Introduction of turbulence modeling

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1 Turbulence modelling

1.1 Introduction

The basic brief introduction of Turbulence modeling is as follows:

Define the Reynolds stresses in terms on known (averaged) quantities

1) **Boussinesq hypothesis**

- simple relationship between Reynolds stresses and velocity gradients through the eddy viscosity (similar to molecular viscosity)

- isotropic (eddy viscosity is a scalar!)

2) **Reynolds stress transport models**

- equations derived directly manipulating the NS equations

- still contain unknown (undetermined) quantities

- no assumption of isotropy

- very complicated and expensive to solve

3) **Non-linear Eddy viscosity models (Algebraic Reynolds stress)**

4) **Model directly the divergence of the Reynolds Stresses**

1.2 Eddy viscosity models

Reynolds stresses,R_{ij} is define in terms of known (averaged) quantities by Boussinesq relationship Wilcox (1993) :

$$R_{ij} = -\overline{u\prime_j u\prime_i} = 2\frac{\mu_t}{\rho}S_{ji} \tag{1.1}$$

where, $S_{ji} = \frac{1}{2}(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i})$

$$\frac{\partial U_i}{\partial t} + U_j\frac{\partial U_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + \frac{\partial}{\partial x_j}\left[\frac{(\mu+\mu_t)}{\rho}\frac{\partial U_i}{\partial x_j}\right] \tag{1.2}$$

where, μ_t is eddy viscosity (m^2/s) .

Guidelines for defining the eddy viscosity:

1) **Dimensional arguments**

- units are [m2/s]

- define 2 out of three scales: velocity, length, time

2) **Physical arguments**

- asymptotic analysis

- consistency with experimental findings

3) **Numerical arguments**

- simple and easy to compute

1.3 Classification of eddy viscosity models

The various models (about 200) are classified in terms of number of transport equations solved in addition to the RANS equations:

1)zero-equation/algebraic models:
Mixing Length, Cebeci-Smith, Baldwin-Lomax, etc
2)one-equation models:
Wolfstein, Baldwin-Barth, Spalart-Allmaras, k-model, etc
3)two-equation ($\kappa - \phi$) models:
$\kappa - \varepsilon$, $\kappa - \omega$,$\kappa - \tau$, $\kappa - L$, etc.
4)three-equation models:
$\kappa - \varepsilon - A$
5)four-equation models:
$v^2 - f$ -model

1.4 Two-equation model ($\kappa - \varepsilon$) for Turbulence modeling

In the present work we are considering $\kappa - \varepsilon$ model of turbulent Wilcox (1993). Reynolds averaged equation of mass and momentum is given as follows:

$$\rho \frac{\partial U_i}{\partial x_i} = 0 \tag{1.3}$$

$$\rho \frac{\partial U_i}{\partial t} + \rho \frac{\partial}{\partial x_j}(U_j U_i) + \overline{u\prime_j u\prime_i} = -\frac{\partial p}{\partial x_i} + \frac{\partial}{\partial x_i}(\tau_{ji}) \tag{1.4}$$

Incorporating contribution of fluctuating velocity contribution in stress term, the above equation can be written as

$$\rho \frac{\partial U_i}{\partial t} + \rho \frac{\partial}{\partial x_i}(U_j U_i) = -\frac{\partial p}{\partial x_i} + \frac{\partial}{\partial x_i}(2\mu(\frac{1}{2}(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}) - \rho \overline{u\prime_j u\prime_i}) \tag{1.5}$$

Reynolds Stress equation is given as

$$\frac{\partial \tau_{ij}}{\partial t} + U_k \frac{\partial \tau_{ij}}{\partial x_k} = -\tau_{ik} \frac{\partial U_j}{\partial x_k} - \tau_{jk} \frac{\partial U_i}{\partial x_k} + \epsilon_{ij} + \prod_{ij} + \frac{\partial}{\partial x_k}\left[\nu \frac{\partial \tau_{ij}}{\partial x_k} + C_{ijk}\right] \tag{1.6}$$

Where $\prod_{ij} = \overline{p\prime(\frac{\partial u\prime_i}{\partial x_j} + \frac{\partial u\prime_j}{\partial x_i})}$, $\epsilon_{ij} = 2\mu \overline{\frac{\partial u\prime_i}{\partial x_k} \frac{\partial u\prime_j}{\partial x_k}}$ and $C_{ijk} = \rho \overline{u\prime_i u\prime_j u\prime_k} + \overline{p\prime u_i}\delta_{jk} + \overline{p\prime u\prime_j}\delta_{ik}$
From above equation for incompressible flow$\prod_{ij}$=0 , the equation for k and ϵ is give as

$$\rho \frac{\partial k}{\partial t} + \rho U_j \frac{\partial k}{\partial x_i} = \tau_{ij} \frac{\partial U_i}{\partial x_j} - \rho \varepsilon + \frac{\partial}{\partial x_j}(\mu \frac{\partial k}{\partial x_j} - \frac{1}{2}\rho \overline{u\prime_i u\prime_i u\prime_k} - \overline{p\prime u\prime_j}) \tag{1.7}$$

The left hand side in above equation represents unsteady and convection term. The right hand side represents the production, dissipation, molar diffusion, rate of turbulence energy transport through fluid turbulence fluctuation respectively. The last term on this side represent pressure diffusion (due to turbulence) velocity and pressure fluctuation.

