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國立臺北大學 113 學年度第 1 學期期 中 考試試卷

National Taipei University Student's Answer Paper

系級/Department & Grade 電機系2年級　　科目/Course Title 工程數學

(該科目所屬系級)/Course Own Department [redacted]

學號/Stu[redacted]

任課教師/Instructor Name 陳建宏　　[redacted] 10 月 Month 29 日 Date

85

1. Solve the following differential equations：(48%)

$y(1+x^2)dy-(3x+xy^2)dx=0, y(1)=3$

$y(1+x^2)dy = x(3+y^2)dx$　　$u=x^2$, $du=2x\,dx$

$\int \frac{x}{1+x^2}dx = \int \frac{3+y^2}{y}dy$

$\int \frac{\frac{1}{2}}{1+u}du = \int(\frac{3}{y}+y)dy$

$\frac{1}{2}\ln|1+u| = 3\ln|y| + \frac{1}{2}y^2 + c$

$\frac{1}{2}\ln|1+x^2| = 3\ln|y| + \frac{1}{2}y^2 + c$

+7

$(2x^2y+3y^3)dx-(x^3+2xy^2)dy=0$

$M = 2x^2y+3y^3 \Rightarrow \frac{\partial M}{\partial y} = 2x^2+9y^2$

$N = -x^3-2xy^2 \Rightarrow \frac{\partial N}{\partial x} = -3x^2-2y^2$

$tM = 2(tx)^2(ty)+3(ty)^3 = t^3[2x^2y+3y^3]$

$tN = -(tx)^3-2(tx)(ty)^2 = t^3[-x^3-2xy^2]$

let $y=ux$　$dy=u\,dx+x\,du$

$[2x^2(ux)+3(ux)^3]dx-[x^3+2x(ux)^2](u\,dx+x\,du)=0$

$(2x^3u+3x^3u^3)dx-(x^3u+2x^3u^3)dx-(x^4+2x^4u^2)du=0$

$(x^3u+x^3u^3)dx = (x^4+2x^4u^2)du$

$x^3(u+u^3)dx = x^4(1+2u^2)du$

$\int\frac{x^3}{x^4}dx = \int\left(\frac{1+2u^2}{u+u^3}\right)du$　　$w=u^2$, $dw=2u\,du$　$\int\frac{u}{1+u^2}du = \int\frac{\frac{1}{2}dw}{1+w} = \frac{1}{2}\ln|1+w| = \frac{1}{2}\ln|1+u^2|$

$\ln|x| = \int\left(\frac{1}{u(1+u^2)}+\frac{2u^2}{u(1+u^2)}\right)du = \int\left(\frac{1}{u}-\frac{u}{1+u^2}+\frac{2u}{1+u^2}\right)du = \ln|u|+\frac{1}{2}\ln|1+u^2|+c$

$\therefore \ln|x| = \ln\left|\frac{y}{x}\right| + \frac{1}{2}\ln\left|1+\left(\frac{y}{x}\right)^2\right| + c$

+8

$y(1+x)dx+x(1+y)dy=0$

$M = y(1+x) \Rightarrow \frac{\partial M}{\partial y} = 1+x$　　$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$

$N = x(1+y) \Rightarrow \frac{\partial N}{\partial x} = 1+y$

$\frac{\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}}{-M+N} = \frac{1+x-1-y}{-y-xy+x+xy} = \frac{x-y}{x-y} = 1 = f(x+y)$

$u = e^{\int 1\,d(x+y)} = e^{x+y}$

$uM = e^{x+y}y(1+x)$

$uN = e^{x+y}x(1+y)$

$xe^x - \int e^x dx = xe^x - e^x$

$\int e^{x+y}y(1+x)dx + g(y) = e^y y\int(e^x + xe^x)dx + g(y) = e^y y(e^x + xe^x - e^x) + g(y)$

$= xye^{x+y} + g(y) = c$

$\frac{\partial f}{\partial y} = xe^x(e^y+e^y y) + g'(y) = e^{x+y}x(1+y) + g'(y) = e^{x+y}x(1+y)$

$g'(y) = 0 \Rightarrow g(y) = 0$　　$\therefore f(x,y) = xye^{x+y} = c$

+8

$y'+(\tan x)y=\sin 2x$

$p(x) = \tan x$　　$\int\tan x\,dx = \int\frac{\sin x}{\cos x}dx = -\ln|\cos x|$

$f(x) = \sin 2x$

$u(x) = e^{\int\tan x\,dx} = e^{-\ln|\cos x|} = \cos^{-1}x = \frac{1}{\cos x}$

$y = \frac{1}{u(x)}\left[\int f(x)\cdot u(x)dx + c\right]$

$= \cos x\left[\int\frac{\sin 2x}{\cos x}dx + c\right]$

$= \cos x\left[\int\frac{2\sin x\cos x}{\cos x}dx + c\right]$

$= \cos x\left(2\int\sin x\,dx + c\right)$

$= \cos x(-2\cos x + c)$

$= -2\cos^2 x + c\cos x$

+7

$y''+2y'+2y=2\sqrt{2}\cos(\sqrt{2}t), y(0)=0, y'(0)=0$

① $\lambda^2+2\lambda+2=0 \Rightarrow \lambda = \frac{-2\pm\sqrt{4-8}}{2} = -1\pm j$

