Module 9

Solving Differential Equations with Simulink

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Overview

This Module will serve to develop Simulink programming skills to solve differential equations and make simulation programs for dynamic systems including zero-, first-, and second-order systems and for control methods.

Learning Outcomes

Upon completion of this tutorial we will be able to

- Simulate dynamic systems using different blocks from Simulink Library Browser including Integrator Block, Transfer Function Block and State Space Model Block;
- Explain dynamics of zero-order systems;
- Explain dynamics of first-order systems and second-order systems;
- Solve first-order ODEs with Simulink and second-order ODEs; and
- Create a subsystem.

Focus Questions

- How are differential equations solved by Simulink with block diagrams?
- How are simulation parameters (start time, stop time, solver, step size) set for dynamic simulation?
- How are simulated results saved to files for later use?

1. Procedure to Solve Engineering Problems with Software

- Step 1: State the problem clearly
- Step 2: Describe input and output information
- Step 3: Work the problem by hand
- Step 4: Develop a MATLAB or Simulink solution (algorithm)
- Step 5: Test the solution with variety of data

With Simulink, ODEs can be solved with block diagrams. The following example is to illustrate how to build block diagram for a Simulink model.

Problem: Solve the following 2nd-order differential equation using the block diagrams

$$a\ddot{y} + b\dot{y} + cy = du \tag{1}$$

Solution

1. Use of integrator blocks: the above equation is rewritten as

$$\ddot{y} = -\frac{c}{a}y - \frac{b}{a}\dot{y} + \frac{d}{a}u \tag{2}$$

This equation is expressed in the block diagram in **Fig. 15**.

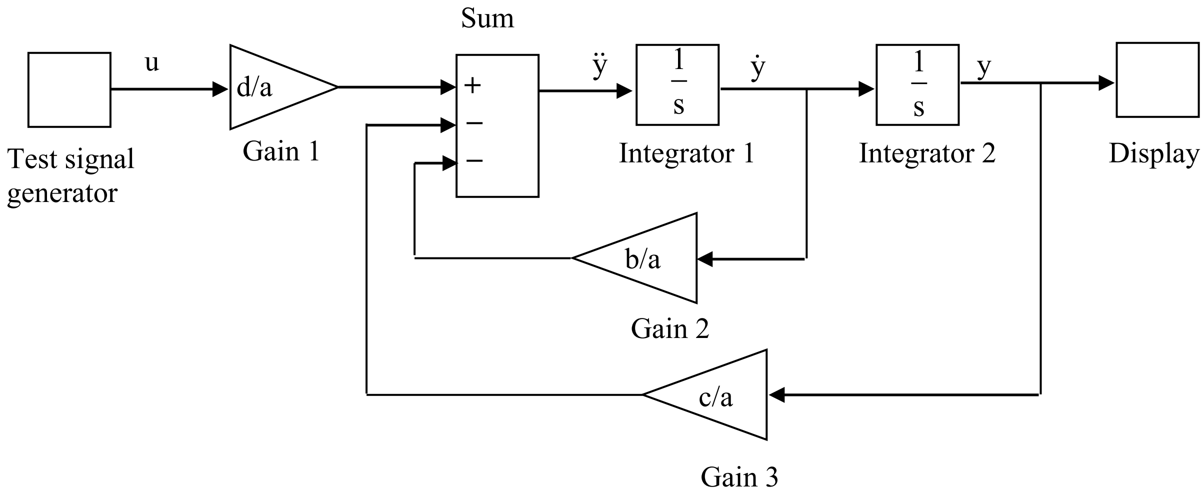


Figure 1 Block diagram of a second-order differential equation using integrators

2. Block Diagram Using Transfer Function Block: The above is be rewritten in the transfer function form:

$$G(s) = \frac{Y(s)}{U(s)} = \frac{d}{as^2 + bs + c} \tag{3}$$

or

$$G(s) = \frac{Y(s)}{U(s)} = \frac{b_0}{a_2s^2 + a_1s + a_0} \quad (b_0 = d,\ a_2 = a,\ a_1 = b,\ a_0 = c) \tag{4}$$

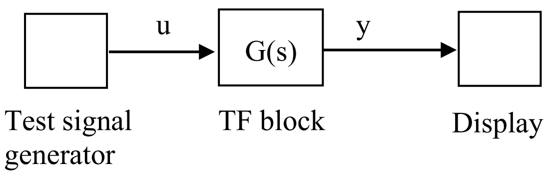


Figure 2 Block diagram of a second-order differential equation using TF block

3. Block Diagram Using State Space Model Block: State variables are defined as follows:

$$
\begin{aligned}
x_1 &= y \\
x_2 &= \dot{y} \\
\mathbf{u} &= u
\end{aligned}
\qquad (5)
$$

Therefore, the above equation is rewritten as:

$$\dot{x}_1 = \underbrace{x_2}_{f_1} \qquad (6)$$

$$\ddot{y} = \underbrace{-\frac{c}{a}y - \frac{b}{a}\dot{y} + \frac{d}{a}u}_{f_2} \qquad (7)$$

The state space model is

$$\underbrace{\begin{bmatrix} \dot{x}_1 \\ x_2 \end{bmatrix}}_{\dot{\mathbf{x}}} = \underbrace{\begin{bmatrix} 0 & 1 \\ -\frac{c}{a} & -\frac{b}{a} \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{\mathbf{x}} + \underbrace{\begin{bmatrix} 0 \\ \frac{d}{a} \end{bmatrix}}_{\mathbf{B}} \underbrace{[u]}_{\mathbf{u}} \qquad (8)$$

$$\underbrace{y}_{\mathbf{y}} = \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{\mathbf{C}} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{\mathbf{x}} + \underbrace{\begin{bmatrix} 0 \\ 0 \end{bmatrix}}_{\mathbf{D}} \underbrace{[u]}_{\mathbf{u}} \qquad (9)$$

A block diagram for the state space model is shown in **Fig. 3**.

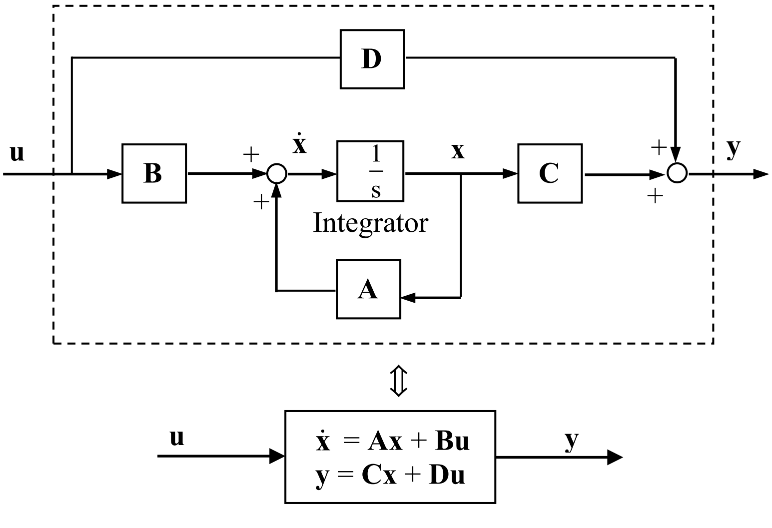


Figure 3 Linear time-invariant state space model

2. Hands-on Exercises

2.1 Exercise 1 Zero-order systems (ideal systems) Using Gain/Slider Gain block

A zero-order system is expressed by the following equation:

$$y(t) = Ku(t) \qquad (10)$$

where K is gain, y(t) is output and u(t) the input. Make a Simulink model to simulate (10).