Stress tensor is given by Boussineq approximation as follows:

$$\tau_{ij} = 2\mu_T(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}) - \frac{2}{3}\rho k \delta_{ij} \tag{1.8}$$

The scalar quantity in turbulence is considered to be transported by gradient diffusion. In analogy to molecular transport process $\overline{-u_j \phi'} = \mu_T \partial \Phi / \partial x_j$. There is no straight forward analogy for pressure diffusion term. The pressure diffusion term is grouped with turbulence transport terms. The sum of turbulence transport terms is considered to be gradient-transport process [equation (1.9].

Table 1.1: The values of closure coefficients

C_μ	C_1	C_2	σ_k	σ_ϵ
0.09	1.44	1.92	1.0	1.3

$$\frac{1}{2}\rho\overline{u\prime_i u\prime_i u\prime_k} + \overline{p\prime u\prime_j} = -\frac{\mu_T}{\sigma_k}\partial k/\partial x_j \tag{1.9}$$

Substituting the above equation 1.9 in equation 1.7.

$$\rho\frac{\partial k}{\partial t} + \rho U_j\frac{\partial k}{\partial x_j} = \tau_{ij}\frac{\partial U_i}{\partial x_j} - \rho\epsilon + \frac{\partial}{\partial x_j}\left[(\mu + \mu_T/\sigma_k)\frac{\partial k}{\partial x_j}\right] \tag{1.10}$$

The equation for turbulence dissipation energy, ϵ is given by following equation

$$\rho\frac{\partial \epsilon}{\partial t} + \rho U_j\frac{\partial \epsilon}{\partial x_j} = C_{\epsilon 1}\frac{\epsilon}{k}\tau_{ij}\frac{\partial U_i}{\partial x_j} - C_{\epsilon 2}\rho\frac{\epsilon^2}{k} + \frac{\partial}{\partial x_j}\left[(\mu + \mu_T/\sigma_\epsilon)\frac{\partial \epsilon}{\partial x_j}\right] \tag{1.11}$$

Where, C_μ , C_1 , C_2 ,σ_k andσ_ϵ are closure coefficients. The values of closure coefficients Wilcox (1993)are given in given in Table 1.1.

1.5 Estimation of $\kappa - \varepsilon$

Shear stress of flow is through the annulus in term of outer radius, R. The shear stress at $r = R$ and $r = \lambda R$ are calculated as follows (Bird R. B (2006)).

Shear stress at $r = R$:

$$\tau_{R,z} = \frac{4\overline{u}}{R}\left(\frac{1 - a^2}{\frac{1-\lambda^4}{1-\lambda^2} - \frac{1-\lambda^2}{ln(1/\lambda)}}\right) \tag{1.12}$$

Shear stress at $r = \lambda R$:

$$\tau_{\lambda R,z} = \frac{4\overline{u}}{R}\left(\frac{\lambda - \frac{a^2}{\lambda}}{\frac{1-\lambda^4}{1-\lambda^2} - \frac{1-\lambda^2}{ln(1/\lambda)}}\right) \tag{1.13}$$

where,$a^2 = \frac{1-\lambda^2}{2ln(1/\lambda)}$

Knowing the mean velocity of flow $\overline{u}$, the shear stresses $(\tau_{\lambda R,z})_{inner-wall}$ and $(\tau_{R,z})_{outer-wall}$ can be calculated from equations 1.12 and 1.13 .

We have considered upper wall, $(\tau_{R,z})_{outer-wall} \approx \tau_w$ in present study. We have considered radius, $R = 25$ mm and constant ,$\lambda = 0.02$.

Friction velocity ($\overset{*}{U}$), turbulence length scale (l_s) and distance from the boundary (y) can be defined as follows (Wilcox (1993) and McCabe and Harriott (1993))

$$\overset{*}{U} = \sqrt{\frac{\tau_w}{\rho}} \tag{1.14}$$

Length scale, l_s

$$l_s = \frac{\nu}{\overset{*}{U}} \tag{1.15}$$

The distance (y) from the boundary can be calculated using friction velocity ($\overset{*}{U}$) and turbulence length scale (l_s) as follows

$$y = y^+ l_s \tag{1.16}$$

If $y^+ = 1$, the distance (y) from the boundary becomes $y = l_s$.

The constitutive relations for $\kappa - \varepsilon$ are given by equations 1.17 and 1.18 Wilcox (1993) and McCabe and Harriott (1993).

$$\kappa(y) = \frac{\overset{*}{U}^2}{C_\mu^{0.5}} \tag{1.17}$$

$$\varepsilon(y) = \frac{\overset{*}{U}^3}{Ky} \tag{1.18}$$

where, $K = 0.42$ is Karman constant, $C_\mu = 0.09$ is constant, $\kappa(y)$ turbulence kinetic energy (m^2/s^2) and $\varepsilon(y)$ turbulence dissipation energy (m^2/s^3).

Reference

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Wilcox, D.C., 1993. Turbulence Modeling for CFD. First Edition, DCW Industries.