Classical Euler Equations in Newtonian Limit

1. **Classical Euler's Equation**

2. **GR to Newtonian limit**

Classical Euler's Equation

Classical Perfect Fluids

- Continuum approximation: system size $\gg$ mean free path.
- Mean free path (order of magnitude)

$$\ell \sim \frac{1}{n\,\pi d^2},$$

  with number density n and molecular diameter d.

- Use a fixed control volume V with boundary $S = \partial V$ and outward unit normal $\hat{\mathbf{n}}$.

Equation of continuity & mass conservation:

- outgoing mass = density × outgoing volume:

$$\rho\,(\mathbf{v}\cdot\hat{\mathbf{n}})\,dS$$

- $\hat{\mathbf{n}}$: outward normal vector of dS
- total mass of fluid leaving from volume V in Δt:

$$m_{\text{tot}} = \Delta t \oint_S \rho\,\mathbf{v}\cdot\hat{\mathbf{n}}\,dS = \Delta t \int_V \nabla\cdot(\rho\,\mathbf{v})\,dV$$

- Gauss theorem:

$$\oint_S \mathbf{F}\cdot\hat{\mathbf{n}}\,dS = \int_V \nabla\cdot\mathbf{F}\,dV$$

Mass conservation equation

- mass conservation equation:

$$\frac{d}{dt}\int_V \rho \, dV \;=\; -\int_V \nabla \cdot (\rho \, \vec{v}) \, dV$$

  decrease in mass $=$ outgoing mass

- $$\int_V \left[\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \, \vec{v})\right] dV = 0$$

- holds for any V $\Rightarrow$ equation of continuity

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \, \vec{v}) = 0$$

  time evolution of the density of the fluid

Momentum conservation: acceleration of fluid

- particle:

$$\frac{d\vec{v}}{dt} = \lim_{\Delta t \to 0} \frac{\vec{v}(t + \Delta t) - \vec{v}(t)}{\Delta t}$$

- $$\frac{\partial \vec{v}}{\partial t} \equiv \lim_{\Delta t \to 0} \frac{\vec{v}(\vec{x}, t + \Delta t) - \vec{v}(\vec{x}, t)}{\Delta t}$$

  Not particle acceleration.

- Define particle acceleration (velocity change along its trajectory $\vec{x}(t)$, $\vec{x}_0$):

$$\frac{D\vec{v}}{Dt} \equiv \lim_{\Delta t \to 0} \frac{\vec{v}(\vec{x} + \vec{v}\,\Delta t,\, t + \Delta t) - \vec{v}(\vec{x}, t)}{\Delta t}$$

Momentum conservation: acceleration of fluid

- Lagrangian derivative: the time derivative defined along the flow; with Taylor expansion

$$v(x + v\,\Delta t,\, t + \Delta t) = v(x,t) + \frac{\partial v}{\partial t}\Delta t + \frac{\partial v}{\partial x}\, v\,\Delta t \quad \text{(1-Dim)}$$

$$\vec{v}(\vec{x} + \vec{v}\,\Delta t,\, t + \Delta t) = \vec{v}(\vec{x},t) + \frac{\partial \vec{v}}{\partial t}\Delta t + (\vec{v}\cdot\nabla)\vec{v}\,\Delta t \quad \text{(3-Dim)}$$

Material derivative (Eulerian form)

- $$\frac{D\vec{v}}{Dt} = \frac{\vec{v}(\vec{x} + \vec{v}\,\Delta t,\, t + \Delta t) - \vec{v}(\vec{x}, t)}{\Delta t}$$
  $$= \frac{\vec{v}(\vec{x}, t) + \frac{\partial \vec{v}}{\partial t}\Delta t + (\vec{v}\cdot\nabla)\vec{v}\,\Delta t - \vec{v}(\vec{x}, t)}{\Delta t}$$
- $$\boxed{\frac{D\vec{v}}{Dt} = \frac{\partial \vec{v}}{\partial t} + (\vec{v}\cdot\nabla)\vec{v}} \quad \text{Eulerian form}$$

Euler's Equation

- pressure force on a surface element:

$$d\vec{F} = p\,\hat{\vec{n}}\,dS$$

- total force due to the external pressure acting on the whole surface of volume V:

$$-\oint_S p\,\hat{\vec{n}}\,dS \;=\; -\int_V \nabla p\,dV \qquad \text{(Gauss's divergence theorem)}$$