SOLUTION

- Open a Simulink model and save as "M9_Exercise1_Zeroorder_Ver1.mdl"

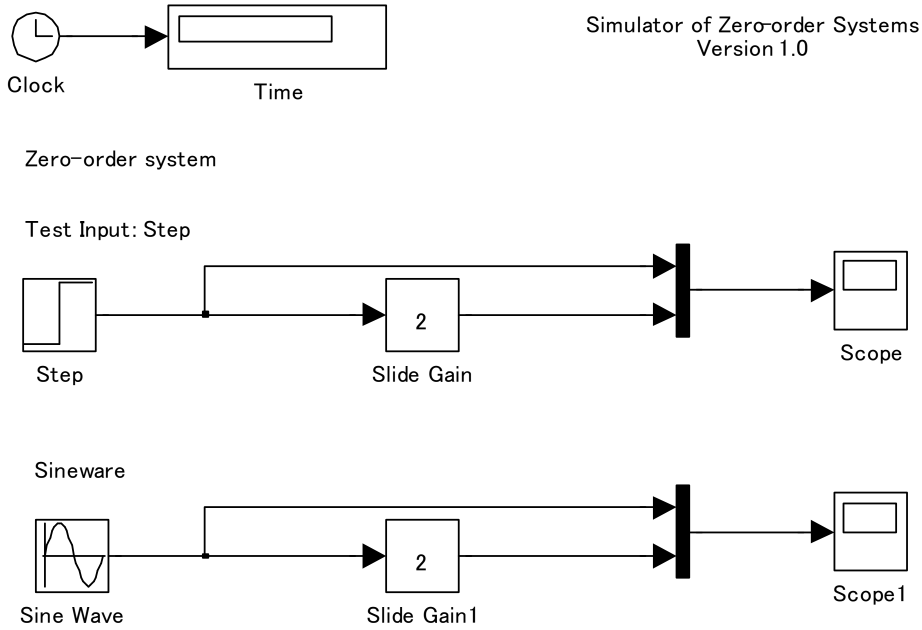
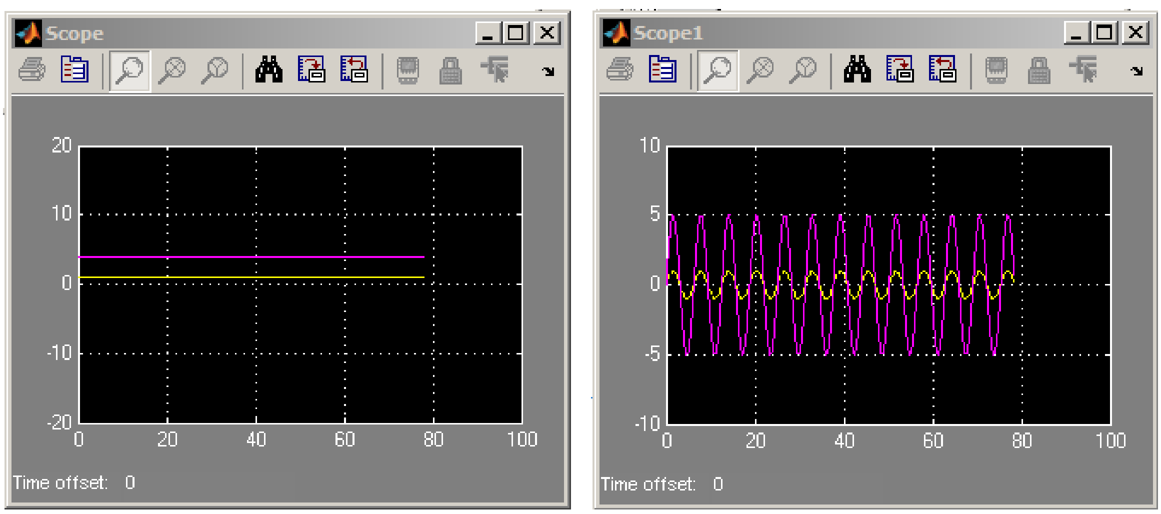


Figure 4 Simulink models for zero-order systems

The resulting Scopes for K = 4 (step function) and K = 5 (sine wave input) are shown in **Fig. 5**.



(a) Step function response (b) Frequency response

Figure 5 Zero-order system responses

2.1 Exercise 2 First-order Systems (Using Integrator Block)

State of Problem: A storage tank system is shown in **Fig. 6**. By applying the mass balance principle in Fluid Mechanics the relationship between the level (h) and the inlet flow rate (m^3/min) is derived as follows:

$$A\dot{h} + K_v\sqrt{h} = q_{in} \tag{11}$$

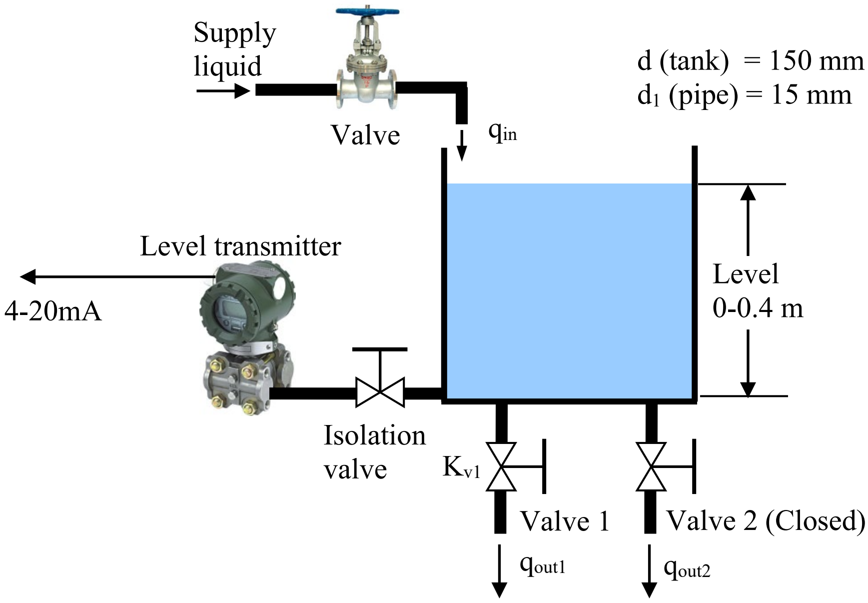


Figure 6 Tank level system

Equation (2) is a non-linear differential equation. It is hard to solve ODEs with analytical methods. With the aid of computer and software a non-linear differential equation can be solved using numerical integration methods. Make a simulation program with Simulink to solve Equation (2). Use the following numerical values: $K_{v1} = 0.000187$, d = 150 mm, level in range of 0 to 400 mm, q_{in} in range of 0 to 0.0071 m^3/min. In the Simulink model, do the following tasks:

- Convert the flow rate q_{in} from m^3/min to m^3/s
- Convert the level h from m to mm
- Set low alarm limit (20 mm) and high alarm limit (380 mm) for the level
- Simulate a level transmitter providing a level signal in range of 4 to 20 mm.

SOLUTION

Block Diagram Algorithm

Using the SI units the tank level system is represented by the block in **Fig. 7**.

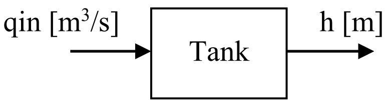


Figure 7 Block diagram of the tank level system (open-loop system)

Equation (11) for the tank level system can be rewritten as follows for a Simulink program:

$$\dot{h} = \frac{1}{A}\left(-K_{v1}\sqrt{h} + q_{in}\right) \qquad (12)$$

In order to solve equation (12) we develop a block diagram algorithm below:

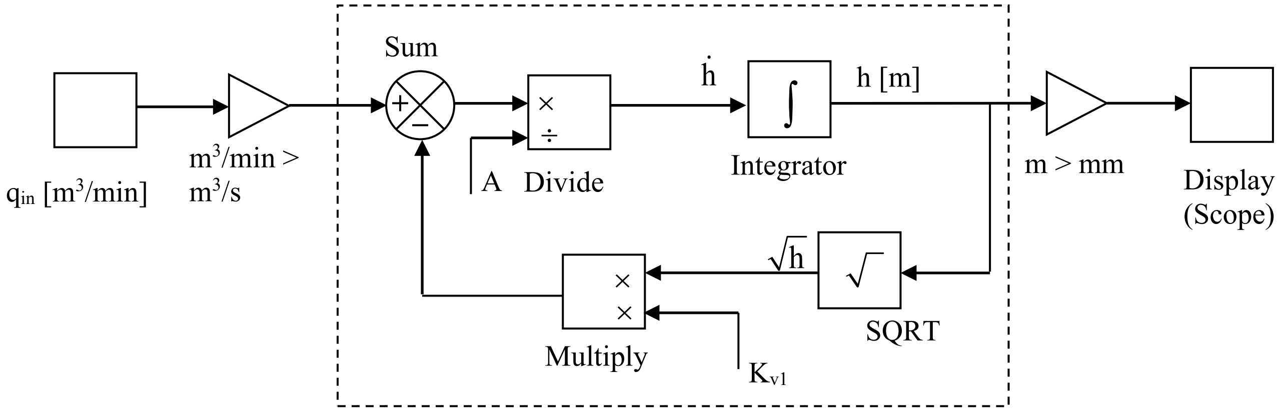


Figure 8 Block diagram algorithm for solving a first-order differential equation with Simulink

The cross-sectional of the tank is calculated by the following equation:

$$A = \pi\frac{d^2}{4} \ [m] \qquad (13)$$

Programming

- Open a Simulink model and save as "M9_Ex2_FirstOrder_V1.mdl".
- Using the above block diagram and make a Simulink model for solving equation (12). The Simulink model may look like that in **Fig 9**.

