National Taipei University Student's Answer Paper

系級/Department & Grade 電機系2年級　　科目/Course Title 工程數學

(開科目所屬系級)/Council Given Department 電機

☑學士班 Bachelor Program　☐碩士班 Master Program

[redacted]

1. 週期函數 $f(x)$ 的傅利葉級數(以複數型來表示)：$f(x)=\begin{cases}1, & 0\le x\le \pi/2\\ 0, & \text{otherwise}\end{cases}$　$-\pi\le x\le \pi$，$f(x)=f(x+2\pi)$ (10%)

$f(x)=C_0+\sum_{n=1}^{\infty}\{C_n e^{jnx}\}=\frac{1}{4}+\sum_{n=1}^{\infty}\frac{-1}{2\pi nj}(e^{-jn\frac{\pi}{2}}-1)e^{jnx}$

$a_0=\frac{2}{2\pi}\int_0^{\frac{\pi}{2}}1\,dx=\frac{1}{\pi}\cdot\frac{\pi}{2}=\frac{1}{2}\Rightarrow C_0=\frac{a_0}{2}=\frac{1}{4}$

$C_n=\frac{1}{2\pi}\int_0^{\frac{\pi}{2}}1\,e^{-jnx}dx$

$=\frac{1}{2\pi}\frac{1}{-jn}e^{-jnx}\Big|_0^{\frac{\pi}{2}}=\frac{-1}{2\pi nj}(e^{-jn\frac{\pi}{2}}-1)=\frac{-1}{2\pi jn}[\cos(\frac{-n\pi}{2})+j\sin(\frac{-n\pi}{2})-1]=\begin{cases}\frac{1}{2\pi} & n=1,5,9\\ 0 & \text{else}\end{cases}$

+10

2. 以傅利葉餘弦積分式 $f(x)=\frac{2}{\pi}\int_0^{\infty}F_c(\omega)\cos\omega x\,d\omega$，其中 $F_c(\omega)=\int_0^{\infty}f(x)\cos\omega x\,dx$，計算 $\int_0^{\infty}\frac{\cos\omega x}{1+\omega^2}d\omega=?$ (15%)

$\frac{\pi}{2}f(x)=\int_0^{\infty}F_c(\omega)\cos\omega x\,dx=\int_0^{\infty}\frac{1}{1+\omega^2}\cos\omega x\,d\omega$

$F_c(\omega)=\int_0^{\infty}e^{-x}\cos\omega x=\int_0^{\infty}e^{-x}\,\frac{e^{j\omega x}+e^{-j\omega x}}{2}dx$

$=\frac{1}{2}\int_0^{\infty}e^{-(1-j\omega)x}+e^{-(1+j\omega)x}dx$

$=\frac{1}{2}\left[\frac{-1}{1-j\omega}e^{-(1-j\omega)x}+\frac{-1}{1+j\omega}e^{-(1+j\omega)x}\right]_0^{\infty}$

$=\frac{1}{2}\left(\frac{1}{1-j\omega}+\frac{1}{1+j\omega}\right)=\frac{1}{1+\omega^2}$

$\therefore\int_0^{\infty}\frac{1}{1+\omega^2}\cos\omega x\,d\omega=\frac{\pi}{2}f(x)=\frac{\pi}{2}e^{-x}$

+13

3.(a) 給 $\overline{E}=E_x(x,y,z)\overline{a_x}+E_y(x,y,z)\overline{a_y}+E_z(x,y,z)\overline{a_z}$，證明 $\nabla\cdot\overline{E}=\lim_{\Delta v\to 0}\frac{\oint_s\overline{E}\cdot\overline{dS}}{\Delta v}=\frac{\partial E_x}{\partial x}+\frac{\partial E_y}{\partial y}+\frac{\partial E_z}{\partial z}$ (10%)，(b)寫出球座標下的 $\nabla\cdot\overline{E}$ 公式(5%)

(a) $\nabla\cdot\vec{E}=\nabla\cdot\vec{E_x}+\nabla\cdot\vec{E_y}+\nabla\cdot\vec{E_z}$

$\nabla\cdot\vec{E_x}=\lim_{\Delta v\to 0}\frac{\oint_s\vec{E_x}\cdot d\vec{s}}{\Delta v}=\int\int[E_x+\frac{\partial E_x}{\partial x}(\frac{\Delta x}{2})]\vec{a_x}dydz\,\vec{a_x}+[E_x+\frac{\partial E_x}{\partial x}(-\frac{\Delta x}{2})]\vec{a_x}dydz(-\vec{a_x})$