$y_h = e^{-t}(c_1\cos t + c_2\sin t)$

② $y_p = A\cos\sqrt{2}t + B\sin\sqrt{2}t$　　$\times 2$

$y_p' = -\sqrt{2}A\sin\sqrt{2}t + \sqrt{2}B\cos\sqrt{2}t$　　$\times 2$

+) $y_p'' = -2A\cos\sqrt{2}t - 2B\sin\sqrt{2}t$　　$\times 1$

$(2A+2\sqrt{2}B-2A)\cos\sqrt{2}t + (2B-2\sqrt{2}A-2B)\sin\sqrt{2}t = 2\sqrt{2}\cos\sqrt{2}t$

$2\sqrt{2}B = 2\sqrt{2} \Rightarrow B=1$, $A=0$

$\therefore y_p = \sin\sqrt{2}t \Rightarrow y = e^{-t}(c_1\cos t + c_2\sin t) + \sin\sqrt{2}t$

$y' = -e^{-t}(c_1\cos t + c_2\sin t) + e^{-t}(-c_1\sin t + c_2\cos t) + \sqrt{2}\cos\sqrt{2}t$

$y(0) = 0 = c_1 = 0$, $y'(0) = 0 = c_2 + \sqrt{2} \Rightarrow c_2 = -\sqrt{2}$

$\therefore y = -\sqrt{2}e^{-t}\sin t + \sin\sqrt{2}t$

+8

$x^2y''-2xy'+2y=x^2+2$

① $m(m-1)-2m+2=0$　　$y''-\frac{2}{x}y'+\frac{2}{x^2}y = 1+\frac{2}{x^2}$

$m^2-m-2m+2=0$

$m^2-3m+2=0$

$(m-2)(m-1)=0$

$\therefore m = 2$ or $1 \Rightarrow y_h = c_1x + \cancel{c_2(\ln x)}x^2$

② $y_{p_1} = k \Rightarrow 2k = 2 \Rightarrow k = 1$

$y_{p_2} = Ax^2+Bx+C$　　$\times 2$

$y_{p_2}' = 2Ax+B$　　$\times(-2)$

+) $y_{p_2}'' = 2A$　　$\times 1$

$\cancel{2Ax^2+(2B-4A)x+(2A-2B+2C) = x^2}$

$2A = 1 \Rightarrow A = \frac{1}{2}$

$2B-2 = 0 \Rightarrow B = 1$

$1-2+2C = 0 \Rightarrow C = \frac{1}{2}$

$\therefore y_p = \frac{1}{2}x^2+x+\frac{1}{2}+1 = \frac{1}{2}x^2+x+\frac{3}{2}$

$\Rightarrow y = c_1x + c_2(\ln x)x^2 + \frac{1}{2}x^2 + x + \frac{3}{2}$

+2

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2. Prove the solution formula for linear first-order differential equations $y'+p(x)y=f(x)$,

$y=\frac{1}{u(x)}[\int u(x)f(x)dx+C]$, $u(x)=e^{\int p(x)dx}$ (10%)

$y'+p(x)y=f(x)$

$u(x)[y'+p(x)y]=f(x)u(x)$

$\frac{d}{dx}[u(x)\cdot y]=f(x)u(x)$

$\int d(u(x)\cdot y)=\int f(x)u(x)\,dx$

$u(x)\cdot y=\int f(x)\cdot u(x)\,dx+C$

$y=\frac{1}{u(x)}\left[\int f(x)u(x)dx+C\right]$

$u(x)y'+u(x)p(x)y=[u(x)\cdot y]'=u(x)y'+u'(x)y$

$\Rightarrow u(x)p(x)=u'(x)=\frac{du(x)}{dx}$

$u(x)p(x)\,dx=du(x)$

$\int p(x)dx=\int\frac{du(x)}{u(x)}$

$\int p(x)dx=\ln|u(x)|$

$u(x)=e^{\int p(x)dx}$

+10

3. Find the particular solution of the following differential equations (12%)

(a) $y''-y=e^{-x}$ (Use differential operators) (b) $y''-2y'+2y=2e^{x}\cos x$

(a) ① $y^2-1=0\Rightarrow y=1\text{ or }-1\Rightarrow y_h=c_1e^{x}+c_2xe^{-x}$

② $(D^2-1)y=e^{-x}$

$y_p=\frac{1}{(D+1)(D-1)}e^{-x}$

$\frac{1}{D-1}e^{-x}=e^{x}\int e^{-x}e^{-x}dx=\cancel{e^{x}(e^{-2x}\ldots)}\;e^{x}(-\frac{1}{2}e^{-2x})=-\frac{1}{2}e^{-x}$