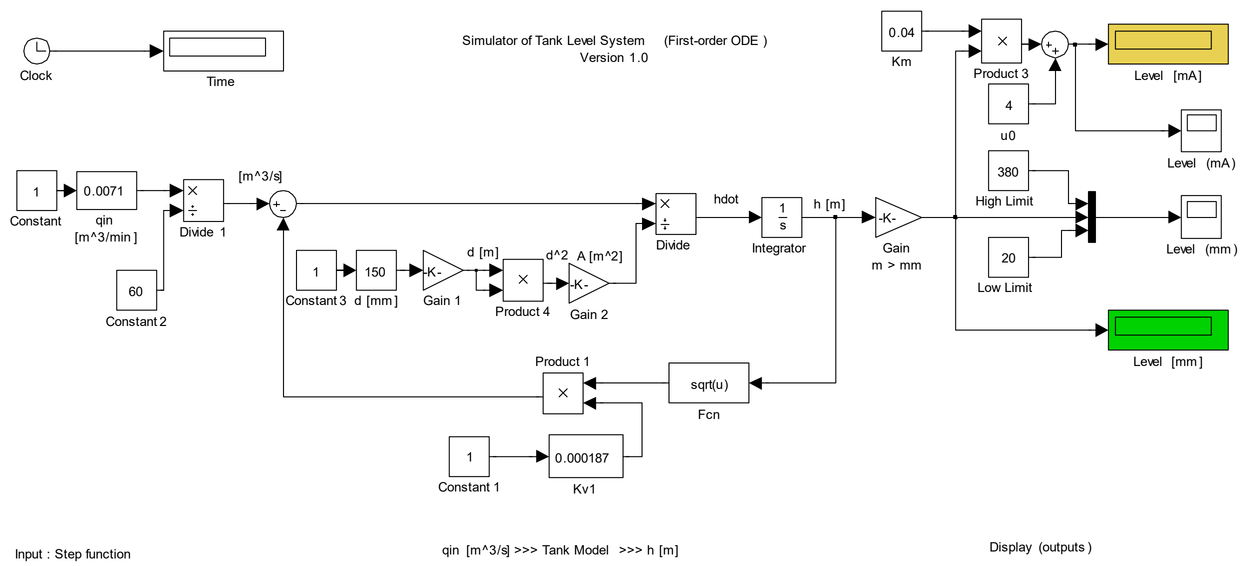


Figure 9 Simulink model to solve a first-order ODE

- Save the model.

System Parameters

- Set all system parameters for the model.
- Set initial conditions for the Integrator block by double-clicking it and set:
  - h(0) = 0, check Limit output (upper: 0.4 m, lower: 0.0)

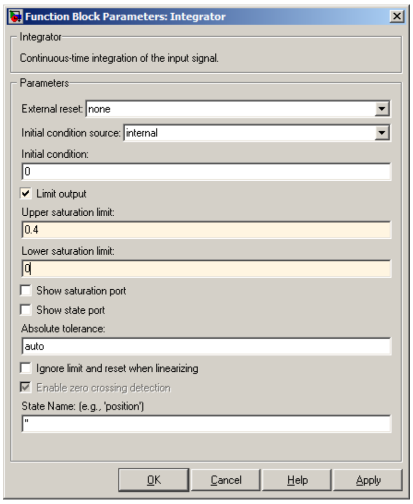


Figure 10 Set initial conditions for the Integrator block

- Save the model. The resulting program looks like...

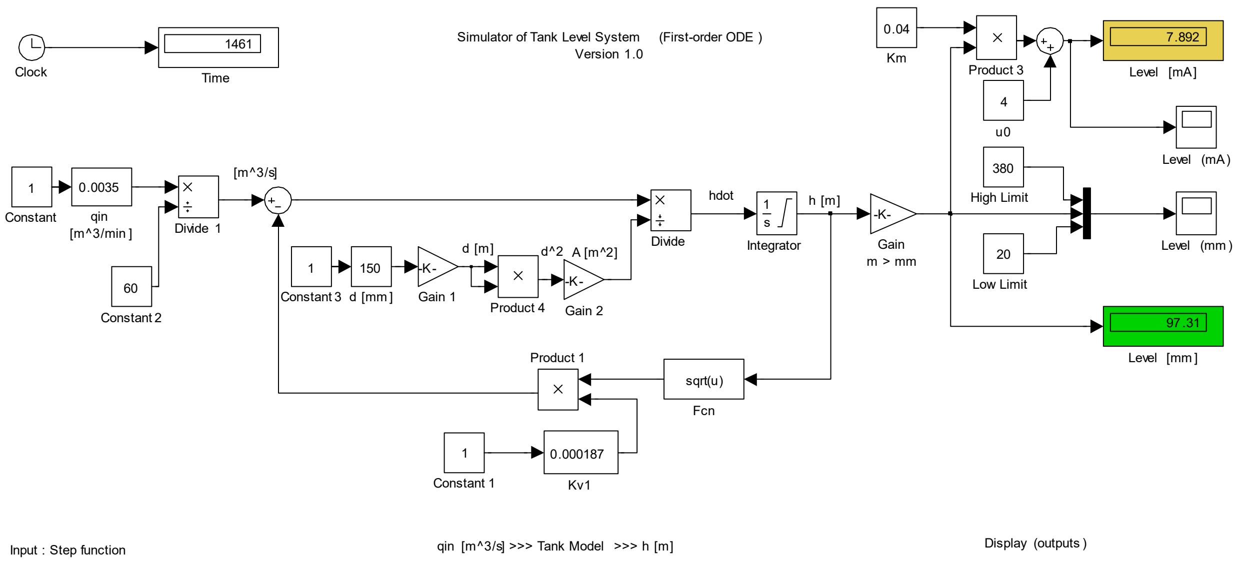


Figure 11 Simulink model after setting the Integrator block

Simulation Parameters

- Set the simulation parameters as in the following figure
- Click the OK button.

