Derivation of Friedmann Equations

1. **Energy Momentum Tensor**

2. **Perfect Fluid**

3. **Friedmann Equations**

Energy Momentum Tensor

General Figure of Energy Momentum Tensor

Any component in $T^{\alpha\beta}$ shows how much α-momentum is flowing through it per unit time per unit area.
$T^{\alpha\beta} =$ flux of α-momentum across a surface of constant x^β.

Key features:

- $\alpha : p^\alpha = [E, p^1, p^2, p^3]$ α give which component we are talking.
- β :through which 'surface' we are measuring transport. (A surface which spacetime coordinate x^β is constant.
- Symmetric: $T_{\alpha\beta} = T_{\beta\alpha}$
- Conserved: $\nabla^\alpha T_{\alpha\beta} = 0$

Components

$$
\begin{aligned}
T^{00} &= \text{flux of 0-momentum, i.e. energy, across surfaces } t = \text{const} \\
&= \text{energy density,} \\
T^{0i} &= \text{energy flux across surface } x^i = \text{const,} \\
T^{i0} &= \text{flux of momentum in the } x^i \text{ direction across surfaces } t = \text{const} \\
&= x^i\text{-momentum density,} \\
T^{ij} &= \text{flux of } x^i \text{ momentum across surfaces } x^j = \text{const,}
\end{aligned}
$$

Something else but related

All fluxes are measured by an observer momentarily at rest in a local inertial frame comoving with the matter element at point p. Often, this follows the covariance principle.

- Start with normal coordinates, find the energy momentum tensor from the local laws in special relativity.
- Generalize the tensor to arbitrary coordinates using the coordinate invariance of tensors.

Perfect Fluid

Definitions of Perfect Fluid

A perfect fluid is a continuous matter distribution with no viscosity and no heat conduction in the locally comoving frame.
The form of the energy momentum tensor for this type of matter follows from looking more closely at the meaning of “no viscosity” and “no heat conduction”.
*Viscosity is defined as a force component exerted by one particle on another that perpendicular to the line of sight of 2 particles.

Features

Key Features:

- No heat conduction: E_{total} of a particle contains some internal energy. In this scenario, the internal energy is not transferred to another particle.
  So, the energy can only flow if the particle flows.
- No viscosity: the force between 2 particles only changes the momentum in the direction along their line of sight. (Forces only have radial component)

No Viscosity

Key Features:

- Rotating the coordinate to the direction coincides with $x^i$for some fixed i, which will not loss generality. So, the only momentum that can flow this direction is then the p^i component.
- $T^{ij} \neq 0$ only for $i = j$.

  No direction preferred $\Rightarrow \mathsf{T}^{11} = T^{22} = T^{33} =: P$

The Equation

- in special relativity in the locally comoving frame $u^\alpha = [1, 0, 0, 0]$

$$T^{\alpha\beta} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & P & 0 & 0 \\ 0 & 0 & P & 0 \\ 0 & 0 & 0 & P \end{pmatrix} \overset{!}{=} (\rho + P) u^\alpha u^\beta + P \eta^{\alpha\beta}.$$

- The general relativistic expression follows from replacing $\eta^{\alpha\beta}$ with $g^{\alpha\beta}$ according to the covariance principle, so that

$$\boxed{T^{\alpha\beta} = (\rho + P) u^\alpha u^\beta + P g^{\alpha\beta}} \tag{1}$$

Back to Classical Fluid Mechanics

- V is an arbitrary vector:

$$\hat{\mathbf{n}} \cdot \mathbf{V} = V_{\parallel}$$

-

$$\mathbf{V}_{\parallel} = (\hat{\mathbf{n}} \cdot \mathbf{V})\, \hat{\mathbf{n}} \,.$$

- So we now have orthogonal projection operators:

$$P_{ij}^{\parallel} = \hat{n}_i \hat{n}_j , \qquad P_{ij}^{\perp} = \delta_{ij} - \hat{n}_i \hat{n}_j \,.$$

- Compare to the relativistic operator:

$$h^{\alpha}{}_{\beta} = g^{\alpha}{}_{\beta} + u^{\alpha} u_{\beta}$$

It is completely analogous!