$=\frac{\partial E_x}{\partial x}\Delta y\Delta z$

$\nabla\cdot\vec{E_y}=\lim_{\Delta v\to 0}\frac{\oint_s\vec{E_y}\cdot d\vec{s}}{\Delta v}=\int\int[E_y+\frac{\partial E_y}{\partial y}(\frac{\Delta y}{2})]\vec{a_y}dxdz\,\vec{a_y}+[E_y+\frac{\partial E_y}{\partial y}(-\frac{\Delta y}{2})]\vec{a_y}dxdz(-\vec{a_y})$

$=\frac{\partial E_y}{\partial y}\Delta x\Delta z$

$\nabla\cdot\vec{E_z}=\lim_{\Delta v\to 0}\frac{\oint_s\vec{E_z}\cdot d\vec{s}}{\Delta v}=\int\int[E_z+\frac{\partial E_z}{\partial z}(\frac{\Delta z}{2})]\vec{a_z}dxdy\,\vec{a_z}+[E_z+\frac{\partial E_z}{\partial z}(-\frac{\Delta z}{2})]\vec{a_z}dxdy(-\vec{a_z})$

$=\frac{\partial E_z}{\partial z}\Delta x\Delta y$　　$\therefore\nabla\cdot\vec{E}=\frac{\partial E_x}{\partial x}+\frac{\partial E_y}{\partial y}+\frac{\partial E_z}{\partial z}$

(b) $\vec{a_r}=\sin\theta\cos\phi\,\vec{a_x}+\sin\theta\sin\phi\,\vec{a_y}+\cos\theta\,\vec{a_z}$

$\vec{a_\theta}=\cos\theta\cos\phi\,\vec{a_x}+\cos\theta\sin\phi\,\vec{a_y}-\sin\theta\,\vec{a_z}$

$\vec{a_\phi}=-\sin\phi\,\vec{a_x}+\cos\phi\,\vec{a_y}$

4.(a) $\phi(x,y)=\frac{-1}{(x+y)}$，$\nabla\times(\nabla\phi(x,y))=?$ (10%)，(b) $\overline{F}(x,y,z)=x^2\overline{a_x}-2x^2y\overline{a_y}+2yz^4\overline{a_z}$，求點P=(1,-1,1)之 $\nabla\bullet\overline{F}=?$ (5%)

(a) $\nabla\phi(x,y)=(\frac{\partial\phi}{\partial x},\frac{\partial\phi}{\partial y})=(\frac{1}{(x+y)^2},\frac{1}{(x+y)^2})$

$\nabla\times(\nabla(\phi(x,y)))=0$

(b) $\nabla\cdot\vec{F}\big|_{(1,-1,1)}=(2x-2x^2+8yz^3)\big|_{(1,-1,1)}$

$=2-2-8=-8$

+7　　+5

8

國立臺北大學 113 學年度第 2 學期期 中 考試試卷
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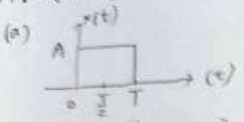
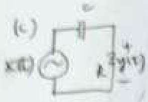
5.(a)寫出非週期輸入信號 $x(t)=A\Pi\left(\frac{t-T/2}{T}\right)$ 的傅利葉轉換，(b)畫出 $X(f)$振幅頻譜及相位頻譜，(c)透過反傅利葉轉換算出一階 RC 高通濾波器的脈衝響應 $h(t)$，(d)求輸出響應 $y(t)$，(e)如果輸入 $f(t)=10\cos 3t$，輸出 $z(t)$ 為何? (30%)

(a)



(c)



$y(t)=\frac{R}{\frac{1}{sC}+R}x(t)$

$Y(s)=\frac{R}{\frac{1}{sC}+R}X(s)$

$H(s)=\frac{R}{\frac{1}{sC}+R}=\frac{sCR}{1+sCR}=\frac{s}{\frac{1}{RC}+s}$

(d) ∵ x(t) 脈衝響應 $=\delta(t)$

$Y(s)=H(s)\,\delta(s)$

$\therefore y(t)=h(t)=e^{-\frac{1}{RC}t}$

$x(t)=C_0+\sum_{n=1}^{\infty}C_n e^{jn\omega t}$　　$\frac{1}{s+a}\xrightarrow{\mathcal{F}^{-1}}e^{at}$　　$h(t)=e^{-\frac{1}{RC}t}$ ※