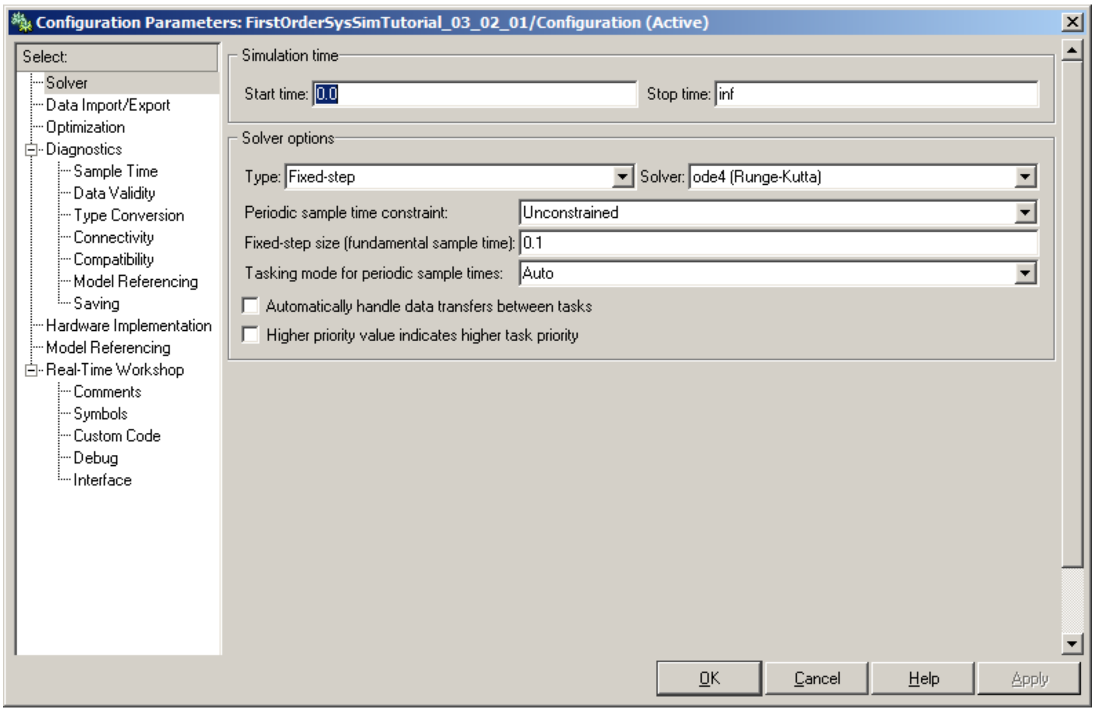


Figure 12 Configuration parameters

- Save the model.
- Run the model and test its functionality. The results (for q_{in} = 0.0035 m^3/min) are shown in **Fig. 13** and **Fig. 14**.

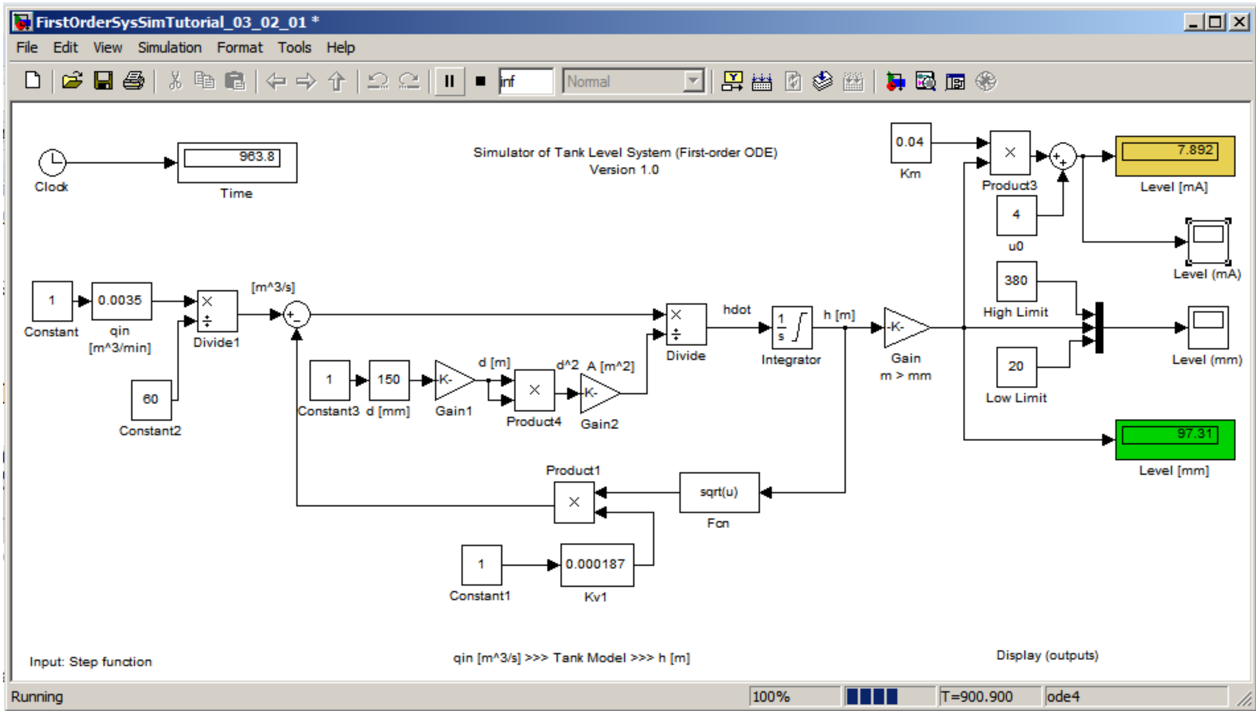


Figure 13 The Simulink model is running for q_{in} = 0.0035 m^3/min

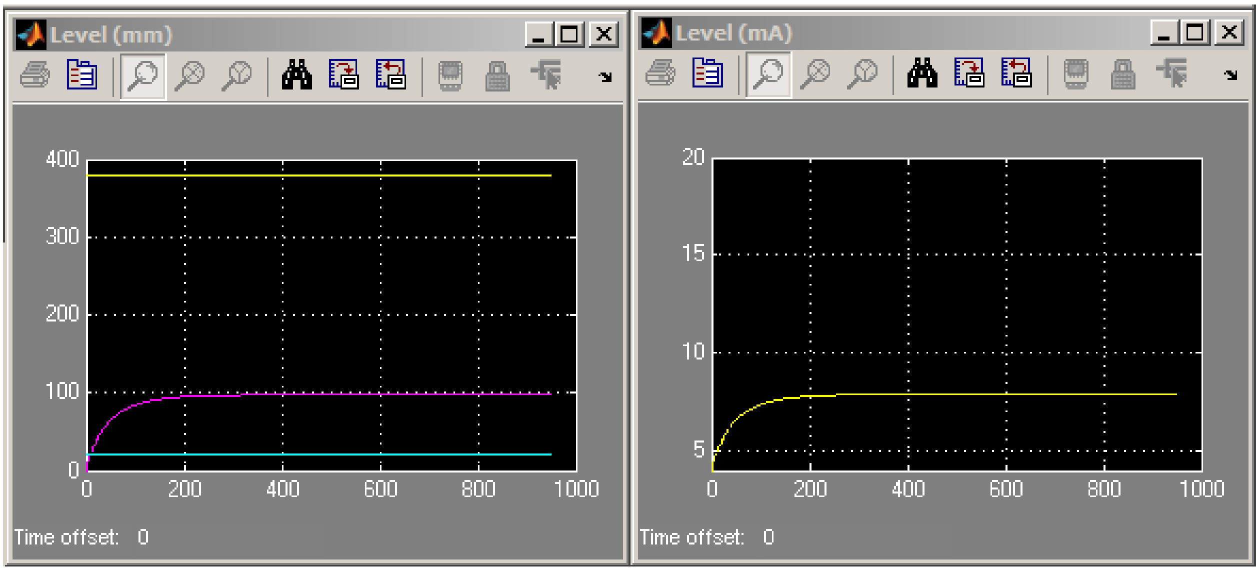


Figure 14 Resulting Scopes (Level in mm and mA)

Create a Subsystem (for cross-sectional area calculating algorithm)

- Select blocks as in **Fig. 15**.
- Edit menu > Create Subsystem (or Right-click the selected blocks > Create Subsystem).

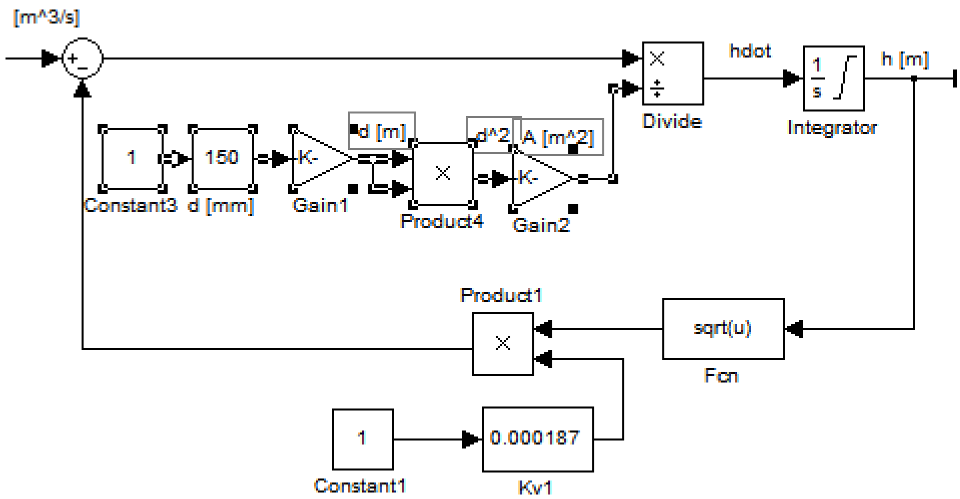


Figure 15 Selecting blocks

The resulting program looks like...

