Radial Light Rays, Conformal Time and Redshift

1. **Radial Light Rays**

2. **Conformal Time**

3. **Redshift**

Radial Light Rays

Radial Light Rays in FRW Space–time

- Metric in χ form:
$$ds^2 = dt^2 - a^2(t)\left[d\chi^2 + S_K^2(\chi)\, d\Omega^2\right]$$
- Studying light propagation. For a radial photon, $d\Omega^2 = 0$ and $ds^2 = 0$, so
$$0 = dt^2 - a^2(t)\, d\chi^2 \quad \Longrightarrow \quad \frac{d\chi}{dt} = \pm\frac{1}{a(t)}$$
- Comoving radial distance. The comoving interval χ that a photon covers is obtained by integrating $d\chi = \pm\dfrac{dt}{a(t)}$ over time.

Still Light Rays

- Interpretation of coordinates:
  χ is a pure comoving radial coordinate, adapted to the spatial curvature.
  Angular and radial labels describe the shape of each spatial slice; they are fixed for comoving objects and do not depend on cosmic time.
- Scale factor $a(t)$ governs how the Universe expands. Any comoving interval $d\chi$ converts to proper length via $d\ell = a(t)\, d\chi$.
- Curvature determines how much transverse distance corresponds to a given χ, encoded in $S_K(\chi)$.

Still still Light Rays

- The coordinate χ marks *your* comoving position; it follows the Hubble flow and is the natural radial label that matches the spatial curvature.
- Homogeneity and isotropy ensure that the labels (θ, ϕ, χ) stay fixed for any comoving observer even as the Universe expands.

Scale Factor $a(t)$

- The scale factor $a(t)$ converts a comoving coordinate x^i into a physical coordinate:

$$x^i_{\mathsf{phys}} = a(t)\, x^i.$$

- Spatial line element in χ coordinates:

$$dl^2 = a^2(t)\big[d\chi^2 + S_K^2(\chi)\, d\Omega^2\big].$$

Scale Factor $a(t)$ and Proper Lengths

- Along the radial direction, a small comoving step $d\chi$ becomes a tangential physical distance
$$d_{\|} = a(t)\, d\chi$$
- A transverse arc at fixed χ (like normal distance of $d\theta$) has length
$$dl_{\perp} = a(t)\, S_K(\chi)\, d\theta,$$
where $S_K(\chi)$ encodes curvature while $a(t)$ rescales the entire spatial grid uniformly.
- In short, $S_K(\chi)$ fixes shape; $a(t)$ controls overall size.

Conformal Time

Conformal Time

- Definition of conformal time τ:
$$d\tau = \frac{dt}{a(t)},$$
i.e. cosmic time rescaled by the scale factor.
- Intuitive picture:
dt is the actual, "stretched" physical time.
$d\tau$ behaves like the time measured by the comoving grid; it removes the stretching due to $a(t)$.

Conformal Time

- By integrating

$$\tau(t) = \int^t \frac{dt'}{a(t')},$$

  one can invert to get $t(\tau)$ whenever needed.
- In conformal coordinates, the factor $a(t)$ is pulled outside the metric; light rays then follow 45-degree lines just as they do in flat space, simplifying many calculations.

FRW Metric in Conformal Form

- Put the scale factor back into the metric:

$$ds^2 = a^2(\tau)\Big[d\tau^2 - \big(d\chi^2 + S_K^2(\chi)\,d\Omega^2\big)\Big].$$

- Interpretation: a static Minkowski metric multiplied by the time-dependent factor $a^2(\tau)$.

- For light (null geodesics) we set $ds^2 = 0$. Therefore

$$d\tau = \pm\,d\chi,$$

  which is the *same condition* as in flat Minkowski space.

Redshift

Red-shift of Photons in an Expanding Universe

- For a photon, wavelengthλ is proportional to the scale factor $a(t)$:

$$\lambda \propto a(t).$$

- Comparing emission at timet_{e} to observation at timet_{o},

$$\lambda_{\mathrm{o}} = \frac{a(t_{\mathrm{o}})}{a(t_{\mathrm{e}})}\,\lambda_{\mathrm{e}}.$$

- Because the Universe has expanded

$$a(t_{\mathrm{o}}) > a(t_{\mathrm{e}})$$

  the observed wavelength is longer, which stretches with the cosmic expansion, producing the cosmological red-shift.:

$$\lambda_{\mathrm{o}} > \lambda_{\mathrm{e}}.$$

Classical Waves' Redshift

- Classical waves experience the same cosmological red-shift as light.

- Setup: a galaxy sits at a fixed comoving distance d and emits a signal at cosmic time t_{e}.
  For a radial null path in FRW,

$$d\tau = d\chi, \qquad \chi = d \implies \Delta\tau = \Delta\chi = \Delta d.$$

$$\tau_e = \tau_0 + \Delta d$$

As τ is stretched with the "grid", it is measured with an "incorrect" scale.

Conformal Time and Physical Time

- The emission and observation point both have the conformal time interval $\Delta\tau$.But to obtain the physical time t, we should multiply different scale factor $a(t)$ with t at t_e and t_o.
- $$\Delta t_e = a(t_e)\Delta\tau$$

  is the lasting physical time at the emission point.

  $$\Delta t_o = a(t_o)\Delta\tau$$

  is the lasting physical time at the observed point.

Cosmological Redshift

- Setting $c = 1$, period T equals wavelength λ, so

$$\lambda_{\mathrm{e}} = \Delta t_{\mathrm{e}}$$

$$\lambda_{\mathrm{o}} = \Delta t_{\mathrm{o}}.$$

- Therefore

$$\frac{\lambda_{\mathrm{o}}}{\lambda_{\mathrm{e}}} = \frac{a(t_{\mathrm{o}})\,\Delta\tau}{a(t_{\mathrm{e}})\,\Delta\tau} = \frac{a(t_{\mathrm{o}})}{a(t_{\mathrm{e}})} = 1 + z,$$

  which is the cosmological red-shift relation.