Line Element and FRW Metric

1. Line Element in Flat Space

2. Friedmann-Robertson-Walker Metric

Line Element in Flat Space

Line Element Introduction

- General form:

$$ds^2 = g_{\mu\nu}\, dx^\mu dx^\nu$$

- In flat (Euclidean) 3-space:

$$dl^2 = \delta_{ij}\, dx^i dx^j$$

And it is invariant under spatial translations:

$$x'^i = x^i + a^i$$

$$a^i = \text{const.} \Rightarrow dl^2 \text{unchanged}$$

An Example of Rotation

- Invariance under rotations:
  $x'^i = R^i{}_k \, x^k$
  Example (rotation about z-axis):

$$R_z(\phi) = \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

  - Rotation matrices are orthonormal: $R^T R = \mathbf{1}$
  -
$$\delta_{ij} \, R^i{}_k R^j{}_\ell = \delta_{k\ell}$$
  - Metric is preserved:
$$g'_{k\ell} = g_{k\ell} = \delta_{k\ell}$$
- Geometry unchanged $\implies$ isotropy

Positively Curved Space: the 3-Sphere S^3

- Constant positive curvature $\implies$ 3-sphere embedded in $\mathbb{E}^4$
- Ambient coordinates (x^1, x^2, x^3, u) with Euclidean metric

$$d\ell^2 = (dx^1)^2 + (dx^2)^2 + (dx^3)^2 + du^2$$

- Constraint (radius a) defining the hypersurface

$$(x^1)^2 + (x^2)^2 + (x^3)^2 + u^2 = a^2$$

- Isotropy of $\mathbb{E}^4$ induces isotropy on S^3

Spherical Coordinates on S^3

- Use coordinates (χ, θ, ϕ):

$$u = a\cos\chi, \qquad r = \sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2} = a\sin\chi$$

  with angular unit vector $n^i(\theta, \phi)$ on S^2.
- Embedding: $x^i = r\, n^i(\theta, \phi) \;\; (n^i n_i = 1)$
- Differential relations give the induced metric

$$d\ell^2 = a^2\left[d\chi^2 + \sin^2\chi\,(d\theta^2 + \sin^2\theta\, d\phi^2)\right]$$

- In terms of r:

$$d\ell^2 = \frac{dr^2}{1 - r^2/a^2} + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$$

- Positive curvature $k = +1$ (unit 3-sphere after rescaling)

Friedmann-Robertson-Walker Metric

FRW line element with positive spatial curvature

- Line element for a homogeneous, isotropic, expanding/contracting universe:

$$ds^2 = -c^2 dt^2 + a^2(t)\left[\frac{dr^2}{1 - \frac{r^2}{a^2}} + r^2 d\Omega^2\right], \qquad d\Omega^2 = d\theta^2 + \sin^2\theta\, d\phi^2$$

- Spatial slices ($t = \text{const.}$) are 3-spheres (*constant positive curvature*, curvature radius$a$)
- Scale factor $a(t)$ encodes cosmic expansion or contraction
- Comoving coordinate r labels points fixed in the expanding grid

Physical meaning of “curved” and “expanding”

- **What is curved?** The *geometry of each spatial slice*; curvature $k = +1$
- **What changes with time?** The overall *size* of those slices, set by $a(t)$
- Cosmic expansion $\uparrow a(t)$ increases the distance between freely moving, unbound objects
  - Galaxies that are not gravitationally bound recede
  - Light from distant objects is red-shifted
- **But:** Expansion does *not* stretch objects held together by forces (atoms, rulers, planets, etc. remain the same proper size)
- Comoving length unit stays fixed in r; a physical ruler has constant proper length 1 cm even while its comoving separation changes

Comoving Coordinates

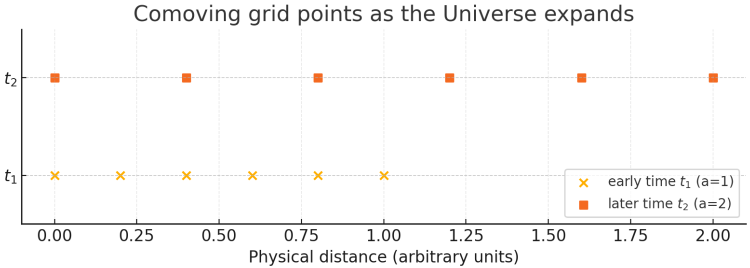

Figure: Comoving Coordinate

Curvature Radius and Spatial Line Element

- R_0 — **curvature radius** (a constant)
  - Think of an embedded sphere of radius R_0; the radius is fixed.
- Metric on the 2-sphere:

$$d\Omega^2 = d\theta^2 + \sin^2\theta\, d\phi^2$$

- General factor $\dfrac{dr^2}{1 - K\, r^2}$ with curvature index $K = +1, 0, -1$
- Define the function

$$S_K(x) = \begin{cases} \sin x, & K = +1, \\ x, & K = 0, \\ \sinh x, & K = -1, \end{cases}$$