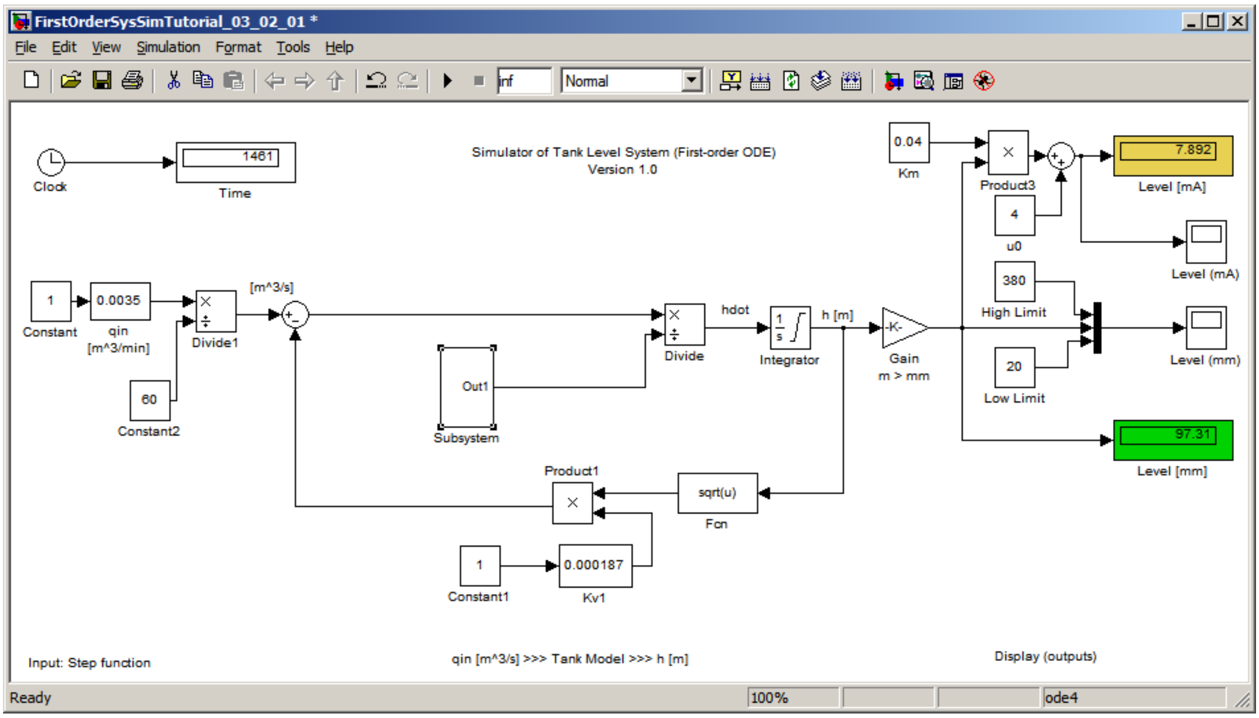


Figure 16 Created a Subsystem

- Change Subsystem to A, double click the A block, change the output port (Out1) to A as the figure below.

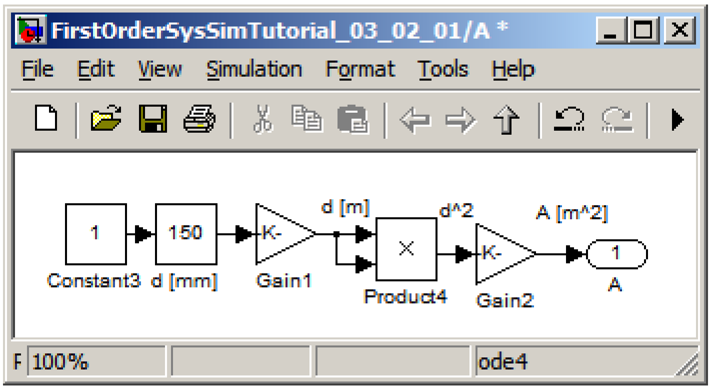


Figure 17 Subsystem A for cross-sectional area

- Save the model as "FirstOrderSysSimTutorial_02_02_02.mdl".
- Run the Simulink model and test its functionality. The same results should be obtained.

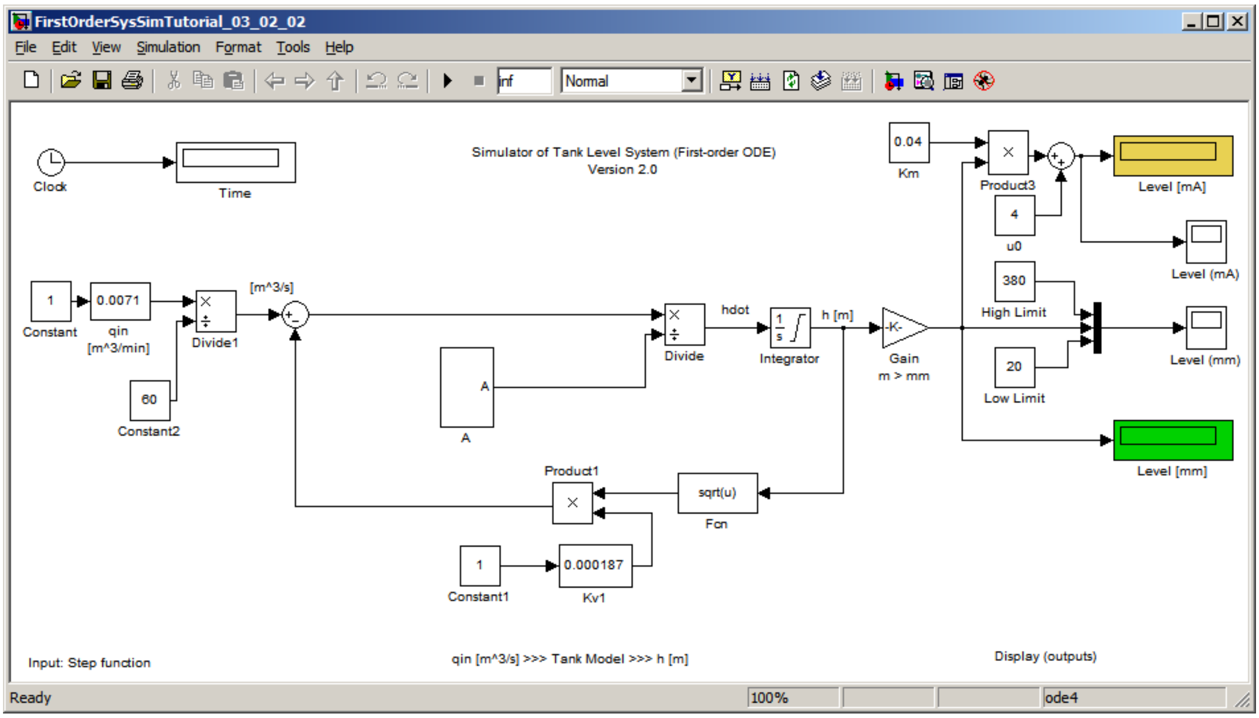


Figure 18 Simulink model with a Subsystem (A)

2.3 Exercise 3 Closed-loop Control System (Tank Level System)

Simulate the closed loop control system as shown in **Fig. 19** and **20** using the simulation program in **Exercise 2**.

