From Poisson Equation to Einstein Equation

1. **Poisson Equation**

2. **Einstein Equation**

Poisson Equation

Solving Poisson's Equation in Spherical Symmetry

- Poisson equation for the Newtonian potential $U(\mathbf{x})$:

$$\nabla^2 U(\mathbf{x}) = 4\pi G\, \rho(\mathbf{x}).$$

- Integral form:

$$U(\mathbf{x}) = -G \int \frac{\rho(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|}\, d^3x'.$$

- For a spherically symmetric mass distribution of radius R:

$$U(r) = \begin{cases} -\dfrac{1}{r} \displaystyle\int_0^R r'^2\, \rho(r')\, dr', & r > R, \\ -\dfrac{1}{r} \displaystyle\int_0^r r'^2\, \rho(r')\, dr' \; - \displaystyle\int_r^R r'\, \rho(r')\, dr', & r < R. \end{cases}$$

Gravitational Potential: Multipole Terms

- Expand the Green's function for $|\mathbf{x}| \equiv r \gg |\mathbf{x}'|$:

$$\frac{1}{|\mathbf{x}-\mathbf{x}'|} = \frac{1}{r} + \sum_k \frac{x^k x'^k}{r^3} + \frac{1}{2}\sum_{k\ell}(3x^k x^\ell - r^2\delta_{k\ell})\frac{x'^k x'^\ell}{r^5} + \cdots$$

- Substitute into the integral formula $U(\mathbf{x})$:

$$U(\mathbf{x}) = -\frac{GM}{r} - \frac{G}{r^3}\sum_k x^k D^k - \frac{G}{2r^5}\sum_{k\ell} Q^{k\ell} x^k x^\ell + \cdots$$

  - $M = \int \rho(\mathbf{x}')\, d^3x'$ — total mass.
  - $D^k = \int x'^k \rho(\mathbf{x}')\, d^3x'$ — mass dipole moment.
  - $Q^{k\ell} = \int (3x'^k x'^\ell - r'^2\delta_{k\ell})\rho(\mathbf{x}')\, d^3x'$ — quadrupole moment (trace-free).

Mass Quadrupole Tensor $Q^{k\ell}$

- Definition
$$Q^{k\ell} = \int (3x'^k x'^\ell - r'^2 \delta_{k\ell}) \rho(\mathbf{x}') \, d^3x'.$$

- Measures how much the mass distribution deviates from spherical symmetry; sometimes called the mass quadrupole moment.

- If the origin is chosen at the centre of mass the dipole moment D^k vanishes, so $Q^{k\ell}$ gives the leading non-spherical contribution to the potential $\frac{1}{r^3}$ and gravitational field $\frac{1}{r^4}$.

- Describes 'how much it doesn't look like a sphere'.

Mass Quadrupole Tensor $Q^{k\ell}$

- Trace-free by construction:

$$Q^k{}_k = 0$$

  Make the tensor record only the asymmetric components. Removing the trace avoids counting the spherical part twice.
- Symmetric:

$$Q^{k\ell} = Q^{\ell k}$$

  A symmetric, trace-free 3×3 tensor has five independent components.
- "Higher-order multipoles look similar"—each successive moment captures finer asymmetries of the mass distribution.

Using J_2 to Quantify Flattening

- Define the dimensionless quadrupole coefficient

$$J_2 \; = \; -\frac{Q^{33}}{2MR_0^2},$$

  where Q^{33} is the zz–component of the mass-quadrupole tensor, M the total mass, and R_0 a reference (mean-equatorial) radius.

- J_2 measures the oblateness of a nearly spherical body:
  - $J_2 = 0$ for a perfect sphere.
  - $J_2 > 0$ indicates an equatorial bulge (flattened at the poles) such as a rotating planet.

Using J_2 to Quantify Flattening

- Example:
  For a uniformly dense spheroid (equatorial bulge due to spinning):
  A non-zero quadrupole $Q^{k\ell}$ leads to:

$$J_2 \neq 0$$

  , quantifying its departure from spherical symmetry.

Einstein Equation

- The source of the gravitational field is the stress–energy tensor $T^{\mu\nu}$.

- The gravitational field itself can be described by a second-rank tensor $E^{\mu\nu}$.

- In General Relativity the Newtonian potential is replaced by the space–time metric $g_{\mu\nu}$; the tensor $E^{\mu\nu}$ is constructed from first and second derivatives of $g_{\mu\nu}$.

- Conservation laws: $\mathsf{T}^{\mu\nu}{}_{;\mu} = 0, \qquad E^{\mu\nu}{}_{;\mu} = 0$, which ensure local energy–momentum conservation and consistency of the field equations.

Form of $E^{\mu\nu}$ from Second Derivatives of $g_{\mu\nu}$

- The tensor $E^{\mu\nu}$ must be *linear* in the second derivatives of the metric $g_{\mu\nu}$.
- The most general such combination is

$$E^{\mu\nu} = R^{\mu\nu} + a\, g^{\mu\nu} R + b\, g^{\mu\nu},$$

  where $R^{\mu\nu}$ is the Ricci tensor and
  $R = g_{\alpha\beta} R^{\alpha\beta}$ the Ricci scalar.

Einstein Field Equation and the Newtonian Limit

- Imposing the conservation law

$$E^{\mu\nu}{}_{;\mu} = 0$$

- Requiring

$$\left(R^{\mu\nu} + a\, g^{\mu\nu} R + b\, g^{\mu\nu}\right)_{;\mu} = 0$$

  fixes $a = -\frac{1}{2}$ and leaves b as a constant Λ.

