1. Consider a box B_1 with two balls and a box B_2 with one ball. A student rolls a *fair* four-sided die. If the die shows one, then the student draws a ball from B_1 to B_2; otherwise, the student draws a ball from B_2 to B_1. The student keeps playing the die *independently* until one box is empty.

(a) What is the probability that the student draws a ball from B_1 in the first round?

(b) What is the probability that the game ends in 3 rounds?

(c) Suppose that eventually B_1 will be empty with probability p. Given that the die shows one and three in the first and second rounds, respectively, what is the probability that B_1 will be empty eventually? (in terms of p)

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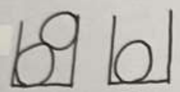
(d) Calculate p in the above question, i.e., the probability that B_1 will be empty eventually.

(e) Let E_1 be the event that the die shows two in the first rounds. Let E_2 be the event that B_1 will be empty eventually. Are E_1 and E_2 independent?

$\frac{9+4+48}{64}$ 16 $\frac{4}{64}$ $\frac{52}{\frac{9}{61}}$



B_1 B_2

① ②③④

(a) $\frac{1}{4}$

(b) case 1 : $B_2 \to B_1 \Rightarrow B_2$ empty

case 2 : $B_1 \to B_2,\ B_1 \to B_2 \Rightarrow B_1$ empty

case 3 : $B_1 \to B_2,\ B_2 \to B_1,\ B_2 \to B_1 \Rightarrow B_2$ empty

$$P[\text{ends in 3 round}] = \frac{3}{4} + \frac{1}{4}\times\frac{1}{4} + \frac{1}{4}\times\frac{3}{4}\times\frac{3}{4}$$

$$= \frac{3}{4} + \frac{1}{16} + \frac{9}{64} = \frac{61}{64}$$ ※

$\frac{3}{16}$ 48 4 9 61 $\frac{13}{\frac{9}{21}}$

(c) $$P(B_1 \text{ empty} \mid ①③) : \frac{\frac{1}{4}\times\frac{3}{4}\times p}{\frac{1}{4}\times\frac{3}{4}} = p$$ ※

(d) $$P[B_1 = 0] = \left(\frac{1}{4}\right)^2 + \frac{1}{4}\times\frac{3}{4}\times\left(\frac{1}{4}\right)^2 + \frac{1}{4}\times\frac{3}{4}\times\frac{1}{4}\times\frac{3}{4}\times\left(\frac{1}{4}\right)^2 + \cdots$$

$$= \sum_{n=1}^{\infty}\left(\frac{3}{16}\right)^{n-1}\left(\frac{1}{16}\right) = \frac{\frac{1}{16}}{1-\frac{3}{16}} = \frac{1}{13}$$ ※

(e) $P[E_1] = \frac{1}{4}$

$P[E_2] = \frac{1}{13}$

$$P[E_1 \cap E_2] = 0 \neq P[E_1] \times P[E_2] \Rightarrow \text{dependent}$$

因為die為2的話，球從 $B_2 \to B_1$，B_2 empty，結束

所以B_1為空的機率 = 0

2. You want to sell an old iPad and ask for 2 dollars. Alex and Bob are interested and bids the iPad at A dollars and B dollars, respectively. Suppose that A and B are independent and follow the discrete uniform $(1,3)$ distribution. That is, the sample space (consisting of all outcomes (A,B)) is $\{(1,1),(1,2),\cdots,(3,3)\}$. You use the *second price auction* as follows.

- If $A \neq B$ and $\max\{A,B\} \geq 2$, then the iPad goes to the highest bidder.
- If $A = B$ and $\max\{A,B\} \geq 2$, then the iPad goes to either one at random.
- Otherwise, no one gets the iPad.

①給誰 ②價格

Moreover, if anyone gets the iPad, the one pays the maximum between the second highest bid and 2 dollars; otherwise, both pay zero. For example, if $A < 2 < B$, then Bob gets the iPad and pays 2 dollars. If $2 < A < B$, then Bob gets the iPad and pays A dollars. If $A < B < 2$, then both does not get the iPad and pay zero. Let Z be the payment. We define the *surplus* S of Alice as follows. If she gets the phone, the surplus is $S = A - Z$; otherwise, the surplus is $S = 0$.

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(a) What is the PMF $f_Z(z)$ of Z?

(b) What is expected payment?

(c) What is the PMF $f_S(s)$ of S?

(d) What is the expected surplus of Alice?

$z=0$: (1,1) (1,2) (1,3)

(2,1) (2,2) (2,3)

(3,1) (3,2) (3,3) $Z=3$

(a) 5

$$f_Z(z) = \begin{cases} \frac{1}{9} & , \text{if } z=0 \\ \frac{7}{9} & , \text{if } z=2 \\ \frac{1}{9} & , \text{if } z=3 \\ 0 & , \text{else} \end{cases}$$

(A,B)	1,1	1,2	1,3	2,1	2,2	2,3	3,1	3,2	3,3
Z	0	2	2	2	2	2	2	2	3
S	0	0	0	0	0	0	1	1	0

(b) 5

$$E[Z] = 2 \times \frac{7}{9} + 3 \times \frac{1}{9} = \frac{17}{9}$$

(c) (2,1) (2,2) (3,1) (3,2) → Alex get

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P: $\frac{1}{9}$ $\frac{1}{18}$ $\frac{1}{9}$ $\frac{1}{9}$