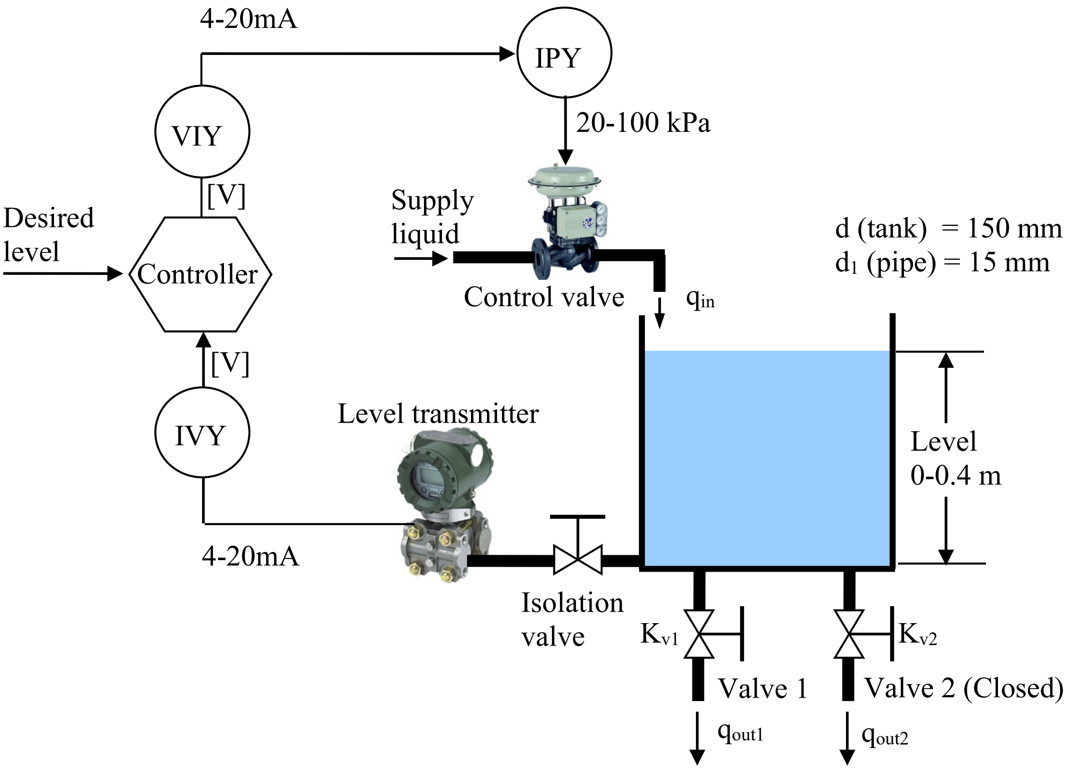


Figure 19 Tank level control system (closed-loop system)

Fig. 20 shows the block diagram for the closed-loop tank level control system with a PID controller (see the sample Simulink model including IVY, VIY and limiters).

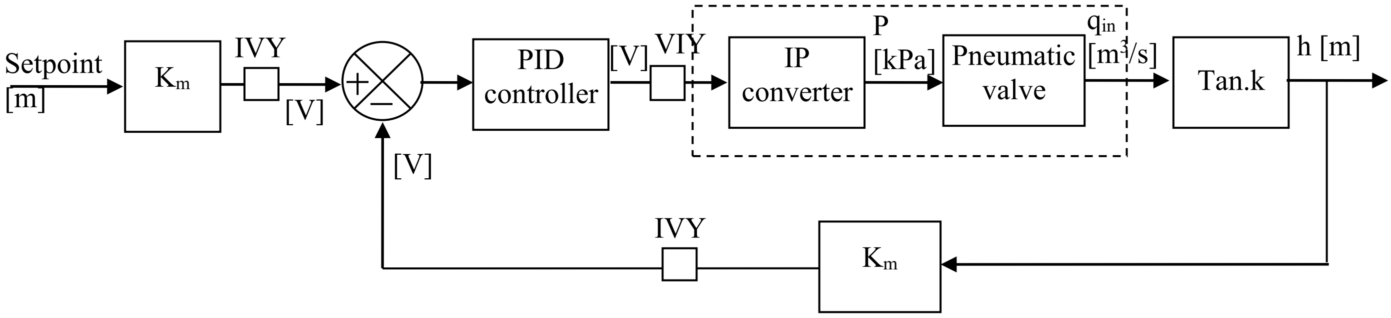


Figure 20 Block diagram of the closed tank level control system (see sample Simulink models for open-loop tank system and closed-loop tank system)

2.4 Exercise 4 Simulation of Second-order Systems (Open-Loop Systems)

Background

A mass-spring-damper system is shown in **Fig. 21** in which the input force u is in N, m is mass [kg], k is spring stiffness, and b is viscous damping coefficient, y is the displacement of the mass. Use these numerical values: m = 20 kg, k = 2.0 N/m, and b = 4.0 N/(m/s) and u = -8/+8 N.

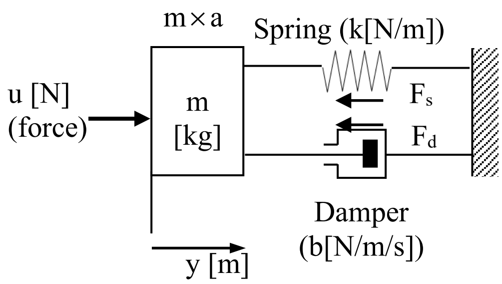


Figure 21 Mass-spring-damper system (typical mechanical system)

The mass-spring-damper is expressed by a second-order ODE below:

$$m\ddot{y} + b\dot{y} + ky = u \tag{14}$$

Equation (1) can be rewritten as follows:

$$\ddot{y} = \frac{1}{m}\left(-ky - b\dot{y} + u\right) \tag{15}$$

A block diagram algorithm for Equation (6) is shown in **Fig. 19**.

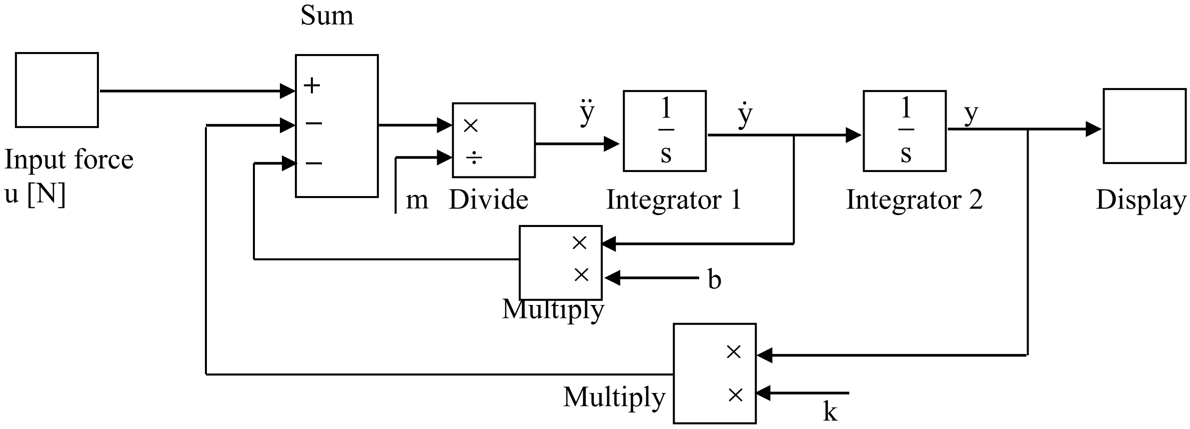


Figure 22 Block diagram of a second-order differential equation using integrators

Programming

- Open a Simulink model and save as “M9_Ex4_SecondOrder_V1.mdl”.
- Add necessary block and wire them as in the following figure.