Relativistic Fluid

- With implication of energy conservation law:
  $\nabla_\alpha T^{\alpha\beta} = 0$.
- With a comoving frame,
  $u^\alpha = [1, 0, 0, 0]$ and $u_\alpha = [1, 0, 0, 0]$.
- For any 4-vector V^β, we can have inner product
  $u_\beta \cdot V^\beta = (-1) \cdot V^0 + 0 \cdot V^i = -V^0$,
  which is the time component. Similarly, we can use this method to find the 'parallel component' of $\nabla_\alpha T^{\alpha\beta}$.
- $u_\beta \cdot \nabla_\alpha T^{\alpha\beta} = 0$ describes the parallel component, just like the dot product in $\mathbb{E}^3$.

Relativistic Fluid

- With some operations on $u_\beta \cdot \nabla_\alpha T^{\alpha\beta} = 0$ we can have this equation:

$$\boxed{u^\alpha \nabla_\alpha \rho \ + \ (\rho + P)\, \nabla_\alpha u^\alpha \ = \ 0}$$

  This equation is obtained from the energy conservation implication, so it is the law of mass conservation.

- With some further operations, we can have this equation:

$$\boxed{(\rho + P)\, u^\alpha \nabla_\alpha u^\beta \ = \ -(g^{\alpha\beta} + u^\alpha u^\beta)\, \nabla_\alpha P}$$

  This equation is called Euler equation of fluid dynamics in general relativity.

Friedmann Equations

Robertson-Walker Metric

- The FRW metric:

$$ds^2 \;=\; -\,dt^2 \;+\; a(t)^2\left[\frac{dr^2}{1-kr^2} + r^2\big(d\theta^2 + \sin^2\theta\,d\phi^2\big)\right].$$

- It can be written in this form:

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu$$

$$g_{\mu\nu} \;=\; \mathrm{diag}\big(-1,\; a^2\gamma_{ij}\big), \qquad \gamma_{ij}\,dx^i dx^j \;=\; \frac{dr^2}{1-kr^2} \;+\; r^2\big(d\theta^2 + \sin^2\theta\,d\phi^2\big).$$

- $$g^{00} = -1,\; g^{0i} = 0,\; g^{ij} = \tfrac{1}{a^2}\gamma^{ij}.$$

  Only the scale factor $a(t)$ carries time dependence.

Einstein Equation

- To be honest I don't know where the Einstein equations comes from, but it is given:

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}.$$

- To derive the Friedmann equations from the Einstein equations, we first need to calculate the components of the geometry tensor $G_{\mu\nu}$.
- And the $G_{\mu\nu}$ is given in this form:

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}$$

Ricci Tensor

- The equation of the Ricci tensor is:

$$R_{\mu\nu} = \partial_\rho \Gamma^\rho_{\mu\nu} - \partial_\nu \Gamma^\rho_{\mu\rho} + \Gamma^\rho_{\mu\nu} \Gamma^\sigma_{\rho\sigma} - \Gamma^\rho_{\mu\rho} \Gamma^\sigma_{\nu\sigma}.$$

- So, to calculate the Ricci tensor, we firstly need to calculate the non-zero components of the Christoffel symbol $\Gamma^\rho_{\mu\nu}$ with its equation:

$$\Gamma^\rho_{\mu\nu} = \frac{1}{2} g^{\rho\sigma} (\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu}).$$

Christoffel symbol

- Here are some useful derivatives:

$$\partial_0 g_{ij} = 2a\dot{a}\,\gamma_{ij}, \quad \partial_k g_{ij} = a^2 \partial_k \gamma_{ij}, \quad \partial_\alpha g_{00} = \partial_\alpha g_{0i} = 0.$$

