cualquiera de sus entidades asociadas o filiales, sus directores, ejecutivos, agentes, empleados y clientes de todo reclamo de cualquier tipo surgido a causa de dicho uso.

Nombres completos: _____

Dirección: _____

Firmas: _____

Fecha: _____



Appendix 16. Figure/Multimedia Parental Permission Form—Participants <18 Years of Age



NATIONAL COUNCIL OF
TEACHERS OF MATHEMATICS

RELEASE FORM FOR USE OF STUDENT'S IMAGE OR WRITTEN WORK

Permission is necessary when a student can be identified in photograph(s) or in a video or audio recording, or if their name appears on their written work. Similar forms used by student's school or author's institution may be substituted for form below.

I give my permission for _____'s image to be photographed, videotaped, audiotaped. I also authorize these photographs, videos, audio records, and/or work samples to be used in print, and electronic, and web-based media by the National Council of Teachers of Mathematics (NCTM).

I hereby release NCTM, and any of its associates or affiliates, their directors, officers, agents, employees, and customers from all claims of every kind on account of such use.

Author: _____

Name: _____
(Parent or Guardian)

Home address: _____

Signature: _____

Date: _____



Appendix 17. Figure/Multimedia Parental Permission Form—Participants <18 Years of Age (in Spanish)



NATIONAL COUNCIL OF
TEACHERS OF MATHEMATICS

FORMULARIO DE AUTORIZACIÓN PARA EL USO DE LA IMAGEN O EL TRABAJO ESCRITO DEL ESTUDIANTE

Es necesario contar con una autorización cuando un estudiante se puede identificar en fotografías o en una grabación de audio o video o si su nombre apareciera en el trabajo escrito. Los formularios similares utilizados por la escuela del estudiante o la institución del autor pueden ser sustituidos por el formulario siguiente.

Doy mi autorización para que se grabe en video, se grabe en audio o se tomen fotografías de la imagen de _____. También autorizo que el National Council of Teachers of Mathematics (NCTM) utilice estas fotografías, videos, grabaciones de audio o muestras de trabajo en medios impresos, electrónicos y web.

Por medio de este documento, exono a NCTM y a cualquiera de su asociados o filiales, sus directores, ejecutivos, agentes, empleados y clientes de todo reclamo de cualquier tipo generado por dicho uso.

Revista: _____

Autor: _____

Nombre: _____
(Padre o tutor)

Dirección residencial _____

Firma: _____

Fecha: _____

1906 ASSOCIATION DRIVE
RESTON, VA 20191-1502
TEL: (703) 620-9840
FAX: (703) 476-2970
WWW.NCTM.ORG



Appendix 18. Sample Manuscript Submission with Cover Letters

Appendix 18. Sample Manuscript Files

COVER LETTER SAMPLE

Mathematics Teacher: Learning and Teaching Pre-K–12

Cover Letter: MTLTPK12-2020-0062

Thank you for considering our manuscript. It is not under review by any other journals at this time.

A. Barlow

**REVISED MANUSCRIPT AUTHOR RESPONSE LETTER
SAMPLE**

May 12, 2020

To Whom It May Concern:

Notice that the authors sign the letter anonymously; additionally, they have responded to editor and reviewer comments in a point-by-point summary response.

Thank you for the opportunity to revise our manuscript, “Maximizing the Impact of Mobile Devices.” We believe the reviewer feedback has resulted in a stronger manuscript.

In response to the feedback, the following changes have been made.

- We have inserted a table that has questions to ask related to each of the features.
- We have added a statement to the “Why Attend . . .” section to indicate that it is not necessary to have all three features in a single lesson.
- We made the requested change to wording and added supporting citations (Process Standards, Common Core (SMP), Mathematics Teaching Practices, and Strands of Mathematical Proficiency (NRC)).
- We added a sentence to bring clarity to why this addition app video case is an example of personalization.
- The AE comment regarding value added helped us to better articulate the role of mobile technology and the descriptions in the framework. Therefore, we have utilized the “value-added” phrase throughout. We feel this helps not only to connect back to Janet’s story at the beginning but supports the reader’s understanding of the purpose of the framework.
- The presentation of the three categories in the text has been changed to match the introductory video.
- We added language to the statement connected to the first video to clarify that its intent is to explain what is meant by mediating the student’s experience.
- Statements have been added to personalization and authenticity to highlight why they are examples of the associated feature.