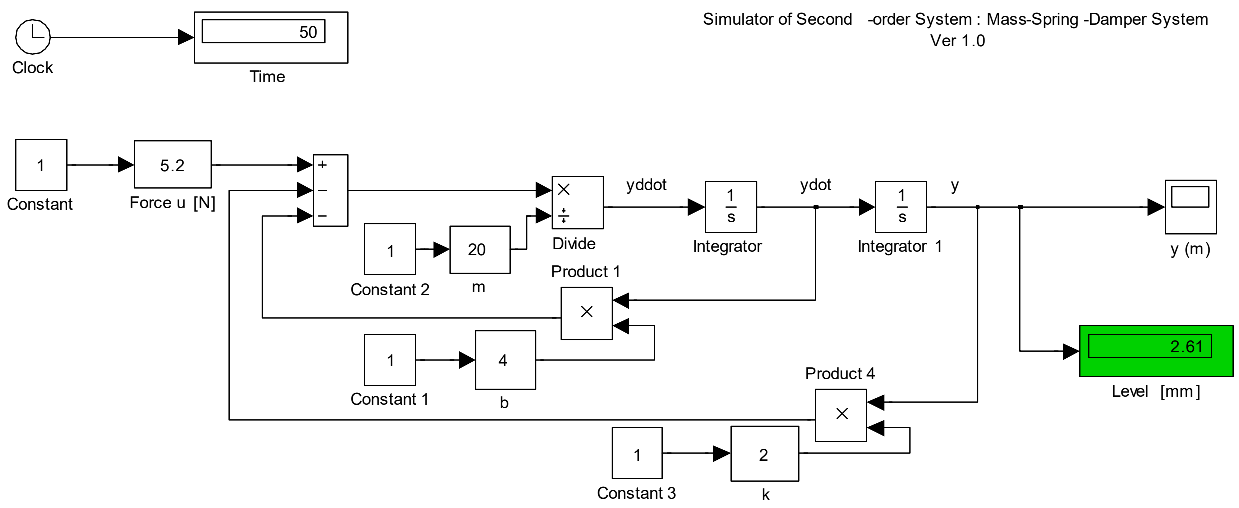


Figure 23 Simulink model for the second-order system (open-loop system)

System Parameters

U = 5.2 [N], m = 20 [kg], b = 4 [N/(m/s)] and k = 2 [N/m]

Initial Conditions

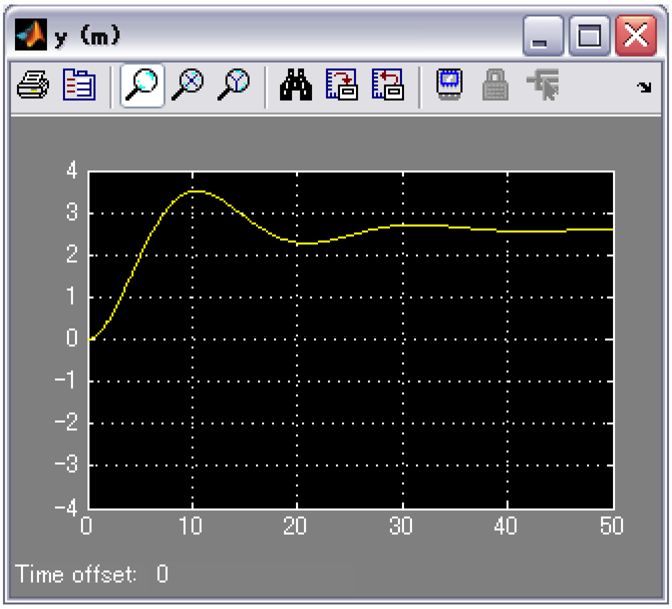
- Double click Integrator, ydot(0) = 0
- Double click Integrator 1, y(0) = 0.
- Save the program.

Simulation Parameters

- Fixed step: 0.1
- Run the simulation program.

The result (**Fig. 21**) is obtained.

You can test functionality by changing values of input force u.



- Save the program (if it is not).

Figure 24 Scope of y [m]

2.5 Exercise 5 Principal Mechanical Vibration Control System

The following figure shows a principal vibration control system that consists of:

- Process/plant: mass spring damper system (input force u [N] and displacement y [m] (see Exercise 4);
- Sensor: displacement y [m] and measured displacement y_m [V]

- Controller/comparator: setpoint [m] > [V], measured displacement [V], and control signal uc [V];
- Actuator (electro-hydraulic machine): control signal uc [V], manipulated input u [N].

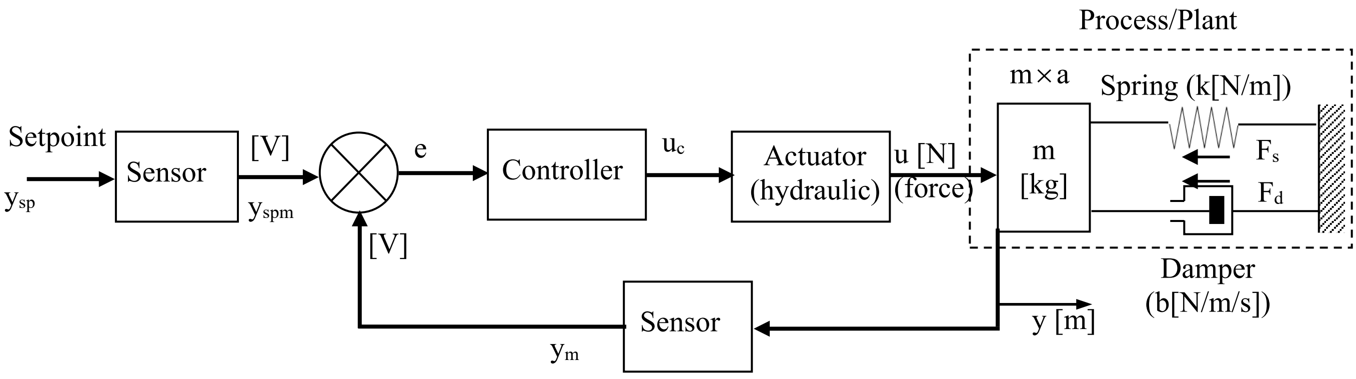


Figure 25 Principal vibration control system

Solution
Equations:

Plant: $m\ddot{y} + b\dot{y} + ky = u$ $(m = 20, b = 4, k = 2)$ or $G(s) = \dfrac{Y(s)}{U(s)} = \dfrac{1}{ms^2 + bs + k}$

Sensor: $K_m = \dfrac{dy_m}{dy}$

Controller (PID): $e = y_{spm} - y_m$

$$G_{PID} = K_P + \frac{K_I}{s} + K_D s$$

Actuator: $K_a = \dfrac{du}{du_c}$

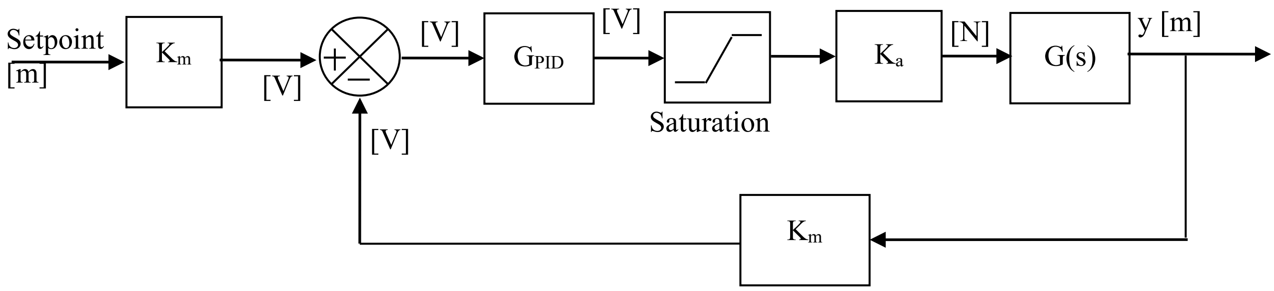


Figure 26 Block diagram of the vibration control system

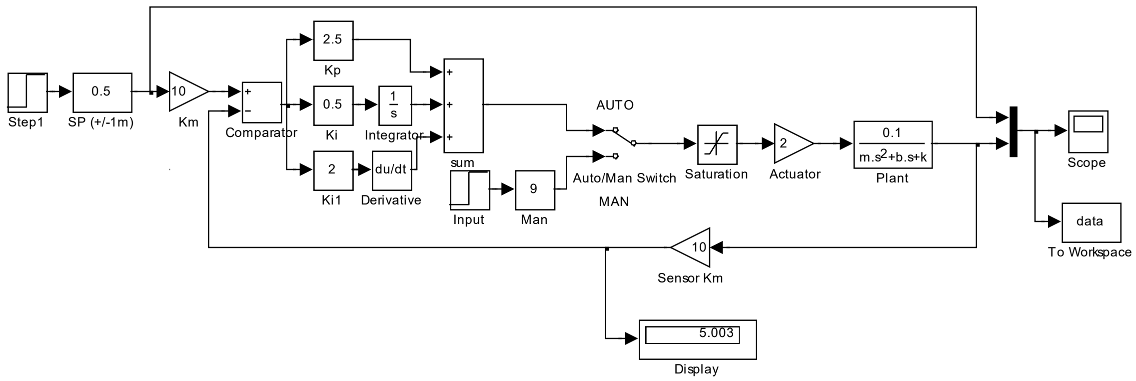


Figure 27 Simulink model for the vibration control system (see the sample Simulink model for this exercise)

