國立臺北大學 112 學年度第 1 學期期中考試試卷

National Taipei University Student's Answer Paper

系級/Department & Grade 電機系2年級 科目/Course Title 工程數學

(設科目所屬系級)/Course Given Department 1

☐學士班 Bachelor Program ☐碩士班 Master Program ☐博士班 Ph.D. Program

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任課教師/Instructor Name 陳建宏 2023 年 Year 10 月 Month 31 日 Date

1.解下列微分方程式：(42%)

$yy' = 2xe^{y^2}$

$y\frac{dy}{dx} = 2xe^{y^2}$

$ye^{-y^2}dy = 2x\,dx$

$\int ye^{-y^2}dy = \int 2x\,dx$

$-\frac{1}{2}e^{-y^2} = x^2 + C$

A: $-\frac{1}{2}e^{-y^2} = x^2 + C$

+7

$(2x^2y + 3y^3)dx - (x^3 + 2xy^2)dy = 0$

M N

$M(tx,ty) = 2t^3x^2y + 3t^3y^3 = t^3M(x,y)$

$N(tx,ty) = t^3x^3 + 2t^3xy^2 = t^3N(x,y)$ ⇒ 齊次

設 $y = ux$, $dy = x\,du + u\,dx$

$(2x^3u + 3x^3u^3)dx = (x^3 + 2x^3u^2)(u\,dx + x\,du)$

$x^3(2u + 3u^3)dx = x^3(1 + 2u^2)(u\,dx + x\,du)$

$\frac{u(2+3u^2)}{1+2u^2} = \frac{u\,dx + x\,du}{dx} = u + \frac{x\,du}{dx} \Rightarrow \frac{2+3u^2}{1+2u^2} = 1 + \frac{x\,du}{u\,dx}$

$\frac{2+3u^2}{1+2u^2} - 1 = \frac{x\,du}{u\,dx}$

$\frac{1+u^2}{1+2u^2} = \frac{x\,du}{u\,dx}$

$(1+u^2)u\,dx = x(1+2u^2)du$

$\int \frac{1}{x}dx = \int \frac{1+2u^2}{u+u^3}du$

$\ln|x| = \ln|u+u^3| + C$

A: $\ln|x| = \ln|u+u^3| + C$

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$(x^2y^3 + 2y)dx + (2x - 2x^3y^2)dy = 0$

M N

$\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = 3x^2y^2 + 2 - (2 - 6x^2y^2) = 9x^2y^2$

$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{Ny - Mx} = \frac{9x^2y^2}{-3x^3y^3} = \frac{-3}{xy}$

$u(x) = e^{\int -\frac{3}{xy}d(xy)} = e^{-3\ln xy} = (xy)^{-3}$

$\frac{1}{x^3y^3}(x^2y^3 + 2y)dx + \frac{1}{x^3y^3}(2x - 2x^3y^2)dy = 0$

$\left(\frac{1}{x} + \frac{2}{x^3y^2}\right)dx + \left(\frac{2}{x^2y^3} - \frac{2}{y}\right)dy = 0$

$\int\left(\frac{1}{x} + \frac{2}{x^3y^2}\right)dx + g(y) = C \Rightarrow \ln x - \frac{1}{x^2y^2} + g(y) = C$

$\frac{2}{x^2y^3} + g'(y)$

$= \frac{2}{x^2y^3} - \frac{2}{y}$

$g'(y) = -\frac{2}{y}$

$\Rightarrow g(y) = -2\ln y$

A: $\ln x - \frac{1}{x^2y^2} - 2\ln y = C$

+7

$y' + (\tan x)y = \sin 2x$

$u(x) = e^{\int \tan x\,dx} = e^{\ln|\sec x|} = \sec x$

$y = \cos x\int \frac{1}{\cos x}\sin 2x\,dx + \cos x$

$= \cos x\int \frac{2\sin x\cos x}{\cos x}dx + \cos x$

$= -2\cos^2 x + \cos x \cdot C$

A: $y = -2\cos^2 x + \cos x \cdot C$

+7

$y'' + 4y' + 5y = 3e^{-2x}$

$\lambda^2 + 4\lambda + 5 = 0 \Rightarrow \lambda = \frac{-4 \pm \sqrt{16-20}}{2} = -2 \pm i$

$y_h = e^{-2x}(C_1\cos x + C_2\sin x)$

$y_p = Ae^{-2x}$ (×5)

$y_p' = -2Ae^{-2x}$ (×4)

$y_p'' = 4Ae^{-2x}$ (×1)

$Ae^{-2x} = 3e^{-2x} \Rightarrow A = 3$

A: $y = 3e^{-2x} + e^{-2x}(C_1\cos x + C_2\sin x)$

+7

$x^2y'' + xy' - y = 0$

$m(m-1)(m-2) + m - 1 = 0$

$m(m^2 - 3m + 2) + m - 1 = 0 \Rightarrow m^3 - 3m^2 + 3m - 1 \Rightarrow m = 1, 1, 1$

$(m-1)(m^2 - 2m + 1) = (m-1)^3 \Rightarrow y_h = C_1x + C_2x\ln x + C_3x(\ln x)^2$

$W = \begin{vmatrix} x & x\ln x & x(\ln x)^2 \\ 1 & \ln x + 1 & (\ln x)^2 + 2\ln x \\ 0 & \frac{1}{x} & \frac{2}{x}\ln x + \frac{2}{x} \end{vmatrix} = \cancel{(\ln x)^2} + 2(\ln x + 1)^2 - \ln x(2\ln x + 2) - \cancel{(\ln x)^2} \ldots = 2$