$C_0=\frac{1}{T}\int_0^T A\,dt=\frac{1}{T}AT=A$

(e) ∵ $\omega=3$

$\therefore |H(j3)|=\frac{3}{\sqrt{(3CR)^2+1}}$

$\angle H(j3)=\angle 90^\circ-\tan^{-1}\frac{3CR}{1}$

$\Rightarrow z(t)=\frac{30CR}{\sqrt{(3CR)^2+1}}\cos(3t-\tan^{-1}3CR)$ ※

+6

$C_n=\frac{1}{T}\int_0^T Ae^{-jn\omega t}dt$

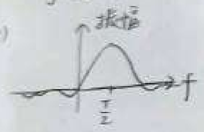
$=\frac{A}{T}\frac{-1}{jn\omega}e^{-jn\omega t}\Big|_0^T$

$=\frac{A}{jn\omega T}(1-e^{-jn\omega T})$

$=\frac{A}{jn2\pi}(1-e^{-jn2\pi})=\frac{A}{jn2\pi}(1-[\cos 2\pi n+j\sin 2\pi n])$

$=\frac{-A}{jn2\pi}(1-\cos 2\pi n)$

(b)



+2

6. (a). $\overline{F}=\left(\frac{4x^2}{x^2+y^2}-6y\right)\overline{a_x}+\left(\frac{4xy}{x^2+y^2}+6x\right)\overline{a_y}+9\overline{a_z}$，以圓柱座標表示$\overline{F}$?，(b)畫出球座標$(r,\theta,\phi)$示意圖，寫出$\left[\overline{a_r},\overline{a_\theta},\overline{a_\phi}\right]^T$ V.S. $\left[\overline{a_x},\overline{a_y},\overline{a_z}\right]^T$ 關係式，(c) 球座標 $\overline{a_\theta}dS$ 如何由$\overline{a_r}$，$\overline{a_\phi}$ 之單位長度來計算? (15%)

(a)

$\vec{a_x}=\cos\phi\,\vec{a_r}-\sin\phi\,\vec{a_\phi}$

$\vec{a_y}=\sin\phi\,\vec{a_r}+\cos\phi\,\vec{a_\phi}$

$\vec{a_z}=\vec{a_z}$

$\vec{F}=\left(\frac{4x^2}{x^2+y^2}-6y\right)(\cos\phi\,\vec{a_r}-\sin\phi\,\vec{a_\phi})+\left(\frac{4xy}{x^2+y^2}+6x\right)(\sin\phi\,\vec{a_r}+\cos\phi\,\vec{a_\phi})+9\vec{a_z}$

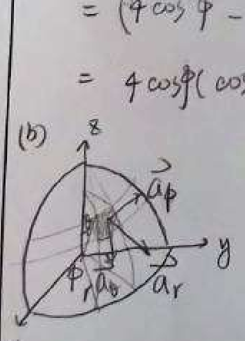
$=(4\cos^2\phi-6r\sin\phi)(\cos\phi\,\vec{a_r}-\sin\phi\,\vec{a_\phi})+(4\cos\phi\sin\phi+6r\cos\phi)(\sin\phi\,\vec{a_r}+\cos\phi\,\vec{a_\phi})+9\vec{a_z}$

$=(4\cos^3\phi-6r\sin\phi\cos\phi+4\cos\phi\sin^2\phi+6r\cos\phi\sin\phi)\vec{a_r}+(-4\cos^2\phi\sin\phi+6r\sin^2\phi+4\cos\phi\sin\phi+6r\cos^2\phi)\vec{a_\phi}+9\vec{a_z}$

$=4\cos\phi(\cos^2\phi+\sin^2\phi)\vec{a_r}+6r\vec{a_\phi}+9\vec{a_z}=4\cos\phi\,\vec{a_r}+6r\vec{a_\phi}+9\vec{a_z}$ ※

+5

(b)

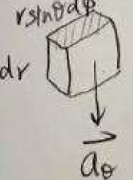


$\vec{a_r}=\sin\theta\cos\phi\,\vec{a_x}+\sin\theta\sin\phi\,\vec{a_y}+\cos\theta\,\vec{a_z}$

$\vec{a_\theta}=\cos\theta\cos\phi\,\vec{a_x}+\cos\theta\sin\phi\,\vec{a_y}-\sin\theta\,\vec{a_z}$

$\vec{a_\phi}=-\sin\phi\,\vec{a_x}+\cos\phi\,\vec{a_y}$

(c) $\vec{a_\theta}dS=(r\sin\theta d\phi)\vec{a_\phi}\times(dr)\vec{a_r}$



$=r\sin\theta\,d\phi\,dr\,\vec{a_\theta}$

+5.