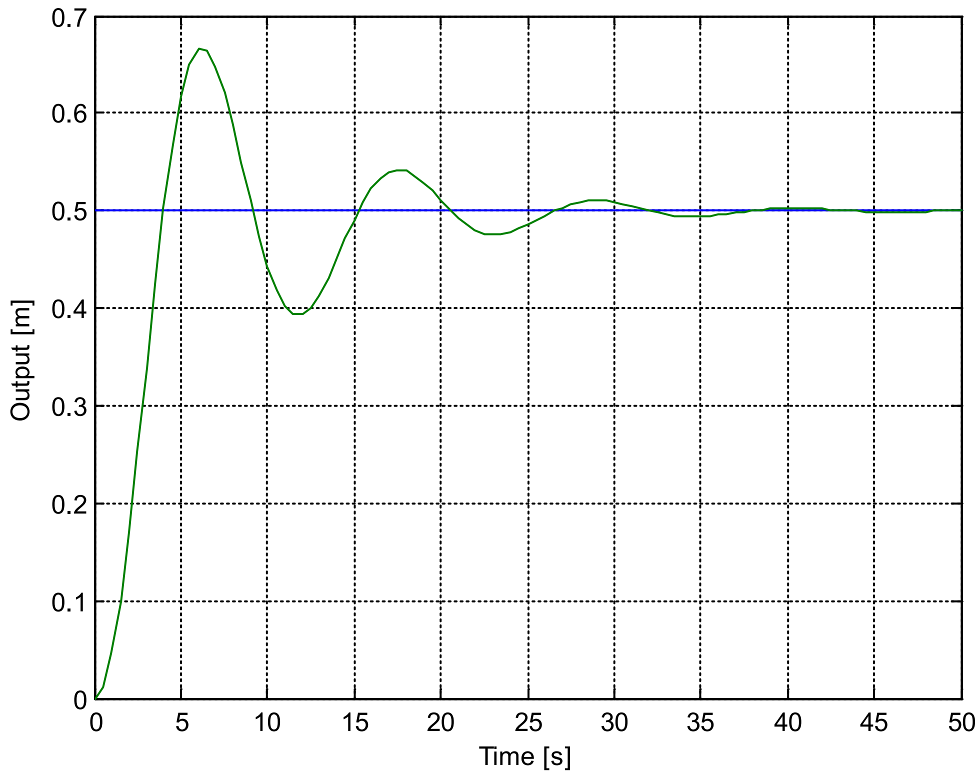


Figure 28 Setpoint and response

3. Conclusions

At this point, the following LOs have been met:

- Explain dynamics of zero-order systems;
- Explain dynamics of first-order systems and second-order systems;
- Solve first-order ODEs with Simulink and second-order ODEs; closed-loop control system; and
- Create a subsystem.

Follow-Up Activities

Activity 1 RC Network

Make a Simulink model to simulate the following RC Circuit expressed by the first-order differential equation:

$$RC\dot{e}_o + e_o = e_i$$

where e_o and e_i are output and input voltages, R and C are resistance and capacitance, respectively. This equation is rewritten in a standard first-order ODE below:

$$T\dot{e}_o + e_o = Ke_i$$

where T (= RC) is *time constant* (seconds), and K (=1) is *sensitivity* (*gain*, V/V).

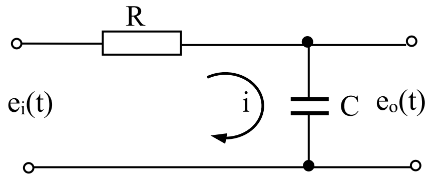


Figure 29 An RC network

Use these numerical values: R = 10 kΩ ; C = 10^{-4} μF ; $e_i(t) = A_a\sin(2\pi f_a) + A_b\sin(2\pi f_b) + V_i$

Note: Use may compare results with those of the following RC circuit simulator:
http://techteach.no/simview/rc_circuit/index.php

Activity 2 RLC Circuit

The following RLC network is represented by a second-order ODE below:

$$LC\ddot{e}_o + RC\dot{e}_o + e_o = e_i$$

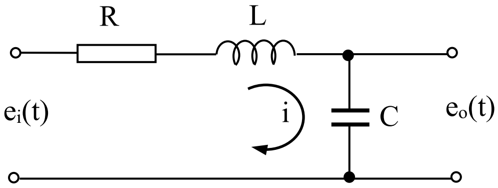


Figure 30 An RLC network

- Make a Simulink model to simulate the network. Use these numerical values: R = 100 Ω, L = 0.2 H and C = 2000 μF, $e_i(t)$ = 5 V.
- Modify the above program for a closed-loop control system with a PID controller to maintain the output voltage at a desired value or a predefined waveform.

<u>**Hints**</u>: You may want to use the Transfer Function block or State Space Model block

Activity 3 Rudder Steering Marine

Make Simulink program to simulate the following steering machine (T_{rud} = 11.9, a = 1)

$$\dot{\delta} = \frac{\delta_c - \delta}{|\delta_c - \delta| T_{rud} + a}$$

Activity 4 Storage Tank with a DC Motor Driven Valve

Let's recall the tank level system with motor-driven valve in Module 1. Use the following numerical values: $K_m = 10$, $J = 6E\text{-}3\ kgm^2$, and equations in Module 1. Make Simulink program to simulate this system.

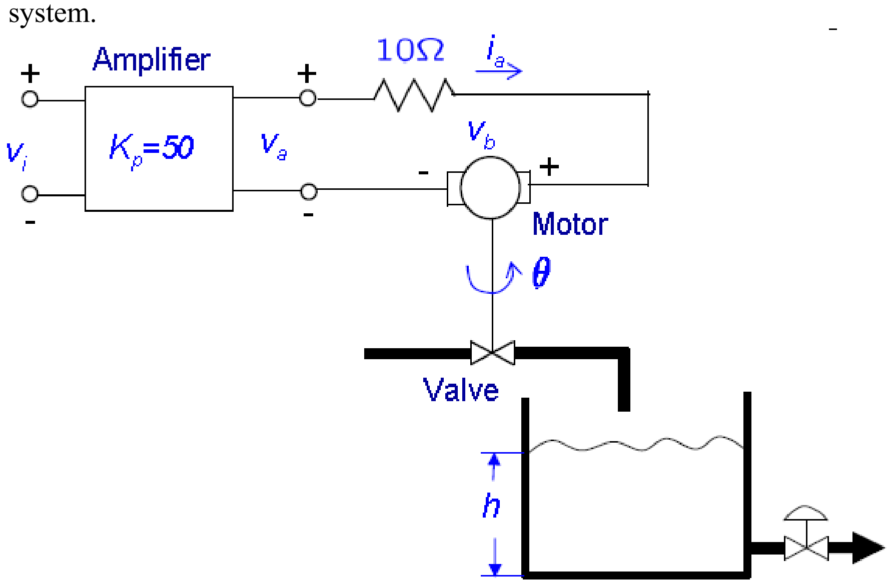


Figure 31 Level control system (as in marine boiler)

Activity 5 Two Tank System

Let's recall the system in Activity 1 Module 1 and draw a block diagram to solve differential equations for the two tank system.

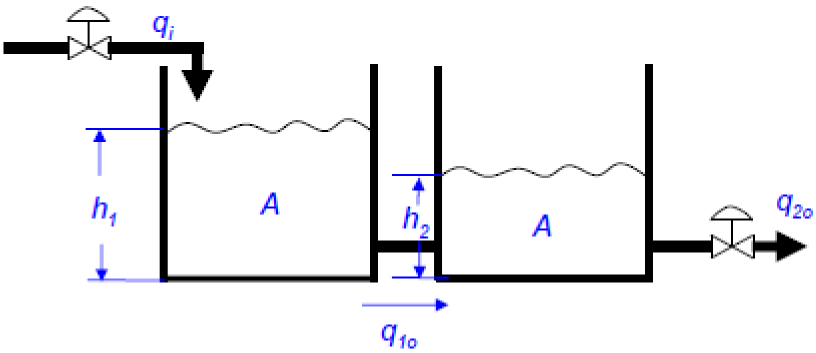


Figure 32 Two tank system