$y_p = 0$, $y = y_h$

A: $y = C_1x + C_2x\ln x + C_3x(\ln x)^2$

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證明線性一階微分方程式 $y' + p(x)y = f(x)$ 的公式：$y = \frac{1}{u(x)}\left[\int u(x)f(x)dx + C\right]$, $u(x) = e^{\int p(x)dx}$ (10%)

$y' = -p(x)y + f(x)$

$y = \int -p(x)y\,dx + \int f(x)dx$

$= \frac{1}{u(x)}\left(\int u(x)f(x)dx + C\right)$

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3. 求微分方程式的特解(14%)

(a) 以微分運算子求 $y''+4y'+4y=x^3e^{-2x}$ (b) $y''-2y'+2y=2e^x\cos x$ (求特解即可)

$\lambda^2-2\lambda+2=0 \Rightarrow \lambda=\frac{2\pm\sqrt{-4}}{2}=1\pm i$

$y_h=e^x(A\cos x+B\sin x)$

(a) $(D^2+4D+4)Y=x^3e^{-2x}$

$(D+2)^2Y=x^3e^{-2x}$

$Y_p=\frac{1}{(D+2)(D+2)}x^3e^{-2x}$ A: $Y_p=\frac{1}{20}e^{-2x}x^5$

$Y_p=e^{-2x}\int\left(e^{2x}e^{-2x}\int e^{2x}e^{-2x}x^3dx\,dx\right)$

$=e^{-2x}\int\left(\int x^3dx\right)dx$

$=e^{-2x}\int \frac{x^4}{4}dx=\frac{1}{20}e^{-2x}x^5$

(b) $Y_p=xe^x(A\cos x+B\sin x)$ (×2)

$Y_p'=(1+x)e^x(A\cos x+B\sin x)+xe^x(-A\sin x+B\cos x)$ (×-2)

$Y_p''=2e^x(A\cos x+B\sin x)+2(1+x)e^x(-A\sin x+B\cos x)+xe^x(-A\cos x-B\sin x)$ (×1)

$2e^x(-A\sin x+B\cos x)=2e^x\cos x \Rightarrow A=0,\ B=1$

A: $Y_p=xe^x(\sin x)$

4.(a).寫出 RC 高通濾波電路的微分方程式，(b).當輸入為直流電壓 $E_0u(t)$，電容的初始狀態為零，求流過的電阻 R 的電壓 $v_R(t)$，(c).畫出 $v_R(t)$波形，(d).說明 $v_R(t)$暫態響應及穩態響應的區間與輸入的關係. (12%)

(a) A: $v(t)=Ri(t)+\frac{1}{C}\int i(t)dt$

(b) 電容一開始會分到較多的分壓，所以 $V_R(0)=0$，但後面會慢慢變大

(c) $V_R(t)$

(d) $V_{(R)}t$ 的暫態響應跟輸入無關，穩態響應與輸入有關

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5. 計算下列的 Laplace transform (a) t^3 (b) $e^{-3t}\sin 5t$ (c) $u(t-t_0)$ (d) $k[\delta(t-2)-\delta(t-4)]$ (12%)

(a) $\mathcal{L}[t^3]=\int_0^\infty t^3e^{-st}dt=\int_0^\infty \frac{t^3}{e^{st}}dt$

(b) $\mathcal{L}[e^{-3t}\sin 5t]=\int_0^\infty e^{-3t}\sin 5t\,e^{-st}dt=\int_0^\infty e^{-(3+s)t}\sin 5t\,dt=\frac{1}{s+3}\left(\frac{-3}{s^2-9}\right)=\frac{-3}{s^3+3s^2-9s-27}$, A: $\frac{-3}{s^3+3s^2-9s-27}$

(c) $\mathcal{L}[u(t-t_0)]=\int_0^{t_0}0\cdot e^{-st}dt+\int_{t_0}^\infty 1\cdot e^{-st}dt=0+\frac{1}{-s}e^{-st}\Big|_{t_0}^\infty=\frac{1}{s}e^{-st_0}$, A: $\frac{1}{s}e^{-st_0}$

(d) $\mathcal{L}[k(\delta(t-2)-\delta(t-4))]=k\left[\int_0^\infty\delta(t-2)e^{-st}dt-\int_0^\infty\delta(t-4)e^{-st}dt\right]=k\left[e^{-s(t-2)}\Big|_{t=2}-e^{-s(t-4)}\Big|_{t=4}\right]=k(1-1)=0$

A: 0

6. 證明 (a) $L\{\sin at\}=\frac{a}{s^2+a^2}$, (b) $L\{\cosh at\}=\frac{s}{s^2-a^2}, s>|a|$ (10%)

(a) $\mathcal{L}[e^{jat}]=\mathcal{L}[\cos at+j\sin at]=\frac{1}{s-ja}=\frac{s+ja}{s^2+a^2}$

$=\frac{s}{s^2+a^2}+j\frac{a}{s^2+a^2}$

$\mathcal{L}[\cos at]=\frac{s}{s^2+a^2}$, $\mathcal{L}[j\sin at]=j\frac{a}{s^2+a^2}$

$\mathcal{L}(\sin at)=\frac{a}{s^2+a^2}$ 得證

(b) $\mathcal{L}[\cosh at]=\mathcal{L}\left[\frac{e^{at}+e^{-at}}{2}\right]=\frac{1}{2}\left(\frac{1}{s-a}+\frac{1}{s+a}\right)=\frac{1}{2}\left(\frac{2s}{s^2-a^2}\right)$

$=\frac{s}{s^2-a^2}$, $s>|a|$

$\therefore \mathcal{L}[\cosh at]=\frac{s}{s^2-a^2}$, $s>|a|$ 得證

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