As previously given:

$$g^{00} = -1,\ g^{0i} = 0,\ g^{ij} = \tfrac{1}{a^2}\gamma^{ij}.$$

- So, for any pure time or space entry, it should be equals to zero:

$$\Gamma^0_{00} = 0, \quad \Gamma^0_{0i} = 0, \quad \Gamma^k_{00} = 0.$$

- Pure-space entries

$$\Gamma^i_{jk}(g) = \Gamma^i_{jk}(\gamma),$$

i.e. identical to the Christoffels of the 3-metric γ_{ij}

Mixed Entries (time–space) of Christoffel symbol

- $\Gamma^0_{ij} = a\dot{a}\,\gamma_{ij}$

$$\Gamma^\rho_{\mu\nu} = \tfrac{1}{2} g^{\rho\sigma}\left(\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu}\right)$$

  Choose $\rho = 0,\ \mu = i,\ \nu = j$:

$$\Gamma^0_{ij} = \tfrac{1}{2}\, g^{00}\left(-\partial_0 g_{ij}\right) = \tfrac{1}{2}(-1)\left[-2a\dot{a}\,\gamma_{ij}\right] = a\dot{a}\,\gamma_{ij}.$$

  $\partial_i g_{0j} = \partial_j g_{0i} = 0$, $\partial_0 g_{ij} = 2a\dot{a}\,\gamma_{ij}$.

- $\Gamma^i_{0j} = \dfrac{\dot{a}}{a}\,\delta^i_j$ Choose $\rho = i,\ \mu = 0,\ \nu = j$ ($g^{i\sigma} = a^{-2}\gamma^{i\sigma}$):

$$\Gamma^i_{0j} = \tfrac{1}{2}\, g^{i\sigma}\partial_0 g_{\sigma j} = \tfrac{1}{2}\,\frac{\gamma^{ik}}{a^2}\,\partial_0\!\left(a^2\gamma_{kj}\right) = \tfrac{1}{2}\,\frac{\gamma^{ik}}{a^2}\,(2a\dot{a})\gamma_{kj} = \frac{\dot{a}}{a}\,\delta^i_j.$$

  Other terms $\partial_i g_{\sigma 0}$ and $\partial_\sigma g_{0j}$ vanish.

Time–Time Component of Ricci tensor R_{00}

$$R_{00} = \partial_\rho \Gamma^\rho_{00} - \partial_0 \Gamma^\rho_{0\rho} + \Gamma^\rho_{00}\Gamma^\sigma_{\rho\sigma} - \Gamma^\rho_{0\sigma}\Gamma^\sigma_{0\rho}.$$

- $\Gamma^\rho_{00} = 0$ for *all* $\rho \implies$ first & third terms vanish.
- $\partial_\rho \Gamma^\rho_{00} = 0$ (same reason).
- Non-zero piece sits in the last term:

$$-\Gamma^\rho_{0\sigma}\Gamma^\sigma_{0\rho} = -\Gamma^i_{0j}\Gamma^j_{0i} = -\left(\tfrac{\dot a}{a}\right)^2 \delta^i_j \delta^j_i = -3\,\frac{\dot a^2}{a^2}.$$

- But $\partial_0 \Gamma^\rho_{0\rho} = \partial_0 \Gamma^i_{0i} = \partial_0\big(3\tfrac{\dot a}{a}\big) = 3\big(\tfrac{\ddot a}{a} - \tfrac{\dot a^2}{a^2}\big)$.

Combine:

$$\boxed{R_{00} = -3\,\frac{\ddot a}{a}}$$

Space–Space Component R_{ij}

$$R_{ij} = \partial_k \Gamma^k_{ij} - \partial_j \Gamma^k_{ik} + \Gamma^k_{ij}\Gamma^\ell_{k\ell} - \Gamma^k_{i\ell}\Gamma^\ell_{jk}.$$

Split into two parts:

- purely spatial contribution
  Since $\Gamma^k_{ij}(g) = \Gamma^k_{ij}(\gamma)$, these four terms equal the 3-dimensional Ricci tensor:

$$^{(3)}R_{ij} = 2k\,\gamma_{ij}.$$

- time-dependent correction
  From mixed symbols:

$$-\partial_0 \Gamma^0_{ij} - \Gamma^0_{i\ell}\Gamma^\ell_{0j} - \Gamma^0_{0\ell}\Gamma^\ell_{ij} = 0 \quad (\because \Gamma^0_{0\ell} = 0).$$