Thank you for this opportunity to revise and resubmit. We look forward to hearing from you soon.

Sincerely,
The Authors

MANUSCRIPT SUBMISSION SAMPLE

Notice the inclusion of digital assets; these are not required but are preferred. Be sure anonymity is maintained when adding digital links. If that is not possible, remove links and place in a separate file categorized as "Supplemental files NOT for review"

Maximizing the Impact of Mobile Devices

Janet's elementary school recently purchased a mobile cart of iPads. Eager to use them with her students, Janet began searching for mathematics lessons that incorporated iPads. She summarized her findings by saying, "I found a lot of lessons that used iPads. I wasn't sure, though, if the iPad was influencing the students' learning. Most of the lessons that I found could have been taught without the technology --- like students completing a worksheet on the iPad."

Like Janet, we wondered how to utilize iPads and other mobile devices in elementary mathematics instruction in impactful ways. This led us to the M-Learning Framework (Kearney, Schuck, Burden, and Aubusson 2012), which considers the use of mobile technology as a mediator of students' learning experiences. In this article, our goal is to highlight the M-Learning Framework as a way to view effective use of mobile devices, such as iPads, in the elementary mathematics classroom. Following an overview of the framework, we provide three video cases as examples of the framework's key features along with a description of why attending to these features is important.

The M-Learning Framework: An Overview

When discussing mobile learning (or m-learning), many educators tend to focus on the affordances of the technology (Traxler 2007). From the earlier example of accessing a worksheet via an iPad, such affordances include ease of distribution/collection of the worksheet, which is an affordance related to the teacher's perspective regarding classroom management. In contrast, the M-Learning Framework (Kearney et al. 2012) focuses on the technology from the learner's perspective. Specifically, attention is given to how the mobile device mediates the students' learning experiences (see video 1 M learning w music.mov). The framework includes three key

features: authenticity, collaboration, and personalization. The features are discussed in the following sections.

Authenticity

Mobile technology can be used to provide students with access to meaningful practices in the real world. This feature of mobile technology is referred to as authenticity and can be considered at three levels: problems set in real-word contexts (task authenticity); realistic task details (factual authenticity); and true-to-life engagement in the practices of a community (process authenticity). Regardless of the level, the key is that the mobile device mediates the authentic experience.

As an example, consider the video case involving the Paper Chain Problem ([insert video 2 Paperchain3.mov](#)). In this lesson, the teacher encouraged students to engage in the practices of mathematicians, specifically creating representations of problems. The mobile technology provided the tools for students to engage in this practice, as students chose to use virtual base-ten blocks. In this way, the use of the mobile device exemplified the m-learning feature of process authenticity.

Collaboration

A second key feature identified in the M-Learning Framework is collaboration. Often involving conversation, dialogue, and/or feedback, collaboration is an aspect of a student's learning experience that can be mediated by mobile technology. As an example, the Maximum Area Challenge video case ([insert video 3 Maximum Area.mov](#)) features students receiving feedback from a local preservice teacher. The communication that occurred between the students and the preservice teacher was made possible by the mobile device and, therefore, represented the m-learning feature of collaboration.

Use standard (and consistent!) formatting in MS Word to distinguish between heading levels. MTLT allows up to four levels of headings.

Personalization

Finally, mobile devices can provide learners with opportunities for customization of their learning experiences as well as learner choice. Further, by giving the learner control over the pace and goals of learning activities, the mobile device provides opportunities for agency and self-regulation. These features (i.e., customization, learner choice, agency, and self-regulation) represent aspects of personalization.

The Addition App video case ([insert video 4 new Addition App.mov](#)) shows students working through addition problems via an app. As demonstrated in the video case, the students' correct and incorrect answers determined the next problem to be presented to the student. In this way, the mobile device provided a customized learning experience for the students, which represented the key feature of personalization.

Why Attend to these Features

Mathematics lessons that are characterized by authenticity, collaboration, and/or personalization generally represent strong lessons, regardless of whether these characteristics are mediated by a mobile device. For example, collaboration among students that results from the proximity of the students is just as beneficial for the learning process as collaboration that results from the use of a mobile device. However, when the collaboration is mediated by a mobile device, it opens conversations that might not be possible otherwise and can broaden and enrich the learning process. Imagine examining the solution processes of students from another part of the country – or the world! Imagine students receiving feedback from a mathematician or other professional. Mobile devices can be used to virtually transport students beyond the classroom to access contexts to be modeled (authenticity) or to access learning supports that are tailored to the needs of the individual student (personalization). These key features of the M-Learning

All references should be listed in the text and vice versa. Self-citations should be referred to as "Author (year)" in text as in the reference list. No additional publication information should be given in the reference list.

