Metric (Proper) Distance and Redshift

1. **Metric Distance (Proper Distance)**

2. **Redshift**

Metric Distance (Proper Distance)

Metric Distance in an FRW Space–time

- The *metric distance* (often called the proper distance) is the shortest spatial separation between two objects, calculated directly from the metric.
- i.e. At a fixed time t, the proper distance is calculated by integrate along the shortest separation of objects in FRW space.
- In general space–time

$$ds^2 = g_{\mu\nu}\, dx^\mu dx^\nu .$$

Metric Distance in an FRW Space–time

- Choose a slice of constant cosmic time $t = t_0$ so that $dt = 0$. The spatial line element becomes
$$dl^2 = g_{ij}(t_0, \mathbf{x})\, dx^i dx^j.$$
- Proper distance between two points A and B at the *same* cosmic time:
$$D_{\mathrm{metric}}(t_0; A \to B) \;=\; \int_A^B \sqrt{g_{ij}(t_0, \mathbf{x})\, dx^i dx^j} \;=\; \int_A^B dl.$$

Radial Proper Distance in an FRW Universe

- FRW metric (set $c = 1$):

$$ds^2 = dt^2 - a^2(t)\Big[\frac{dr^2}{1 - Kr^2} + r^2\, d\Omega^2\Big].$$

- Slice at the same cosmic time $t = t_0$ ($dt = 0$):

$$dl^2 = a^2(t_0)\Big[\frac{dr^2}{1 - Kr^2} + r^2\, d\Omega^2\Big].$$

- Because FRW space is isotropic, any radial line from a reference point is itself a spatial geodesic, so the above integral already gives the shortest path between the two comoving positions. (Just straightforwardly connect 2 points by the geodesic.)

Proper Distance along a Radial Geodesic

- On a radial line the angular part vanishes:

$$d\Omega^2 = d\theta^2 + \sin^2\theta \, d\phi^2 = 0.$$

- The spatial line element reduces to

$$dl_{\mathrm{rad}}^2 = a^2(t) \, \frac{dr^2}{1 - Kr^2}.$$

- Integrating from the origin to radius r at fixed cosmic time t_0 gives the metric (proper) radial distance:

$$D_{\mathrm{metric}}(t_0; 0 \to r) = \int dl_{\mathrm{rad}} = a(t_0) \int_0^r \frac{dr'}{\sqrt{1 - Kr'^2}}.$$

Proper Radial Distance Using χ

- Starting point: proper (metric) distance from the origin to radius r at cosmic time t,

$$D_{\text{metric}}(t; 0 \to r) = a(t) \int_0^r \frac{dr'}{\sqrt{1 - Kr'^2}}.$$

- Introduce the curvature-adapted radial coordinate

$$r = S_K(\chi), \qquad S_K(\chi) = \begin{cases} \sin\chi & K = +1, \\ \chi & K = 0, \\ \sinh\chi & K = -1. \end{cases}$$

Proper Radial Distance Using χ

- In a space of constant curvature one finds (*Somehow in a sphere)

$$d\chi = \frac{dr}{\sqrt{1 - Kr^2}},$$

  so the integral simplifies to $D_{\mathrm{metric}} = a(t)\,\chi$.
- Spatial line element expressed in χ:

$$dl^2 = a^2(t)\left[d\chi^2 + S_K^2(\chi)\,d\Omega^2\right].$$

- With $d\Omega^2 = 0$, χ measures the comoving radial distance, and $a(t)\,\chi$ is the proper radial distance at time t.

Redshift

Connecting χ to the Observed Red-shift

- Scale factor at the time of emission,

$$a_{\mathrm{e}} \equiv a(t_{\mathrm{e}}) = \frac{1}{1+z},$$

where z is the red-shift measured today.

- A photon travels along a null geodesic, $ds^2 = 0$, in the FRW line element

$$ds^2 = dt^2 - a^2(t)\left[d\chi^2 + S_K^2(\chi)\, d\Omega^2\right].$$

Connecting χ to the Observed Red-shift

- For a radial ray, $d\Omega^2 = 0$ and thus

$$dt^2 = a^2(t)\, d\chi^2, \qquad d\chi = d\tau = \frac{dt}{a(t)}.$$

- Integrating from emission (t_e) to observation (t_0) gives the comoving distance:

$$\chi = \int_{t_\mathrm{e}}^{t_0} \frac{dt}{a(t)}.$$

- Therefore the observed red-shift determines $a(t_\mathrm{e})$ and, through the integral above, the comoving separation between us and the emitting galaxy.