- Full Einstein equation (with cosmological term):

$$R^{\mu\nu} - \tfrac{1}{2}\, g^{\mu\nu} R + \Lambda\, g^{\mu\nu} = \kappa\, T^{\mu\nu}, \qquad \kappa = \frac{8\pi G}{c^4}.$$

The Newtonian Limit

- Newtonian (weak-field, slow-motion) limit reproduces Poisson's equation for the gravitational potential Φ:
$$\nabla^2\Phi = 4\pi G\,\rho.$$
- Thus the Einstein tensor reduces to the Newtonian potential in the appropriate limit, ensuring consistency with classical gravity.

Recovering Poisson's Equation from Linearised GR

- Newtonian potential obeys
$$\nabla^2 \Phi = 4\pi G\,\rho.$$

- Weak-field approximation:
$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \qquad |h_{\mu\nu}| \ll 1.$$

- Time–time component relates to the Newtonian potential:
$$g_{00} = -(1 + 2\Phi/c^2) \quad \Longrightarrow \quad h_{00} = -\frac{2\Phi}{c^2}.$$

Recovering Poisson's Equation from Linearised GR

- Define the trace-reversed perturbation

$$\bar{h}_{\mu\nu} = h_{\mu\nu} - \tfrac{1}{2}\,\eta_{\mu\nu}\,h, \qquad h = \eta^{\alpha\beta} h_{\alpha\beta}.$$

- Linearized Einstein tensor in Lorentz gauge:

$$G_{\mu\nu} = -\tfrac{1}{2}\,\eta^{\alpha\beta}\partial_\alpha\partial_\beta\,\bar{h}_{\mu\nu}.$$

- Setting $\mu = \nu = 0$ reproduces Poisson's equation in the static, non-relativistic limit, confirming consistency with Newtonian gravity.

Field Source and Local Conservation

- *Slow-motion (Newtonian) limit* of the stress–energy tensor:

$$T^{00} \simeq \rho\, c^2, \qquad T^{0i} \simeq 0, \qquad T^{ij} \simeq 0.$$

  The rest-mass energy dominates; momentum flow and pressure are negligible.

- This $T^{\mu\nu}$ serves as the *source term* in the field equations.

- Local energy–momentum conservation must hold: $\nabla_\mu T^{\mu\nu} = 0$.

- In linearised gravity the metric is written $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, and retains covariance under the infinitesimal gauge transformation

$$h_{\mu\nu} \;\longrightarrow\; h_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu.$$

Unique Role of the Einstein Tensor $G_{\mu\nu}$

- Among all second-derivative combinations of the metric, *only* the Einstein tensor $G_{\mu\nu}$ satisfies the differential identity

$$\nabla_\mu G^{\mu\nu} = 0,$$

- Linearised around flat space, the Einstein tensor reduces to

$$G_{\mu\nu} = -\tfrac{1}{2}\,\eta^{\alpha\beta}\partial_\alpha\partial_\beta\,\bar{h}_{\mu\nu},$$

  where $\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h$ is the trace-reversed perturbation.

- Working in the **Lorenz gauge**

$$\partial^\mu \bar{h}_{\mu\nu} = 0$$

GR–EM Analogy in Lorenz Gauge

- In electromagnetism the four–potential $A^\mu = (\phi/c, \mathbf{A})$ satisfies the Lorenz condition $\partial_\mu A^\mu = 0$.

- In linearised gravity (Lorenz gauge) we have the equation

$$G_{\mu\nu} = \kappa\, T_{\mu\nu}, \qquad \kappa = \frac{8\pi G}{c^4},$$

with $G_{\mu\nu}$ playing the role of the field operator, and $T_{\mu\nu}$ acting as the mass-energy source.

GR–EM Analogy in Lorenz Gauge

- In the static, weak-field limit $G_{\mu\nu} \approx \kappa T_{\mu\nu}$ reproduces Poisson's equation $\nabla^2\Phi = 4\pi G\,\rho$.
- Write the perturbed metric $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$; for Newtonian potentials,

$$h_{00} = -\frac{2\Phi}{c^2}, \qquad h_{ij} = -\frac{2\Phi}{c^2}\delta_{ij},$$

  while cross terms h_{0i} vanish for slowly moving sources.
- Thus the metric perturbation $h_{\mu\nu}$ plays a role analogous to the electromagnetic potential A^μ, with Φ acting as the gravitational "scalar potential".

Recovering Poisson's Equation

- Linearised Lorenz-gauge field equation

$$\nabla^2 \bar{h}_{00} = -2\kappa\, \rho\, c^2.$$

- For the Newtonian potential $h_{00} = -\,2\Phi/c^2$ we have

$$h = -h_{00} + h_{ii} = -\frac{2\Phi}{c^2} + 3\left(-\frac{2\Phi}{c^2}\right) = -\frac{8\Phi}{c^2},$$

$$\bar{h}_{00} = h_{00} - \tfrac{1}{2}h = -\frac{4\Phi}{c^2}.$$

Recovering Poisson's Equation

- Substitute in static limit:

$$\nabla^2\left(-\frac{4\Phi}{c^2}\right) = -2\kappa\,\rho\,c^2 \quad\Longrightarrow\quad \nabla^2\Phi = \frac{\kappa c^4}{8}\,\rho.$$

- Matching with Poisson's equation $\nabla^2\Phi = 4\pi G\,\rho$ fixes

$$\boxed{\kappa = \frac{8\pi G}{c^4}}.$$

- Therefore the full Einstein field equation is

$$G_{\mu\nu} = \frac{8\pi G}{c^4}\,T_{\mu\nu},$$

consistent with Newtonian gravity in the weak-field limit.

Thank You!!!!!!!!!!!!

Thank you so much for reading this slide:)))))