  Compute:

$$-\partial_0 \Gamma^0_{ij} = -\partial_0(a\dot{a})g_{ij} = -(a\ddot{a} + \dot{a}^2)g_{ij},$$
$$-\Gamma^0_{i\ell}\Gamma^\ell_{0j} = -(a\dot{a})g_{i\ell}(\tfrac{\dot{a}}{a}\delta^\ell_j) = -\dot{a}^2 g_{ij}.$$

Ricci Tensor Components

- Time Part:

$$\boxed{R_{00} = -3\,\frac{\ddot{a}}{a}}$$

which encodes cosmic acceleration: $\ddot{a}/a$.

- Spacial Part

$$R_{ij} = {}^{(3)}R_{ij} - g_{ij}\left[a\ddot{a} + 2\dot{a}^2\right] = \left[a\ddot{a} + 2\dot{a}^2 + 2k\right] g_{ij}.$$

$$\boxed{R_{ij} = \left[a\,\ddot{a} + 2\dot{a}^2 + 2k\right]\gamma_{ij}}$$

which splits into intrinsic 3-curvature $2k$ and kinematic pieces $a\ddot{a}$ and $\dot{a}^2$.

Ricci Scalar $R = g^{\mu\nu} R_{\mu\nu}$

- Inverse metric:

$$g^{00} = -1, \qquad g^{0i} = 0, \qquad g^{ij} = a^{-2}\gamma^{ij}.$$

- Plug in the tensor components

$$R = g^{00} R_{00} + g^{ij} R_{ij} = (-1)\left[-3\,\tfrac{\ddot{a}}{a}\right] \; + \; a^{-2}\gamma^{ij}\left[a\,\ddot{a} + 2\dot{a}^2 + 2k\right]\gamma_{ij}.$$

$$\gamma^{ij}\gamma_{ij} = 3 \quad \Longrightarrow \quad R = 3\,\frac{\ddot{a}}{a} + \frac{3}{a^2}\left[a\,\ddot{a} + 2\dot{a}^2 + 2k\right].$$

$$\boxed{R = 6\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2 + k}{a^2}\right)}$$

Interpretation: the first term measures cosmic acceleration, the second contains kinetic ($\dot{a}^2$) and spatial-curvature (k) parts.

Geometry Tensor Components

- We have the equation of $G_{\mu\nu}$:

$$G_{\mu\nu} = R_{\mu\nu} \; - \; \tfrac{1}{2}\, R\, g_{\mu\nu}$$

- The time part:

$$G_{00} = R_{00} - \tfrac{1}{2} g_{00} R = 3\left(\frac{\dot{a}}{a}\right)^2 + \frac{3k}{a^2}$$

- The space part:

$$G_{ij} = R_{ij} - \tfrac{1}{2} g_{ij} R = -\left(2\,\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{k}{a^2}\right) g_{ij}$$

$$G^{i}{}_{j} = -\big[\,2\ddot{a}/a + \dot{a}^2/a^2 + k/a^2\big]\,\delta^i_j$$

Perfect Fluid

- And we have these equations from the perfect fluid part:

$$T^{\alpha\beta} = (\rho + P)u^{\alpha}u^{\beta} + Pg^{\alpha\beta}$$

$$u^{\alpha} = (1, 0, 0, 0)$$

$$T^{00} = -\rho$$

$$T^{ij} = P\delta^{i}_{j}$$

The First Friedmann Equation

- By plugging G_{00} and T^0_0 into the Einstein equation, we can have the following equation:
$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2} = \frac{8\pi G}{3}\rho + \frac{\Lambda}{3}$$
- Take $\mu = \nu = 0$ in (E).
Because $g_{00} = -1$,
$$G_{00} + \Lambda g_{00} = G_{00} - \Lambda = 8\pi G\, T_{00} = 8\pi G\, \rho.$$
- Insert G_{00}:
$$3\left(\frac{\dot{a}}{a}\right)^2 + \frac{3k}{a^2} - \Lambda \;=\; 8\pi G\, \rho.$$
- Divide by 3 and move the $-\Lambda/3$ term to the right:
$$\boxed{\left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2} = \frac{8\pi G}{3}\rho + \frac{\Lambda}{3}}.$$