Differential Relation between Cosmic Time and Red-shift

- Red-shift definition:

$$1 + z = \frac{1}{a(t)}.$$

- Differentiate (with $H(z) = \dot{a}/a$):

$$\frac{dz}{dt} = -\,a^{-2}(t)\,\dot{a}(t) = -(1+z)\,H(z),$$

- Rearranging:

$$dz = -(1+z)\,H(z)\,dt \quad \Longrightarrow \quad dt = -\frac{dz}{(1+z)\,H(z)}.$$

Differential Relation between Cosmic Time and Red-shift

- Sometimes one needs $dt/a(t)$:

$$\frac{dt}{a(t)} = \frac{dt}{1/(1+z)} = (1+z)\,dt = -\,\frac{dz}{H(z)}.$$

- These relations are useful when rewriting time integrals in terms of red-shift, for example when computing comoving distances or look-back time.

Comoving Distance and Curvature Function $S_K(\chi)$

- Comoving radial coordinate expressed in terms of red-shift:

$$\chi(z) = \int_{t_e}^{t_0} \frac{dt}{a(t)} = \int_z^0 \frac{dz'}{H(z')} = \int_0^z \frac{dz'}{H(z')}.$$

- Spatial line element in χ coordinates:

$$dl^2 = a^2(t)\,\left[d\chi^2 + S_K^2(\chi)\,d\Omega^2\right],$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta\,d\phi^2$.

Comoving Distance and Curvature Function $S_K(\chi)$

- Curvature function $S_K(\chi)$:

$$S_K(\chi) = \begin{cases} \sin\chi & \text{for } K = +1 \text{ (closed)} \\ \chi & \text{for } K = 0 \text{ (flat)} \\ \sinh\chi & \text{for } K = -1 \text{ (open)} \end{cases}$$

- $S_K(\chi)$: sometimes the comoving transverse radius; it encodes how curvature bends transverse distances at χ.
- χ itself is the comoving radial distance (one integrates $dt/a(t)$) , $dz/H(z)$ to convert red-shift into distance in comoving space.

Proper and Comoving Distances

- With $d\Omega = 0$ (straight-line radial separation) the proper distance between two points at cosmic time t is

$$D_{\text{proper}}(t;\chi) = a(t)\,\chi \quad \text{(exact for a flat spatial slice).}$$

- Including the transverse part we define the *transverse comoving distance*

$$D_M(\chi) \equiv S_K(\chi),$$

where $S_K(\chi) = \sin\chi,\ \chi,\ \sinh\chi$ for $K = +1,\ 0,\ -1$.

- The corresponding *proper* transverse distance at time $t$is simply scaled by the factor $a(t)$:

$$D_{\text{proper, trans}}(t;\chi) = a(t)\,S_K(\chi).$$

How Proper Distance Scales with $1 + z$

- Today ($t = t_0$, choose $a_0 = 1$):

$$D_{\text{proper}}(t_0; \chi) = a_0\, S_K(\chi) = S_K(\chi).$$

- At emission ($t = t_{\rm e}$, red-shift z):

$$D_{\text{proper}}(t_{\rm e}; \chi) = a_{\rm e}\, S_K(\chi) = \frac{a_0\, S_K(\chi)}{1+z} = \frac{S_K(\chi)}{1+z}.$$

- The physical separation between us and the emitter has grown by the same factor $(1+z)$ as that stretches the wavelength of the photons!!!

Proper (Metric=Comoving) Distance vs. Red-shift

- Spatially flat universe ($K = 0$): the metric distance equals the comoving distance χ.
- For an observer at t_0 and a galaxy of red-shift z, the comoving distance is

$$\chi(z) = \int_{t_e}^{t_0} \frac{dt}{a(t)} = \int_0^z \frac{dz'}{H(z')}.$$

- Because $a_0 = 1$ in the usual convention, the proper distance *today* is simply

$$D^0_{\text{proper}}(z) = \chi(z).$$

Thank You!!!!!!!!!!!!

Thank you so much for reading this slide:)))))