- conservation of momentum for a volume V:

$$\int_V \rho\,\frac{D\vec{v}}{Dt}\,dV \;=\; -\int_V \nabla p\,dV$$

Euler's Equation

- total momentum:

$$\vec{p} = \int_V \rho\,\vec{v}\,dV$$

- net force:

$$\vec{F}_{\text{net}} = \frac{d\vec{p}}{dt} \;\Rightarrow\; \frac{d}{dt}\int_V \rho\,\vec{v}\,dV = -\oint_S p\,\hat{\vec{n}}\,dS$$

- from $\displaystyle\int_V \rho\,\frac{D\vec{v}}{Dt}\,dV = -\int_V \nabla p\,dV \Rightarrow$

$$\rho\,\frac{D\vec{v}}{Dt} = -\,\nabla p$$

Local form and external force

- material derivative:

$$\frac{D\vec{v}}{Dt} = \frac{\partial \vec{v}}{\partial t} + (\vec{v}\cdot\nabla)\vec{v}$$

- Euler's equation:

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v}\cdot\nabla)\vec{v} = -\frac{1}{\rho}\nabla p$$

- if there is an external force $\vec{f}$ acting on unit mass of fluid other than pressure:

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v}\cdot\nabla)\vec{v} = -\frac{1}{\rho}\nabla p + \vec{f}$$

Conservation form of Euler equation

- conservation form:

$$\frac{\partial}{\partial t}(\rho\,\vec{v}) = \vec{v}\,\frac{\partial \rho}{\partial t} + \rho\,\frac{\partial \vec{v}}{\partial t}$$

- continuity equation:

$$\frac{\partial \rho}{\partial t} = -\,\nabla\cdot(\rho\,\vec{v}) \quad\Rightarrow\quad \vec{v}\,\frac{\partial \rho}{\partial t} = -\,\vec{v}\,\nabla\cdot(\rho\,\vec{v})$$

- Euler equation:

$$\frac{\partial \vec{v}}{\partial t} = -\,\frac{1}{\rho}\nabla p - (\vec{v}\cdot\nabla)\vec{v}$$

Conservation form of Euler equation

- combine:

$$\frac{\partial}{\partial t}(\rho\,\vec{v}) = -\,\vec{v}\,\nabla\cdot(\rho\,\vec{v}) - \rho\,(\vec{v}\cdot\nabla)\vec{v} - \nabla p$$

- component form:

$$\frac{\partial}{\partial t}(\rho v_i) = -\,v_i\,\frac{\partial}{\partial x_j}(\rho v_j) - \rho\,v_j\,\frac{\partial v_i}{\partial x_j} - \frac{\partial p}{\partial x_i}$$

$$-\,v_i\,\frac{\partial}{\partial x_j}(\rho v_j) - \rho\,v_j\,\frac{\partial v_i}{\partial x_j} = -\,\frac{\partial}{\partial x_j}(\rho v_i v_j)$$

$$\frac{\partial p}{\partial x_i} = \frac{\partial}{\partial x_j}(p\,\delta_{ij})$$

- final conservative form:

$$\boxed{\frac{\partial}{\partial t}(\rho v_i) + \frac{\partial}{\partial x_j}(\rho v_i v_j + p\,\delta_{ij}) = 0}$$

Momentum flux density tensor

- Define
$$\Pi_{ij} = \delta_{ij}\, p + \rho\, v_i v_j$$
  - $\delta_{ij}\, p$: momentum flow due to pressure (isotropic)
  - $\rho\, v_i v_j$: momentum flow due to fluid motion directly past the surface
- Π_{ij}: flow of i-component of the momentum across the surface perpendicular to basis vector $\hat{e}_j$

Momentum flux density tensor

- Conservation form

$$\frac{\partial}{\partial t}(\rho\, v_i) + \frac{\partial}{\partial x_j}\Pi_{ij} = 0$$

- integral form

$$\frac{d}{dt}\int_V \rho\, v_i\, dV = \int_V \frac{\partial}{\partial t}(\rho\, v_i)\, dV = -\oint_S \Pi_{ij}\, n_j\, dS$$

(total momentum flowing out surface S)

GR to Newtonian limit

Perfect fluids in GR (Newtonian limit)

- $T^{\mu\nu} = (\rho + p)\, U^\mu U^\nu + p\, g^{\mu\nu}$
- let $c = 1$, $v \ll 1$
  - $X^\mu = (ct, x^i)$, $\quad U^\mu = \dfrac{dX^\mu}{d\tau}$
  - $U^\mu = \left(\dfrac{dt}{d\tau}, \dfrac{dx^i}{d\tau}\right) = (\gamma, \gamma v^i)$
  - $v \ll c \;\Rightarrow\; \dfrac{dt}{d\tau} \simeq 1,\;\; U^\mu \simeq (1, v^i)$