The Acceleration Equation

- Einstein equation:

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G\, T_{\mu\nu}, \qquad T_{\mu\nu} = (\rho + P) u_\mu u_\nu - P\, g_{\mu\nu}.$$

- For the FRW metric

$$G_{ij} = -\Big[2\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{k}{a^2}\Big] g_{ij}, \qquad T_{ij} = P\, g_{ij}.$$

- Write the ij component and cancel g_{ij}:

$$-\Big[2\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{k}{a^2}\Big] + \Lambda \;=\; 8\pi G\, P.$$

- Use the 00 component to get

$$\frac{\dot{a}^2}{a^2} + \frac{k}{a^2} \;=\; \frac{8\pi G}{3}\rho + \frac{\Lambda}{3},$$

and substitute $(\dot{a}^2/a^2 + k/a^2)$ back into the previous line to obtain the acceleration equation.

Fluid Equation

- FRW metric:

$$ds^2 = -dt^2 + a^2(t)\,\gamma_{ij}dx^i dx^j$$

- Perfect-fluid:

$$T^{\mu\nu}: \quad T^{\mu\nu} = (\rho + P)u^\mu u^\nu - P\,g^{\mu\nu},$$

$$u^\mu = (1, 0, 0, 0).$$

- Start with covariant conservation:

$$\nabla_\mu T^{\mu 0} = 0.$$

-

$$\nabla_\mu T^{\mu\nu} = \partial_\mu T^{\mu\nu} + \Gamma^\mu_{\ \lambda\mu}\,T^{\lambda\nu} + \Gamma^\nu_{\ \lambda\mu}\,T^{\mu\lambda} = 0.$$

$$\partial_\mu T^{\mu 0} + \Gamma^\mu_{\ \lambda\mu}\,T^{\lambda 0} + \Gamma^0_{\ \lambda\mu}\,T^{\mu\lambda} = 0.$$

Fluid Equation

- T^{i0} vanishes due to isotropy.
- Only have $\lambda = 0$ entry, and $T^{00} = \rho, \;\; T^{\mu}{}_{\lambda} = P\delta^{\mu}_{\lambda}$
- $$\frac{d\rho}{dt} + \Gamma^{\mu}_{\;0\mu}\,\rho - \Gamma^{\lambda}_{\;\mu 0}\,T^{\mu}{}_{\lambda} = 0.$$
- $$\Gamma^{\mu}_{\;0\mu} = 3\frac{\dot{a}}{a}, \qquad \Gamma^{\lambda}_{\;\mu 0} = \frac{\dot{a}}{a}\,\delta^{\lambda}_{\mu}$$

  $\Gamma^{\lambda}_{\;\mu 0}$ vanishes unless μ and λ are spatial indices equal to each other, which gives $\frac{\dot{a}}{a}$.
- $$\dot{\rho} + \left(3\frac{\dot{a}}{a}\right)\rho - \left(\tfrac{\dot{a}}{a}\right)3P = 0 \implies \boxed{\dot{\rho} + 3\frac{\dot{a}}{a}(\rho + P) = 0}.$$

Friedmann Equations

- The first equation is derived purely from the ρ component, which represents the law of energy conservation:

$$\boxed{\left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2} = \frac{8\pi G}{3}\rho + \frac{\Lambda}{3}}.$$

- The second equation, the acceleration equation, is derived by mixing ρ and P components:

$$\boxed{\frac{\ddot{a}}{a} \;=\; -\,\frac{4\pi G}{3}\,(\rho + 3P) \;+\; \frac{\Lambda}{3}}$$

- The third equation, the fluid equation, is derived from the perfect fluid properties.

$$\boxed{\dot{\rho} + 3\frac{\dot{a}}{a}(\rho + P) = 0}.$$

Thank You!!!!!!!

Thank you so much for reading this slide:)