- Spatial line element:

$$dl^2 = a^2(t)\, R_0^2 \Big[\, dx^2 + S_K^2(x)\, d\Omega^2 \Big]$$

where $a(t)$ is the scale factor.

FRW Metric and Coordinate Types

- **Spatial line element** (fixed cosmic time):

$$dl^2 = a^2(t)\,\gamma_{ij}\,dx^i dx^j$$

- **Full space–time metric:**

$$ds^2 = dt^2 - a^2(t)\,\gamma_{ij}\,dx^i dx^j$$

- **Comoving coordinates** $x^i \equiv \{x^1, x^2, x^3\}$ stay fixed with the expanding grid.
- **Physical coordinates**

$$x^i_{\text{phys}} = a(t)\,x^i$$

- Example: if $x^i = 1$cm and the scale factor doubles, x^i_{phys} likewise doubles to 2cm.

Velocities: Hubble Flow vs. Peculiar Motion

- **Physical velocity**

$$V^i_{\text{phys}} = \frac{dx^i_{\text{phys}}}{dt} = a(t)\frac{dx^i}{dt} + \dot{a}\,x^i \;=\; V^i_{\text{pec}} + H\,x^i_{\text{phys}}$$

where $H \equiv \dot{a}/a$.

- **Peculiar velocity**

$$V^i_{\text{pec}} \equiv a(t)\,\dot{x}^i$$

measured by a locally comoving (non-cosmic) observer.

Velocities: Hubble Flow vs. Peculiar Motion

- Hubble flow $H\,x^i_{\text{phys}}$ — recession caused solely by expansion of the grid.
- *Hubble flow*: large-scale, uniform expansion.
- *Peculiar velocity*: motion relative to the comoving grid (e.g. galaxy orbit in a cluster).
- A local ruler does not stretch; physical separations grow only through $a(t)$.
- If V_{pec} dominates over V_{phys}, objects can move inward locally even while the Universe expands globally.

Bound Objects Do Not Stretch with Expansion

- Cosmic expansion changes *grid spacing* but leaves the **proper length of bound objects** (rulers, atoms, galaxies, solar systems) unchanged.
- Example: a ruler of proper length $L = 3$ m:
  Comoving (peculiar) separation: $x_{\text{pec}} = 3$ m, $\dot{x}_{\text{pec}} = 0$.
  No internal stretching; length is measured locally, not by the cosmic scale factor.

Bound Objects Do Not Stretch with Expansion

- Physical velocity of any point on the ruler

$$V^i_{\mathsf{phys}} = H\, x^i_{\mathsf{phys}} \quad \text{when} \quad V^i_{\mathsf{pec}} = 0,$$

  i.e. it only participates in the global Hubble flow as a rigid body.
- Illustrated at right: the surrounding grid expands, but the ruler remains 3 m long

FRW Metric and a Useful Rescaling

- **Standard form**

$$ds^2 = dt^2 - a^2(t)\left[\frac{dr^2}{1 - K\,r^2} + r^2 d\Omega^2\right]$$

- **Rescaling symmetry** If we scale the coordinates and parameters as

$$a(t) \longrightarrow \lambda\, a(t)$$

$$r \longrightarrow \frac{r}{\lambda}$$

$$K \longrightarrow \lambda^2 K,$$

then the line element ds^2 remains unchanged.

FRW Metric and a Useful Rescaling

- Alternative comoving radial variable Define χ by $r = S_K(\chi)$; the metric becomes

$$ds^2 = dt^2 - a^2(t)\left[\, d\chi^2 + S_K^2(\chi)\, d\Omega^2\right],$$

$$S_K(\chi) = \begin{cases} \sin\chi & (K = +1), \\ \chi & (K = 0), \\ \sinh\chi & (K = -1). \end{cases}$$

- Motivation: this form is more convenient when studying the propagation of light (null geodesics) in an FRW universe.

Thank You!!!!!!!!!!!!!!!!

Thank you so much for reading this slide:))))