Perfect fluids in GR (Newtonian limit)

- spatial part:
$$T^{ij} = (\rho + p)U^iU^j + p\,g^{ij} \;=\; (\rho + p)\,v^iv^j + p\,g^{ij}$$
- $g^{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1) \;\Rightarrow\; g^{ij} = \delta^{ij}$
-
$$T^{ij} = (\rho + p)\,v^iv^j + p\,\delta^{ij}$$
- Non-relativistic: $\rho \simeq \rho_{\mathsf{matter}},\; p \ll \rho \Rightarrow p\,v^iv^j \simeq 0$
$$T^{ij} \simeq \rho_{\mathsf{matter}}\,v^iv^j + p\,\delta^{ij} \;=\; \Pi_{ij}$$

Conservation of $T^{\mu\nu}$

- As $T^{\mu\nu}$ is conserved:
$$\nabla_\mu T^{\mu\nu} = 0$$
-
$$\nu = 0: \; \nabla_\mu T^{\mu 0} = 0$$
$$\nu = i: \; \nabla_\mu T^{\mu i} = 0$$
-
$$T^{\mu\nu} = (\rho + p)\, U^\mu U^\nu + p\, g^{\mu\nu}$$
-
$$\nabla_\mu T^{\mu\nu} = \partial_\mu T^{\mu\nu} + \Gamma^\mu{}_{\mu\lambda} T^{\lambda\nu} + \Gamma^\nu{}_{\mu\lambda} T^{\mu\lambda}$$

Conservation of $T^{\mu\nu}$

- in flat spacetime: $\Gamma = 0$

$$\nabla_\mu T^{\mu\nu} = \partial_\mu T^{\mu\nu} = \partial_t T^{0\nu} + \partial_i T^{i\nu}$$

- nonrelativistic ($\gamma \simeq 1$, $v^i v^i \ll 1$):

$$T^{00} \simeq (\rho + p)\, 1 \cdot 1 - p = \rho, \qquad T^{i0} \simeq (\rho + p)\, v^i$$

-

$$\partial_t \rho + \partial_i \big[(\rho + p) v^i\big] = 0$$

$$(p \ll \rho \Rightarrow \partial_t \rho + \nabla \cdot (\rho\, \vec{v}) = 0)$$

-

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho\, v_i) = 0$$

From $\nabla_\mu T^{\mu i} = 0$ to Euler equation

- $$\nabla_\mu T^{\mu i} = 0$$
- $$T^{\mu i} = (\rho + p)\, U^\mu U^i + p\, g^{\mu i}$$
- $$T^{0i} \simeq (\rho + p)\, v^i + p\, g^{0i} = (\rho + p)\, v^i$$
  momentum density (or energy/ mass flux) in the non-relativistic limit ($\gamma \simeq 1$; if $p \ll \rho$ then $\simeq \rho v^i$).
- $$T^{ji} \simeq (\rho + p)\, v^j v^i + p\, g^{ji}$$
  momentum flux / stress: convective transport $\rho v^j v^i$ plus isotropic pressure $p\, \delta^{ji}$ (since $g^{ji} = \delta^{ji}$ in flat space).

From $\nabla_\mu T^{\mu i} = 0$ to Euler equation

- in flat space:
$$\nabla_\mu T^{\mu i} = 0 \;=\; \partial_t T^{0i} + \nabla \cdot T^{ji}$$
-
$$\partial_t T^{0i} = \partial_t\big[(\rho + p)v^i\big] \simeq \partial_t(\rho v^i)$$
-
$$\nabla \cdot T^{ji} = \nabla \cdot \big[(\rho + p)v^j v^i + p\, g^{ji}\big] \simeq \nabla \cdot \big(\rho v^j v^i + p\, g^{ji}\big)$$
-
$$\partial_t(\rho v^i) + \nabla \cdot \big(\rho v^j v^i + p\, g^{ji}\big) = 0$$
- with $v^i = v_i,\ \ \Pi^{ij} = \rho v^i v^j + p\, g^{ij},\ \ g_{ij} = \mathrm{diag}(1,1,1) \Rightarrow \Pi^i{}_j = \Pi_{ij}$
- final:
$$\boxed{\partial_t(\rho v_i) + \nabla \cdot \Pi_{ij} = 0} \quad \text{(Euler equation)}$$

Thank You!!!!!!!!!!!!!

Thank you so much for reading this slide:)))))