Framework (Kearney et al. 2012) can guide teachers in evaluating the use of mobile devices in mathematics lessons.

Conclusion

The M-Learning Framework (Kearney et al. 2012) provides a means for thinking about how to use mobile devices, such as iPads, in ways that positively influence students' learning experiences. The three key features (i.e., authenticity, collaboration, and personalization) represent aspects of effective mathematics instruction that have been described in documents such as the Standards for Mathematical Practice (NGA Center and CCSSO 2010) and the Mathematics Teaching Practices (NCTM 2014). The framework, though, brings attention to the use of mobile devices in mediating these aspects of effective mathematics instruction and, in doing so, expanding the opportunities for students' learning. Our hope is that as teachers examine mathematics lessons that incorporate mobile devices, they will consider this framework so as to maximize the technology's affordances and positively influence students' learning.

References

- Kearney, Matthew, Sandra Schuck, Kevin Burden, and Peter Aubusson. 2012. "Viewing Mobile Learning from a Pedagogical Perspective." *Research in Learning Technology*, 20: 14406.
- National Council of Teachers of Mathematics (NCTM). 2014. *Principles to Actions: Ensuring Mathematical Success for All*. Reston, VA: NCTM.
- National Governors Association Center for Best Practices (NGA Center) and the Council of Chief State School Officers (CCSSO). 2010. *Common Core State Standards for Mathematics*. Washington, DC: NGA Center and CCSSO. <http://coresstandards.org>.



Appendix 19. MTLT Reference Style Examples

Book

Shute, William, William Shirk, and George Porter. 1957. *Solid Geometry*. New York: American Book.

Electronic Book

Shaughnessy, J. Michael, Beth Chance, and Henry Kranendonk. 2009. *Focus in High School Mathematics: Reasoning and Sense Making in Statistics and Probability*. Reston, VA: National Council of Teachers of Mathematics. PDF e-book.

For a downloaded e-book, indicate format as the last part of the citation. For a book consulted online, indicate the URL or the DOI as the last part of the citation. For a freely available electronic edition of an older work, include the URL as the last element.

Chapter in a Book

Oakes, Jeannie. 1995. "Opportunity to Learn: Can Standards-Based Reform Be Equity-Based Reform?" In *Seventy-Five Years of Progress: Prospects for School Mathematics*, edited by Iris M. Carl, pp. 78-98. Reston, VA: National Council of Teachers of Mathematics.

PhD Dissertation

Akdag, Fusun Semiha. 1985. "The Effects of Computer Programming on Young Children's Learning." PhD diss., The Ohio State University.

McGalliard, William A. 1982. "Selected Factors in the Conceptual Systems of Geometry Teachers." Doctoral diss., University of Georgia. Dissertation Abstracts International 44 (1983): 1364A.

Paper Presented at a Conference

Lehrer, Richard, and Paul Smith. 1986. "Logo Learning: Is More Better?" Paper presented at the Annual Meeting of the American Educational Research Association, San Francisco, April 16-20, 1986.

Journal Article

Craine, Timothy, and Rheta Rubenstein. 1993. "A Quadrilateral Hierarchy to Facilitate Learning in Geometry." *Mathematics Teacher* 86, no. 1 (January): 30-36. Please note that we do not include doi numbers.

Electronic Journal Article

Battista, Michael T. 2002. "Building Properly Structured Mental Models for Reasoning about Volume." *ON-Math: Online Journal of School Mathematics* 1 (Winter).

<https://www.semanticscholar.org/paper/Building-Properly-Structured-Mental-Models-for-Battista/54f8fd68c2f1dd15b40db52e2d9e5e4926b7754>.

Newspaper Article

Braun, R. J. 1993. "Schools That Flunk." *Star-Ledger* Special Report series. Newark (NJ) *Star-Ledger*, October 24-31, 1993.

Tyler, Marshall. 1985. "200th Birthday." *Los Angeles Times*. March 15, 1985, sec. 1A, p. 3.