$S=0$ $S=0$ $S=1$ $S=1$

$$\frac{6}{9} + \frac{1}{18} = \frac{13}{18}$$

$$f_S(s) = \begin{cases} \frac{13}{18} & , S=0 \\ \frac{2}{9} & , S=1 \\ 0 & , \text{else} \end{cases}$$

4 (d)

$$E[S] = 0 \times \frac{13}{18} + 1 \times \frac{2}{9} = \frac{2}{9}$$

3. Consider a CDF of a random variable X,

$$F_X(x) = \begin{cases} 0 & x < 0; \\ 0.5 & 0 \le x < 1; \\ 1 & x \ge 1. \end{cases}$$

(a) Is X a discrete random variable or continuous random variable?

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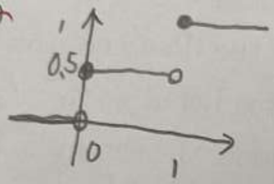
(b) What is $P[-1 \le X \le 0.1]$?

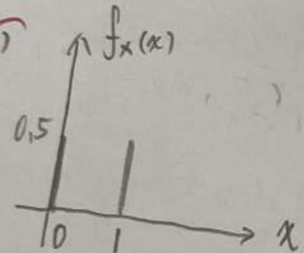
(c) What is $E[X]$?

5 (a)



discrete random variable

5 (b)



$P[-1 \le X \le 0.1] = 0.5$

5 (c)

$E[X] = 0 \times 0.5 + 1 \times 0.5 = 0.5$ ※

4. Consider a CDF of a continuous random variable X,

$$F_X(x) = \begin{cases} 0 & x < 0; \\ a - e^{-x} & x \geq 0. \end{cases}$$

for some constant a.

(a) what is a?

(b) What is $P[-1 \leq X \leq 0.1]$?

(c) What is the PDF of X?

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(a) $x \to \infty \Rightarrow F_X(\infty) = 1 \Rightarrow a = 1$ ※

5

(b) $P[-1 \leq X \leq 0.1] = F(0.1) - F(-1)$

$= 1 - e^{-0.1} - 0 = 1 - e^{-0.1}$

5

(c) $\frac{dF(x)}{dx} = e^{-x}$

$$f_X(x) = \begin{cases} 0 & , \text{if } x < 0 \\ e^{-x} & , \text{if } x \geq 0 \end{cases}$$

5. Observe the number of packets at a router for one unit time during [0, 1]. Suppose that the number of packets arriving at the router is a Poisson (1) random variable.

(a) What is the probability that one packet arrives during time [0, 1/2] and no packet arrives during time [1/2, 1]?

(b) Given there is a packet arriving at the router in the unit time. What is the probability that the packet arrives during time [0, 1/2]?

(c) Suppose that the server spends 2^x dollars for handling x packets, i.e., zero packet for 2^0 dollar, one packet for 2^1 dollar, two packets for 2^2 dollars, $\cdots$. Let Y be the total cost in the unit time. What is the PMF of Y?

(d) Follow Sub-problem (c). What is the expected total cost?

(e) Let T be the time when the first packet arrives at the router. What is the PDF of T?

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Poisson (1) $f_X(x) = \begin{cases} \frac{e^{-1} 1^x}{x!} & , x = 0, 1, 2, \dots \\ 0 & , \text{else} \end{cases}$

(a) 5

$$P = \frac{e^{-\frac{1}{2}} \frac{1}{2}^1}{1!} \times \frac{e^{-\frac{1}{2}} \frac{1}{2}^0}{0!} = \frac{1}{2} e^{-\frac{1}{2}} \cdot e^{-\frac{1}{2}} = \frac{1}{2} e^{-1}$$

(b) 0 $P[1 \text{ in } [0, \frac{1}{2}) \mid 1 \text{ in } [0, 1]] = \frac{1}{2}$

$P[N[0,\frac{1}{2})=1 \mid N[0,1]=1]$

$= \frac{\cdots}{\cdots} = \frac{1}{2}$

(c) 5

$P(Y=1) = P(X=0) = \frac{e^{-1} 1^0}{0!}$

$P[Y=2] = P[X=1] = \frac{e^{-1} 1^1}{1!}$

$P[Y=4] = P[X=2] = \frac{e^{-1} 1^2}{2!}$

$$f_Y(y) = \begin{cases} \frac{e^{-1} 1^x}{x!} & , \text{if } Y = 2^x \ (x = 0, 1, 2, \dots) \\ 0 & , \text{else} \end{cases}$$

$e^x = 1 +$

$\frac{2^n}{n!} =$

(d) 5

$$E(Y) = 1 \cdot \frac{e^{-1}}{1!} + 2 \cdot \frac{e^{-1}}{2!} + 4 \cdot \frac{e^{-1}}{3!} + 8 \cdot \frac{e^{-1}}{4!} + \cdots$$

$$= e^{-1} \sum_{n=1}^{\infty} \frac{2^n}{n!} = e^{-1} \cdot e^2 = e$$

(e) 5

$$f_T(t) = \begin{cases} e^{-t} & , \text{if } t \geq 0 \\ 0 & , \text{else} \end{cases}$$