Energy Momentum Tensor and Perfect Fluid

1. **Energy Momentum Tensor**

2. **Perfect Fluid**

Energy Momentum Tensor

General Figure of Energy Momentum Tensor

Any component in $T^{\alpha\beta}$ shows how much α-momentum is flowing through it per unit time per unit area.
$T^{\alpha\beta} =$ flux of α-momentum across a surface of constant x^{β}.

Key features:

- $\alpha : p^{\alpha} = [E, p^1, p^2, p^3]$ α give which component we are talking.
- β :through which 'surface' we are measuring transport. (A surface which spacetime coordinate x^{β} is constant.
- Symmetric: $T_{\alpha\beta} = T_{\beta\alpha}$
- Conserved: $\nabla^{\alpha} T_{\alpha\beta} = 0$

Components

$$
\begin{aligned}
T^{00} &= \text{flux of 0-momentum, i.e. energy, across surfaces } t = \text{const} \\
&= \text{energy density}, \\
T^{0i} &= \text{energy flux across surface } x^i = \text{const}, \\
T^{i0} &= \text{flux of momentum in the } x^i \text{ direction across surfaces } t = \text{const} \\
&= x^i\text{-momentum density}, \\
T^{ij} &= \text{flux of } x^i \text{ momentum across surfaces } x^j = \text{const},
\end{aligned}
$$

Something else but related

All fluxes are measured by an observer momentarily at rest in a local inertial frame comoving with the matter element at point p. Often, this follows the covariance principle.

- Start with normal coordinates, find the energy momentum tensor from the local laws in special relativity.
- Generalize the tensor to arbitrary coordinates using the coordinate invariance of tensors.

Perfect Fluid

Definitions of Perfect Fluid

A perfect fluid is a continuous matter distribution with no viscosity and no heat conduction in the locally comoving frame.
The form of the energy momentum tensor for this type of matter follows from looking more closely at the meaning of “no viscosity” and “no heat conduction”.
*Viscosity is defined as a force component exerted by one particle on another that perpendicular to the line of sight of 2 particles.

Features

Key Features:

- No heat conduction: E_{total} of a particle contains some internal energy. In this scenario, the internal energy is not transferred to another particle.
  So, the energy can only flow if the particle flows.
- No viscosity: the force between 2 particles only changes the momentum in the direction along their line of sight. (Forces only have radial component)

No Viscosity

Key Features:

- Rotating the coordinate to the direction coincides with $x^i$for some fixed i, which will not loss generality. So, the only momentum that can flow this direction is then the p^i component.
- $T^{ij} \neq 0$ only for $i = j$.

  No direction preferred $\Rightarrow \mathsf{T}^{11} = T^{22} = T^{33} =: P$

The Equation

- in special relativity in the locally comoving frame $u^\alpha = [1, 0, 0, 0]$

$$T^{\alpha\beta} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & P & 0 & 0 \\ 0 & 0 & P & 0 \\ 0 & 0 & 0 & P \end{pmatrix} \overset{!}{=} (\rho + P)u^\alpha u^\beta + P\eta^{\alpha\beta}.$$

- The general relativistic expression follows from replacing $\eta^{\alpha\beta}$ with $g^{\alpha\beta}$ according to the covariance principle, so that

$$\boxed{T^{\alpha\beta} = (\rho + P)u^\alpha u^\beta + Pg^{\alpha\beta}} \tag{1}$$

Back to Classical Fluid Mechanics

- V is an arbitrary vector:

$$\hat{\mathbf{n}} \cdot \mathbf{V} = V_{\parallel}$$

-

$$\mathbf{V}_{\parallel} = (\hat{\mathbf{n}} \cdot \mathbf{V})\,\hat{\mathbf{n}}\,.$$

- So we now have orthogonal projection operators:

$$P^{\parallel}_{ij} = \hat{n}_i \hat{n}_j, \qquad P^{\perp}_{ij} = \delta_{ij} - \hat{n}_i \hat{n}_j\,.$$

- Compare to the relativistic operator:

$$h^{\alpha}{}_{\beta} = g^{\alpha}{}_{\beta} + u^{\alpha} u_{\beta}$$

It is completely analogous!

Relativistic Fluid

- With implication of energy conservation law:
  $\nabla_\alpha T^{\alpha\beta} = 0$.
- With a comoving frame,
  $u^\alpha = [1, 0, 0, 0]$ and $u_\alpha = [1, 0, 0, 0]$.
- For any 4-vector V^β, we can have inner product
  $u_\beta \cdot V^\beta = (-1) \cdot V^0 + 0 \cdot V^i = -V^0$,
  which is the time component. Similarly, we can use this method to find the 'parallel component' of $\nabla_\alpha T^{\alpha\beta}$.
- $u_\beta \cdot \nabla_\alpha T^{\alpha\beta} = 0$ describes the parallel component, just like the dot product in $\mathbb{E}^3$.

Relativistic Fluid

- With some operations on $u_\beta \cdot \nabla_\alpha T^{\alpha\beta} = 0$ we can have this equation:

  $$\boxed{u^\alpha \nabla_\alpha \rho \;+\; (\rho + P)\,\nabla_\alpha u^\alpha \;=\; 0}$$

  This equation is obtained from the energy conservation implication, so it is the law of mass conservation.

- With some further operations, we can have this equation:

  $$\boxed{(\rho + P)\,u^\alpha \nabla_\alpha u^\beta \;=\; -(g^{\alpha\beta} + u^\alpha u^\beta)\,\nabla_\alpha P}$$

  This equation is called Euler equation of fluid dynamics in general relativity.