$\frac{1}{D+1}(-\frac{1}{2}e^{-x})=e^{-x}\int e^{x}(-\frac{1}{2}e^{-x})dx=e^{-x}\int(-\frac{1}{2})dx=e^{-x}(-\frac{1}{2}x)$

$\therefore y_p=-\frac{1}{2}xe^{-x}$ ※

+6

(b) ① $\lambda^2-2\lambda+2=0\Rightarrow\lambda=\frac{2\pm\sqrt{4-8}}{2}=1\pm j\Rightarrow y_h=e^{x}(c_1x\cos x+c_2\sin x)$

② $y_p=Axe^{x}\cos x+Bxe^{x}\sin x$ ×2

$y_p'=Ae^{x}\cos x+Axe^{x}\cos x+(-Axe^{x}\sin x)$ ×(-2)

$+Be^{x}\sin x+Bxe^{x}\sin x+Bxe^{x}\cos x$

$y_p''=Ae^{x}\cos x+(-Ae^{x}\sin x)+Ae^{x}\cos x+Axe^{x}\cos x+(-Axe^{x}\sin x)$ ×1

$-Ae^{x}\sin x-Axe^{x}\sin x-Axe^{x}\cos x$

$+Be^{x}\sin x+Be^{x}\cos x+Be^{x}\sin x+Bxe^{x}\sin x+Bxe^{x}\cos x$

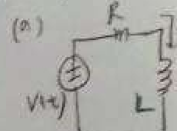
+) $+Be^{x}\cos x+Bxe^{x}\cos x-Bxe^{x}\sin x$

$-2A+2A+2B=2\Rightarrow B=1,\ -2B-A-A+2B=0\Rightarrow A=0\quad\therefore y_p=xe^{x}\sin x$ ※

+6

4.(a). Write the differential equation for a series RL circuit. (b). When the input is a DC voltage E_0, and the initial condition of the inductor is zero, find the current $i(t)$ flowing through the resistor R? (c). Draw the waveform of $i(t)$. (d). Explain the relationship between the transient and steady-state response intervals of $i(t)$ and the input signal. (15%)

(a)

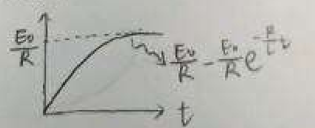


$v(t)=Ri(t)+L\frac{di(t)}{dt}$ ※

(b) $i(t)=i(\infty)+[i(0)-i(\infty)]e^{-\frac{R}{L}t}$

$=\frac{E_0}{R}+(0-\frac{E_0}{R})e^{-\frac{R}{L}t}=\frac{E_0}{R}-\frac{E_0}{R}e^{-\frac{R}{L}t}$

(c) $i(t)$



(d) $i(t)=\frac{E_0}{R}-\frac{E_0}{R}e^{-\frac{R}{L}t}$

input(E_0) ↕ steady-state

system (R, L) ↕ transient-state

+15

5. Write down the Laplace transform for the following sub-problems. (a) t^3 (b) $e^{-4t}\cos 5t$ (c) $\delta(t-t_0)$ (d) $[u(t-2)-u(t-4)]$ (e) $\sinh at$ (15%)

(a) $\mathcal{L}[t^3]=\int_0^\infty t^3e^{-st}dt=\frac{\Gamma(3+1)}{s^{3+1}}=\frac{3!}{s^4}=\frac{6}{s^4}$ ※

(b) $\mathcal{L}[\cos 5t]=\frac{s}{s^2+5^2}$

$\mathcal{L}[e^{-4t}\cos 5t]=\frac{s+4}{(s+4)^2+5^2}=\frac{s+4}{s^2+8s+41}$ ※

(c) $\mathcal{L}[\delta(t-t_0)]=\frac{1}{s}e^{-s}$ ※

(d) $\mathcal{L}[u(t-2)-u(t-4)]=\mathcal{L}[u(t-2)]-\mathcal{L}[u(t-4)]$

$=\frac{1}{s}e^{-2s}-\frac{1}{s}e^{-4s}$ ※

(e) $\mathcal{L}[\sinh at]=\mathcal{L}\left[\frac{e^{at}-e^{-at}}{2}\right]$

$=\frac{1}{2}(\mathcal{L}[e^{at}]-\mathcal{L}[e^{-at}])$

$=\frac{1}{2}(\frac{1}{s-a}-\frac{1}{s+a})=\frac{1}{2}\left(\frac{s+a-s+a}{s^2-a^2}\right)=\frac{a}{s^2-a^2}$ ※

+12